Kolmogorov and the General Theory of Problems

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Abstract This essay is our modest contribution to a volume in honor of our dear friend and fellow logician Peter Schroeder-Heister. The objective of the article is to reexamine Kolmogorov's problem interpretation for intuitionistic logic and the basics of a general theory of problems. The task is developed by first examining the interpretation and presenting a new elucidation of it through Reduction Semantics. Next, in view of Kolmogorov's intentions concerning his problem interpretation, Reduction Semantics is employed in an brief epistemological analysis of Euclidean Geometry and its construction problems. Finally, on the basis of the previous steps, some theses are raised concerning intuitionistic logical constants and concerning proofs and hypotheses in Euclid's *Elements*.

1 Introduction

The epistemology of mathematics has gone through an important change in the last centuries: from focus in problems to focus in theorems. It is time for the pendulum to move back. A landmark was *The Foundations of Geometry* (Hilbert, 1899) which transformed *Propositiones* of Euclid's *Elements* into theorem assertions, while a good amount of them where the statement of a problem. This is the case of the first three *Propositiones* in book I.

The paper investigates *Kolmogorov's problem interpretation*. The objective is to unfold the basics of a general theory of problems, answering some very simple questions like: what are problems, why they are not assertions and how problems in geometry were structured.

Kolmogorov (1932, p. 151) states his purposes as follows:

> Along with the development of theoretical logic, which systematizes the schemes of proofs of

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T. Piecha and K. F. Wehmeier (eds.), *Peter Schroeder-Heister on Proof-Theoretic Semantics*, Outstanding Contributions to Logic 29,
https://doi.org/10.1007/978-3-031-50981-0_5

> theoretical truths, it is also possible to systematize the schemes of solution of problems, for example, geometrical construction problems [. . .] The second section in which the general intuitionistic presuppositions are accepted, presents a critical analysis of intuitionistic logic. It is shown that this logic should be replaced by the calculus of problems, since the objects under consideration are in fact problems, rather than theoretical propositions.

Years later, in a comment about his paper, Kolmogorov evaluated it as follows (Tikhomirov, 1991, p. 452):

> *On the interpretation of intuitionistic logic* (IIL) was written with the hope that the logic of solutions of problems would later become a regular part of courses on logic. It was intended to construct a unified logical apparatus dealing with objects of two types — propositions and problems.

Kolmogorov (1932) belongs to a context where the nature of intuitionistic logic was being discussed. A crisis concerning some basic mathematical concepts had opposed Hilbert's school with his foundational program for mathematics and Brouwer's intuitionistic school. Intuitionists rejected certain means of proof in traditional mathematics like the proof of existential sentences over infinite collections by deducing and absurd from a negated universal hypotheses (*reductio*), as also the use of third middle excluded principle. Since the criticisms were directed to logical principles, it became important to elucidate what would be an acceptable intuitionistic logic. Heyting (1930) is now assumed as the standard formulation of intuitionistic logic. Kolmogorov (1932) is a later publication where the issue of how to interpret logical constants was still being debated. It is considered as a part of the so called BHK[1] interpretation of intuitionistic logical constants.

In the quotes above Kolmogorov remarks that logic has been historically occupied with schemes of theoretical truths. Next, he states some surprising claims. First, there is the claim about logic being occupied with schemes of solutions of problems, with geometrical constructions as a relevant example. Second, the claim that intuitionistic logic should be replaced by a calculus of problems. Third the statement about the unified logical apparatus for dealing with objects of two distinct types: propositions and problems. As declared, he had hopes that the logic of solutions of problems could become a regular part of courses on logic. Kolmogorov's problem conceptualization has usually been explained through the paradigm of propositions-as-types (Coquand, 2007). Here, a different elucidation will be developed.

The investigation of Kolmogorov's problem semantics explores each of these claims. His problem semantics is going to be reformulated as Reduction Semantics. The adequacy of this elucidation is dealt with in two separate steps. First, the formulation of the semantics and the issues of soundness and of completeness for intuitionistic logic are examined in the Section 2. Second, a partial anatomy of Euclid's Geometry by means of Reduction Semantics trying to unfold the schemes of solutions of problems in geometrical constructions is investigated in Section 3. To these two sections is added a third section focusing two specific subjects. In one

[1] Kolmogorov's interpretation is the third element in what has been conventionally called BHK interpretation of intuitionistic logical constants. The abbreviation corresponds to the initials of Brouwer, Heyting and Kolmogorov.

hand, the questions of what is a logical constant and if the usual set of intuitionistic propositional logical constants should be extended or not. On the other hand, a partial discussion concerning the role of hypotheses inside geometry.

Reduction Semantics is the further development of an initial investigation of problem interpretation in de Campos Sanz (2012). Reduction semantics turns to be similar to Hypo Semantics (de Campos Sanz, 2019) with a new dressing, since the basic objects are now problems.

2 A semantics of problems

2.1 Kolmogorov on problems

Kolmogorov's article is schematic and it does not contain an explanation of what is a problem, only a few carefully chosen examples (Kolmogorov, 1932, pp. 151–152):

1. Find four integers x, y, z and n such that $x^n + y^n = z^n$, $n > 2$.
2. Prove that Fermat's theorem is false.
3. Construct a circle passing through three given points (x, y, z).
4. Given one root of the equation $ax^2 + bx + c = 0$, find the other root.
5. Assuming that the number π has a rational expression, $\pi = m/n$, find a similar expression for the number e.

According to Coquand (2007, § 2.5), the perspective of interpreting intuitionistic logic as a calculus of problems is an important antecedent of a later distinction: that between formulae-as-types and λ-terms which would then correspond to a separation of problems and solutions, respectively. Although this can be a productive way of interpreting Kolmogorov about the semantics of problems, it is not clear that this author would subscribe such a sharp separation.

Kolmogorov (1932, p. 151) characterized his semantics of problems for logical molecular operators as follows:

> If a and b are problems, then $a \wedge b$ denotes the problem "solve both problems a and b", while $a \vee b$ stands for "solve at least one of the problems a and b". Further, $a \supset b$ is the problem [(FIRST)] "given a solution to problem a, solve problem b" or, which is the same, [(SECOND)] "reduce the solution of problem b to the solution of problem a. [...] $\neg a$ denotes the problem "assuming that there is a solution to problem a, derive a contradiction".[2]

In the five problems quoted above, the main verb is imperative. They are to be considered as the command of an action. The semantic explanations, in their turn, notoriously contain the command: "solve". Implication is the only exception. It has two explanations: (FIRST) and (SECOND). And the author regards them as equivalent. The expression "solution" is used in both explanations for naming, and the expression "reduce" in the (SECOND) explanation is used for commanding.

[2] Square brackets were added by us.

Some assumed that the problem interpretation required two different representational structures for problems and solutions. But there are reason for adopting another point of view, given that the inner nature of problems and solutions seems to be the same: they are about actions.

Consider Kolmogorov's description of what is a conjunction. Given any two problems a and b, $a \wedge b$ is the problem "solve both problems a and b". The action "solve" being commanded is iterated in order to form the description of a problem. When there is plain knowledge of how to proceed in order to bring about an action being commanded, then solving the problem corresponds to do the required action. Hence, in case a and b are problems that one knows how to solve immediately, and problem $a \wedge b$ represents the problem of solving a and solving b, then to solve $a \wedge b$ consists in doing what solves a and doing what solves b.

An example might help to clarify the point. Consider the following simple problem: construct a circle *BCD* of center A with radius AB and construct a circle *ACE* of center B with radius BA. This is a conjunctive problem. Employing Kolmogorov's description, it denotes the problem of solving both problems, that is, solve the problem of constructing *BCD* and solve the problem of constructing *ACE* as required. Both problems are immediately solvable. The solution consists in doing the construction of the circle *BCD* and doing the construction of the circle *ACE*.

Concerning implication now, it is obscure how the semantics could be formulated by using the command "solve" in the antecedent. By saying "first solve a, then solve b" another logical connective distinct of the usual implication seems to be employed once "solve" is an action and the expression "first solve ..., then solve ..." indicates a temporal ordering for the actions. Actually, in order to establish an intuitionistic implication, if we start by solving b, $a \supset b$ can be considered solved, but this violates the meaning explanation using the word "solve" which is truly a before-after conjunction. We will come back to this question later.

In the reverse direction, an attempt to employ the expression "solution" for explaining conjunction and disjunction would require the use of another command verb in place of "solve", as in: "produce the solution of ..." or just "do ...". This seems to be no real progress. Thus, a strange heterogeneity appears in the above semantical explanations. In one side we have one kind of explanation for disjunction and conjunction employing the verb "solve", in the other side another kind for implication employing the concept "solution" and/or the verb "reduce".

Actually, the (SECOND) explanation of implication above is partially misleading since the expression "reduce *the* solution of b ..." seems to make reference to a specific solution. But a problem could possess distinct solutions. Observe that the expression "reduce a solution of b ..." is also unclear, since it could be understood as meaning: "for some solution of b reduce it ...". The statement would be better rendered if it were formulated as "reduce *solvability* of b ...", which in turn might bring it closer to the (FIRST) explanation. Similar issues concerning the article "the" also affect the complement part of the (SECOND) explanation. It is important to stress that the usual intuitionistic interpretation of logical constants requires, for any construction of a, a corresponding construction of b, in order to have a true implication.

The (FIRST) explanation of implication can be seen as a characterization where solvability/demonstrability is transmitted, similar to what happens in the case of admissible rules. Under this interpretation the expression "suppose" is to be assumed as a hidden preamble like: "[suppose] given a solution to problem a ...".

The simplest and best way for rendering the (SECOND) explanation, in our opinion, would be to explain it as "reduce problem b to problem a", characterizing a reducibility junction between problems. But this is acceptable only under the proviso that a solution for b must be obtainable whenever a solution for a is provided. This seems the best explanation for implication from the point of view of problem semantics. It does not suffer from the issues pointed in (SECOND) and it can also be extended to other constants, as we are going to argue bellow. But it is distinct from the (FIRST).

