Frege's Class Theory and the Logic of Sets

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Abstract We compare Fregean theorizing about sets with the theorizing of an ontologically non-committal, natural-deduction based, inferentialist. The latter uses free Core logic, and confers meanings on logico-mathematical expressions by means of rules for introducing them in conclusions and eliminating them from major premises. Those expressions (such as the set-abstraction operator) that form singular terms have their rules framed so as to deal with *canonical identity statements* as their conclusions or major premises. We extend this treatment to *pasigraphs* as well, in the case of set theory. These are defined expressions (such as 'subset of', or 'power set of') that are treated as basic in the *lingua franca* of informal set theory. Employing pasigraphs in accordance with their own natural-deduction rules enables one to 'atomicize' rigorous mathematical reasoning.

1 Introduction

Our honoree Peter has had abiding and deep interests both in Frege's work in logic, and in proof-theoretic semantics, a field in which he has played an important founding role. I thought it fitting, then, to combine a bit of both in this paper in his honor.

In his recent study Schroeder-Heister (2016), Peter's abstract reads as follows:

> I present three open problems the discussion and solution of which I consider relevant for the further development of proof-theoretic semantics: (1) The nature of hypotheses and the problem of the appropriate format of proofs, (2) the problem of a satisfactory notion of proof-theoretic harmony, and (3) the problem of extending methods of proof-theoretic semantics beyond logic.

This study will address (3), by venturing beyond logic to set theory. In seeking to provide a *natural and free* logic of sets, we shall also have some things to say about

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(1) and (2). The first part of our journey will involve revisiting Frege, to examine why such a logic is called for, and how to set it up. We shall find that certain proof-theoretic constraints will make that ‘it’ — that logic — unique . . . or so the line of exposition and development on offer here should lead one to believe.

The eventual goal, which will be reached by the end of this study, is to show how the logic of sets (which consists entirely of natural-deduction rules of inference) can be made *ontologically non-committal*. Its rules of inference will nevertheless be fully *meaning-conferring*. This observation applies not only to the central primitive notions — the variable-binding term-forming operator $\{x \mid \ldots x \ldots\}$ for set abstraction, and the binary relation $\in$ for membership — but also to all those ancillary expressions such as $\subseteq$ (‘subset of’), $\bigcup$ (‘union of’), $\mathscr{P}$ (‘power set of’), etc. The latter notions — though of course logically definable in terms of the primitive notions — are so familiar and ‘practically primitive’ in the *lingua franca* of informally rigorous set theory that they call for a more focused rule-theoretic treatment. We shall call them *pasigraphs*, and furnish rules for them. Those rules will be meaning-conferring, but *still* incur no ontological commitments at all.

This means that we can furnish set theorists with a framework of logical rules for set-theoretic notions without committing them to an ontology. We can leave to them the job of specifying which sets exist outright, and which sets exist conditionally on the existence of which other sets.

Our foray in this study into the logic of sets is a protean, if rather ambitious, first step in a more general and unifying study of both *natural deduction and truthmaker semantics for pasigraphs*. The motivating idea is that every pasigraph will have introduction and elimination rules in a system of natural deduction governing one’s deductive reasoning to and from sentences with the pasigraph in question suitably dominant.[1] In addition, every pasigraph will have (model-relative) verification- and falsification-rules for constructing logical truthmakers and falsitymakers of the kind described in Tennant (2018) and Tennant (2010). Such rules afford the pasigraphs what is essentially a proof-theoretic semantics. In the language of the pasigraphs, the notion of logical consequence will be defined in terms of how verifications for premises may be transformed into verifications for conclusions. The aim will then be to show how proofs in Classical Core Logic $\mathbb{C}^+$ afford ‘quasi-’effective methods for carrying out such transformations of (model-relative) truthmakers. (The scare-quoted prefix will be able to be dropped in the constructive case where Core Logic $\mathbb{C}$ affords all the transformations required.)

[1] ‘Suitable dominance’ is plain dominance in the case of sentence-forming operators such as connectives and quantifiers. In the case of term-forming operators @, such as the set-abstraction operator $\{x \mid \Phi(x)\}$, the natural-deduction rules will govern inferences to and from ‘canonical identity statements’ of the form $t = @x\Phi(x)$. We shall expand on this below.

2 The natural and free logic of sets

There can be an analytically valid logic of sets, even if sets themselves are not *logical* objects. For the purposes of this study, the words 'set' and 'class' will be treated as synonyms. No von Neumann–Bernays–Gödel distinction will be countenanced, according to which sets are those classes small enough to be able to belong to yet other sets or classes, whereas (proper) classes are too big to do so, even though they exist.

