Disjunctive Syllogism without *Ex falso*

Luiz Carlos Pereira, Edward Hermann Haeusler and Victor Nascimento

Abstract The relation between *ex falso* and *disjunctive syllogism*, or even the justification of *ex falso* based on disjunctive syllogism, is an old topic in the history of logic. This old topic reappears in contemporary logic since the introduction of *minimal logic* by Johansson. The disjunctive syllogism seems to be part of our general non-problematic inferential practices and superficially it does not seem to be related to or to depend on our acceptance of the frequently disputable *ex falso* rule. We know that the acceptance of the *ex falso* is a sufficient condition for the acceptance of the disjunctive syllogism, but the interesting question is: is the *ex falso* a necessary condition for the acceptance of the disjunctive syllogism? The aim of the present paper is to discuss some possible ways to define systems that combines the preservation of the disjunctive syllogism with the rejection of the *ex falso*.

1 Introduction

The relation between *ex falso* and *disjunctive syllogism*, or even the justification of *ex falso* based on disjunctive syllogism, is an old topic in the History of Logic. This old topic reappears in contemporary logic since the introduction of *minimal logic* by Johansson. The disjunctive syllogism seems to be part of our general non-problematic inferential practices and superficially it does not seem to be related to or to depend on our acceptance of the *ex falso* rule; on the other hand, the general validity of the *ex falso* has been subjected to dispute. We know that the acceptance of the *ex falso* is

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T. Piecha and K. F. Wehmeier (eds.), *Peter Schroeder-Heister on Proof-Theoretic Semantics*, Outstanding Contributions to Logic 29,
https://doi.org/10.1007/978-3-031-50981-0_6

a sufficient condition for the acceptance of the disjunctive syllogism, as the following simple derivation in an intuitionistic natural deduction system shows:

$$\dfrac{(A \vee B) \qquad [A]^1 \qquad \dfrac{\dfrac{[B]^2 \qquad \neg B}{\bot}\,\neg_{\mathrm{E}}}{A}\,\bot_i}{A}\,\vee_{\mathrm{E}}\ 1, 2$$

The interesting question is: is the *ex falso* a necessary condition for the acceptance of the disjunctive syllogism?

As it was said, the relation between *ex falso* and the disjunctive syllogism has a long history. A form of the disjunctive syllogism appears in Stoic Logic[1] as the fifth type of *undemonstrated* argument, "an argument which, having an exclusive disjunction and the contradictory of one of the disjuncts as premises, infers the other disjunct as its conclusion"[2]. Diogenes Laertius, in *Lives of Eminent Philosophers* (VII, 49), gives the following example:

Either it is day or it is night.
It is not night.
Therefore, it is day.

A medieval argument from the 12th century, attributed to William of Soissons, shows how to derive the *ex falso* from the disjunctive syllogism and other "non-problematic" rules. The argument in natural language is[3]:

> I wonder that certain men oppose the thesis that from a per se impossibility anything whatsoever follows . . .. For doesn't it follow that if Socrates is a man and not a man, then Socrates is a man, but if Socrates is a man, then Socrates is man or a stone. Therefore, if Socrates is a man and not a man, then Socrates is a man or a stone. But if Socrates is a man and Socrates is not a man, then Socrates is not a man. Therefore, if Socrates is a man and Socrates is not a man, then Socrates is a stone.

We can reconstruct this argument axiomatically[4] as:

1. $(A \wedge \neg A) \rightarrow A$ Conditional Simplification,
2. $(A \rightarrow (A \vee B))$ Conditional Addition,
3. $((A \wedge \neg A) \rightarrow (A \vee B))$ 1, 2, Transitivity,
4. $((A \wedge \neg A) \rightarrow \neg A)$ Conditional Simplification,
5. $((A \wedge \neg A) \rightarrow ((A \vee B) \wedge \neg A))$ 3, 4, Conditional Adjunction,
6. $(((A \vee B) \wedge \neg A) \rightarrow B)$ Conditional Disjunctive Syllogism,
7. $(A \wedge \neg A) \rightarrow B)$ 5, 6, Transitivity.

[1] It is worth noticing that, while stoic disjunction is exclusive, all the new systems examined in this paper use inclusive disjunctions. Even though changes are promoted in elimination rules, we are always allowed to use the standard rules of conjunction elimination and disjunction introduction to show that $A \wedge B \vdash A \vee B$.

[2] Benson Mates (1953), p. 73.

[3] See Martin (1986), p. 571.

[4] We prefer the axiomatic style here as it looks closer to the text.

This argument is considered a precursor to the argument known as *Lewis' argument*[5]:

(1) Assume $p \;\; \sim p$.

(2) $(1) \strictif p$

If p is true and p is false, then p is true.

(3) $(1) \strictif \sim p$

If p is true and p is false, then p is false.