In order to exemplify the reducibility junction, let's consider again the above example concerning circles. A new problem of finding a point C that is distant of points A and B as much as A and B are among themselves is a problem that is reducible to the problem of constructing circle BCD and constructing circle ACE as described above. That is, constructing both circles as described is a way of obtaining a solution for the problem of finding the required point C.

Concerning the notions of negation and contradiction we find in Kolmogorov's paper the following passages. The first is the text of a footnote accompanying the word "contradiction" in the last quotation above, (*ibid.*, p. 151):

> [...] $\neg a$ should not be read as the problem "prove the unsolvability of problem a". In the general case, if the "unsolvability of problem a" is considered as a completely defined notion, we only obtain that $\neg a$ implies the unsolvability of a, and not the converse assertion. If, for example, it were proved that a realization of the well-ordering of the continuum is beyond our possibilities, it would not be possible to assert that the existence of such a well-ordering implies a contradiction.

p. 156:

> It should also be mentioned that if $\vdash p$ is false in classical propositional logic, then the corresponding problem $\vdash p$ cannot be solved. Indeed, in view of the earlier accepted formulas and rules of the calculus of problems, this formula $\vdash p$ readily implies the contradictory formula $\vdash a \wedge \neg a$.

p. 157:

> [...] Brouwer suggests a new definition of negation, namely "a is false" should be understood as "a leads to a contradiction". Thus, the negation of a proposition a is transformed into the *existential sentence* "there exists a chain of logical inferences leading to a contradiction if a is assumed to be true".

Kolmogorov does not give further explanations of what he takes negation and contradiction to mean in the context of problems. So, what we have is the conflation of the concept of contradiction with the usual contradictory formula. But this formula already uses a negation as can be observed in the second quotation of the three above. We believe that in a good measure the concept was assumed as unproblematic by the author. In a similar attitude, when explaining the intuitionistic logical constants, Heyting (1956, p. 102) explicitly says that he takes it to be a primitive notion and complements by pointing that it is easy to recognize a contradiction when we are in front of one.

2.2 Reduction semantics

We claim that the explanations of Kolmogorov can be made homogeneous by adopting a new organization of the background concepts by taking the concept of reduction as the basic relation in the semantical explanation of problem constants.

The expression $\Gamma \Vdash b$ will mean from now on "problem b has been reduced to the multiset of problems Γ" but only if it is guaranteed that a solution of b is obtained in case of having solutions whatever for each problem in Γ. This proviso concerning how to define reduction is stated in Veloso (1984, p. 26) as a requirement for the acceptability of problem transformations.[3]

Based on the reduction relation, every logical constant that Kolmogorov was considering can be defined by semantical clauses, where the logical constant "$\perp$" represents a basic problem–the *impossible* problem.[4] Negation is explicitly defined as: $\neg a \equiv a \supset \perp$. The clauses are:

Clauses for using a logical constant in the repertoire:

$(\wedge^l)$: $\Gamma, c \wedge d \Vdash e \cong \Gamma, c, d \Vdash e$;
$(\vee^l)$: $\Gamma, c \vee d \Vdash e \cong \Gamma, c \Vdash e$ and $\Gamma, d \Vdash e$;
$(\supset^l)$: $\Gamma, c \supset d \Vdash e \cong$ given any problem $a : (a, c \Vdash d \Rightarrow \Gamma, a \Vdash e)$;
$(\perp^l)$: $\Gamma, \perp \Vdash a \cong$ always (where a is a basic problem).

Clauses for using a logical constant in the focus:

$(\wedge^r)$: $\Gamma \Vdash c \wedge d \cong \Gamma \Vdash c$ and $\Gamma \Vdash e$;
$(\vee^r)$: $\Gamma \Vdash c \vee d \cong$ given any problem a $((\Gamma, c \Vdash a$ and $\Gamma, d \Vdash a) \Rightarrow \Gamma \Vdash a)$;
$(\supset^r)$: $\Gamma \Vdash c \supset d \cong \Gamma, c \Vdash d$;
$(\perp^r)$: $\Gamma \Vdash \perp \cong$ given any basic problem $a : \Gamma \Vdash a$.

The *repertoire* of problems is in left side of the semantical symbol, and the *focal* problem is in the right side of the semantical symbol. Clauses are of two kinds: (i) those explaining the use of logical constants in repertoires, that is, finite multisets[5]; (ii) those explaining the use of logical constants as a focal problem. The symbol "$\cong$"

[3] He suggests that we should not consider arbitrary links between problems, but only those guaranteeing that solvability of b is obtained from solvability of a.

[4] Kolmogorov did not use it in his paper, we here follow a late tradition in intuitionistic logic.

[5] Finite sets admitting multiple copies of an element.

expresses a necessary and sufficient condition[6], that is, an explanation of meaning which has *explicandum* in the left side and *explicans* in the right side.[7]

As an exemplification, the reading of the clauses for implication are presented next. Consider clause $(\supset^r)$. As a semantical rule, it is read as follows: in order to reduce the reducibility problem $c \supset d$ to the multiset of problems Γ it is necessary and sufficient to show that d reduces to the multiset $\Gamma \cup \{c\}$. The reading of clause $(\supset^l)$ is a little bit more complex. As a semantical rule, it has to be read as follows: in order to reduce problem e to the multiset of problems $\Gamma \cup \{c \supset d\}$ it is necessary and sufficient to show that, for any given problem a, on the supposition that d reduces to the multiset of problems $\{a, c\}$ it follows that d reduces to the multiset of problems $\Gamma \cup \{a\}$.

Some expressions with logical content are unavoidable in the metalanguage, and they must be interpreted in their constructive sense. The other clauses are read in a similar way to those two just considered, with *explicans* quantified or not.

The clauses can be seen in action in the following proof.

Theorem 2.1 $\Vdash \neg(a \wedge \neg a)$.

Proof Suppose that $b, a \Vdash \bot$. Thus, $b, a \Vdash \bot \Rightarrow a, b \Vdash \bot$. Hence, for any given problem b, $(b, a \Vdash \bot \Rightarrow a, b \Vdash \bot)$. Now, by clause $(\supset^l)$, $a, \neg a \Vdash \bot$ according to the definition of negation. By clause $(\wedge^l)$, $a \wedge \neg a \Vdash \bot$. By the definition of negation and by clause $(\supset^r)$ we have finally $\Vdash \neg(a \wedge \neg a)$. □

Given any problem, we can fairly say that either it has been solved or it has not. If it was solved, then either it has been *positively* solved — when a correct solution is given to the problem — or it has been *negatively* solved — when a positive solution has shown to be impossible. The expression "to be solvable" is ambiguous and it can be employed in a strict sense or in large sense. The distinction is relevant for the discussion coming next.

Although the command "solve" could mean in a large sense either positively solve or negatively solve[8], the intended reading of the expression in the case of Kolmogorov is probably the strict sense, that of positively solve. Since the notion of reduction is at hand, the command "solve" can be dismissed. That is, categorically and positively solve $\neg(a \wedge \neg a)$ is the same as to show that it has been reduced to the empty repertoire of problems. There is an infinite set of problems that can also be reduced to the empty repertoire.[9] Negatively solve a problem means to show that the impossible problem has been reduced to it, like in $a \wedge \neg a \Vdash \bot$. An implication $a \supset b$ is positively solved categorically when b has been reduced to a according to clause $(\supset^r)$. The expression "has been reduced" means that an action took place.

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[6] Harmony is an inherent property of explanations stated in terms of necessary and sufficient conditions.

[7] Thus, it is not an explicit definition once the constants being explained occur also in the metalanguage used for stating the clauses.

[8] Which then means that one of two possible conclusions is expected.

[9] This must be taken into account when reading the clauses containing a quantification, i.e., $(\supset^l)$ and $(\vee^r)$

A true simplification seems to be achieved in Reduction Semantics, since both verbs "solve" and "do" mentioned before can be substituted by the verb "reduce".

The fictitious impossible ($\bot$) problem is considered as negatively solved. Thus, since negation is defined via implication, $\neg a$ is the problem of reducing the impossible to problem a. Hence, at the same time that a is negatively solved, $\neg a$ is also positively solved by ($\supset^r$). The fictitious impossible problem is assumed as a basic problem itself and it is semantically characterized as a problem to which all basic problems reduce. That is, its solution is a *panacea*. Hence, the concept of contradiction originally used in Kolmogorov's paper can also be dismissed. As such, when the author says that a conditional problem is meaningless and then consider it as solved, this is equivalent to say that the antecedent problem is impossible and the whole conditional problem is reduced to the empty repertoire, since every basic problem reduces to the impossible problem by definition.

The above clauses must be complemented by the following *structural* semantical principles making explicit the properties of the semantical relation of reduction:

(basic problem Identity)	for basic a	$a \Vdash a$;
(Load problem)		$\Gamma \Vdash d \Rightarrow \Gamma, c \Vdash d$;
(Drop basic problem)	for basic a	$(\Gamma, a \Vdash d \;\&\; \Gamma \Vdash a) \Rightarrow \Gamma \Vdash d$.

The above semantics of problems, that is, the structural principles plus the clauses is called *Reduction Semantics*. When the variables a, b, c etc. are interpreted as sentences we call it *Hypo[thesis] Semantics* (de Campos Sanz, 2019).

Theorem 2.2 *(i) The full (Identity) principle holds, i.e., for any c,* $c \Vdash c$. *(ii) The full (Drop) principle holds, i.e., for any c,* $(\Gamma, c \Vdash d \;\&\; \Gamma \Vdash c) \Rightarrow \Gamma \Vdash d$.

Proof (i) and (ii) by induction in the logical degree of problem c, by using both clauses for the repertoire and clauses for the focus.[10] $\square$

Semantical variations or extensions of the above set of problem constants given by the clauses are envisageable. In some cases it might well occur that for a new constant there are restrictions for the application of full drop. We think this might well be the case of some modal operators, and this is a reason for preferring the weaker and not the full formulation of the structural principles.