Frege is the natural starting point for our study. His legacy of complete formalization, both of his logical resources and of the proofs he provided for his results, is invaluable when it comes to considering exactly what the *logic* of sets really is. Two other Fregean themes are of great importance here too.

1. Is this logic to be framed for a so-called 'logically perfect' language in Frege's sense (the most important feature of which, for the purposes of this study, is that every singular term denotes)? — or should one use instead a free logic that can handle non-denoting singular terms?
2. Is this logic to be formulated in such a way as to allow for the possibility that the universe of discourse also contains *Urelemente*?[2]—or should it concern only the more restricted mathematicians' universe of (hereditarily) *pure* sets?

Frege, as is well known, plumped for the first option in each of these cases. And as is also well known, his system suffered the disaster of Russell's paradox. That (in our view) was entirely owing to Frege's answer to question (1) — that every singular term denotes. His answer to question (2) — that we should allow for *Urelemente* — threatens no inconsistency at all, and is well worth implementing in any universally applicable logic of sets that recognizes that some things are *not* sets, and that some sets can have non-sets as members.

It will be argued here that the disaster of Russell's paradox stemmed solely from the misguided choice of a 'logically perfect' language for theorizing about sets, regardless of whether one speaks only of pure sets or allows for the 'impurities' of *Urelemente* in their membership pedigrees.

It will also be argued that the logic of sets that emerges for the revisionary Fregean who adopts a free logic is optimally formulated in terms of introduction and elimination rules (in natural deduction) for the set-abstraction operator

$$\{x \mid \ldots x \ldots\}.$$

Such rules will be stated in due course. This pulls one from the set-theoretic frontiers of Zermelo (1908), back to Fregean origins. The usual story about set theory is one

[2] *Urelemente* — if one's theory permits them — are individuals in the domain of discourse that are *not* sets (or classes). Simple examples would be ordinary physical objects, such as Hilbert's beer mugs, chairs and tables; or, in more sophisticated vein, the fundamental particles of subatomic physics. Not all *Urelemente*, however, have to be concrete individuals. They can be abstract, without being sets (or classes). One could, for example, treat the *natural numbers* as *sui generis* mathematical (or *logical*) objects, not to be identified with any 'set-theoretical surrogates' such as the finite von Neumann ordinals. One could then 'build sets' on top of them, as Weyl sought to do.

of the logicist being utterly vanquished, and the transition being made to a purely *mathematical* (synthetic *a priori*, at best) theory of abstract objects known as pure sets, characterized (as had been the natural, rational and real numbers) by an appropriate effectively decidable set of axioms and axiom schemata. The main implication of the investigation that will unfold here is that this 'mathematization' of set theory by Zermelo and his followers can be regarded as overly precipitous. It abandoned too early, and too pessimistically, the logicist's aim of characterizing at least the *logic* of our talk about sets. This logic embodies just the constraints governing or constituting the *concept* of set, rather than the existential or ontological commitments of any particular set theories.

At the very least, Zermelo's set theory makes it impossible to deal with *Urelemente* alongside sets. This is because its Axiom of Extensionality identifies any two things that have no members. The empty set (which Zermeloan set theory says exists) has no members; and no *Urelement* can have any members. Thus every *Urelement is* the empty set. But no *Urelement* is a set. So there are no *Urelemente*. Zermelo can be talking only about (hereditarily) *pure* sets. And it would remain a mystery how his set theory can find application in our talk about 'the real world' of physical objects, which are the paradigm examples of *Urelemente*. Another such example would be the natural numbers taken as objects *sui generis*, as they are in Reverse Mathematics. These too are really *Urelemente*, a subtlety often overlooked.

There is a line to be drawn between what is logico-analytically valid in our theorizing about sets in general, and which of them have to be specifically postulated, outright or conditionally, as existing. We shall learn that the natural-deduction theorist who is sympathetic to the pursuit of a logic of sets can make a distinctive contribution by taking a very careful look at what was going on in Frege's first systematic stab at the problem. The *Core Logicist* can sharpen the tools Frege left us in a way that is interestingly and significantly short of total mathematizing surrender to the disaster that was Russell's paradox.

The Core Logicist is the theorist who follows the methodological maxim that rules of inference serving to fix the meanings of primitive logico-mathematical expressions have their natural niche in the constructive and relevant deductive reasoning characterized by Core Logic. Conceptual interconnections articulated by definitional rules of inference are constructive and relevant. The aforementioned 'logic of sets' will be generated by using the rules of (free) Core Logic for the usual logical operators, along with well-chosen rules of natural deduction governing set-abstraction.