(4) $(2) \,.\, \strictif \,.\, p \lor q$

If, by (2), p is true, then at least one of the two, p and q, is true.

$(3) \,.\, (4) \colon \strictif \,.\, q$

If, by (3), p is false, and by (4), at least one of the two, p and q, is true; then q must be true.

We can also easily show that the disjunctive syllogism axiom implies the *ex falso* theorem:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[B]^1}{(A \lor B)} \quad [\neg B]^2}{((A \lor B) \land \neg B)} \qquad (((A \lor B) \land \neg B) \to A)}{A}}{(B \to A)}\;1}{(\neg B \to (B \to A))}\;2$$

And the same result can be obtained if we add the *disjunctive-syllogism rule* (DS)

$$\dfrac{(A \lor B) \qquad \neg A}{B}\;\mathrm{DS}$$

to minimal logic[6]

$$\dfrac{\dfrac{\dfrac{\dfrac{[B]^1}{(A \lor B)} \qquad [\neg B]^2}{A}\;\mathrm{DS}}{(B \to A)}\;\to_{\mathrm{I}}\,1}{(\neg B \to (B \to A))}\;\to_{\mathrm{I}}\,2$$

which allows a simple reconstruction of Soissons' argument in natural deduction:

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[5] See Lewis and Langford (1959, p. 250).

[6] Rodolfo Ertola-Biraben called our attention to the fact that it is enough to add the following particular case $\mathrm{DS}_\neg$ of DS

$$\dfrac{(A \lor \neg A) \qquad \neg\neg A}{A}\;\mathrm{DS}_\neg$$

in order to obtain the full power of *ex-falso*.

$$\dfrac{\dfrac{\dfrac{(A \wedge \neg A)}{A}}{(A \vee B)} \qquad \dfrac{(A \wedge \neg A)}{\neg A}}{B}\ \text{DS}$$

It is interesting to observe that all these arguments and proofs are *not normal* in the proof-theoretical sense, as some occurrences of disjunctive formulas are both the conclusion of an introduction rule and the *major premise* of an application of the disjunctive syllogism, which has the shape of an elimination rule.

But are we really committed to the *ex falso* if we accept the disjunctive syllogism? Is it really necessary to resort to the *ex falso* in order to justify the *disjunctive syllogism*? Could we not try some sort of *admissibility argument* to justify the disjunctive syllogism?

2 An admissibility argument in minimal logic

An *admissibility* strategy was considered by Tim van der Molen in the paper "The Johansson/Heyting letters and the birth of minimal logic". In the very beginning of the paper we find the following interesting passage:

> The provability of Formula 4.41 $[(((A \wedge \neg A) \vee B) \rightarrow B)]$ in minimal logic is a desideratum because it stems from the disjunction property. The disjunction property is a property shared by all the usual intuitionistic formal systems. It states that if we can produce a proof of $(A \vee B)$, then we can also produce a proof of A or a proof of B. So, if $((A \wedge \neg A) \vee B)$ (the antecedent of 4.41) has been proved, then, by the disjunction property, $(A \wedge \neg A)$ is provable or B is. In a consistent system like minimal logic $(A \wedge \neg A)$ is not provable. Therefore, B (the consequent of 4.41) must be provable. This indicates that Formula 4.41 should hold in minimal logic. (van der Molen, 2016, p. 2)

The argument used by van der Molen has the form of an *admissibility argument*: in order to show that the rule

$$\frac{A_1 \quad \ldots \quad A_n}{B}$$

is admissible, we show that if $\vdash A_1, \ldots, \vdash A_n$, then $\vdash B$. If we try to apply this kind of *admissibility argument*[7] to the disjunctive syllogism we obtain:

[7] An alternative admissibility argument could be obtained by means of the so-called Dummett's fundamental assumption, according to which every proof (i.e., closed derivation) in intuitionistic logic — and *a fortiori* in minimal logic — can be reduced to a canonical proof (i.e., a closed derivation using an introduction rule in the last step). Both the disjunctive property and the consistency of minimal logic are immediate corollaries of applying the fundamental assumptions to $\vdash A \vee B$ and $\vdash \bot$ (due to the shape of the introduction rule for disjunction and to the absence of introduction rules for $\bot$). Then, since we must have either a proof of A or of B and (by assumption) we have a proof of $\neg A$, the fact that a proof of A could then lead us to a proof of $\bot$ and that it is impossible to obtain a proof of $\bot$ allows us to use a meta-application of the disjunctive syllogism and conclude $\vdash B$. (We would like to thank one of the anonymous reviewers for drawing our attention to this formulation of the admissibility argument).

1. Assume $\vdash (A \vee B)$ and $\vdash \neg A$;
2. By the disjunctive property, we have that $\vdash A$ or $\vdash B$;
3. But then, assuming that minimal logic is consistent, $\nvdash A$;
4. Thus, $\vdash B$.

