The Validity of Inference and Argument*

Dag Prawitz

It has been common in contemporary logic and philosophy of logic to identify the validity of an inference with its conclusion being a (logical) consequence of its premisses. This identification pays attention to at most a necessary condition for an inference being acceptable in a deductive argument or proof. An inference is not acceptable unless the conclusion becomes evident because of being supported by the premisses. Can we define this condition in a stringent way so that we get a concept of valid inference allowing us to characterize a proof or valid deductive argument as a chain of valid inferences? This is the main question that I shall be concerned with in this essay. By the validity of an inference I understand henceforth a concept of that kind, which I shall strive to explicate here.

1 The concept of inference

Before entering the main discussion of what should be required of a valid inference, we should pay attention to the concept of inference. It is reasonable to think that the validity of an inference should be connected with the activity of making inferences

Dag Prawitz
University of Stockholm, Sweden, e-mail: dag.prawitz@philosophy.su.se

* It is a great pleasure to me to submit this essay to a volume devoted to Peter Schroeder-Heister in the series of “Outstanding Contributions to Logic”. I have had the privilege to follow part of Peter’s career from his doctoral dissertation. His work in proof-theory has influenced my own in substantial ways. In the last years we have had intensive discussions on the topic of this essay. As indicated in the essay, I am now following a path partly different from the one he argues for. The concept of valid inference is however a difficult one to explicate, and as can be seen from this essay, I have tried several blind alleys in the past. I hope for a continued discussion and co-operation with Peter on these issues and a deepened understanding of validity.

T. Piecha and K. F. Wehmeier (eds.), *Peter Schroeder-Heister on Proof-Theoretic Semantics*, Outstanding Contributions to Logic 29,
https://doi.org/10.1007/978-3-031-50981-0_4

and in particular with what we expect to achieve when we perform this activity and infer a conclusion from some premisses.[1]

Regarded as a mental act, an inference comprises a number of judgments and consists in a transition to one of them, the conclusion, from the other ones, the premisses. In this transition, the premisses are held to support the conclusion, which thereby is taken to be justified. In a *deductive* inference the conclusion is held to get a *conclusive* support or, as one also says, to be provided with a *binding* ground; these attributes will usually be left out since we are concerned here with deductive inference exclusively.

When an inference is verbalized, it becomes a compound speech act comprising a number of assertions. They can be seen as manifestations of judgements, performed by uttering sentences with assertive force. That the conclusion is taken to be supported by the premisses is then typically indicated by inserting a prefix, like “therefore” or “hence”, in front of the conclusion, or when the conclusion is stated first, by beginning the premisses with a word like “since” or “because”. Inferences will here be seen primarily as speech acts of that kind.

An inference is thus not just a succession of assertions. Its crucial feature is that one of the assertions, the conclusion, is held to be supported by the other assertions, the premisses.[2] To support the assertion that appears as conclusion and to justify it thereby is also the very aim of the inference. An individual inference may of course be driven by all kinds of different individual aims. But the *characteristic aim* (to use a term from speech act theory) of inference seen as an act-type is to obtain a support for the conclusion. When one holds the conclusion to be supported by the premisses, one thus understands the inference act as having been successful in attaining its aim. What precisely it amounts to for an inference to provide its conclusion with a conclusive support is a question that we have to come back to when trying to explicate the concept of valid inference; the person who makes an inference need not have an answer to this question but may nevertheless be right in holding the premisses to support the conclusion.

This view of inferences as transitions from categorical assertions already justified to conclusions that become justified agrees essentially with how Frege saw inferences. Established deductive practice knows however plenty of inferences that do not conform to Frege’s picture. Reductio ad absurdum, frequently used already at the time of the Greek antiquity, is an example. It presupposes reasoning from assumptions not considered by Frege (although his two-dimensional way of writing formulas could be said to be a way to represent assertions made under assumptions[3]). Such reasoning got an explicit and regimented form only later with Gentzen (1935) and Jaśkowski (1934).

A full account of deductive inferences should pay attention also to such reasoning. We shall therefore allow not only categorical assertions but also hypothetical ones, even called assertions made under assumptions. Furthermore, we shall allow assertions

[1] That inferences are primarily acts has been emphasized in contemporary logic by Martin-Löf and Sundholm in particular; see for instance Martin-Löf (1985) and Sundholm (1998).

[2] This feature of inferences is also stressed by Boghossian (2014).

[3] See Tichý (1988), von Kutchera (1996), and Schroeder-Heister (2014).

and assumptions that are unsaturated or open, expressed by sentences containing free variables.

This makes inferences more complicated as compared to what was said above, since in addition to being transitions from premisses to conclusions they may also discharge or, as I shall say, *bind* assumptions that the premisses depend on. They may also bind variables that occur free in asserted sentences. The terminology is to hint to how we understand reasoning with assumptions and variables that are free in the argument, not bound by any inference, namely as a kind of schematic reasoning intended to remain correct if free variables are replaced by closed terms and free assumptions are replaced by valid arguments for them; we shall return to this later to make it clear.

2 Arguments

To describe how inferences bind variables and assumptions we have to consider *arguments*, by which we shall understand reasoning that proceeds by making a number of inferences chained to each other so that the conclusion of one inference also becomes a premiss of another.

The validity of an inference or argument should of course not depend on who makes the inference or in what situation or at what time the inference is made, provided no indexicals are involved,[4] which I presuppose here. We may therefore restrict ourselves here to *generic acts* where we have abstracted from such features of individual inferences or arguments.

An argument, that is, a generic argument act, is determined by its inferences, their ordering, and information concerning initial premisses about whether they are asserted outright, categorically, or occur as assumptions, a category of speech acts of its own. For convenience I shall sometimes count initial premisses that are asserted outright as inferred by inferences from zero premisses.

The inferences of the argument are in turn determined by their premisses and conclusions and by the variables and assumption occurrences that they bind; they may be seen as determined by yet other factors, but here I restrict myself to the mentioned ones.

When we make an argument its inferences become of course ordered linearly by time, but for the purpose of logic it is sufficient and in fact more to the point to require that the ordering is a strict partial order such that for each inference, except for one last inference, its conclusion is also the premiss of the immediately succeeding inference. This ordering gives rise to a strict partial ordering of the assertion occurrences of the argument, too. There is thus one last conclusion of an argument, called its *final conclusion*. Each occurrence of an assertion determines a *subargument*, namely the initial part of the argument that has the occurrence as its final conclusion. By

[4] I owe this proviso to Cesare Cozzo.

the *immediate subarguments* of a given argument are understood the subarguments determined by the premisses of the last inference of the argument.

In ordinary deductive practice, an inference that binds an assumption binds all its occurrences. But when studying inferences on a meta-level, considering among other things operations on them, it is important to allow an inference to bind only some occurrences of an assumption.

The dependency on assumption occurrences is defined inductively: An assumption occurrence *depends* on itself; the conclusion of an inference *depends on* every assumption occurrence that a premiss depends on and is not being bound by the inference. An occurrence of an assertion in an argument depending on the set of assumptions Γ is said to be an *assertion under the (set of) assumptions* Γ. An argument Π whose final conclusion is an occurrence of $\mathcal{A}$ depending on the set of assumptions Γ is said to be an *argument for* the assertion $\mathcal{A}$ *from (the set of assumptions)* Γ; when the final conclusion does not depend on any assumption, Π is said to be an *argument* for $\mathcal{A}$.

When an inference binds a variable, it binds all occurrences of the variable that are free in the assertions of the subargument determined by a premiss and are not bound by an inference in the subargument. Such binding is not to occur if the variable has free occurrences in an assumption that the conclusion depends on; a restriction imposed in order that the replacement described at the end of the previous section is to give the desired result.

An argument is said to be *closed* when all its assumption occurrences as well as all the variables that occur free in an assertion of the argument are bound. It is said to be *open* otherwise. Note that an occurrence of a variable in an assertion of an argument may be free *in the assertion but bound in the argument*.