Going back to Kolmogorov's paper in order to determine if the above picture fits in, notice that the author intends with his calculus "... to systematize the schemes of solution of problems", as those in geometrical constructions. Then next, two issues are going to be considered. The first is the adequacy of Kolmogorov's calculus of problems with respect to Reduction Semantics. The second is the adequacy of Kolmogorov's notion of problem for describing constructions and proofs in ancient geometry. A case study already envisaged by him.

[10] In case the full principles were assumed from the beginning, it would be enough to possess one clause for each logical constant, once the other clause could be obtained by the use of the full principles. But, of course, in that case, we would miss the fact that the weaker structural principles are enough to formulate the semantics.

2.3 Adequacy of Reduction Semantics with respect to Kolmogorov's interpretation

The second part of Kolmogorov's paper presents a calculus of problems and describes the validation of its axioms and theorems. Here are the axioms:

[group] A

2.1 $\vdash a \supset a \wedge a$;
2.11 $\vdash a \wedge b \supset b \wedge a$;
2.12 $\vdash (a \supset b) \supset (a \wedge c \supset b \wedge c)$;
2.13 $\vdash (a \supset b) \wedge (b \supset c) \supset (a \supset c)$;
2.14 $\vdash (b \supset (a \supset b)$;
2.15 $\vdash a \wedge (a \supset b) \supset b$;
3.1 $\vdash a \supset (a \vee b)$;
3.11 $\vdash a \vee b \supset b \vee a$;
3.12 $\vdash (a \supset c) \wedge (b \supset c) \supset (a \vee b \supset c)$;
4.1 $\vdash \neg a \supset (a \supset b)$;
4.11 $\vdash (a \supset b) \wedge (a \supset \neg b) \supset \neg a$.

Kolmogorov gives as an example an argument containing a general method for solving the problem 2.12, valid for any a, b, c. That is, he is basically showing an intuitive semantical justification based on the problem interpretation for the validity of axiom 2.12 above:

> For example, in problem 2.12, assuming that the solution of b has already been reduced to the solution of a, one should reduce the solution of $b \wedge c$ to that of $a \wedge c$. Let a solution of $a \wedge c$ be given. This means that we are given both a solution of a and a solution of c. By the hypothesis, we can derive a solution of b from that of a, and, since a solution of c is known, we obtain solutions of both problems b and c and hence a solution of problem $b \wedge c$.

The expression "the solution of", we remember, has been dismissed as being inappropriate. The above argumentation can be reformulated as a validity proof of the above axioms with respect to Reduction Semantics.

Lemma 2.3 *All axioms in the calculus of problems are valid with respect to Reduction Semantics.*

Proof The axiom 2.12 with all parentheses becomes: $\vdash (a \supset b) \supset ((a \wedge c) \supset (b \wedge c))$. It is validated as follows. (1) $a \wedge c \Vdash a \wedge c$ by (Id); (2) $a \supset b, a \wedge c \Vdash a \wedge c$ from 1 by (Load); (3) $a \supset b, a \wedge c \Vdash a$ from (2) by the necessary condition of $(\wedge^r)$; (4) $a \supset b, a \wedge c \Vdash c$ from (2) by the necessary condition $(\wedge^r)$; (5) $a \supset b \Vdash a \supset b$ by (Id); (6) $a, a \supset b \Vdash b$ from (5) by the necessary condition of $(\supset^r)$; (7) $a, a \supset b, a \wedge c \Vdash b$ by (Load) from (6); (8) $a \supset b, a \wedge c \Vdash b$ from (3) and (7) by (Drop); (9) $a \supset b, a \wedge c \Vdash b \wedge c$ from (4) and (8) by the sufficient condition of $(\wedge^r)$; (10) $a \supset b \Vdash (a \wedge c) \supset (b \wedge c)$ from (9) by the sufficient condition of $(\supset^r)$; and finally (11) $\Vdash (a \supset b) \supset ((a \wedge c) \supset (b \wedge c))$ by the sufficient condition of $(\supset^r)$. That is, axiom 2.12 is semantically valid. The other axioms are validated in a similar fashion. □

As remarked before, each expression of form $\Gamma \Vdash e$ is to be read as "problem e reduces to the multiset of problems in Γ". This reading does not mention the word "solution" at all. Hence, the above proof can be read without using the word "solution".

Next, Kolmogorov presents three rules for extending the set of solved problems in the calculus. The equivalence between this calculus and the intuitionistic propositional calculus is now explicitly pointed in the passage:

> [Group B] We can now formulate the rules of our calculus of problems.
> 1. First, we include the problems of group (A) in the list of solved problems.
> 2. If the list includes $\vdash p \wedge q$, then we are allowed to replace it by $\vdash p$.
> 3. If both formulas $\vdash p$ and $\vdash p \supset q$ are in the list, then we can replace them by $\vdash q$.
> 4. If $\vdash p(a, b, c, \ldots)$ is in the list and $q, r, s, \ldots$ are arbitrary problem functions, then we are allowed to replace it by $\vdash p(q, r, s, \ldots)$ in the list.[11]
>
> Based on the above Postulates, it is easily seen that the formal calculus does in fact guarantee the solution of the corresponding problems.
>
> We are not going to develop this calculus further here, since all formal rules and a priori formulas above coincide with the computational rules and axioms suggested by [Heyting (1930)].[12] Hence, we can interpret all formulas of this paper as problems and assume that all problems are solved.

Validity of 2, 3 and 4 in Reduction Semantics requires that the following principles be proved:

Lemma 2.4 *The following principles hold in Reduction Semantics:*
2. $\Vdash p \wedge q \Rightarrow\ \Vdash p$*;*
3. $(\Vdash p$ *and* $\Vdash p \supset q) \Rightarrow\ \Vdash q$*;*
4. $\Vdash p \Rightarrow\ \Vdash p[^{a,b,c,\ldots}_{q,r,s,\ldots}]$*, where* $p[^{a,b,c,\ldots}_{q,r,s,\ldots}]$ *is the result of substituting* $a, b, c, \ldots$ *by* $q, r, s, \ldots$ *inside* p*, respectively.*

Proof Principle 2 is obtained by the necessary condition of clause $(\wedge^r)$ $\Vdash p \wedge q \Rightarrow (\Vdash p$ and $\Vdash q)$ and, *a fortiori*, $\Vdash p \wedge q \Rightarrow\ \Vdash p$. Principle 3 is obtained by using clauses full (Id) and $(\supset^r)$. Principle 4 holds in virtue of the clauses being closed under homogeneous substitution of problems. □

Theorem 2.5 *Kolmogorov's calculus of problems is sound with respect to Reduction Semantics, that is:* $\vdash c \Rightarrow\ \Vdash c$.

Proof By Lemmas 2.3 and 2.4. □

There are reasons to believe that the usual set of propositional intuitionistic logical constants is incomplete for dealing with problems. It probably should be extended with other constants, among them the before-after conjunction. If this is correct,

[11] We suspect that the word "replace" employed in 2, 3 and 4 might be a poor translation of the original, since "add" would be a better way to formulate it. A question that might be raised is: does the passage suggest that "problem" and "function" are related concepts? We do not think so. We interpret the passage as just stating that $q, r, s, \ldots$ are variables for problems which can be substituted by problems.

[12] The square brackets substitutes the bibliographic note [1] in the original paper.

then completeness is not a so meaningful property. The before-after conjunction will be used and discussed ahead when problems in geometry are discussed. Finally, we notice that Kolmogorov did not give a completness proof, probably because his semantics was formulated in an intuitive way.

Completeness can be shown if the set of logical constants is restricted to $\wedge, \vee, \supset, \perp$. To say that Kolmogorov's calculus of problems is complete with respect to Reduction Semantics means that $\Vdash c \Rightarrow \vdash c$. The recipe for proving it is the following. First, define a sequent calculus with rules for the semantic principles of reduction — full (Id), (Load) and full (Drop) — and double inferences, that is, left to right and right to left inferences corresponding to the following clauses: $(\vee^l)$, $(\wedge^r)$, $(\supset^r)$ and $(\perp^r)$. Second, prove that each of the semantical clauses and principles of Reduction Semantics is a valid metatheoretical property of this calculus, that is, substitute the symbol "$\vdash$" in place of the semantical symbol "$\Vdash$" and prove each Reduction Semantics principle, be it a clause or a structural principle, as a metalanguage property of the calculus just defined.[13] This is almost immediate. Third, prove that the calculus just described is theoremhood-equivalent to Kolmogorov's calculus of problems.

The meaning of problem logical constants was explained in Reduction Semantics through two distinct uses: in the repertoire and in the focus. The fact that the full (Drop) principle as the full (Identity) principle holds for the set of logical constants given in the clauses of the semantics above does not mean that it will hold for any extension of the set of logical constants.

Reduction Semantics is our way to elucidate, clarify, the problem interpretation of intuitionistic logic as proposed by Kolmogorov.

2.4 Why problem semantics and intuitionistic logic?

Problem semantics was the conceptual way in which Kolmogorov interpreted intuitionistic logic. We think that there are two reasons why Kolmogorov adopted such approach.

The first is the fact that *tertium non datur* ($\vdash a \vee \neg a$) is not a valid principle in the calculus of problems and in intuitionistic logic. Kolmogorov (1932, p. 156) interprets the sign "$\vdash$", differently from Heyting, as meaning generality:

> For a function $p(a,b,c,\ldots)$ of undefined problems $a,b,c,\ldots$ we simply write $\vdash p(a,b,c,\ldots)$ instead of $(a)(b)(c)\ldots\ p(a,b,c,\ldots)$. Hence, $p(a,b,c,\ldots)$ denotes the problem "*find a general method for solving the problem $p(a,b,c,\ldots)$ for each individual choice of the problems $a,b,c,\ldots$*".[14]

His interpretation is constructive in the measure it considers the problem of proving logical formulas as requiring a general method of solution. In this sense intuitionistic logic is a part of a theory of problems. In particular, $\vdash a \vee \neg a$ would be valid if

[13] Observe that $(\supset^r)$ is the semantical version of deduction theorem.