Could then an intuitionist accept the disjunctive syllogism without accepting the general validity of *ex falso*? What would be Brouwer's own position concerning the *ex falso* and the disjunctive syllogism? According to van Atten, Brouwer would reject the irrestricted validity of the *ex falso*:

> In his dissertation from 1907, Brouwer gave an account of the hypothetical judgement that served him all his life. On that account, hypothetical judgements may in certain cases have false antecedents, but there is no justification of the general principle Ex Falso Sequitur Quodlibet. Neither is the familiar derivation of Ex Falso using the disjunctive syllogism acceptable on Brouwer's view of logic. A systematic conclusion, then, is that Brouwer's logic is a relevance logic. (van Atten, 2009, p. 123)

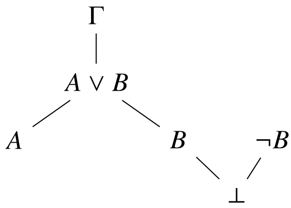
But again according to van Atten, a kind of "admissibility argument" in defense of the disjunctive syllogism can be attributed to Brouwer:

> The application of the disjunctive syllogism is not problematic either. For if $A \vee B$ is a description that applies to a mathematical construction, this means that we have a mathematical method that, when carried out, will show that the description A applies, or that the description B applies; a proof of $\neg B$ then simply tells us that the outcome of that method will be a proof of A. But then we also know that we would have obtained A as a description of the mathematical construction in question if no independent proof of $\neg B$ had been available to us. The disjunctive syllogism, then, accompanies the mathematical operation of leaving the construction described by A as is. (van Atten, 2009, p. 124)

From the way the argument is formulated, it is obvious that it seems circular: the last step of the argument is an explicit application of the very rule we are trying to justify, to wit, the *disjunctive syllogism*, even if only a *meta-application*! But is it necessary to understand this *meta*-application of disjunctive syllogism as dependent on a previous acceptance of ex falso? Let us consider the following scenario where we explore a comparison between our argument paths and trails we can follow in a promenade.

3 An informal account

Suppose that John is hiking in a forest and that at some point of the trail he finds a bifurcation point marked $A \vee B$. From this point, John could take the path marked by A or the path marked by B. But assume now that there is a sign (an extra information) $\neg B$ that indicates that the path B will lead to a dead-end (the $\perp$). In this case, the only path open to John is the path marked A. This situation can be graphically represented as:



The sign $\perp$ here indicates that John *can no longer go* along path B. In a certain sense, this scenario reminds an interesting passage in the third chapter of Brouwer's thesis where he says:

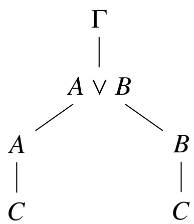
> 'But', the logician will retort, 'it might have happened that in the course of these reasonings a contradiction turned up between the newly deduced relations and those that had been kept in store. This contradiction, to be sure, will be observed as a logical figure, and this observation will be based upon the principium contradictionis.' To this I can reply: 'The words of your mathematical demonstration merely accompany a mathematical construction that is effected without words. At the point where you enounce the *contradiction*, I simply perceive that the construction no longer *goes*, that the required structure cannot be imbedded in the given basic structure.' And when I make this observation, I do no think of a principium contradictionis. (Brouwer, 1975, pp. 72–73)

The idea is that the traveller, as Brouwer says, *simply perceives that he can no longer consider path B, when he finds the contradiction*. It is true that the *ex falso* may be "secretly" used in the process, but our first impression is that this scenario would be acceptable to a *minimal logician*!

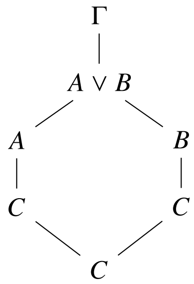
Just another point: this scenario also reminds Gentzen's interesting remark on the form of disjunction elimination. Gentzen says:

> In this example the tree form must appear somewhat artificial since it does not bring out the fact that it is after the enunciation of $X \vee (Y \vee Z)$ that we distinguish the cases X, Y and Z. (Gentzen's first example (1.1) on page 79 of Gentzen, 1969)

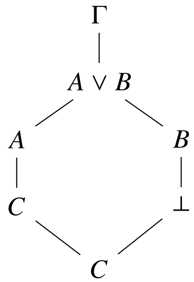
The point of this remark is that the fact that disjunction elimination has the form it has in the usual natural deduction systems is a kind of *artificial* adaptation to the general tree-form of derivations, but that, truly, the assumptions discharged come after the major premiss, as we do in some multiple-conclusion versions of natural deduction, as the following figure shows:



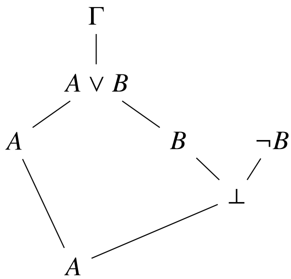
But if we consider a disjunction as a *branching point*, we need some kind of *synchronization mechanism* to bring paths together again!