If Π is an argument for the assertion $\mathcal{A}$ and Σ is an argument in which $\mathcal{A}$ occurs as a free assumption, we understand by a *composition of* Π *and* Σ a result of putting the two arguments together by letting one or several free occurrences of the assumption $\mathcal{A}$ in Σ come after an occurrence of Π; these assumption occurrences in Σ are in this way replaced by arguments for them.

3 Representations of inferences and arguments

We have to distinguish between generic inferences considered in isolation and inferences occurring in an argument. For inferences that do not bind assumptions or variables — let us call them *simple inferences* — there is no difference: A simple inference is determined by its premisses and conclusion. It can be represented by a figure of the form

$$\frac{A_1 \quad A_2 \quad \ldots \quad A_n}{B}$$

where B is the sentence asserted by the conclusion and $A_1, A_2, \ldots, A_n$ are the sentences asserted by the premisses of the inference. (This common way of representing generic inferences introduces an order between the premisses that is insignificant but will actually be used sometimes as a way of reference.)

In the general case, a generic inference is determined also by the variables that it binds and the assumptions that it may bind. It can be represented by a figure of the form

$$\dfrac{\begin{array}{cccc} [\Gamma_1] & [\Gamma_2] & \ldots & [\Gamma_n] \\ \Big| v_1 & \Big| v_2 & \ldots & \Big| v_n \\ A_1 & A_2 & \ldots & A_n \end{array}}{B}$$

where $A_1, A_2, \ldots, A_n$, and B are again sentences, v_i is a set of variables, and Γ_i is a set of sentences. When applied within an argument, all free occurrences of the variables of v_i in assertions of the argument for A_i not already bound by other inferences become bound, and furthermore occurrences of assumptions that have the shape of a sentence in Γ_i and have A_i in its scope may become bound ($i \leq n$); not until the generic inference is applied in a specific argument is it determined which assumption occurrences become bound — the generic inference determines only which assumptions *may* become bound. For convenience, I presume that the variables in v_i do not occur in B.

Figures of the form exhibited above that represent generic inferences can be called *inference figures*. An *inference rule* or *schema* is like an inference figure but instead of containing sentences, predicates, individual variables, and individual terms it may contain schematic letters for them. An *instance* of an inference rule is obtained by replacing the schematic letters by specimens of the kind that they stand for, and is thus a generic inference (figure).

A generic argument act can be represented conveniently by a tree of assertions, which in turn may be represented by writing Frege's assertion sign in front of the sentences asserted — or, dropping the assertions sign, by just the sentences asserted. At the top of the tree are put sentences representing the initial premisses with information about whether they represent categorical assertions or assumptions; in the latter case the sentence represents both the assumption made and the assertion of the sentence under that assumption. Going down in the tree, we put successively sentences that represent the assertions inferred. The binding of variables and assumptions are to be marked at the inference where it occurs (e.g., one may attach the same numeral to an assumption and to the inference that binds it).

Note that when an argument is represented in this way by a tree of sentences, an assertion of a sentence A under the assumptions Γ is represented by just the sentence A; thus, it is only A that appears as a premiss or conclusion of an inference — the assumptions Γ that A depends on are easily read off from the tree. This is Gentzen's original way of arranging his natural deductions; arguments differ from them only in the respect that its inferences need not be instances of predetermined rules but can be of any kind.[5,6]

When we want to distinguish the representations of arguments from the arguments that they represent, we may call them *argument figures*. They may be seen as protocols of argument acts in which all the features of the acts that matter logically are noted down.

The inferences of an argument can be seen as *applications* of generic inferences. We shall allow that a substitution σ of terms for free variables is made at such applications; as usual it is here taken for granted that the terms do not contain free variables that become bound in the result A^σ of carrying out σ on A. Let $\mathcal{G}$ be a generic inference represented by the inference figure exhibited above. An inference of an argument is an *application of* $\mathcal{G}$, if and only if, for some (possibly empty) substitution σ of terms for variables different from the ones of v_i and occurring free in B, A_i or sentences of Γ_i it holds: 1) the premisses and conclusion of the inference are $A_1^\sigma, A_2^\sigma, \ldots, A_n^\sigma$, and B^σ, respectively, 2) the inference binds all occurrences of the variables of v_i that stand free in assertions of the argument for the premiss A_i^σ and 3) the inference binds at most some occurrences of assumptions of the form of a sentence of Γ_i^σ in the argument for A_i^σ that A_i^σ depends on ($i \leq n$). Note that there can be several different generic inferences of varying generality that an inference of an argument is the application of; replacing an individual term of a sentence of a generic inference with a variable, we get a new more general generic inference that has as applications all the applications of the first more specific generic inference. By a *result of applying* $\mathcal{G}$ to a sequence or a set of arguments $\{\Pi_1, \Pi_2, \ldots, \Pi_n\}$ we shall understand an argument whose last inference is an application of $\mathcal{G}$ and whose immediate subarguments are Π_1, Π_2, ..., and Π_n; their final conclusions have to be $A_1^\sigma, A_2^\sigma, \ldots, A_n^\sigma$ for some substitution σ.

Since all particular features of generic acts of inferences or arguments that are logically significant are present also in the figures that represent them, we may as well make the syntactical representations instead of the acts themselves the object of

[5] An alternative is to represent the assertions of the argument by sequents $\Gamma \Longrightarrow A$ where Γ is a sequence of sentences representing the assumptions that A is asserted under. Premisses and conclusions of inferences will then be represented by sequents instead of sentences. In some later publications Gentzen adopted this way of writing natural deductions. At this level it is only a question of alternative representations of one and the same argument act. However, if we follow Sundholm (2006) and understand a sequent $A_1, A_2, \ldots, A_n \Longrightarrow B$ as representing an assertion saying "if A_1 is true, A_2 is true, ..., and A_n is true, then B is true", the tree of sequents will primarily represent not reasoning from assumptions but reasoning starting from axioms of the form "If A is true, then A is true". The inferences too take different forms; cf.

$$\frac{Pc \quad \text{(assumption)}}{\exists x Px} \qquad\qquad \frac{Pc \Longrightarrow Pc}{Pc \Longrightarrow \exists x Px}.$$

[6] The Curry-Howard isomorphism suggests that there is also an alternative representation of arguments by terms in an extended lambda calculus containing parameters for functions corresponding to different arbitrary inferences of an argument. However, to establish really that this is a possible way to represent arguments, we have to pin down what it is to argue from assumptions, which is what is attempted here, partly by using a representation that is closer at hands.

study. The figures may simply be called inferences and arguments, respectively, as is customary in logic, the representation of an argument for the assertion of a sentence A may for simplicity be called an argument for A, and so on. It remains however that when discussing their validity, one should recall that the syntactical objects are representations of acts with aims; this general feature of the generic acts is of course lost when they are represented by figures.

4 Soundness and validity of inferences. A first approximation of validity

As remarked in the introduction, the validity of an inference has commonly been identified with the holding of the relation of (logical) consequence. The inferences or rather inference figures that one has in mind here are the simple ones at which no assumptions or variables are bound. Such an inference figure

$$\frac{A_1 \quad A_2 \quad \ldots \quad A_n}{B}$$

where A_i and B are closed sentences, is valid, one has said, when the inference is necessarily truth preserving, spelled out by saying either that it is impossible that all the premisses A_i $(i \leq n)$ are true while the conclusion B is false, or that necessarily if all A_i $(i \leq n)$ are true, then so is B. These two conditions are of course equivalent classically and have also been used alternatively in the traditional definition of the relation of entailment or consequence.

With Bolzano and Tarski the modal notion of necessity or impossibility is replaced with a variation of the meaning of the non-logical terms of the sentences involved and of the individual domain. We then get the well-known definition saying that a sentence A is a logical consequence of a set of sentences Γ, when A is true under each variation of assignments to the non-logical terms and of the domain of the individual variables under which all the sentences of Γ are true. This has become the dominant definition also of the validity of an inference from Γ to A in contemporary philosophy and logic.