As explained in Tennant (2017), Core Logic, in its natural deduction formulation, has all its elimination rules in 'parallelized' form. Moreover, their major premises always *stand proud*, with no non-trivial proof-work above them. This ensures two important features: (i) all core natural deductions are in normal form; and (ii) they are also, in a naturally definable sense, isomorphic to the corresponding sequent-calculus proofs. In Core Logic, sequent proofs use Reflexivity as their only structural rule; and otherwise consist only of applications of Right rules and/or Left rules for the operators involved. So core sequent proofs are both cut-free and thinning-free. Right rules in sequent calculus correspond to introduction rules in natural deduction; while Left rules correspond to elimination rules.

A simple example of a natural deduction and its corresponding sequent proof will serve to fix these ideas. Note how the step of ∧-Elimination labeled (2) (and with major premise $A \wedge B$) is in parallelized form, and discharges the conjuncts A and B at their assumption occurrences.

Natural Deduction

$$\dfrac{\dfrac{{\scriptstyle(i)}\,\overline{A \wedge B} \qquad \dfrac{\neg A \vee \neg B \qquad \dfrac{\overline{\neg A}^{\,(3)} \quad \overline{A}^{\,(2)}}{\bot} \qquad \dfrac{\overline{\neg B}^{\,(3)} \quad \overline{B}^{\,(2)}}{\bot}}{\bot}\,{\scriptstyle(3)}}{\bot}\,{\scriptstyle(2)}}{\neg(A \wedge B)}\,{\scriptstyle(1)}$$

Sequent Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{A : A}{\neg A, A :} \qquad \dfrac{B : B}{\neg B, B :}}{\neg A \vee \neg B, A, B :}}{\neg A \vee \neg B, A \wedge B :}}{\neg A \vee \neg B : \neg(A \wedge B)}$$

3 A look at some Fregean basics

Consider this formal sentence in Frege’s now archaic notation:

$$\Delta \frown \text{ἐ}\Phi(\varepsilon)$$

Today it would be written

$$\Delta \in \{x \mid \Phi(x)\}.$$

For Frege, Δ stood for an individual, and Φ for a first-level concept. Frege stipulated in his *Grundgesetze* that the sentence of his displayed form above was to be co-referential[3] with

$$\Phi(\Delta).$$

This would mean, for the modern inferentialist, that Frege would regard as logically or analytically valid the two inference rules[4]

$$\text{F1} \quad \dfrac{\Phi(t)}{t \in \{x \mid \Phi(x)\}} \qquad \text{and} \qquad \text{F2} \quad \dfrac{t \in \{x \mid \Phi(x)\}}{\Phi(t)}.$$

Here we use t instead of Frege’s Δ as a placeholder for singular terms. We shall do this throughout, when couching things in natural-deduction terms.

Frege wanted also to have his first-order binary membership relation

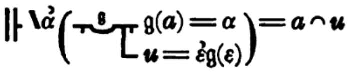
$$\xi \frown \zeta$$

explained ‘für alle möglichen Gegenstände als Argumente’. The explanatory definition he offered was as follows (here, for $a \frown u$):[5]

[3] The German term was ‘gleichbedeutend’ (Frege, 1893, §34, at p. 52). All English translations of material quoted from Frege are taken from Frege (2013).

[4] See, for example, the Appendix in Prawitz (1965).

[5] *Ibid.*, p. 53.

(Def.$\frown$) 

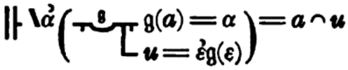
We seek to render (Def.$\frown$) in notation we use today. In preparing to do so, we need to remind ourselves that any singular term of the form

$$\backslash\dot{\varepsilon}\Phi(\varepsilon)$$

was Frege's version of a definite description ('the x such that $\Phi(x)$'), but with the strange twist — in fulfillment of Frege's strict self-imposed requirement that all well-formed singular terms should denote — that, should there *not* be exactly one Φ, the denotation of the displayed term is the class of all Φs. So, if Φ is an empty concept, then the denotation of the displayed term is the empty class; while if more than one object falls under the concept Φ, then the denotation is the class (or set) that they form.[6] In contemporary notation (using iota as the modern definite-description operator) we may render Frege's definition of the displayed term as follows:

$$\backslash\dot{\varepsilon}\Phi(\varepsilon) =_{df} \begin{cases} \iota x\Phi(x) & \text{if } \exists x\forall y(x = y \leftrightarrow \Phi(y)); \\ \{x \mid \Phi(x)\} & \text{otherwise.} \end{cases}$$