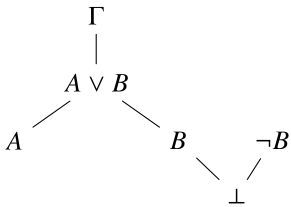
The point now is that the case of the disjunctive syllogism seems to require a new kind of *synchronization mechanism*, as the following figure shows:



Maybe this is just an extravagant idea, but now what seemed to be an application of a disputable rule, the *ex falso*, is just a kind of *synchronization mechanism* required by disjunction. This idea applied to the disjunctive syllogism yields:



But as we shall see, this representation is not free of problems. In a certain sense, this representation suggests that we could go from the *end-point* $\perp$ to A and this path may hide an application of *ex falso*. Maybe a more faithful representation would be



But this representation would inevitably leave us in the realm of multiple-conclusion systems.

4 The system $\mathsf{M}_\vee$

If we do not want to go *multiple-conclusion*, we could try to define a set of new disjunction eliminations ($\vee_\perp$-eliminations) as follows[8]:

1. $\vee_\perp$-elimination-1

$$\dfrac{\begin{matrix}\Gamma \\ \Pi \\ A \vee B\end{matrix} \quad \begin{matrix}[A]^m \\ \Pi_1 \\ C\end{matrix} \quad \begin{matrix}[B]^n \\ \Pi_2 \\ C\end{matrix}}{C}\; m,\, n$$

2. $\vee_\perp$-elimination-2

$$\dfrac{\begin{matrix}\Gamma \\ \Pi \\ A \vee B\end{matrix} \quad \begin{matrix}[A]^m \\ \Pi_1 \\ C\end{matrix} \quad \begin{matrix}[B]^n \\ \Pi_2 \\ \perp\end{matrix}}{C}\; m,\, n$$

3. $\vee_\perp$-elimination-3

$$\dfrac{\begin{matrix}\Gamma \\ \Pi \\ A \vee B\end{matrix} \quad \begin{matrix}[A]^m \\ \Pi_1 \\ \perp\end{matrix} \quad \begin{matrix}[B]^n \\ \Pi_2 \\ C\end{matrix}}{C}\; m,\, n$$

Let us the consider the natural deduction system $\mathsf{M}_\vee$ that is obtained from the propositional part of the Gentzen-Prawitz natural deduction system M for minimal logic through the replacement of the usual disjunction-elimination rule by this new set of disjunction eliminations ($\vee_\perp$-elimination-1, $\vee_\perp$-elimination-2 and $\vee_\perp$-elimination-3). It is clear that the system M is a proper subsystem of the new system $\mathsf{M}_\vee$ and that $\mathsf{M}_\vee$ is a subsystem of the propositional part of the Gentzen-Prawitz natural deduction system I for intuitionistic logic, but is $\mathsf{M}_\vee$ a proper subsystem of I? Is it possible to prove the full power of *ex falso*, to prove $(A \rightarrow (\neg A \rightarrow B)))$, in $\mathsf{M}_\vee$? If it is not possible, then the system $\mathsf{M}_\vee$ could be a good candidate to be an *intermediate* system, lying between the minimal system M and the intuitionistic system I. But consider now the following simple derivations:

$$\dfrac{\dfrac{\dfrac{\dfrac{[A]^2}{(A \vee B)} \quad \dfrac{[A]^3 \quad [\neg A]^1}{\perp} \quad [B]^4}{B}\;\vee\, 3,\, 4}{(\neg A \rightarrow B)}\;\rightarrow_{\mathrm{I}}\ 1}{(A \rightarrow (\neg A \rightarrow B))}\;\rightarrow_{\mathrm{I}}\ 2$$

[8] This modification of the disjunction-elimination rule was first proposed by Neil Tennant (1979).

$$\dfrac{\dfrac{\dfrac{[(B \wedge E)]^4}{B}}{(A \vee B)} \qquad \dfrac{\dfrac{[(A \vee B)]^3 \quad [A]^1 \quad \dfrac{[B]^2 \quad \neg B}{\bot}}{A}\,{\scriptstyle 1,\,2}}{((A \vee B) \to A)}\,{\scriptstyle 3}}{\dfrac{A}{(B \wedge E) \to A}\,{\scriptstyle 4}}$$

The first derivation is a correct non-normal proof of $(A \to (\neg A \to B))$ in $\mathsf{M}_\vee$ using the detour $A \vee B$; the second example is a correct derivation of $\{\neg B\} \vdash ((B \wedge E) \to A)$ using the *detour* $((A \vee B) \to A)$. How can we avoid these problematic derivations in $\mathsf{M}_\vee$?