The concept of valid inference that we are concerned with in this essay must obviously be very different from that notion of valid inference. Even if one considers only simple inferences, that notion demands both too much and too little from the perspective of this essay. Although clearly a valid inference cannot have true premisses and a false conclusion, inferences by for instance mathematical induction come out as non-valid when it is required that a valid inference preserve truth under all variations of the meaning of the non-logical terms and of the domain of the variables. More importantly, the prevalent notion demands too little. To establish that something is a logical consequence of something else we usually need a proof, often a long proof with many inferences. What is to be required of these inferences cannot then be that the sentence asserted in conclusion is a logical consequence of the sentences asserted in the premisses; if that was sufficient, there would never be a need of proofs containing more than one inference step.

The property of inference that has commonly been called validity is nevertheless a significant one, and I propose that it is called *soundness*. This would be in agreement with established terminology in connection with deductive systems, which are called sound when their inference rules preserve truth.

Already at the beginning of logic one was interested in distinguishing a kind of inferences that satisfied stronger demands than soundness. Aristotle distinguished between *syllogisms* in general and *perfect syllogisms* saying:

> A syllogism is a form of speech in which, certain things being laid down, something follows of necessity from them.
>
> A perfect syllogism is one that needs nothing other than the premisses to make the conclusion evident.[7]

Aristotle's general notion of syllogism (not restricted to the particular inferences that he studied in detail) has been a common point of departure for discussions and different proposals about what later became called valid inference. In contrast, there has been little interest in trying to develop his narrower notion of perfect syllogism; the attention it is has received seems mostly to have been of an exegetical kind about what Aristotle intended with that notion. Having an epistemic ingredient, it seems to be in the same direction as the concept that is focused on in this essay.

The term "evident" used in the above translation of Aristotle's definition of perfect syllogism may seem to be natural to use here in view of its etymology: when an inference is valid, it should be "seen" that the conclusion is right given that the premisses are.[8] However, the term is not to be understood here as referring to the actual state of mind of a person when something is obvious to her. We do not want to say that an inference is valid for a person, nor is it likely that Aristotle meant that a syllogism can be perfect for one person but not for another. The term must therefore be understood here not primarily as a psychological term but as referring to an objective property: A valid inference or perfect syllogism gives evidence to the assertion made in the conclusion in the sense that it gives a ground for the assertion, which thereby becomes justified. When understood in this way "evidence of an assertion" may be used interchangeably with "ground for an assertion". It is another thing that the existence of a ground for an assertion can in principle become known and therefore makes the assertion potentially evident to a person.

The notion of ground has of course a broader use. We are here interested in epistemic grounds. A speaker is normally expected to have some kind of epistemic ground for what he or she asserts. The nature of what is counted as such grounds for assertions varies with different kinds of assertions. For assertions with empirical content a ground may be obtained by observations under suitable circumstances. A ground for the assertion of an arithmetical identity may be got from a computation. How good the ground is required to be varies with the context. In some contexts, for

---

[7] Ross (1949, p. 287).

[8] The term is used by Martin-Löf and Sundholm too in their writings on the validity of inference, see, e.g., Martin-Löf (1985) and Sundholm (2004).

instance in mathematics, the ground is expected to be binding, and this is the case that concerns us now.

When we make an inference, it is understood that the ground for the conclusion comes from the premisses. But how? The premisses of an inference are sometimes called grounds for the conclusion, but clearly the premisses themselves do not constitute grounds for the premisses. The ground for the conclusion must rather come somehow from their grounds, which we take implicitly to exist since they are asserted.

For the inference to be valid, there must be some immediacy in how the ground for the conclusion comes from the grounds for the premisses. No further inferences should be needed to obtain the ground. This is a point that Aristotle perhaps wants to make when he says that the perfect syllogism "needs nothing other than the premisses".

The validity of an inference thus requires that a ground for the conclusion appears directly given any grounds for the premisses; "given" in the sense of being at least assumed to exist. One should expect furthermore that the meanings of the sentences involved could be crucial for the validity of the inference.

The concept of validity is to be tied primarily to generic inferences. What has been said so far may be put together as a first rough approximation of their validity:

> For a generic inference to be valid it must be required that in virtue of the meanings of the involved sentences, it appears directly, without any further inferences, that given any grounds for the premises, there is a ground for the conclusion.

If the stated requirement is not satisfied, the assertion that occurs in the conclusion is made without a ground, or at least not with a ground coming from the premisses, and the inference cannot then be valid. The requirement is also sufficient for the validity of an inference: when it is satisfied a person who makes the inference is being provided with a ground for the conclusion, or can at least easily provide herself with such a ground, given that she knows the meaning of the involved sentences and has grounds for the premisses; the assertion appearing as the conclusion of the inference thereby becomes justified and what was aimed at when making the inference is thus achieved. An argument consisting of inferences that all satisfy the proposed requirement gives a ground for its final conclusion, and if closed it can then be called a proof. To call such inferences valid is thus in accordance with my introductory declaration.

One may object to the requirement of directness and remark that a challenge of an inference is typically met by inserting a number of other inferences between the premisses and the conclusion. If these inferences are accepted as valid, one normally accepts as valid also the challenged inference. But this objection is built on another concept of validity than the one we are now concerned with. We want to clarify what may be called *immediate validity* where the point is that the inference should satisfy certain requirements as it stands (without adjuncts). When that has been clarified, one can easily define what it is for a simple inference to be *mediately valid*: there is an argument for the conclusion of the inference from its premisses that uses only immediately valid inferences. The problem is to explicate immediate validity, and I shall continue to refer to it simply as validity, using the term mediate validity for the property that can then be defined in terms of it.

To get on with this task we must especially inquire what constitute grounds for assertions in the present context. In mathematics we expect since the time of Greek antiquity that grounds for categorical assertions come in the form of deductive proofs, valid closed arguments. At least in the case of categorical assertions of logically compound sentences, we know no other way to obtain conclusive justifications. In case the assertions are not categorical but hypothetical or open, the grounds take the form of valid open arguments.

However, if we explain proofs as valid closed arguments and valid arguments as arguments consisting of valid inferences, as I have suggested above, and then explain the validity of inferences in terms of grounds explicated as valid arguments, we are of course moving in a circle.[9] This may seem disastrous for the proposed explanation of valid inference and valid argument that I have just begun.

5 Other concepts of proof

Is there a way to avoid this circularity problem? To turn to other ways of understanding the concept of proof may be thought to be a possibility. In the philosophy of intuitionistic mathematics, a proof has been seen not as a chain of valid inferences but as a mathematical construction. A sentence is taken to express the intention of a construction and to prove the sentence is to realize this intention.[10] More precisely, a proof is the construction process that results in the intended construction expressed by the sentence.[11]

In the so-called BHK-interpretation[12] as usually understood, a shift occurs so that a proof becomes rather the intended construction itself, not the realization process that establishes the existence of the intended construction. The proofs are there defined by recursion over the build-up of the proved sentences; for instance, "a proof of $A \to B$ is a construction that permits us to transform any proof of A into a proof of B".[13] However, the construction intended by a sentence cannot in itself in general constitute a ground for asserting the sentence. To have defined a construction that in fact transforms any proof of A into a proof of B does not justify the assertion of

[9] This circularity problem was noted in several lectures by Martin-Löf in the last decade. The problem is noted in one of his earlier papers too (Martin-Löf, 1985), where he saw it as a mistake to take the concept of (valid) immediate inference as conceptually prior to the concept of proof and concluded: "inference and proof are the same".

[10] Heyting (1934).

[11] As Heyting (1958) puts it: "The steps of the proof are the same as the steps of the mathematical construction." See also Sundholm (1983) concerning the ambiguity of the term construction.

[12] Stated by Troelstra and Dalen (1988).