[14] Our emphasis.

(a) $(a \vee \neg a)$ were valid, which means that we should have a general method for solving problems of form $a \vee \neg a$ for each individual choice of a.[15]

The observation concerning the requirement of a method of solution brings us to the second reason. It concerns the interpretation of existential propositions, when the proof of existence does not exhibit the object. The quotation is curious since the problem calculus presented is considering only propositional constants (Kolmogorov, 1932, p. 157):

> Brouwer does not, however, intend to exclude existential propositions from mathematics completely. He only explains that an existential proposition should not be stated without presenting the corresponding construction. At the same time, according to Brouwer, an existential proposition is not a mere indication of the fact that we have already found the desired element of K. In this case the existential proposition would be false prior to the invention of the construction and true after that. Thus, propositions of a completely new type arise, which, although their content does not change in time, can nevertheless be stated only under certain conditions.
>
> The natural question which can arise is whether this specific type of proposition is a mere fiction. Indeed, the problem "find an element of a set K possessing a property A" is posed. This problem actually has a certain sense independent of the state of our knowledge. If this problem has been solved, that is, if the corresponding element x is found, we obtain the empirical proposition "our problem is now solved". Thus, Brouwer's existential proposition is partitioned into two elements: an objective component (problem) and a subjective component (its solution).

Intuitionists do not accept [classical] negation of an universal to be a basis for the inference of an existential judgement involving an infinite collection. According to the conceptual background set by Kolmogorov, the meaning of an existential problem cannot depend on the possession of a construction exhibiting the required element because a previous understanding of the problem is needed in order to even look for the solution. Thus problems have an objectivity that solutions might lack. Problems have to be understood in order to be solved, they must be meaningful. Solutions require ingenuity in order to be obtained, and meaning cannot be made to depend on them.

3 Problems and solutions in geometry

After offering an analysis of Kolmogorov's problem interpretation in terms of Reduction Semantics, it is time to consider more closely the concepts of problem and solution taking as a reference for the analysis that piece of mathematical knowledge where problems seem to play a central role: Euclidean Geometry.

[15] Kolmogorov (1932, p. 156): "formula [$\vdash a \vee \neg a$] reads as follows: find a general method which for any problem a allows one either to find its solution or to derive a contradiction from the existence of such a solution." Since a formalized language has to be previously presented before defining its semantics, all basic problems in a calculus of problems would also be exhaustively enumerated beforehand. The only hope of validating *tertium non datur* principle occurs when language is such that we possess a general method for solving all basic problems enumerated on it. For a non-specific language such a general method is obviously impossible.

3.1 Problems and practical principles

The first three Postulates of Euclidean Geometry have been regarded historically as practical principles. There is a modern tradition culminating in Kant going in this sense (Critique of Pure Reason, A234/B287, our emphasis):[16]

> Now in mathematics *a postulate is the practical proposition* that contains nothing except the synthesis through which we first give ourselves an object and generate its concept, e.g., to describe a circle with a given line from a given point on a plane; and *a proposition of this sort cannot be proved*, since *the procedure that it demands is precisely that through which we first generate the concept of such a figure*

On Kant's interpretation, the same procedure that is going to generate a circle under the Postulate I.3 is also the procedure behind the definition of what is a circle. And it is fair to suppose that he holds a similar opinion about straight-lines. Each of the first three Postulates involves a practical proposition establishing some of the most basic elements of geometry. Although his description of what is a postulate can be understood for the first three, it is not clear that the same holds for the fourth and the fifth Postulates.

Actually, *problemata* were seen by Kant as practical propositions too (*Logik Hechsel*, 1992, p. 88):

> [. . .], and problemata, practical propositions which require a solution.

Moreover (*Jäsche Logik*, 1800/1904, § 38, our emphasis):

> A postulate is a practical, immediately certain proposition, or a principle that determines a possible action, in the case of which it is presupposed that the way of executing it is immediately certain. Problems (*problemata*) are demonstrable propositions that require a directive, or ones that express an action, the manner of whose execution is not immediately certain. [. . .] Note 2: A problem involves (1.) the *question*, which contains what is to be accomplished, (2.) the *resolution*, which contains the way in which what is to be accomplished can be executed, and (3.) the *demonstration* that when I have proceeded thus, what is required will occur.

Beyond any doubt, the treatment of problems as practical questions had already been considered in the history of philosophy. And the canonical examples of problems Kant is refereeing are those in geometry. He underlines three elements involved in problems in the footnote accompanying the quotation above. This is a good starting point for considering the nature of problems. They involve: (1) the question, or what is asked; (2) the resolution seen as something that is going to be executed, (3) a proof that the resolution accomplishes what was asked. The first two elements are common to all problems. The third is characteristic of mathematical problems. These three elements are present in many of the geometric *Propositiones* in Euclid's *Elements*, mainly in those asking a construction like I.1, for example.

[16] See Lassalle-Casanave (2019).

3.2 What is a problem?

As remarked before, Kolmogorov did not answer the question of what is a problem. We can try to organize some ideas departing from Kant's considerations.

Many of the Euclid's *Propositiones* are construction problems. One example is problem III.1: *to find the center of a given circle*.[17] According to Definition I.15, a circle is a plane figure contained by a single line, the circumference, such that all of the straight-lines radiating towards the circumference from one point amongst those lying inside the figure are equal to one another. Thus, historically at least, problem III.1 is not the problem of showing the existence of the center of the circle, since this is already guaranteed to exist according to Euclid's definition of a circle. Also, strictly speaking, there is no construction to be effected, once the circle is already given. The problem for which a solution is then asked is that of finding or determining the center when this is not clear from the figure given, supposing it to be a circle.

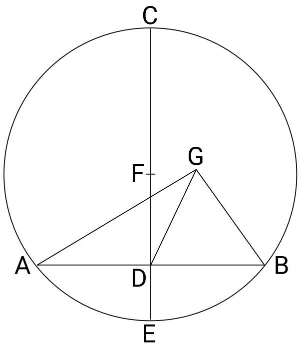

Fig. 1 Prop. III.1

By a series of actions the center of a circle can be found. Why do we say actions? Because the problem is to be solved in the end by employing the first three Postulates which, as Kant pointed, are practical principles. Indeed, the starting step in providing a solution to problem III.1 consists in the production of a chord AB of the circle, that is, the production of a straight-line according to Postulate I.1. And anyone of the endless chords would serve for the purpose.

Actually, the construction given as a solution to problem III.1 is only a description of which actions and in which order they should be taken for determining the center of the circle. It constitutes then a recipe of construction, or resolution in kantian terms.

What is problem III.1 about? It is about "finding the center" and the verb "to find" is a verb used to describe an action. How the problem can be put to someone, a student let's say? It can be put by uttering a command or asking an action: Find the center of circle ABC! Only actions can be commanded. Clearly, it does not make

[17] All quotations of Euclidean Geometry are taken from Fitzpatrick's (2008) translation of Heiberg (1885).

sense to command a usual proposition or a sentence. We assert sentences. The two speech acts are of different nature.

The same terminology that Kant used for describing the content of a command or invitation involving an action is going to be employed here: *practical proposition*. A *problem* is stated by asking or commanding an action whose result and/or execution is unknown or assumed to be unknown. We can separate two aspects of a problem: one corresponding to the speech act involved in uttering the problem as such, asking or commanding; and another concerning the content or practical proposition formulating the action being asked. This distinction seems to be suggested in Kant's passages quoted above. Kolmogorov is consistent in using the concept of problem in connection with actions, but he does not make explicit the two aspects just pointed.

To formulate a problem is the same as to ask or to command an action: to find something, to draw something and — we have reasons to think also that — to prove something. A *solution* to a problem is in consequence a certain organized sequence of actions accomplishing the action commanded or asked. In particular, Postulates can be interpreted as problems of a very specific nature: they are such that the action being asked is assumed to be immediately feasible or supposed to be immediately solvable. This explains why the first three Postulates are not proved. They are the simplest problems to which others are reduced.

But Euclid's *Elements* contain other kinds of *Propositiones* that do not ask a construction. Most of them are what we could call proving problems. *Propositio* I.47 — Pythagora's theorem — is an example. This is a curious case, since I.47 could also have been stated as a construction problem asking to obtain a square equivalent to the sum of other two given squares.

Geometry, in the hands of Hilbert (1899), has received a definite assertional turn in which construction problems as those of Euclid's *Elements* were left out. Nonetheless, the opposite movement of traveling back home might be considered, giving way to a *problemational turn*. That is, a turn in the focus of epistemology making it more sensitive to the original formulation of Euclidean Geometry.

Solutions can be divided in two kinds, which implies that problems are themselves divided in two kinds: token-problems and type-problems. For example, a problem like (1) in Kolmogorov's quotation enumerating problems, the one asking to find four numbers, is a token-problem. It requires a specific action. Its solution, when there is one, involves an act of exhibition of a result or a state-of-affairs. A problem like (3) is a type-problem, since the solution expected is a recipe/algorithm showing how to fulfill what is being commanded or asked for given x, y and z, as parameters.

Actually, all solutions can be characterized as recipes. Any token-problems can be assimilated for simplification to a type-problem requiring a recipe containing a final act of exhibition. In other terms, here the recipe is to be thought as being recoverable from the succession of actions that produced the result being exhibited.

From a philosophical point of view, the investigation of problems belongs at the same time to theoretical philosophy and to practical philosophy. A solution is a recipe describing actions which when realized will solve a problem, while a problem is equivalent to ask or command an action. This seems to be the borning star of mathematics.

3.3 The nature of solutions and problems in Kolmogorov's perspective

Reduction Semantics above has been proposed as an elucidation of the problem interpretation by Kolmogorov. In Reduction Semantics, problems and solutions are treated in an unified fashion and this fact is going to be used for explaining bellow the relative order among Postulates and *Propositiones* in ancient geometry.