The system $\mathsf{M}_\vee$ is clearly related to the intuitionistic relevant system IR defined by Tennant (1987). He recognizes that without further restrictions, the *ex falso* would be derivable, and he considers the following derivation:

$$\dfrac{\dfrac{A}{(A \vee B)}\,\vee_{I_1} \qquad \dfrac{[A]^1 \quad \neg A}{\bot}\,\neg\mathrm{E} \qquad [B]^2}{B}\,\vee_{\mathrm{E}}\ 1,\,2$$

As we saw, Brouwer would have nothing against the use of the disjunctive syllogism in the derivation above; his qualms would be related to the *composition of derivations*:

> The problem is rather with the *composition* of these two inferences. The first inference requires that the mathematical construction being described is one for $\bot$; the second that it is one for A. As in general A and $\bot$ will not be equivalent descriptions, there is no general guarantee that when $\bot$ describes a mathematical construction, A describes it as well. This means that there is no guarantee that the linguistic figures in Lewis' argument accompany a mathematical procedure. (van Atten, 2009, p. 124)

What are "these two inferences" to which van Atten is referring? The recognition that there is a *composition problem* and that some restriction on the composition of derivations is needed is exactly what Tennant does: in any application α of an elimination rule, the major premiss of α cannot be the conclusion of an introduction rule. The derivations

$$\dfrac{A}{(A \vee B)}$$

$$\dfrac{(A \vee B) \qquad \dfrac{[A]^1 \quad \neg A}{\bot} \qquad [B]^2}{B}\,{\scriptstyle 1,\,2}$$

are correct, but the derivation (the result of the *composition*)

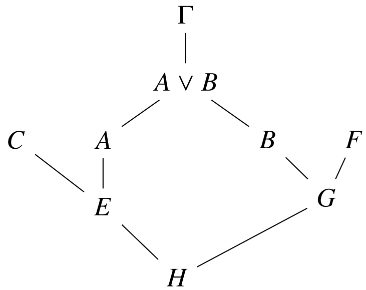
$$\dfrac{\dfrac{A}{(A \vee B)} \qquad \dfrac{[A]^1 \quad \neg A}{\bot} \qquad [B]^2}{B}\,{\scriptstyle 1,\,2}$$

is not.

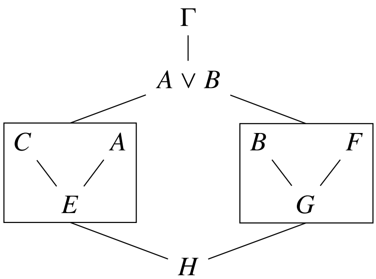
We can certainly use Tennant's idea (forgetting everything about *relevance*) and impose *normality* by stipulation: only *normal derivations* are accepted as legitimate derivations. Obviously the system $\mathsf{M}_\vee$ is Tennant's system IR without the *relevance*-restrictions. As in the case of IR, we can say that:

1. The propositional part of the natural deduction system M for minimal logic is a subsystem of $\mathsf{M}_\vee$ (M satisfies normalization);
2. The disjunctive syllogism is derivable in $\mathsf{M}_\vee$; and
3. $\mathsf{M}_\vee$ does not have the *ex falso*.

It is true one could say that to impose *normality* is a too high price to pay in order to preserve the disjunctive syllogism without preserving the *ex falso*. In order to have a better understanding of what is happening with *detours* of the form $(A \vee B)$ in the derivation above, let us go back to our initial scenario where John is hiking in a trail. Suppose that John is hiking with a map and that John arrives at the same bifurcation point marked $(A \vee B)$. Let us suppose that path A with some extra information C leads to point E and that path B together with extra-information F leads to a point G, and that from the points E and G we can go to point H. We could try to represent this situation with the following figure:



In the case of classical logic, if we are at point E, we can access point C, point F and all points in Γ: no *visibility/accessibility restrictions*. But in case of intuitionistic logic the situation is completely different: after the bifurcation point, *visibility/accessibility* restrictions are required. A more faithful representation of the situation is as follows:



According to this new picture, at point E we have only access to what is inside its box (and the same holds for point G with respect to its box). In the next section we define a new system whose aim is to incorporate these *visibility/accessibility* restrictions.

5 The system $\mathsf{M}_\bot$

Let the natural deduction system $\mathsf{M}_\bot$ be obtained from the propositional part of the Gentzen-Prawitz natural deduction system M for minimal logic by the replacement of the usual disjunction elimination by the following new set of disjunction-elimination rules:

1. $\vee_\bot$-elimination-1

$$\dfrac{\begin{array}{c}\Gamma\\ \Pi\\ A \vee B\end{array} \qquad \begin{array}{c}[A]^m\\ \Pi_1\\ C\end{array} \qquad \begin{array}{c}[B]^n\\ \Pi_2\\ C\end{array}}{C}\ \vee_{\bot\text{E-1}}\ m, n$$