[13] Troelstra and Dalen (1988, p. 9).

the implication $A \to B$ unless we have some ground for holding that the defined construction does effect the transformation in question.[14]

When Heyting's general idea of proofs as the realizations of intended constructions is further developed as is done in Martin-Löf's type theory, we are lead to a construction process in which at each step it is also demonstrated that the construction obtained is of the right, intended kind.[15] The process will thus contain a chain of inferences. We are then back to a notion of proof that presupposes the notion of valid inference.

We thus find that a proof in intuitionistic mathematics is either an intended construction of a proposition or a demonstration establishing that a given construction is the intended construction of a certain proposition. In the first case it does not constitute in itself a ground for asserting the propositions, and in the second case it is a chain of valid inferences. In neither case does it offer a solution of our problem.[16]

6 The conceptual order between valid inference and valid argument

Sticking to the idea of proofs as chains of valid inferences, one may contemplate as a way out of the circularity problem the possibility of first defining the concept of valid argument without referring to the validity of inference and then the concept of valid inference in terms of it, thus turning up-side down what has seemed to be the natural conceptual order.[17] A proposal for how to take the last step from valid argument to valid inference is to define an inference as valid when any application of it to valid arguments is a valid argument.

The proposed defining condition should in fact be a necessary condition for the validity of an inference in accordance with the basic idea concerning how open

[14] For this reason, in the BHK-interpretation first presented by Troelstra (1977, p. 977), the proof of an implication $A \to B$ did not consist of just a construction c that in fact transforms any poof of A into a proof of B but contained also "the insight that c has the property: d proves $A \implies cd$ proves B". To include such an "insight" into the proofs is of course difficult to make compatible with the intuitionistic idea that proofs are mathematical objects, but by dropping this element from the proofs as was done in the later and more well-known BHK-interpretation presented by Troelstra and Dalen (1988), the BHK-proofs lost the general epistemic power to justify the assertion of sentences that they are proof of. See also fn. 18.

[15] See for instance Martin-Löf (1984).

[16] In several papers (see for instance Prawitz 2015b; 2019b), I have discussed in a positive vein the possibility of identifying the grounds for asserting sentences with the intended construction that the sentences are taken to express when understood in an intuitionistic sense, although noting that the way they are defined must be restricted. I do not anymore consider this approach as promising since I know no way of making the right restriction. It is true that when the constructions are restricted to be what can be defined in certain extended lambda calculi it can be seen directly from well-formed terms that they denote the intended constructions of certain sentences. But the constructions intended by sentences of sufficient complexity cannot be exhausted by what can be defined in formal systems.

[17] I proposed a definition of such a notion of valid argument at a fairly early stage (Prawitz, 1973). It was later taken up and somewhat modified by Dummett (1991) and Schroeder-Heister (2006) and was modified more recently and more radically by myself (Prawitz, 2019a).

valid arguments are to be understood as schematic reasoning, presented at the end of Section 1. That idea can be stated more precisely in the form of a principle as formulated below. To get a short formulation let us say that a *regular instance* of an argument is obtained by first replacing a number of variables that are free in the argument by terms and then, in the resulting argument, replacing a number of free assumptions by valid arguments for them; the order is important in case one wants to obtain a closed regular instance.

Principle concerning open arguments and their instances
All regular instances of valid arguments are valid.

Since an inference can be seen as a one-step open argument whose premisses are assumptions and the result of applying it to valid arguments is a regular instance of the argument, we find in particular that it is a necessary condition for the validity of an inference that applications of the inference to valid arguments result in valid arguments.

However, the condition is not sufficient. If there is no valid argument for the assertion of A, the inference from the assertion of A to any other assertion satisfies the condition vacuously.[18] For instance, since, as we know from the proof of Fermat's Theorem, there is no closed valid argument for the premiss of the following inference

$$\frac{\exists x \exists y \exists z\ x^3 + y^3 = z^3}{\perp}$$

(the variables being supposed to range over positive integers), the inference comes out as valid according to the proposed definition of validity. Of course, the inference should come out as mediately valid, but certainly not as (immediately) valid; contrary to what Fermat thought, we seem to need a quite long and complicated argument to refute the assumption $\exists x \exists y \exists z\ x^3 + y^3 = z^3$.

The proposed definition of valid inference in terms of valid argument thus fails, and I see no way of attaining such a definition. The validity of arguments seems best explained by saying that a valid argument is one whose inferences are all valid. If the validity of inferences in turn is explained in terms of grounds as proposed above and grounds for an assertion consist in valid arguments for them, we must conclude that the concepts of valid inference and valid argument depend on each other and cannot be defined in isolation. If so we have to be satisfied with stating principles about how they are related to each other and to some other concepts.[19]

[18] The converse of the principle, in particular the idea that an open argument is valid if all its closed, regular instances are, therefore fails too. It was a substantial part of the definition of valid argument mentioned in fn. 17. The notion of hypothetical proof proposed by Martin-Löf (1985) suffers from the same problem: any argument from the assumption of a false sentence satisfies vacuously his defining condition for being such a proof. Similarly every false sentence A comes out as having a BHK-proof: There is a construction that permits us (vacuously) to transform any proof of a false sentence A into a proof of $\perp$, namely the (empty) function that is defined for all proofs of A and, for each such proof, assumes as value a proof of $\perp$.

[19] In the sequel I shall take this mutual dependency between valid inference and valid argument as a working hypothesis. I want nevertheless to keep open that the validity of inferences could be

7 Principles and heuristic ideas on the validity of inferences and arguments

A basic principle, which expresses intuitions already referred to, is:

Principle 1. The relation between validity of inferences and validity of arguments
An argument is valid, if and only if, all its inferences are valid.

This principle establishes a relation between the validity of an argument and the validity of the inferences that the argument consists of. We have to relate the latter validity to the validity of generic inferences that the concept of validity is primarily tied to, as suggested above. An inference of an argument can always be seen as an application of a generic inference, but as noted above (Section 3), it may be the application of several different generic inferences of varying generality. To be counted as valid it should be sufficient that it is an application of one valid generic inference (which is the same as saying that the least general generic inference that it is an application of is valid). We thus define:

Definition 1.
An inference of an argument is valid, if and only if, it is an application of a valid generic inference.

Principle 1 does not amount to a definition of the concept of valid argument as long as we lack a definition of the concept of validity of generic inference not depending on the concept of valid argument, but it is still informative about the involved concepts and has several immediate corollaries. Some of them are noted below for later use:

Corollary 1.
All results of applying a valid generic inference to a set of valid arguments are valid arguments.

Proof. Consider a valid generic inference $\mathcal{G}$ and let Π be the result of an application of it to a set S of valid arguments. By Principle 1 all inferences of the arguments of S are valid, and by the definition above so is the application of $\mathcal{G}$. Since all the inferences of Π are hence valid, Π is valid by Principle 1, now used in the other direction.

Corollary 2.
A subargument of a valid argument is valid.

Proof. Let Π be a valid argument and let Π' be a subargument of Π. By Principle 1 all the inferences of Π are valid and hence so are all inferences of Π'. The validity

explained in other ways without reference to grounds for the involved assertions. For example, in discussions about the validity of inferences that Peter Schroeder-Heister and I have had, he has suggested that one should demand more of a valid inference than I have done here. It should not only give a ground for the conclusion in the form of a valid argument for it when valid arguments for the premisses are given, but should more generally guarantee an argument for the conclusion, good or bad, of the same quality as the given arguments for the premisses. This stronger requirement should be possible to express without referring to valid arguments, he suggests.

of the Π' follows by using Principle 1 in the other direction. By the same kind of reasoning we get:

Corollary 3.
A composition of two valid arguments is valid.

The part of the principle stated in the previous section that concerns the result of replacing free assumptions of a valid argument by valid arguments for them follows as a corollary of Principle 1 since they can be seen as the effect of an iterated composition — the result is thus valid by Corollary 3. The other part is an independent principle, which we note down here:

Principle 2. The relation between an argument and its substitution instances
The result of substituting terms for variables that are free in a valid argument is valid.

The principle of the previous section is thus obtained as a corollary of Principles 1 and 2:

Corollary 4.
All regular instances of valid arguments are valid.