Nonetheless, there is a question that requires our attention before proceeding with the analysis of problems. The question concerns the two kinds of objects that Kolmogorov's calculus of problems should deal with: propositions and problems.

Problem (5) in the list of examples quoted above is a conditional problem asking to find a way to express number e as a rational expression. It has in the condition position a proposition: "that the number π has a rational expression". Problems are uttered with imperative mood while sentences and propositions are uttered with the indicative mood. Example (5) has a condition in the indicative mood and the conditioned in the imperative mood. No doubt, (5) states a problem and its main verb is in the imperative mood. How propositions should be dealt with in the context of problems?

As mentioned before, typed λ-calculus disposes of two different structures for representing problems and solutions in one same formalism. Formulas are types for λ-terms. Solutions are λ-terms. Problems are regarded as formulas, that is, as types. Propositions or sentences are normally considered a subset of the formulas. Did Kolmogorov intented this classification? We have reasons to doubt it, among other things because different moods are used when stating problems and propositions.

Propositions or sentences can be dealt with in the context of problem interpretation inside some specific kinds of problems. Actually, Kolmogorov stated, although only in an implicit way, what looks like a unifying perspective. It appears when he is discussing the principle of excluded middle. Recall that for him the expression $\vdash a \vee \neg a$ should be read as asking to (*ibid.*, p. 156):

> [...] find a general method which for any problem a allows one either to find its solution or to derive a contradiction from the existence of such a solution! In particular, *if the problem a consists of proving a proposition*[18], one must have a general method which allows one either to prove each proposition or to reduce it to a contradiction.

Proving propositions then seems to be one important kind of problems and the answer to our question.

Assuming that proving propositions is a legitimate distinct kind of problems, then the concept of a problem becomes more general than the concept of theorem. Now, all *Propositiones* in Euclidean Geometry can be regarded as problems, some are mainly construction problems and others are mainly proving problems. In this sense, a calculus of problems would embrace all *Propositiones* without requiring any substantial change in the way they were formulated, instead of transforming construction *Propositiones* into existence statements or something similar.

Problems which consist in proving a proposition p are going to be represented by the expression "(prove p)". Next, assuming also that the expression "(deduce q

[18] Our emphasis.

from Ω)" represents a basic kind of problem, where q is a proposition and Ω a set of hypotheses, the special case where Ω is empty constitutes a way for characterizing what is to prove a proposition p, that is:

(†) (prove p) $\equiv$ (deduce p from $\emptyset$).

Thus, if it has been shown that (prove p) $\Vdash$ (prove q), then $\Vdash$ (prove p) $\supset$ (prove q), by clause ($\supset^r$). This last means $\Vdash$ (deduce p from $\emptyset$) $\supset$ (deduce q from $\emptyset$), according to (†). This corresponds to a transmission of demonstrability, similar to an admissible rule. Observe that it is distinct from $\Vdash$ (prove if p, then q) since this last means (deduce if p, then q from $\emptyset$) according to (†), and this last means (deduce q from p).[19]

Verbs commanding an action were used for talking about deductions and proofs since the expression of problems and solutions require practical propositions. Deductions and proofs become then just the trace of actions conveyed in the practical proposition. Two examples in Section 3.6 illustrate the point.

After examining in a few words what might be proving problems, we can deal with problem (5) given as an example by Kolmogorov. Recall that it is formulated as :"assuming that the number π has a rational expression, $\pi = m/n$, find a similar expression for the number e". Two distinct interpretations seem to be possible.

First, the problem can be understood as meaning: (prove $\pi = m/n$) $\supset$ (find ...). This is a solvable problem, by the mere fact that (prove $\pi = m/n$) $\Vdash \bot$. Euclidean Geometry contains conditional problems where the condition is a proving problem. *Propositio* I.22 is an example. The construction of the triangle as required depends on supposing the provability of a certain relation between the three given straight-lines, A, B and C. Any of them must be shorter than the sum of the other two. The mere supposition that three straight-lines are in this relation does not guarantee that the procedure can be carried adequately.

Second possible interpretation, (5) could also be understood in a more general sense as being a problem to be solved under certain conditions. That π is not a rational number is a fact. Nonetheless, nothing forbidden us of supposing contrary to the fact that π were equal to m/n for m and n rational numbers. However, to derive a contradiction from this supposition is to misunderstand the point of the counterfactual. The condition in (5) could be interpreted as a counterfactual supposition. And, if this were what Kolmogorov intended, then the meaning of (5) would be different from the meaning considered in the precedent paragraph. It would be equivalent to state the following problem: find natural numbers x and y such that $(x/y)^{im/n} = -1$, supposing m and n to be natural numbers and i to be the imaginary number.

---

[19] The action of deduction can be defined by reflecting inside the object language the content of the focus clauses for Reduction Semantics. They can be characterized relative to the deduction problem as follows:

1. (deduce p and q from Ω) $\equiv$ (deduce p from Ω) $\wedge$ (deduce q from Ω)
2. (deduce p or q from Ω) $\equiv$ (r)(((deduce r from Ω, p) $\wedge$ (deduce r from Ω, q)) $\supset$ (deduce r from Ω))
3. (deduce if p, then q from Ω) $\equiv$ (deduce q from Ω, p)
4. (deduce absurd from Ω) $\equiv$ for any atomic p (deduce p from Ω).

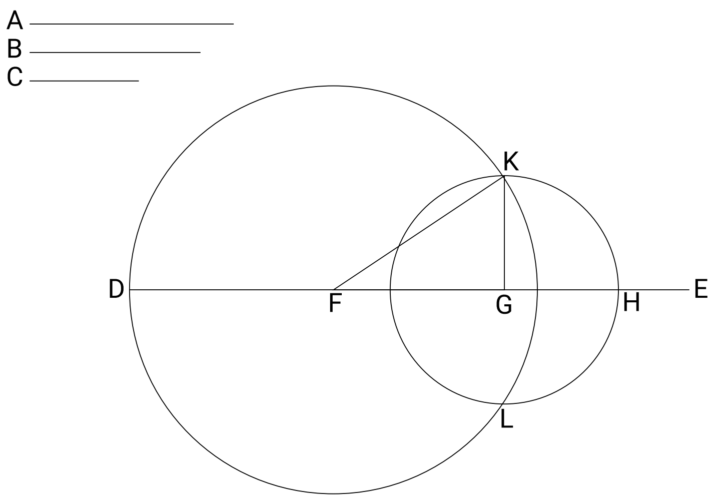

Fig. 2 Prop. I.22

We claim that a second kind of conditional problems should be considered in a calculus of problems, one in which the condition is not itself a problem, not even a proving problem. The condition involves just a description of a putative situation. There are related examples in Euclidean Geometry. One is the correctness-proof for the solution to the problem of finding the center of a circle in *Propositio* III.1. It starts by a contrary to fact supposition: that the point F is not the center. This supposition is the beginning of a *reductio* proof. It is not a supposition of having proved that F is not the center, even because exactly the opposite had been just showed by the construction. We come back to this issue in Section 4.2 when discussing the relation between hypotheses and problems.

Apparently, Kolmogorov did not consider the second interpretation above with respect to example (5) or, at least, he assumed it to be resolved with the first interpretation. And since our subject here is his calculus of problems, we put the alternative interpretation aside for the time being.

Next we turn to the analysis of solutions in ancient geometry. That is, we proceed to the second task concerning the adequacy question relative to Reduction Semantics.

3.4 Problems in ancient geometry

Now turning to Book I of Euclid's *Elements*, let's proceed with the epistemological analysis of problems on the basis of Reduction Semantics. What should be expected from it is a certain homogeneity in treating *Propositiones*, Postulates and Common Notions as problems.

Traditionally, logic has been used to offer an analysis, even if partial, of the proof steps and the structure of Euclidean Geometry. Some believe that since Aristotle, at least, proofs are taken to be discursive sequences of assertions or of declarative sentences. But this perspective does not correspond either with what we read in the three first Postulates or many other *Propositiones*, in particular the first three in Book I. Hence, it is a striking fact that the logical analysis through declarative sentences of Euclid's *Elements of Geometry* starts with an inadequate concept. When Hilbert (1899) came to light things changed, but then all *Propositiones* became declarative statements. Points, lines and planes became three distinct system of things related by declarative axioms.

Together with other authors, we think that it is possible to provide a logical analysis of ancient geometry maintaining its original formulation with all richness of its *Propositiones*. From our perspective it is Kolmogorov's merit to have devised an approach that can be fruitfully applied to ancient geometry, although it remained largely non-explored. We clearly mean his problem interpretation. So, next, the problem interpretation is employed as an epistemological tool for the analysis of the *Elements* via Reduction Semantics.

One attempt in such direction can be found in von Plato and Mäenpää (1990). The authors develop an investigation of Euclidean Geometry based on Martin-Löf's Type Theory assuming it to be an elucidation of that interpretation. They claim that the first three Postulates can be assimilated to constructive functions. According to them, (*ibid.*,p.281):

> The construction postulates lay down the permitted means of producing finite straight-lines, [...]. The functionality of postulates suggests a way of rendering them into the general pattern of natural deduction rules used in intuitionistic type theory. Its inferences may be viewed as functions from premises to conclusions. This proof functionality is explicitly recorded in proof objects, that is, in the objects given in the left side of judgements of the form a:A. It is judgements, not propositions, which figures as premises and conclusion in an inference rule.

This interpretation is further discussed and criticized in Naibo (2018). More recently, a deeper exegesis of Euclidean Geometry partially based on von Plato and Maenpaä's approach has been proposed by Sidoli (2018). But Sidoli does not focus the concept of problems or its elucidation. Also, together with von Plato and Maenpaä, he assumes that points are parameters of constructions. But there are reasons to disagree with them.

Postulate I.1 is interpreted by von Plato and Maenpaä as a constructive function producing a straight-line once two points are given as parameters, similar to a natural deduction rule. But it is doubtful that straight-lines depend on points as parameters in order to be produced. There are at least two reasons for doubting.