2. $\vee_\bot$-elimination-2

$$\dfrac{\begin{array}{c}\Gamma^*\\ \Pi\\ A \vee B\end{array} \qquad \begin{array}{c}[A]^m\\ \Pi_1\\ C\end{array} \qquad \begin{array}{c}[B]^n \quad \Gamma_2^*\\ \Pi_2\\ \bot\end{array}}{C}\ \vee_{\bot\text{E-2}}\ m, n, \Gamma^*, \Gamma_2^*$$

3. $\vee_\bot$-elimination-3

$$\dfrac{\begin{array}{c}\Gamma^*\\ \Pi\\ A \vee B\end{array} \qquad \begin{array}{c}[A]^m \quad \Gamma_1^*\\ \Pi_1\\ \bot\end{array} \qquad \begin{array}{c}[B]^n\\ \Pi_2\\ C\end{array}}{C}\ \vee_{\bot\text{E-3}}\ m, n, \Gamma^*, \Gamma_1^*$$

The notation Γ^* and Γ_i^* $(i = 1, 2)$ indicates that the assumptions in Γ and in Γ_i $(i = 1, 2)$ are *frozen*, i.e., that they cannot be discharged below the conclusion of the application of the corresponding $\vee_\bot$-elimination.[9]

The non-normal derivation of $(A \rightarrow (\neg A \rightarrow B))$ obtained in $\mathsf{M}_\vee$ is clearly not a correct derivation in $\mathsf{M}_\bot$: the application of disjunction elimination that was used is an application α of $\vee$-elimination-2 and the restriction demanded by the new $\vee_\bot$-elimination-2 rule is not satisfied in α, since the hypothesis $\neg A$ is discharged below the conclusion of α. The restrictions on Γ forbid the second problematic example given above. Although these problematic cases are not theorems of $\mathsf{M}_\bot$, we still have a non-normal derivation of $\{\neg A^{*2}, A^{*1}\} \vdash B$!

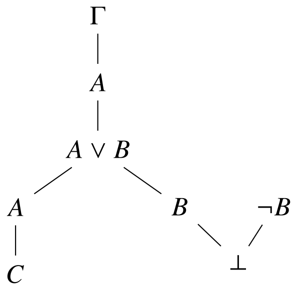
[9] The intuitionistic multiple succedent sequent calculus FIL defined in de Paiva and Pereira (2005) incorporates this idea by means of devices that control dependency relations between formulas in the antecedent and formulas in the succedent of a sequent.

$$\frac{\dfrac{A^{*1}}{(A \vee B)} \qquad \dfrac{[A]^1 \quad \neg A^{*2}}{\bot} \qquad [B]^2}{B} \ \vee_{\bot \text{E-3}} \ 1,\ 2,\ A^{*1},\ \neg A^{*2}$$

In order to avoid these problematic derivability relations, let us examine the normalization problem for $\mathsf{M}_\bot$.

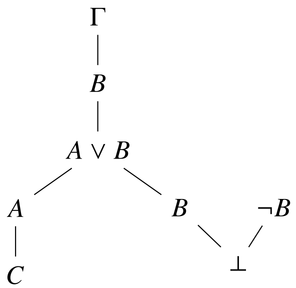
6 Normalization for $\mathsf{M}_\bot$

Let us assume now that our hiker when he arrives at the bifurcation point $(A \vee B)$ his map indicates that he should take path A. This situation can be pictured as:



After taking the path marked by A, John's *promenade* looks as follows:

If John had found the indication to take path B, we would have the following picture:



And after taking the path B, the situation is:

$$\begin{array}{ccc} \Gamma & & \\ | & & \\ B & & \neg B \\ & \searrow \; \swarrow & \\ & \bot & \end{array}$$

The new $\vee_\bot$ reductions corresponding to these figures are:

1. A derivation of the form

$$\dfrac{\dfrac{\begin{array}{c}\Gamma\\ \Pi\\ A\end{array}}{A \vee B} \quad \begin{array}{c}[A]^m\\ \Pi_1\\ C\end{array} \quad \begin{array}{c}[B]^n\\ \Pi_2\\ C_2\end{array}}{\begin{array}{c}C\\ \Pi_3\end{array}} \vee_{\bot\text{E-1}}\ m,\ n$$

where C_2 is either C or $\bot$, reduces (as usual) to

$$\begin{array}{c}\Gamma\\ \Pi\\ [A]\\ \Pi_1\\ [C]\\ \Pi_3\end{array}$$

2. A derivation of the form

$$\dfrac{\dfrac{\begin{array}{c}\Gamma\\ \Pi\\ B\end{array}}{A \vee B} \quad \begin{array}{c}[A]^m\\ \Pi_1\\ C_1\end{array} \quad \begin{array}{c}[B]^m\\ \Pi_2\\ C\end{array}}{\begin{array}{c}C\\ \Pi_3\end{array}} \vee_{\bot\text{E-1}}\ m,\ n$$