I shall presuppose that the languages in which inferences are verbalized have closed terms for all individuals in the domain that the variables are intended to range over; as usual the domain is supposed to be non-empty. Consider under this presupposition the following somewhat strengthened converse of Principle 2: An open argument $\mathcal{A}(v)$ with free variables v is valid, if all results $\mathcal{A}(t)$ of substituting closed terms t for v are valid. Is it a reasonable principle?

The answer must clearly be no since that would be contrary to the idea of valid inference discussed here: the fact that all the arguments $\mathcal{A}(t)$ are valid cannot be sufficient for the validity of the open argument $\mathcal{A}(v)$, unless this fact appears directly from made inferences and the meanings of the involved sentences. We shall return to this issue when now returning to what I called a first approximation of the validity of inferences (Section 4).

This first approximation now amounts to another basic idea concerning the relation between validity of inference and validity of argument when having acknowledged that grounds for assertions consist of valid arguments for them. It now reads as follows when put in the form of an equivalence and restricted to simple inferences (which do not bind anything) where the premisses and conclusions are closed, valid arguments for them therefore amounting to proofs:

> A simple generic inference whose premisses and conclusion are closed is valid, if and only if, in virtue of the meanings of the involved sentences, it appears directly, without any further inferences, that given any proofs of the premisses, there is a proof of the conclusion.

It follows from Principle 1 that a necessary condition for the validity of a generic inference is the existence of a valid argument for the conclusion given valid arguments for the premisses; indeed, the result of applying the inference to the valid arguments for the premisses is such a valid argument for the conclusion according to Corollary 1. The equivalence above states a necessary condition for the validity of a simple generic inference (whose premisses and conclusion are closed) that is in some respect weaker and in some respect stronger than the condition of Corollary 1. It is weaker since it requires only the existence of *some* valid argument for the conclusion. It is stronger since it requires that this existence appear directly in virtue of the meaning of the involved sentences.

Furthermore, the equivalence provides a sufficient condition for the validity of an inference. On the meaning theory presented in the next section there are cases, namely so-called introduction inferences, where the very application of a generic inference to valid arguments results in an argument that is valid in virtue of the meaning of the conclusion, and where the inference is thus valid according to the equivalence above. But in other cases we shall have to find another valid argument for the conclusion in order to establish the validity of an inference via the sufficient condition stated by the equivalence.

It is to be recalled that we are now concerned with the validity of generic inferences, in terms of which the validity of the inferences of an argument is defined. A valid argument for the conclusion of a particular inference in a given argument may appear directly from the arguments for the premisses, given that they are valid, but this is not sufficient for the validity of the inference; otherwise the inference from $A \vee B$ to A would come out as valid when $A \vee B$ has been inferred after having obtained a proof of A. The above condition of validity requires that given *any* proofs of the premisses, a proof of the conclusion appear.

When an application of the generic inference binds variables and may bind occurrences of assumptions at a premiss, the ground for that premiss assumed to exist in the condition for the inference to be valid takes the form of a valid argument for the premiss from the set of assumptions whose occurrences it may bind. However, when the premiss is an open assertion, the condition for validity must require more. Consider the inference represented by the figure

$$\frac{A(x)}{B(x)}$$

where $A(x)$ and $B(x)$ are sentences that contain one free variable x. For it to be valid, it must appear directly that given any proof of the assertion of $A(t)$, there is a proof of the assertion of $B(t)$, where t is any closed individual term. This condition is also sufficient for the validity of the inference under the presupposition made above that each individual in the intended domain is denoted by some term. That a valid argument for asserting $B(x)$ appears directly given a valid argument for asserting $A(x)$ is a weaker condition that would not guarantee that there is a proof of $B(t)$ given a proof o $A(t)$.

This means that the general condition for the validity of a generic inference is most conveniently formulated in terms of applications of the inference, as that notion was defined in Section 3: for any application of the inference to a set S of valid arguments resulting in a closed argument Π, there is to appear a proof of the final conclusion of

Π. We then arrive at the following reformulation of the first approximate explication of the validity of inference:

> *Heuristic idea on the validity of inference in terms of valid arguments*
> A generic inference is valid, if and only if, in virtue of the meanings of the involved sentences, it appears directly without any further inferences that, for any application of the inference to a set of valid arguments resulting in a closed argument Π, there is a proof of the final conclusion of Π.

I call it a heuristic idea because of the vagueness of the expression "it appears directly". A condition of that kind is needed for at least two reasons. One is again the need to avoid the problem of vacuity discussed in the previous section: We do not want inferences to come out valid vacuously just because there are no valid arguments for their premisses. Such an outcome is meant to be blocked when it does not appear directly from the meaning of the premisses that there are no valid arguments for them. To illustrate again with Fermat's theorem: Although it is right that when a generic inference has $\exists x \exists y \exists z\, x^3 + y^3 = z^3$ as premiss any result of applying it to valid arguments satisfies vacuously whatever condition we choose (there being no valid argument for the premiss), this fact does not appear directly in virtue of the meaning of $\exists x \exists y \exists z\, x^3 + y^3 = z^3$. In contrast, the generic inference

$$\frac{\perp}{A}$$

comes out as valid according to the heuristic idea above as it should if we have explained the meaning of $\perp$ by saying that there is no proof of $\perp$; it can then be said rightly that it appears directly in virtue of the meanings of the involved sentences that, for any application of the inference to a valid argument for $\perp$, whatever condition we choose is satisfied.

The requirement of directness is also meant to block that the mere existence of a closed valid argument for the conclusion is sufficient for the validity of a generic inference. For an illustration, consider the generic inference

$$\frac{\exists x \forall y A(x, y)}{\forall y \exists x A(x, y)}$$

Given any valid argument for the assertion of a sentence $\exists x \forall y A(x, y)$ (not depending on assumptions), an argument for the assertion of $\forall y \exists x A(x, y)$ can easily be constructed by using inferences that should certainly come out as valid, but since the construction uses additional inferences, it does not appear directly.

A heavy burden is thus put on the meaning of the vague term directness. In spite of these shortcomings, the stated equivalence will serve as a heuristic guide when searching for additional, more precise principles about how valid inferences and valid arguments are related to each other.

8 Meaning of assertions and validity of inferences

There are inferences whose validity is independent of the assertions involved. An example is the generic inference, represented by the inference figure

$$\dfrac{A \qquad \begin{matrix}[A] \\ \vert \\ B\end{matrix}}{B}$$

(sometimes called the rule of explicit substitution when taken as an inference rule). For any application of this inference to valid arguments that results in a closed argument Π, there is a proof of its final conclusion, because the composition of the two immediate subarguments of Π is such a proof when each free occurrence of the assumption A in the second subargument is replaced by the first subargument; a composition of two valid arguments being valid by Corollary 3 of Section 7. Provided the appearance of this proof by forming a composition of the two valid arguments to which the inference is applied is counted as direct, the inference is thus valid according to the heuristic idea formulated above regardless of what sentences A and B are.

It is surely more common that the validity of an inference depends on what the assertions involved mean. For instance, whether the two generic inferences represented by the figures

$$\frac{A}{A \vee B} \qquad\qquad \frac{B}{A \vee B}$$

are valid cannot but depend on the meaning of sentences of the form $A \vee B$.

As argued by Michael Dummett, the meaning theory for a language should account for all features of the use of the language that depend on the meanings of its sentences, including the acceptance of inferences like the ones above as valid. For it to fulfil this task it is essential how the meanings of sentences are given. A truth-conditional meaning theory of the usual kind states the condition for a sentence to be true in such a way that it may not be possible to derive from that on what kind of grounds an assertion is accepted as justified. In contrast, what Dummett calls a verificationist or justificationist meaning theory explains the meaning of a sentence directly in terms of what counts as a ground or valid argument for asserting the sentence.