First, because the production of a straight-line can just start in a place whatever not determined beforehand and it can stop in a point not determined with antecedence. This is the case of the starting act of resolution of problem III.1 which consists in finding the center of a given circle as in Figure 1. This act is that of producing chord AB at random for the given circle ABC. It is a non-deterministic act. If conceived as a constructive function, it is doubtful that points A and B were previously established when Postulate I.1 was supposed to be applied. And nothing hinders one to have picked two points whatever over the circumference before drawing the line, similar to what was done in the resolution of *Propositio* I.9: *let the point D have been taken at random on AB* . . .. Most striking yet is that, after the chord is drawn, the perpendicular over the middle point D in the chord is raised and, although belonging to the circumference since the beginning, point C cannot be pinpointed before the perpendicular has been produced in Figure 1.

Second, Postulate I.2 — . . . *to produce a finite straight-line continuously in a straight-line* — does not even mention points, much less a stopping point.

A mental image could be of some help here. Think about the points as being mainly the end or the start of a drawing/production. This is in accordance with Definition I.3. "Ends", as such, have no parts. Also, without an actual drawing there is no starting or stopping points. The first three Postulates strongly suggest such a drawing or action, although in the case of Postulate I.3 the center and the *radius* have to be provided before.

The three beginning *Propositiones* of Book I are naturally read as problems not as theorems. Their resolutions show how to fulfill what is being asked or commanded: a construction. Actually, the resolution text in the *Propositiones* contains two principal parts among others: *kastaskeue* and *apodeixis*. The *kastaskeue* contains a recipe or solution for the construction problem. It describes certain actions to be effected according to the Postulates. To draw a circle, to draw another circle, to draw a straight line, to draw another straight-line. The *apodeixis* contains a proof that this solution really produces what is demanded in the *Propositio*. Once I.1 — . . . *to construct an equilateral triangle on a given finite straight-line* — has been resolved and the solution has been proven correct, it can be used for obtaining a solution for I.2 — . . . *to place a straight-line equal to a given straight-line at a given point (as an extremity)*. Therefore, the *Propositio* that was formerly interpreted as *a problem* is now used as part of a *solution* for a new problem.

The change in perspective means that the distinction between problems and solutions is superficial and it depends of a certain history of accomplishments. First an action is seen as problematic. Then a procedure for effecting the action is offered, and the problematic action now just becomes a piece of knowledge that can be employed in the solution of another problem.

Proving problems were briefly described in the previous section and they must be considered as part of Kolmogorov's calculus of problems. Hence, they are part of the language of problems considered in Reduction Semantics.

Propositiones similar to I.47 — i.e., Pythagoras' — are theorems. Their proofs also contain *kataskeue* and *apodeixis*. When proving I.47, similar kinds of problems are to be solved again: first, a construction and, second, a proving problem. In fact both are intertwined and, although proofs in modern reconstructions are composed of assertions, the original geometrical proofs are never exclusively discursive since they involve practical problems: those involved in the construction.

We claim that Reduction Semantics fits the *Propositiones* of Book I the way they were formulated. Before the analysis goes on, it is necessary to effect an examination of Postulates and Common Notions within the perspective proposed.

3.5 Common Notions and Postulates

Common Notions are generic in the sense that they apply to different elements of geometry: lines, angles, triangles, etc. For example, in Common Notion I.2 — *if equal things are added to equal things then the wholes are equal* — the word "things" is a parametric word, which suggests that it can be interpreted as a schematic deductive rule. The same holds for Common Notions I.1 to I.4. Common Notion I.5 is an schematic axiom — *the whole is greater than the part*. Common Notions are either a proving problem assumed to be solved, as in I.5, or they are basic reduction rules as in all other cases. For example, I.2 can be rendered for things A, B and C as: $\Vdash$ (prove if $(A = B)$, then $(A + C = B + C)$). From it follows: (prove $A = B$) $\Vdash$ (prove $A + C = B + C$).

Postulates are the fundamental principles governing geometry. The first three Postulates of Book I are of a practical nature since they are about actions: ... *to produce a straight-line from any point to any point*; ... *to produce a finite straight-line continuously in a straight-line*; ... *to produce a circle with any center and radius*. We claim that they can easily and fairly be understood as problems, even if a very simple one.[20] But they are problems of a special kind. They are supposed to be solvable. This supposition does not require to suppose together any description of how to proceed or which tools should be employed. The alternative of assuming them to be solved seems to be stronger and unreasonable if it involves to suppose an infinity of acts to have been effected. The first three Postulates are rendered as:

I.1 $\Vdash$ (draw [any] finite straight-line XY), where X and Y are the two extremities of the line and they might be determined *a posteriori*;
I.2 for any straight-line AB: $\Vdash$ (extend AB into AX), where X might also be determined *a posteriori*;
I.3 for any point A and any straight-line AB: $\Vdash$ (draw a circle of center A and radius AB).

Postulate I.5 involves the problem of determining when two straight lines are not parallel. According to Definition I.23, straight-lines are parallels if they do not meet each other when indefinitely extended in any direction. And when another straight-line crossing both form internal angles less than two right angles, then they are not parallels. That is, the condition of making angles less than two right is sufficient for showing that they meet on the side they are less than that by indefinitely extending such straight-lines. This Postulate is then a specific principle for straight-lines guaranteeing a solution to the construction problem of finding the meeting point of two straight-lines when a certain condition is fulfilled. This is indeed a construction problem, hence the concept of postulate as presented by Kant fits here perfectly. It becomes, for three straight-lines AB, CD and EF, such that EF cuts AB in the point G and cuts CD in the point H: (prove $BGH + DHG < 2R\angle$) $\Vdash$ (find the intersection point of AB and

[20] The greek word *êitêsthô* is the first occurring in Postulate I.1. It is an imoerative verb and it means "ask for", "demand", which is exactly what one says when stating a problem.

CD, by extending them from B and D).[21] This practical principle effects a selection of which surfaces are to be considered in geometry: only flat surfaces.

Finally, Postulate I.4 — *that all right angles are equal to one another* — is a declarative statement. It does not look like a practical principle. It can be seen as an answer to the problem: when two distinct right angles are equal? We assume it as a solved proving problem differing from Common Notions, since it is specific, not generic. It states that any angles falling under Definition I.10[22] are equal angles. Depending on the surface being considered, this is not a trivial matter. It holds only for homogeneous curved surfaces, not for conic surfaces, for example. It is rendered, for any two angles ABC and DEF: $\Vdash$ (prove if (ABC is a $R\angle$ and DEF is a $R\angle$), then $ABC = DEF$).

The proofs of the *Propositiones* achieve a problem reduction: from a more complex problem to less complex problems until the bottom — the Postulated problems and the Common Notions, in principle. Postulates and Common Notions are supposed to be solvable. And, of course, it does not make sense to ask a correctness-proof for both. Rephrasing Kolmogorov (1932, p. 151), but avoiding any commitment with the existence of solutions, and keeping in mind that problem reduction involves a specific relation between solutions already pointed before:[23]

> If we can reduce ~~the solution of~~ problem b to ~~the solution of~~ problem a and ~~the solution of~~ problem c to ~~the solution of~~ problem b, then ~~the solution of~~ c can also be reduced to ~~the solution of~~ a.

3.6 Towards a general theory of problems, or how to solve it?

Kolmogorov's point of view plunged intuitionistic logic into a general theory of problems. Effectively, there are reasons to think that the schemes of solutions for geometrical problems offer a model of how to approach logic in terms of problems and their solutions, following the structure of Euclid's *Elements*.

Concerning a general theory of problems, Veloso (1984, p. 29) points out that problem decomposition is one of the main strategies for solving problems:[24]

> A common approach to solving a problem is to partition the problem into smaller parts, find the solutions for the parts, and then combine the solutions for the parts into a solution for the whole.

We claim that logical constants give the patterns according to which problems should be decomposed and solved. From our perspective, it was Kolmogorov who

[21] It must be reminded that Kolmogorov's problem interpretation is silent about conditional construction problems with propositional conditions, which would be our preferred solution for this case. See Section 4.2.

[22] Right-angles are defined as the angles formed by the incidence of two straight lines making equal adjacent angles.

[23] That is, when problem b reduces to problem a, any solution of a gives rise to a solution of b.

[24] The original is in Aho, Hopcroft, and Ullman (1975, p. 60).

first devised it in his problem interpretation. Together, reduction and decomposition are the two main general strategies for solving problems, and they form the core of Reduction Semantics. Decomposition and reduction are going to be illustrated next.

Let's consider now how the resolution of *Propositio* I.1 takes place.

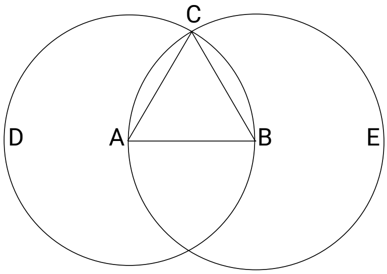

Fig. 3 Prop. I.1

The resolution of I.1 contains a construction solution part — to build the equilateral triangle over straight line AB — which is solved first; and it contains in the sequence a proving solution part — to prove for the three sides of the triangle that $AB = BC = AC$.

The construction part is done in two steps. First, the two circles of radius AB and BA are drawn/produced according to Postulate I.3, and they could be done in any order. Second, and only after the first part is done, the straight lines AC and BC are drawn/produced according to Postulate I.1, and they also could be done in any order. Drawing a circle and drawing a straight line are two problems that we assume to be solvable, since they are postulated. Conjunction ($\wedge$) is a natural way of composing actions when the order is irrelevant: (draw circle BCD of radius AB with center A) $\wedge$ (draw circle ACE of radius BA with center B). Conjunction is used again in the composition of the subsequent actions: (draw straight line AC) $\wedge$ (draw straight line BC). But, the next step requires the distinction of a before and an after, which we are going to represent by "$\rightsquigarrow$", a before-after conjunction:[25]

(‡) $\Vdash$ {(draw circle BCD of radius AB with center A) $\wedge$ (draw circle ACE of radius BA with center B)} $\rightsquigarrow$ {(draw straight line AC) $\wedge$ (draw straight line BC)}.[26]

At this point the construction problem is solved, a triangle was drawn/produced. What is the solution to the problem of drawing/producing triangle ABC? It is the composite action described in (‡). Since each action in (‡) is a problem considered to be solvable, then the whole complex structured action (‡) also shows the problem of drawing/producing an equilateral triangle to be solvable since Reduction Semantics explains how to understand each logical constant employed. Next, the solution of *Propositio* I.1 requires a verification showing that the triangle ABC is equilateral.