where C_1 is either C or $\bot$, reduces (as usual) to

$$\begin{array}{c}\Gamma\\ \Pi\\ [B]\\ \Pi_2\\ [C]\\ \Pi_3\end{array}$$

3. A derivation of the form

$$\dfrac{\dfrac{\begin{array}{c}\Gamma^*\\ \Pi\\ B\end{array}}{A \vee B} \quad \begin{array}{c}[A]^m\\ \Pi_1\\ C\end{array} \quad \begin{array}{c}[B]^n\\ \Pi_2\\ \bot\end{array}}{\begin{array}{c}C\\ \Pi_3\end{array}} \vee_{\bot\text{E-2}}\ m,\ n,\ \Gamma^* \qquad \text{reduces to} \qquad \begin{array}{c}\Gamma^*\\ \Pi\\ [B]\\ \Pi_2\\ \bot\end{array}$$

4. A derivation of the form

$$\dfrac{\dfrac{\begin{matrix}\Gamma^*\\ \Pi\\ A\end{matrix}}{A \vee B} \quad \begin{matrix}[A]^m\\ \Pi_1\\ \bot\end{matrix} \quad \begin{matrix}[B]^n\\ \Pi_2\\ C\end{matrix}}{\begin{matrix}C\\ \Pi_3\end{matrix}} \vee_{\bot\text{E-3}}\ m, n, \Gamma^* \qquad \text{reduces to} \qquad \begin{matrix}\Gamma^*\\ \Pi\\ [A]\\ \Pi_1\\ \bot\end{matrix}$$

We could use here the strategy we used with respect to $\mathsf{M}_\vee$ and impose *normality* by stipulation: only *normal derivations* are accepted as legitimate derivations.

But we can also use a new strategy that was used by Prawitz in *Natural Deduction*. In the appendix A, on set theory, Prawitz introduces the notion of *quasi-derivation*:

> F [F_c] is to be the system that is obtained from I [C'] by the addition of the λI- and λE-rule. A *quasi-deduction* in one of these systems is defined in the same way as a deduction was defined for Gentzen-type systems in general (Chapter I, §2). A deduction is then defined as a quasi-deduction that is normal. (Prawitz, 1965, pp. 94–95)

We could use this idea and define:

Definition 6.1 A *quasi-derivation* in $\mathsf{M}_\bot$ is a derivation as we usually define it. A *derivation* in $\mathsf{M}_\bot$ is a quasi-derivation that is *normal*.

We can now formulate the normalization theorem for $\mathsf{M}_\bot$ as follows:

Theorem 6.2 *Let* Π *be a quasi-derivation of* $\Gamma, \Delta^* \vdash A$. *Then, either* Π *reduces to a derivation* Π' *of* $\Gamma, \Delta^* \vdash A$ *or* Π *reduces to a derivation* Π'' *of* $\Gamma, \Delta^* \vdash \bot$.

The new normalization theorem establishes that a quasi-derivation will always take us either to a normal derivation of the same conclusion or to a normal derivation of $\bot$[10]. We can think of quasi-derivations in $\mathsf{M}_\bot$ as *deduction-maps* that may contain *detours*. Once we follow the map eliminating the detours (the *normalization guide*), we will either reach the marked goal or we will reach a *dead-end*.

Important remark: This idea of *deduction maps* cannot be applied to the system $\mathsf{M}_\vee$, since the system $\mathsf{M}_\vee$ does not satisfy the new normalization theorem. The following simple derivation is a counter-example to the new normalization theorem: This derivation

$$\dfrac{\dfrac{\dfrac{[A]^3}{(A \vee B)} \quad \dfrac{[A]^1 \quad \neg A}{\bot} \quad [B]^2}{B} \vee_{\bot\text{E-3}}\ 1, 2}{(A \to B)} \to_{\text{I}}\ 3$$

reduces to

$$\dfrac{A \quad \neg A}{\bot}$$

We immediately see that the number of open assumptions increases after the reduction (A was not open before the reduction).

[10] We can easily see that in some cases a quasi-derivation can be transformed into different derivations of $\bot$.

7 Conclusion

We have been examining two extensions of the propositional part of the Gentzen-Prawitz natural deduction system M for minimal logic, the systems $\mathsf{M}_\vee$ and $\mathsf{M}_\bot$, that satisfy *in a certain way* the disjunctive syllogism but that do not satisfy the *ex falso*.