Gentzen had an idea about the meanings of logical constants which is a forerunner to Dummett's idea of justificationism. A special feature of Gentzen's system of natural deduction is that for every logical constant c there are a number of inference rules called introduction rules for c, or simply *c-introductions*, where the conclusions are sentences whose outermost sign is c. After having set up his system, Gentzen remarked that the meaning of a logical constant c could be seen as being determined by the c-introductions.[20] It is not obvious how this suggestion is to be understood. One

[20] "The introductions present, so to say, the 'definitions' of the symbols concerned." (Gentzen, 1935, p. 189).

element is of course that, in virtue of the meanings of the logical constants, instances of the introduction rules are to be seen as yielding valid arguments when applied to valid arguments for its premisses. But c-introductions do not constitute the only valid ways of inferring sentences that have c as their outermost sign. So what is special about the introduction rules that gives cause for taking them as meaning constitutive?

One way to answer this question is to say that the c-introductions present the *direct* or *canonical* way of inferring sentences that have the constant c as the outermost sign. This is to be understood as implying, not only that applications of instances of introductions to valid arguments yield valid arguments as results, but also that if the assertion of a sentence is provable at all, then in principle its proof can be put in such a form that the last inference is an instance of an introduction. An argument whose last inference is an instance of an instance of an introduction is said to be in *canonical form*[21] (whether open or closed and irrespective of question of validity).

To exemplify: that the meaning of the disjunction sign is determined by the $\vee$-introductions whose instances are generic inferences of the form exhibited above, is to be understood as saying that the meaning of disjunction is such that 1) the results of applying to valid arguments generic inferences of the kind exhibited above are valid, and 2) if there is a proof of a sentence $A \vee B$, then there is such a proof in canonical form. Hence, if $A \vee B$ is provable, there is a proof of either A or B.[22]

As seen, this fits well with how $\vee$ is understood intuitionistically, but not with how it is understood classically since a disjunction may be provable classically while neither of the disjuncts is provable. Gentzen's $\vee$-introductions can thus be seen as determining the meaning of intuitionistic disjunction, but not of classical disjunction.

Adopting this idea to all the logical constants of the intuitionistic language of first order predicate logic, the meanings of their compound sentences are explained by telling how arguments for the assertions of them have to look to be in canonical form What must be told for different cases of compound sentences is what the immediate subargument or subarguments of an argument Π in canonical form are to consist of. In the case of:

$A \wedge B$, they are to consist of an argument for A and an argument for B;

$A \vee B$, it is to consist of an argument for A or an argument for B;

$A \rightarrow B$, it is to consist of an argument for B from A, where free occurrences of the assumption A may be bound by the last inference of Π;

$\forall x\ A(x)$, it is to consist of an argument for $A(x)$, where x is being bound by the last inference of Π;

$\exists x\ A(x)$, it is to consist of an argument for $A(t)$ for some term t.

This has to be completed by telling what constitute arguments in canonical form for assertions of atomic sentences. In the case of the atomic sentence $\perp$, the explanation

[21] A term earlier used by Brouwer in a different way. Its use in the above sense was proposed by Prawitz (1974) and Dummett (1975).

[22] Although Gentzen never developed his ideas about meaning more precisely, it is clear that he was thinking in this way when remarking: the assertion of "$A \rightarrow B$ attests (German: dokumentiert) the existence of a derivation of B from A". (Gentzen, 1935, p. 189)

is that there is no argument in canonical form for the assertion of $\perp$; there are no $\perp$-introductions. I shall assume that the meanings of all atomic sentences have been explained by telling what the arguments in canonical form for the assertions of them are.[23] One may want to vary what is to count as canonical arguments for atomic sentences of a language. Let us say that a base for a language L specifies the canonical arguments for the atomic sentences and the set of closed individual terms of L. Validity of inferences can then be relativized to such bases.

To extend Gentzen's idea to other languages, one must thus be able to specify the meaning of each different sentence form by giving introduction rules for that sentence form or, in other words, by stating what constitute arguments in canonical form for the assertions of such sentences.[24] It is an open question to what extent this is possible,[25] but there is no problem to give introduction rules that are adequate for the logical constants understood classically.[26]

To summarize the meaning theoretical view adopted here, we can say more generally: To know the meanings of the sentences of a language is to know

1) specifically for each sentence form how arguments in canonical form for the assertions of sentences of that form look, and
2) generally 2a) that an argument in canonical form is valid if its immediate subarguments are and 2b) that all proofs can be put in canonical form.

Knowledge according to clause 1 also determines introduction rules for all forms of sentences. The validity of all their instances is implied trivially by the heuristic idea stated in the previous section. Thus, we get:

Principle 3. Validity of introduction rules
All instances of introduction rules are valid. In particular, all instances of Gentzen's introduction rules for intuitionistic predicate logic are valid.

[23] An early extension of Gentzen's idea to atomic sentences is due to Martin-Löf (1971). He took Peano's first and second axiom for natural numbers as two introduction rules for sentences of the form Nt, one allowing the inference of $N0$ from no premisses, and the other allowing the inference of Nst from the premiss Nt (s standing for the successor operation).

[24] I have left open here general requirements that should be put on meaning explanations to guarantee for instance that they are not circular. They correspond to requirements that introduction rules are to satisfy discussed by Dummett (1991).

[25] If the language contains sentences with empirical content, we may have to broaden the concept of inference and think of introduction inferences as transitions not only from assertions to other assertions but also from other acts such as observations; they deliver what is commonly called direct evidence and may be seen as meaning constitutive.

[26] For instance, a possible introduction rule for classical disjunction is displayed here:

$$\dfrac{\begin{array}{c}[\neg\alpha,\ \neg\beta]\\ \Big| \\ \perp\end{array}}{\alpha \vee \beta}$$

α and β being schematic letters for sentences. We could have a language that contains both classical and intuitionistic logical constants, kept distinct by, e.g., attaching different subscripts to them, and formulate introduction rules for all of them; see further Prawitz (2015a).

9 A precise sufficient condition for an inference to be valid

A more challenging problem is to give precise principles for how inferences can be valid in virtue of the meaning of the involved sentences without being meaning constitutive in the way the introduction rules are according to the previous section. The question is whether and to what extent we can state precise principles for such validity guided by the heuristic idea of Section 7.

Gentzen meant that the elimination rules (E-rules) of his system of natural deduction were valid because of their relation to the meaning constitutive introduction rules (I-rules). He suggested, "It should be possible to display the E-inferences as unique functions of the corresponding I-inferences, on the basis of certain requirements".[27] As a step in that direction and to explain why the elimination rules are valid, I have described them as the inverses of the corresponding introduction rules in the sense of satisfying the following:

Inversion principle
If the last inference of an argument Π for A from Γ is an E-inference whose major premiss is the conclusion of an I-inference, the argument for the major premiss thus being in canonical form, the immediate subarguments of Π already "contain" an argument for A from $\Delta \subseteq \Gamma$.[28]

The expression "contain", which was left undefined when I first stated the principle, can be defined as follows: Let us say that the argument Π is immediately extracted from the set S of arguments when either

(i) Π is an argument of S or is a subargument of an argument of S or
(ii) Π is the result got by substituting terms for free variables in an argument that satisfies (i), or
(iii) Π is the composition of two arguments that satisfy (i) or (ii).

Definition 2. Containment
The argument Π is *contained in* a set S of arguments if and only if there is a sequence $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$, of arguments such that $\Pi = \Sigma_n$ and for each Σ_i ($i \leq n$), Σ_i is immediately extracted from $S \cup \{\Sigma_j \mid j < i\}$.[29]

We note the following corollary:

Corollary 5.
An argument immediately extracted from a set of valid arguments is valid. A fortiori, an argument contained in a set of valid arguments is valid.

The corollary simply combines principles and corollaries stated in Section 7: For the extraction used in clause (i) see Corollary 2, for the one used in clause (ii) see in addition Principle 2, and for the one used in clause (iii) see in addition Corollary 3.

---

[27] Gentzen (1935, p. 189).

[28] Prawitz (1965); the formulation there refers to natural deductions in Gentzen's system instead of arguments as above, otherwise its content is the same.