[25] The before-after conjunction is expressed in programming languages by the semicolon ";". But, as a logical symbol it might cause some confusion, reason why we do not use it.

[26] It can be read as follow: {(draw circle BCD of radius AB with center A) and (draw circle ACE of radius BA with center B)} and after {(draw straight line AC) and (draw straight line BC)}.

Proving that the triangle obtained is equilateral is the proving problem part. For proving problems, Postulates I.4 and I.5, the Common Notions, and the Definitions, all together, contribute to establish the basis of what is considered solvable. Hence the problem (prove $AB = BC = AC$) is decomposed (and solved) into:

(#) $\Vdash$ ((prove $AB = BC$) $\wedge$ (prove $AB = AC$)) $\rightsquigarrow$ (prove $BC = AC$)

Definition I.15 about circles establishes $\Vdash$ (prove $AB = BC$) as positively solvable, since both lines are radius of the same circle *ACE*. A similar reasoning holds for $\Vdash$ (prove $AB = AC$). In third place, from the two precedent equalities by using Common Notion I.1, $\Vdash$ (prove $BC = AC$) is established as positively solvable. That is, (#) makes explicit the decomposition of the problem (prove $AB = BC = AC$) into subproblems until reaching solvable subproblems. Additionally, the expression gives the trace of the proving procedure.[27] The expression (#) is the solution to the problem (prove $AB = BC = AC$) which turns now to be considered positively solvable.

The whole final expression with the solution of *Propositio* I.1 is:

$\Vdash$ {(draw circle of radius AB with center A) $\wedge$ (draw circle of radius BA with center B)} $\rightsquigarrow$ {(draw straight line AC) $\wedge$ (draw straight line BC)}} $\rightsquigarrow$ {((prove $AB = BC$) $\wedge$ (prove $AB = AC$)) $\rightsquigarrow$ (prove $BC = AC$)}.

That is, the solution of a problem can be obtained by decomposition of a problem into subproblems until arriving to solvable subproblems and then composition of these subproblems already considered solved.[28]

The before-after conjunction is usually expressed in mathematics by means of function composition. If $f(x)$ and $g(x)$ are two functions, then $fog(x)$ represents the ordered application $f(g(x))$. Nonetheless, the before-after conjunction can be used when functions cannot be used. In the case of geometry, it is doubtful that drawing/extending a straight-line should be considered a function depending on previous determined parametric points, since the act of drawing/extending a straight-line might create, so to say, its starting and stopping points. That is, the actions corresponding to Postulate I.1 and I.2 cannot be identified with a function, and much less with a constructive function. As remarked, I.2 does not even mention an stopping point. Although powerful, the functional interpretation does not seem to be the best tool for effecting an epistemological analysis of ancient geometry.

After the example of a construction problem, let's examine a theorem *Propositio*, the proving problem I.15: *if two straight-lines cut one another then they make the vertically opposite angles equal to one another*. The proof starts by the following sentence: ($) *Let straight lines AB and CD cut one another at the point E*. It is

[27] All inferential relations can be read from this trace. Of course, this is not a natural deduction tree, but such a tree can be build on the basis of the trace given.

[28] One item might seem to be missing in the given solution: the action of determining the intersection point C. Notice that the Euclidean text contains a reference to point C in both circles drawn: circle *BCD* and circle *ACE*, thus apparently bypassing the issue, but being undeterministic about which intersection to consider, reminding that there are two of them. What is seen in the final solution is a reflection of the text in the *Elements*. If the action of determining the intersection point were added to the solution, we would be falsifying the original text.

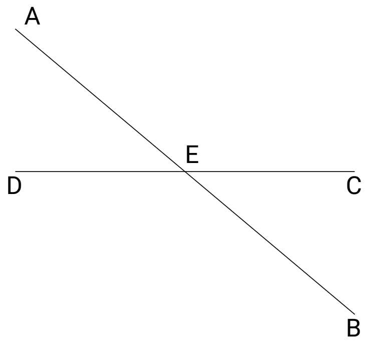

Fig. 4 Prop. I.15

required to show that angle AEC is equal to DEB (and CEB to AED). It is a conditional proving problem whose condition is: *if two straight-lines cut one another . . .*. The problem has to be formulated as: (deduce $AEC = DEB$ from [the supposition that] AE cuts CD). It cannot be formulated as (prove AE cuts CD) $\Vdash$ (prove $AEC = DEB$) because the antecedent problem does not seem to match the sentence ($), "let . . ." does not mean "suppose it has been proved that AE cuts CD". We come back to this point in Section 4.2 discussing the notions of hypothesis and assumption.

Suppose that the straight-line AE cuts CD. Hence, AE stands on the straight-line CD, making angles CEA and AED. The sum of the angles CEA and AED is thus equal to two right-angles according to *Propositio* I.13. That is, the problem $\Vdash$ (deduce $CEA + AED = 2R\angle$ from AE cuts CD) is solved. Also DE stands on AB, hence $\Vdash$ (deduce $AED + DEB = 2R\angle$ from AE cuts CD) is also solved by I.13 too. Next, by the Common Notion I.1, $\Vdash$ (deduce $CEA + AED = AED + DEB$ from AE cuts CD) is then solved. And next, subtracting AED from both sides, by Common Notion I.3, $\Vdash$ (deduce $CEA = DEB$ from AE cuts CD) is then solved. That is, $\Vdash$ (prove if AE cuts CD, then $CEA = DEB$). End of the proof. In this case, no construction is added to the given figure, so there is no *kataskeue*. The *apodeixis* covers the whole resolution of the proving problem and here is the final solution:

$\Vdash$ $\{\{\{$(deduce $CEA+AED = 2R\angle$ from AE cuts CD) $\wedge$ (deduce $AED+DEB = 2R\angle$ from AE cuts CD)$\}$ $\leadsto$ (deduce $CEA + AED = AED + DEB$ from AE cuts CD)$\}$ $\leadsto$ (deduce $CEA = DEB$ from AE cuts CD)$\}$ $\wedge$ $\{\ldots \leadsto$ (deduce $CEB = DEA$ from AE cuts CD)$\}$

It must be kept in mind that this expression is a rough statement of the solution in terms of actions. For example, the action of subtraction is not made explicit, although its use has been made clear. Also the deduction relating the conditions of I.13 and I.15 are not represented above. Solutions are communicated and as such the level of detail is variable and depends on the expertise expected from the audience, if they do not suffer of syntaxism.

As observed, *Propositio* I.15 starts by: *if two straight-lines cut one another . . .*. This is the statement of a condition that selects the situation to be considered and, of course, not every situation lies under this condition. Parallel lines do not, for example. But other configurations like that where with lines AB and FC in Figure 5 do not either, even if they are not parallels.

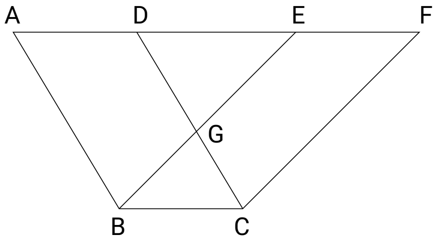

Fig. 5 Prop. I.35

Next, the condition is related to another condition, the condition in the formulation of *Propositio* I.13: *if a straight-line stood on a(nother) straight-line,* .... In other words, any situation satisfying the identifying condition of *Propositio* I.15 has to be already a situation satisfying the identifying condition of *Propositio* I.13.

Now, some words about the before-after conjunction constant are required once both solutions in the two examples above employed it.

4 Logic and problems

4.1 Logical constants and algorithms

The question of what is a logical constant is deeply difficult and interesting. Kolmogorov's paper touches the heart of this question since he states that his calculus of problems should substitute intuitionistic logic. He also expressed he had hoped that the schemes of solutions of problems would become an important part in courses of logic.

There are distinct competing theories of what is logic. Many times, a decision of how to interpret logic already involves a decision of what is a logical constant. Logic has been assumed to be the science of logical truths.[29] However, it also has been assumed to be a science of formal deductions.[30] A third possibility is to conceive it as the central part of a general theory of problems or of a general theory on problems resolution. We think that this alternative differs from the previous two and, at the same time, it extends them once they are limited to proving problems. The third alternative deserves investigation, since the problem approach allows an homogeneous epistemological analysis of items in the history of mathematics the way they were

[29] See Gómez-Torrente (2019).

[30] See Došen (1989, p. 364).

formulated, without twists. After all, the basic sources for logic theorizing are the argumentative practices historically found mainly in mathematics.

Some might wonder if the before-after conjunction could not be defined by the other intuitionist constants. The formula $a \wedge (a \supset b)$ seems to be the candidate that comes to mind. That it is not can be realized when asking which of the two problems a or b should be taken as the "first" problem to be solved. This formula does not distinguish a first or a second element. Indeed, a problem like $a \wedge (a \supset b)$ can be settled by first solving problem b and next solving problem a, but this is unfaithful to the intuitive meaning of the before-after conjunction.

Examples of the use of before-after conjunction were given above. In the solution of *Propositio* I.1, the production of straight-lines AC and BC depends on having beforehand circles ACE and BCD as well as their intersection points. If the circles were not produced, there would be no intersection point C for drawing the two straight-lines. Point C is not determined by a constructive function over circles ACE and BCD as parameters. Only after the circles are produced, there will be two distinct intersection points. Any one can be picked as point C for drawing the triangle, it does not matter which. From this perspective, C is just a stopping point common to both circumferences produced in accordance with Postulate I.3.