The system $\mathsf{M}_\vee$ uses the rules $\vee$-elimination-1, $\vee$-elimination-2, and $\vee$-elimination-3, and *normality* is imposed by construction. In the system $\mathsf{M}_\vee$ we have the following results:

1. By means of the new rules for disjunction, we can easily show that
$$\{(A \vee B), \neg B\} \vdash_{\mathsf{M}_\vee} A.$$
2. Given that there is no restriction on the deduction theorem in $\mathsf{M}_\vee$, we also have
$$\vdash_{\mathsf{M}_\vee} (((A \vee B) \wedge \neg B) \to A).$$
3. The imposition of *normality by contruction* guarantees that
$$\nvdash_{\mathsf{M}_\vee} (\neg A \to (A \to B)).$$
4. $\{\neg A, A\} \nvdash_{\mathsf{M}_\vee} B$.
5. The $\bot$ works as the disjunction unity: $\vdash_{\mathsf{M}_\vee} ((A \vee \bot) \leftrightarrow A)$.
6. A general *composition of derivations* is lost with the imposition of *normality by contruction*. This feature of $\mathsf{M}_\vee$ does not seem to be faithful to our deductive practices, where composition seems absolutely natural, as in the case of proofs of theorems that use lemmas.

The system $\mathsf{M}_\bot$ uses the rules $\vee_\bot$-elimination-1, $\vee_\bot$-elimination-2, and $\vee_\bot$-elimination-3, and these rules impose a more strict control on *dependency relations* between assumptions and derived formulas. *Normality* is not imposed by construction, but it is used to define derivations: $\mathsf{M}_\bot$ works with the concept of *quasi-derivations* and defines derivations as quasi-derivations that are normal. The normalization theorem for $\mathsf{M}_\bot$ guarantees that every quasi-derivation Π of $\Gamma, \Delta^* \vdash A$ either reduces to a derivation Π' of $\Gamma, \Delta^* \vdash A$ or Π reduces to a derivation Π'' of $\Gamma, \Delta^* \vdash \bot$.

In the system $\mathsf{M}_\bot$ we have the following results:

1. $\{(A \vee B)^*, \neg B^*\} \vdash_{\mathsf{M}_\bot} A$. In the application of the new disjunction-elimination rule the assumptions $(A \vee B)$ and $\neg B$ must be *frozen*.
2. $\{(A \vee \bot)^*\} \vdash_{\mathsf{M}_\bot} A$.
3. $\nvdash_{\mathsf{M}_\bot} (((A \vee B) \wedge \neg B) \to A)$.
4. $\{\neg A, A\} \nvdash_{\mathsf{M}_\bot} B$.
5. $\nvdash_{\mathsf{M}_\bot} (\neg A \to (A \to B))$.
6. We can think of the quasi-derivations as *deduction plans* that may contain detours. The main idea now is that the normalization process will guide us through this plan and will lead us either to a derivation that ends with a certain goal, or it will lead us to a dead-end (possibly several dead-ends).

Obviously there is still a lot of work to be done: (1) a more detailed comparison between the systems $\mathsf{M}_\vee$ and $\mathsf{M}_\perp$; (2) a more detailed comparison with Tennant's systems[11,12]; (3) a deeper exploration of the idea of "freezing" hypotheses; and a more in-depth analysis of the proof-theory of $\mathsf{M}_\perp$.

Acknowledgements The work of Peter has been a great source of inspiration for us and this is especially true with respect to this paper: its original motivation was Peter's talk on "Paradoxes, local consequence, and the role and future of deductive logic", given at the Swedish Collegium for Advanced Study (SCAS), Uppsala. We would also like to thank Dag Prawitz, Valeria de Paiva, Wagner de Campos Sanz, Evandro Gomes, Abilio Rodrigues, Rodolfo Ertola-Biraben, Bogdan Dicher, Neil Tennant and the Tübingen group for important suggestions and criticisms. And finally, we would like to give a big thanks to the two anonymous referees for the extremely careful, detailed and generous reviews. Unfortunately, some important suggestions could not be incorporated into the revised version of our text, but they will certainly have to be taken into account in our future work. This research was supported by a CNPq project and by the projects CAPES/PRINT and CAPES/COFECUB.

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[11] Tennant (1987; 2017).

[12] After the text was ready and submitted, Prof. Bogdan Dicher called our attention to the many similarities between our results and those of yet another paper by Neil Tennant (1994). Both Tennant's extraction theorem and our notion of deduction maps heavily rely on procedures which are applied to derivations in order to produce either derivations with the same end formula or derivations of the absurdity constant. However, there are also many important differences between both approaches. Tennant imposes normal form by definition on derivations in his relevant intuitionistic logic, whereas our formulation allows derivations to contain maximal formulas which are later expunged by reduction procedures. Moreover, ours is a logic which stands strictly between minimal and intuitionistic logic, whereas the relation between Tennant's system and those logics is considerably more complex; though it allows one to prove all theorems of intuitionistic logic (even ones such as $\vdash \neg A \to (A \to B)$), it invalidates many of its deducibility relations (it can be shown that $\{\neg A, A\} \nvdash B$, for example). There also seem to be many other interesting differences and similarities between both approaches but, unfortunately, the late discovery of this relationship effectively prevented us from properly addressing the topic, and so further comparisons will have to be left to future work.

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