Let us now turn to the task of translating into modern notation Frege's definition

(Def.$\frown$) 

Remember that this definitional identity is an identity between the truth-value of the left-hand side:

$$\backslash\dot{\alpha}(\ldots\alpha\ldots)$$

and the truth-value of the right-hand side:

$$a \frown u.$$

The parenthetically enclosed material on the left-hand side is Frege's way of rendering the following second-order sentence in modern notation:

$$\neg\forall G(u = \{x \mid G(x)\} \rightarrow \neg G(a) = \alpha).$$

This is classically equivalent to

$$\exists G\neg(u = \{x \mid G(x)\} \rightarrow \neg G(a) = \alpha),$$

[6] See Frege (1893), §11. As Roy Cook puts it in his Appendix in Frege (2013) (at p. A-19),

> [...] '$\backslash\dot{\varepsilon}\Phi(\varepsilon)$' denotes the unique object that is mapped to the True by the concept named by '$\Phi(\xi)$', if there is such, and denotes the object named by '$\dot{\varepsilon}\Phi(\varepsilon)$' otherwise.

which in turn is classically equivalent to

$$\exists G(u=\{x \mid G(x)\} \wedge G(a)=\alpha).$$

So (Def. $\frown$) is asserting an identity between the truth-value of $a \frown u$ and the truth-value denoted by the Fregean definite description (here rendered in mixed notation)

$$\backslash\dot{\alpha}\,(\exists G(u=\{x \mid G(x)\} \wedge G(a)=\alpha)).$$

For this to be the True (*das Wahre*), the term u has to denote an object — the extension (*Werthverlauf*) of some appropriate concept G. Thus the innocent-looking first-order binary predication

$$a \frown u.$$

commits one to the existence of a denotation for the term u. This is of a piece with Frege's insistence that a 'logically perfect' language has all its singular terms denoting.

4 Moving on from Frege

Suppose we abandon that very imperfect conception of logical perfection, and work with a free logic. Let $\exists!t$ be the familiar abbreviation for $\exists x\, x=t$, for any singular term t. Free logic has the Rule of Atomic Denotation for atomic predicates A:[7]

$$\text{RAD} \quad \frac{A(\ldots t \ldots)}{\exists!t};$$

and expresses the Reflexivity of Identity by the rule

$$\text{Ref=} \quad \frac{\exists!t}{t=t}.$$

Note that these are, respectively, logical *weakenings* of the zero-premise rules

$$\overline{\exists!t} \qquad \text{and} \qquad \overline{t=t}$$

to which Frege, with his 'logically perfect' language, was already committed. So one could invite Frege to recognize the validity of these weakened rules (along with those about to be stated) even though one is now working in a logically 'imperfect' language. In the transition to free logic, the rule of Substitutivity of Identicals remains unchanged:

[7] See Tennant (1978), Ch. 7 §10 for a detailed treatment of free logic and the rules for set theory that we are presenting again here. The rule RAD captures the Russellian requirement that an atomic proposition is true only if all its singular terms denote. Of course it is required in addition that the denotations stand in the relation represented by the predicate of the atomic proposition concerned. The rule RAD captures just the existential presuppositions concerning the singular terms involved.

$$\text{Sub.=} \quad \frac{\varphi \qquad t = u}{\psi} \quad \text{where } \psi \text{ results from } \varphi \text{ by at least one substitution of an occurrence of } t \text{ for one of } u \text{ or of an occurrence of } u \text{ for one of } t.$$

The natural-deduction rules about to be stated aim to characterize no more than the *logic* of talk about sets. To this end, one needs to clarify the interrelationships among sethood (i.e., existence of a set), set-abstraction, predication, and membership. This is a theoretical or foundational aim that the natural-deduction theorist can *share* with the Fregean. This study will investigate how the two theorists can pursue that aim, and whether one of them can claim to have achieved it in a more satisfactory fashion. Our answer, of course, will be that the natural-deduction theorist, with her free logic, is the winner in this comparison.

The rules that the two theorists can formulate are for a language with the set-term forming operator $\{x \mid \ldots x \ldots\}$ primitive. Also primitive is the two-place predicate $\in$ of membership. In due course another primitive (but one-place) predicate S will be added to the language (for 'is a set'). The working assumption will be that the same formal-linguistic resources are available to both theorists (Fregean and natural-deduction), so that the comparison of their approaches will be based on the primitive logical rules that they postulate for the same language. Bear in mind that an axiom is here construed as a zero-premise rule.