[29] Due to Peter Schroeder-Heister and me; see Prawitz (2019a).

All instances of Gentzen's elimination rules satisfy the inversion principle when containment is defined as above. We can strengthen the principle in two ways, namely by requiring in clause (i) of the definition of containment that Π is an argument of S or is an *immediate* subargument of an argument of S and by requiring that the argument for the conclusion is not only contained in the set of arguments for the premisses but can in fact be obtained from the set by one immediate extraction. Let us call the result the *strong inversion principle*. It too holds for all instances of Gentzen's elimination rules:

Fact about E-rules
Gentzen's elimination rules satisfy the strong inversion principle.

To illustrate by an example, consider the case where an ∃-introduction is immediately followed by an ∃-elimination. It has the form

$$\dfrac{\dfrac{\begin{matrix}\Pi \\ A(t)\end{matrix}}{\exists x A(x)} \qquad \begin{matrix}\overset{(1)}{[A(x)]} \\ \Sigma(x) \\ B\end{matrix}}{B}\,(1)$$

Another argument for B from no more assumptions is obtained by two operations: first substitute t for x in $\Sigma(x)$ and then form the composition of Π and $\Sigma(t)$ where Π replaces all free assumptions $A(t)$ in $\Sigma(t)$ (the corresponding assumptions $A(x)$ in $\Sigma(x)$ were bound by the considered application of the instance of ∃-elimination). The result of carrying out these two operations is an immediate extraction in the strengthened sense from the set of the immediate subarguments of the argument displayed above.

The elimination rule for $\perp$ is a somewhat special case. Any instance of that rule satisfies the (strong) inversion principle vacuously since according to the meaning of $\perp$ it has no canonical argument.

Invoking the heuristic idea of Section 7 and applying Corollary 5 and the above fact, we can now state

Principle 4 (initial part). Validity of elimination rules
All instances of Gentzen's elimination rules for intuitionistic logic are valid.

Is this principle really implied by the heuristic idea on the validity of inference? Consider an arbitrary instance of an E-rule. Call this generic inference $\mathcal{G}$. To show that $\mathcal{G}$ is valid in accordance with the heuristic idea, we have to show that in virtue of the meanings of the involved sentences it appears directly that, for any application of $\mathcal{G}$ to a set S of valid arguments resulting in a closed argument Σ, there is a proof of the final conclusion of that argument.

To this end we may say the following. Let B be the final conclusion of Σ, let A be the premiss of the last inference of Σ that corresponds to the major premiss of $\mathcal{G}$ (A is thus the result of carrying out on the major premiss of $\mathcal{G}$ the substitution (if any) that yields the application in question), and let Π be the subargument of Σ determined by A. Since the inference does not bind anything in the argument for the major premiss, Π is a closed argument, and is thus a proof since it is assumed to be valid. In virtue of the meaning of A, there is a proof Π^* of A in canonical form. Replacing Π by Π^* in Σ we have an argument Σ^* for B on which the strengthened Inversion Principle has bearing. Hence by a suitable extraction in the strengthened sense from the set S^* of arguments resulting from S by replacing Π by Π^* another argument for B appears. By Corollary 5, this argument for B is valid and hence it is a proof of B.

There are three steps in this little piece of reasoning. As already said, the first step, which gives the argument Π^* for A in canonical form, is immediate in virtue of the meaning of A. In the third and last step we use Corollary 5, an immediate consequence of Principles 1 and 2, which only make explicit basic intuitions about inferences and arguments. What can be questioned is that the appearance in the intermediate step of the argument for B by an immediate extraction is sufficiently direct. Admittedly a person may know the meanings of the involved sentences and have the intuitions made explicit in Principles 1 and 2 without realizing that there is this operation of extraction yielding an argument for B.[30] Nevertheless only knowledge of the meaning of the major premiss A of the inference and a few reflexions are needed to recognize that there is a canonical proof Π^* of A and a proof of the final conclusion B of Σ that uses no other inferences than those already occurring in Π^* or the arguments for the other premisses (if any).

The elimination rule for $\perp$ is again a somewhat special case. In virtue of the meaning of $\perp$, there is no application of an instance of that rule to a valid argument for $\perp$ resulting in a closed argument, since there is no argument for $\perp$ in canonical form and hence no closed valid argument for $\perp$. What has to be shown for all such applications according to the heuristic idea is thus vacuously satisfied directly.

To generalize the Inversion Principle so that it concerns inferences in general, not only elimination rules, and holds more generally for languages given with other introduction rules than Gentzen's, we have to take into account that the inferences whose validity we want to establish may not have one major premiss that can be referred to in the way we did in the statement of the Inversion Principle. Instead of referring to one premiss as the major one, some of the premisses that will play a similar role as the major premiss will be distinguished and will be identified by their ordinal numbers; we shall thus be speaking of the i:th premiss of an inference.[31]

A particular feature of a generic inference $\mathcal{G}$ that satisfies the Inversion Principle is that the following holds for any application of $\mathcal{G}$ to arguments among which the one for the major premiss is in canonical form: another argument for the final conclusion of the result of the application can be obtained whose inferences occur already in the arguments to which $\mathcal{G}$ is applied or are substitution instances of such inferences. Instead of applying $\mathcal{G}$, one can therefore argue for the conclusion by applying those

[30] This is a point stressed by Cozzo (2021), who draws the conclusion that the validity of an elimination inference is synthetic and non-meaning-involved.

[31] This means that I am using the ordering of the premisses of an inference, which otherwise is without significance with respect to their identity (see parenthetical remark in Section 3). In another generalization of the inversion principle proposed by Schroeder-Heister (1983) the idea to distinguish some of the premisses of an inference is crucial in a similar way; instead of referring to them by ordinals he marks them by asterisks.

inferences, which have already been used essentially and have thus been accepted implicitly as valid. What can be argued for by such applications of $\mathcal{G}$ can thus be argued for by using already available inferences. It seems therefore fitting to call inferences that share this feature with those that satisfy the Inversion Principle *non-creative*.

To begin with I shall restrict this notion to inferences whose applications to a set S of arguments are seen to enjoy this feature of non-creativity by extractions from S in the same way as for inferences satisfying the Inversion Principle. It is defined as follows:

Definition 3. Non-creative inferences
A generic inference $\mathcal{G}$ is *non-creative* when it has a number of distinguished premisses at which no binding occurs and it holds for any closed argument Π resulting from an application of $\mathcal{G}$ to a sequence S of arguments such that the ones with the same ordinal numbers as the distinguished premisses are in canonical form that an argument for the final conclusion of Π is contained in S.

With the major premiss as the distinguished one, Gentzen's elimination rules are thus non-creative. So is the generic inference represented by the inference figure

$$\frac{A \qquad A \rightarrow B \qquad B \rightarrow C}{C}$$

with $A \rightarrow B$ and $B \rightarrow C$ as the distinguished premisses. Instances of the rule of explicit substitution considered at the beginning of Section 8 are examples of non-creative inferences with an empty set of distinguished premisses.

We can now state Principle 4 in a more general form:

Principle 4 (completed). Validity of non-creative inferences
All non-creative inferences are valid.

The reasoning to see that the principle is implied by the heuristic principle is essentially the same as that for the first part of the principle stated above, except that we are stretching the requirement of immediate appearance since the argument for the final conclusion of the result of applying the inference may now appear after a series of extractions instead of just one.

It is to be noted that a generic inference $\mathcal{G}$ may be vacuously non-creative because there is one or more distinguished premisses for which it holds that there is no application of $\mathcal{G}$ to a sequence of arguments such that the ones with the same ordinal number as the distinguished ones are in canonical form. But this can only happen in case no argument in canonical form has been specified in the explanation of the meaning of the distinguished premisses, as is the case for $\perp$. In such a case $\mathcal{G}$ is also valid according to the heuristic idea since in virtue of the meaning of the involved sentences the condition stated by the heuristic idea is vacuously satisfied.