The semantical clauses for the before-after conjunction are as follows:

In the repertoire:

($\leadsto^l$): $\Gamma, c \leadsto d \Vdash e \cong$ given any problem a ((first $a \Vdash c$, and after $a \Vdash d$) $\Rightarrow$ $\Gamma, a \Vdash e$)).

In the focus:

($\leadsto^r$): $\Gamma \Vdash c \leadsto d \cong$ (first $\Gamma \Vdash c$, and after $\Gamma \Vdash d$).

The before-after conjunction is not among the usual intuitionistic logical constants, neither among the classical. Function composition has many times been used for obtaining the before-after effect. We repeatedly pointed above why we prefer to consider it a logical constant for problems. Some basic actions do not fit in the role of functions.

More important, with the before-after conjunction the logical relations in the solution of a proving problem can be recovered since this kind of conjunction clearly distinguishes the temporal order in which the actions were resolved or established, thus allowing one to obtain the trace of the proof of an assertion. The following elimination rules partially state the inferential behaviour of the before-after conjunction: $c \leadsto d \vdash c$ and $c, c \leadsto d \vdash d$. Then, clearly, $c \leadsto d \vdash c \wedge (c \supset d)$. But the reverse is not correct as we already argued above.

The logicality of the before-after conjunction is a matter for debate. However, if Kolmogorov's problem interpretation of intuitionistic logic is regarded as a defensible position, then the absence of the before-after conjunction in the intuitionist description of the propositional logical constants would mean that this description is incomplete.

The before-after conjunction is at work everywhere in Euclidean Geometry. It corresponds to a sequential decomposition of a problem in view of producing its solution, one of the basic cases covered by the eternal strategist's *dictum*: divide and conquer. And since ancient geometry is a reference source for the concepts of proof and inference, any attempt at logic theorizing should keep a minimal *adequatio* with respect to what one can find in ancient texts.

Geometric proofs are the communication of how to solve certain problems, of which there are two important kinds: construction problems and proving problems. In general, the *kataskeue* contains the solution of a construction problem; the *apoidexis* contains the solution of a proving problem. But they are often mixed. In both cases what is communicated is a recipe, an algorithm. The communication of a geometric proof is then basically the communication of an algorithm. The structure of such communications has to be enough for understanding and undertaking the acts promoting evidence for a *Propositio*. Observe that the communication of an algorithm may sometimes hide evident or repetitive steps in order to make the communication shorter.

Some parts of the *Propositio* prepare or summarize the communication of the algorithm. The non linguisitc part is the accompanying diagram. It is not part of the algorithm. Diagrams exemplify the data that serve as the object modified, produced or examined in the algorithms being communicated, according to de Campos Sanz (2021). Usually, diagrams represent the final state of the drawing/production and in most cases it does not contain the traces of the actions comprised in the recipe as pointed by Sidoli (2018).

We next consider one particular item found on this argumentative practices: the use of hypotheses/suppositions in the solution/proof communications occurring in Euclid's *Propositiones*.

4.2 Hypotheses, assumptions and problems in geometric proofs

In the history of mathematics, problems seem to be by large the main theoretical concern, being later displaced by the assertion-theoremhood perspective. Since XIX$^{\text{th}}$ logic has become also driven by the assertion-theoremhood perspective. A valid inference is defined as a necessary relation holding among premiss-assertions and a conclusion-assertion. Nonetheless, when suppositions enter the picture things become delicate. Assumptions are sometimes conceived as a special kind of assertion for which we lack a proof and whose provability is then supposed.

In the context of construction problems, there is an exact correlate of assumptions conceived in the way just described. *Propositio* I.22 is a construction problem whose solution depends on supposing that a proving problem has been solved: *to construct a triangle from three straight-lines which are equal to three given. It is necessary for two taken together in any [way] to be greater than the remaining, ...*. Thus, not any trio of straight-lines fits the restriction. What is here being supposed is that a proving problem has been solved: the addition of any two straight-lines must be greater than the third remaining. Indeed, the solution described in the *kataskeue* of I.22 might not work properly if the straight-lines being used were not in the mentioned relation. Thus, here is an example of an assumption in the mould described, that is, the supposition of having a solution to a proving problem as the starting point for the resolution of I.22.

Nonetheless, there are many other cases in which a certain condition is supposed to hold without supposing that a proving problem has been solved. Take *Propositio* I.15: *if two straight-lines cut one another then they make the vertically opposite angles equal to one another.* The resolution of this proving problem starts with a simpler supposition: that they cut one another. It seems wrong to understand it as an assumption. The condition of cutting one another is the mere description of a possible situation. This situation could be produced in a myriad of different ways, it does not matter which. It even might be the case that we do not know how the straight-lines were produced. What matters is if we take the drawings as straight-lines and we suppose them to fall under the condition described, or not. The diagram in Figure 4 that comes together with the *Propositio* is just an exemplification of such a situation and, of course, the reader should suppose that it falls under the condition stated. In this sense, it also does not matter if the straight-lines drawn are not perfect straight-lines. It is enough to suppose that they are, in order to understand the proof establishing the equality of angles.

The solution for I.15 starts by noticing that the situation of cutting one another falls under the condition of *Propositio* I.13 which also describes a situation: *if a straight-line stood on a(nother) straight-line makes angles, it will certainly either make two right-angles, or (angles whose sum is) equal to two right-angles.* In both cases, no assertion is being made in the condition. The situations are merely considered and one is related to the other. In I.15, under the supposition that the situation is of cutting one another, it follows that it should also be a situation of standing on another, thus opening the way for using I.13 in the solution for I.15, as already described previously. Observe that there is no Definition, Common Notion, or Postulate, guaranteeing that any situation of cutting one another falls under the situation of standing on another. The comprehension of such a link seems to depend on perception and on intelect.

Concerning assumptions and hypotheses, it seems important to adopt a distinction. The concept of *assumption* should be reserved for those cases where either a proving problem is supposed to be solved or a construction problem is supposed to be solved. Assumptions are then hypotheses of a specific nature: those where we suppose the validity or the possession of a solution for a problem, I.22 being an example.

The concept of *hypothesis* is more general than that of assumption. There are cases where a situation is being supposed but this cannot be assimilated to the supposition that a proving problem or a construction problem is solvable. We claim that *Propositio* I.15 illustrates it. Hypotheses in a large sense are the supposition that a given situation falls under certain putative identifying conditions.

There are many examples of hypotheses in Euclid's text. For example, the proof of correctness for the solution of finding the center of the circle in *Propositio* III.1 starts by a counterfactual hypothesis: ... *I say that (point) F is the center of the [circle] ABC. For (if) not then, if possible, let G (be the center of the circle), ...*. This hypothesis is the starting point of a *reductio* reasoning. That there should be a center G different of F in the circle just describes a situation that has been deduced from the counterfactual hypothesis and the fact that any circle has a center. The hypothesis does not seem an assumption, since this would be tantamount to suppose the possession of a construction showing that F is not the center of the circle or, at least, showing that a different G is the center of the circle. But what sense has such an assumption since the solution to the construction problem has just determined F as the center? The correctness-proof for F being the center seems to be in fact a proof of the statement that no other point inside the circle can be the center, established by a *reductio* proof.

Most of the hypotheses in Euclidean Geometry seem to be of this more general kind, not assumptions. If this is indeed the case, then the proving problem in which they appear involves a conditional proposition. The point certainly deserves a deeper investigation, but we leave it for future work.

5 Conclusion

Contrary to Kolmogorov's expectation, the calculus of problems has not become a substantial part of contemporary logic courses. Neither his problem interpretation has become widespreadly known in the community. If the above appreciation of both subjects is faithful, then we have reasons to regret that his expectations were deceived. Anyway, his work has been seminal for the development of Intuitionistic Type Theory, as stressed in Coquand (2007).

The above investigation examined the problem interpretation of intuitionistic logic and advanced Reduction Semantics as a way to further elucidate the conceptual structure in this interpretation, with some minor adjustments. Reduction Semantics brings together two main strategies for problem solving, thus giving the basics of general solution schemes. These strategies are reduction among problems and decomposition of problems. The reduction relation has been assumed as the basic semantical relation over which all patterns of problem composition-decomposition are characterized. These patterns happen to be the old well known logical constants, as Kolmogorov seems to have anticipated.

Reduction Semantics was also employed in an epistemological analysis of Euclidean Geometry. Besides construction problems, *Propositiones* like I.15 and many others are seen as proving problems. Again, Kolmogorov seems to have fully realized it, opening the door this way for considering all Euclid's *Propositiones* as problems by unifying construction problems and proving problems in the same approach.

Two points deserve to be stressed from the above investigation. First, intuitionistic logic becomes the core of a general theory of problems if we accept Kolmogorov's thesis that this logic is a calculus of problems. Second, this perspective leaves the way open to extend the usual set of intuitionist logical constants. The before-after conjunction was pointed as an example of a constant that cannot be defined employing the other constants. In the solution of any proving problem it allows one to keep trace of the reasoning.

What has been until now called proofs in Euclid's *Elements* emerge as the communications of a problem resolution. These must be seen as the transmission of an algorithm since virtually all *Propositiones* may be treated as problems, in view of Section 4.1. Diagrams just accompany the communication of the algorithms. They exemplify the objects being dealt with in the algorithm, similar to the role of syntactical data in the tape of a Turing machine, according to de Campos Sanz (2021).

Finally, the epistemological analysis of Euclidean Geometry has noticed that hypotheses are used in distinct roles inside *Propositiones*, the most general being that of identifying criteria for situations. They also appear as the supposition of having a solution. For this last case the word "assumption" has been reserved in order to distinguish it from other uses. These other uses include the case of counterfactual hypotheses, like that in the correctness proof of III.1. The existence of counterfactual hypotheses in Euclid's *Elements* highlights a noticeable fact. Historically, the language of mathematics is more rich than the current regimented languages with which logicians are used to work.

Acknowledgements We thank the anonymous referees and H. Oliveira for their valuable suggestions.

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