One may contemplate extending the term non-creative to inferences that enjoy the feature of non-creativity without this necessarily being seen by making extractions; in other words, conclusions of applications of these inferences to certain sets S of arguments have again arguments using only inferences occurring in S or substitution instances of them but they can be freely combined and do not need to occur in combinations obtained by extractions from S.

10 Concluding remarks

On the basis of an initial discussion of the nature and aim of inferences and inspired by Gentzen's idea about how the meanings of logical constants are determined, we have arrived at some precise principles about the validity of inferences. According to them all instances of the inference rules of natural deduction for intuitionistic logic are valid. The result can be seen as a proposal for how Gentzen's idea is to be understood in order to yield such a result.

A question that naturally arises is whether all non-creative inferences expressible in the language of intuitionistic predicate logic, LIPL, are derivable in that logic, IPL. Whether an inference is non-creative may depend on the base (Section 8); as will be exemplified below, induction inferences are non-creative given a certain condition on the base, although they are of course not derivable in IPL. We should therefore relativize non-creativity to a base and ask whether inferences expressible in IPL that are non-creative relative to all bases is derivable in IPL. This is a precise logical question about a kind of completeness of intuitionistic predicate logic that may be possible to answer.

A more philosophical and vaguer question is whether the heuristic idea about the validity of inferences implies the validity of inferences in the language of intuitionistic predicate logic above what follows from Principles 3 and 4. Here the validity should also be relativized to a base. The question is thus whether inferences expressible in LIPL that are valid relative to all bases according to the heuristic idea are so in force of Principles 3 and 4. A positive answer to this question would be a reason for identifying the logically valid inferences in LIPL with inferences that are either non-creative or are instances of introduction rules.

Going outside of logic, it is of interest to consider the rule of mathematical induction. If the intended individual domain is the set of natural numbers, it may be given the form

$$\frac{A(0) \qquad \begin{array}{c}[A(x)]\\ \Big|\, x\\ A(sx)\end{array}}{A(t)}$$

where t is a schematic letter for individual terms and s is the successor operation (since t may be replaced by a free variable y in instances of the rule, the rule has the whole strength of mathematical induction, allowing us to infer $\forall y\ A(y)$). Provided that the closed individual terms of the language in question consist of numerals, all instances of this inference rule are easily seen to be non-creative, and they are hence valid according to Principle 4. However, they cease to be so when there are closed individual terms other than numerals.

If the intended individual domain contains other elements than natural numbers, the rule of induction has to be qualified by adding Nt (N for the predicate to be a natural number) as a third premiss. In this form the rule has instances that are not non-creative. Since they should of course be counted as valid, going outside of logic one needs to find an extension of Principle 4 if one is to cover by precise principles inferences that we consider valid.

Acknowledgements I am grateful to Cesare Cozzo, Valentin Gorenko, Per Martin-Löf, Luiz Carlos Pereira, Antonio Piccolomini d'Aragona, and Peter Schroeder-Heister for their commenting on an earlier version of this essay, which has helped me to improve several passages. I have also benefitted from remarks made in the discussion after my presenting part of this essay at a Nordic Online Logic Seminar in March 2021.

References

Boghossian, P. (2014). What is inference? *Philosophical Studies* 169 (1), 1–18.

Cozzo, C. (2021). Fallibility and fruitfulness of deductions. *Erkenntnis*. DOI: 10.1007/s10670-021-00487-6.

Dummett, M. (1975). The philosophical basis of intuitionistic logic. In: *Logic Colloquium '73*. Ed. by H. E. Rose et al. Amsterdam: North Holland, 5–40.

– (1991). *The Logical Basis of Metaphysics*. London: Duckworth.

Gentzen, G. (1935). Untersuchungen über das logische Schließen, I. *Mathematische Zeitschrift* 39 (1), 176–210.

Heyting, A. (1934). *Mathematische Grundlagenforschung, Intuitionismus, Beweistheorie*. Berlin: Springer.

– (1958). Intuitionism in mathematics. In: *Philosophy in the Mid-Century*. Ed. by R. Klibansky. Florence: La Nuova Italia, 101–115.

Jaśkowski, S. (1934). On the rules of suppositions in formal logic. *Studia Logica* 1, 5–32.

Martin-Löf, P. (1971). Hauptsatz for the intuitionistic theory of iterated inductive definitions. In: *Proceedings of the Second Scandinavian Logic Symposium*. Ed. by J. E. Fenstad. Amsterdam: North-Holland, 179–216.

– (1984). *Intuitionistic type theory*. Napoli: Bibliopolis.

– (1985). On the meanings of the logical constants and the justifications of the logical laws. In: *Atti degli Incontri di Logica Matematica, Vol. 2*. Ed. by C. Bernardi and P. Pagli. Siena: Scuola di Specializzazione in Logica Matematica, Dipartimento di Matematica, Università degli Studi di Siena, 203–281. Republished in *Nordic Journal of Philosophical Logic 1*, 11–60 (1996).

Prawitz, D. (1965). *Natural Deduction: A Proof-Theoretical Study*. Stockholm: Almqvist & Wiksell. Reprinted Mineola NY: Dover Publications (2006).

– (1973). Towards a foundation of general proof theory. In: *Logic, Methodology and Philosophy of Science IV*. Ed. by P. Suppes et al. Amsterdam: North-Holland, 225–250.

– (1974). On the idea of a general proof theory. *Synthese* 27, 63–77.

– (2015a). Classical versus intuitionistic logic. In: *Why is this a Proof? Festschrift for Luiz Carlos Pereira*. Ed. by E. H. Haeusler et al. London, 15–32.

Prawitz, D. (2015b). Explaining deductive inference. In: *Dag Prawitz on Proofs and Meaning*. Ed. by H. Wansing. Cham: Springer, 65–100.
– (2019a). The fundamental problem of general proof theory. *Studia Logica* 107. *Special Issue: General Proof Theory*. Ed. by T. Piecha and P. Schroeder-Heister. Cham: Springer, 11–29.
– (2019b). The seeming interdependence between the concepts of valid inference and proof. *Topoi* 38, 493–503.
Ross, W. D. (1949). *Aristotle's Prior and Posterior Analytics*. Oxford: Oxford University Press.
Schroeder-Heister, P. (1983). The completeness of intuitionistic logic with respect to a validity concept based on an inversion principle. *Journal of Philosophical Logic* 12, 359–377.
– (2006). Validity concepts in proof-theoretic semantics. *Synthese* 148, 525–571.
– (2014). Frege's sequent calculus. In: *Trends in Logic XIII: Gentzen's and Jaśkowski's Heritage – 80 Years of Natural Deduction and Sequent Calculi*. Ed. by A. Indrzejczak, J. Kaczmarek, and M. Zawidzki. Łódż University Press, 233–245. DOI: 10.15496/publikation-72324.
Sundholm, G. (1983). Constructions, proofs and the meaning of logical constants. *Journal of Philosophical Logic* 12 (2), 151–172.
– (1998). Inferences versus consequence. In: *The LOGICA Yearbook 1998*. Prague: Czech Academy of Sciences, 26–36.
– (2004). Anti-realism and the notion of truth. In: *Handbook of Epistemology*. Ed. by I. Niiniluoto et al. Kluwer Academic Publishers, 437–466.
– (2006). Semantic values for natural deduction derivations. *Synthese* 146, 623–638.
Tichý, P. (1988). *The Foundations of Frege's Logic*. Berlin: De Gruyter.
Troelstra, A. S. (1977). Aspects of Constructive Mathematics. In: *Handbook of Mathematical Logic*. Ed. by J. Barwise. Amsterdam: North-Holland, 973–1052.
Troelstra, A. S. and D. van Dalen (1988). *Constructivism in Mathematics, 2 vol.* Amsterdam: North-Holland.
von Kutchera, F. (1996). Frege and Natural Deduction. In: *Frege: Importance and Language*. Ed. by M. Schirn. Berlin: De Gruyter, 301–304.