4.1 Natural-deduction rules for pure sets

Some notational preliminaries. Where t is a closed term (which of course could be a parameter) and Φ is a formula with just the variable x free, we denote by Φ^x_t the result of replacing every free occurrence of x in Φ with an occurrence of t. Where Φ is a sentence involving at least one occurrence of the parameter a, and with none of those occurrences within the scope of a variable-binding operator applied to the variable x, we denote by Φ^a_x the result of replacing every occurrence of a in Φ with an occurrence of x. Note that every such occurrence of x in Φ^a_x is free.

The rule of introduction (in free logic) for the variable-binding abstraction operator that forms set-terms from predicates is

$$\{\,\}\text{I} \qquad \frac{\begin{array}{c} \underbrace{\overset{(i)}{\overline{\exists! a}}\ ,\ \overset{(i)}{\overline{\Phi^x_a}}} \\ \vdots \\ a \in t \end{array} \qquad \exists! t \qquad \begin{array}{c} \overset{(i)}{\overline{a \in t}} \\ \vdots \\ \Phi^x_a \end{array}}{t = \{x \mid \Phi\}}\ {}^{(i)} \qquad \text{, where } a \text{ is parametric.}^{8}$$

Note how the canonical conclusion

[8] Note that since $\in$ is an *atomic* binary predicate, the assumption $a \in t$ in the rightmost subordinate proof implies $\exists! a$ (by the rule RAD). So is it not necessary to have $\exists! a$ as a further dischargeable assumption in that subordinate proof.

$$t = \{x \mid \Phi\}$$

of { }I has t on its left-hand side, as a placeholder for *any singular term whatsoever*, including the *parameters* (conventionally a, b, c, ...) that can be used for reasoning involving existentials and universals. On the right-hand side of the identity is a set-abstraction term, formed by means of a *dominant* occurrence of the variable-binding abstraction operator $\{x \mid \ldots x \ldots\}$. This operator may be applied to a formula Φ to form the set-abstraction term $\{x \mid \Phi\}$ if, but only if, the variable x has a free occurrence in Φ.

The elimination rules corresponding to the introduction rule stated above for { } are the following three, each one employing the canonical identity statement

$$t = \{x \mid \Phi\}$$

as its major premise (to the left, immediately above the inference stroke). The minor premises (or subproofs) of the first and third rules correspond, respectively, to the first and third immediate subproofs of the introduction rule. This is a convincing sign that the elimination rules are in harmony with the introduction rule that begets them. Bear in mind that the major premise $t = \{x \mid \Phi\}$ stands proud in any application of an elimination rule for { }.

$$\{\,\}\mathrm{E}_1 \quad \frac{t = \{x \mid \Phi\} \qquad \exists! v \qquad \Phi^x_v}{v \in t}$$

$$\{\,\}\mathrm{E}_2 \quad \frac{t = \{x \mid \Phi\}}{\exists! t}$$

$$\{\,\}\mathrm{E}_3 \quad \frac{t = \{x \mid \Phi\} \qquad v \in t \qquad \begin{array}{c} \overline{\Phi^x_v}^{\,(i)} \\ \vdots \\ \theta \end{array}}{\theta}\,(i)$$

The rule { }E_1 has an atomic conclusion, so it is not necessary to parallelize it for the purposes of Core Logic. This is because no atomic conclusion can feature as the major premise of any elimination. The rule { }E_2 is a special case of the Rule of Atomic Denotation. The rule { }E_3 needs to be parallelized, in order to avoid having non-trivial proof-work above Φ^x_v should it happen to stand as the major premise of an elimination.[9]

The introduction rule and the elimination rules just stated for { } are, as just intimated, in harmony. Harmony requires that there at least be reduction procedures that will eliminate from proofs 'maximal sentence occurrences' — conclusions of introductions that are also major premises of eliminations. (Whether harmony requires *more* than this is a more complicated issue that we shall not broach here.) What follow now are the three reduction procedures that are required for harmony (one for each

[9] This risk would be incurred if { }E_3 were to be stated in the serial form $\dfrac{t = \{x \mid \Phi\} \qquad v \in t}{\Phi^x_v}$.

elimination rule). We state them in the notation of Tennant (2017)[10] rather than in our original format in Tennant (1978). We shall refrain from providing reduction procedures for any other Introduction-Elimination pairs, since (as a referee was kind enough to observe) they ‘write themselves’.

$$
\left[\dfrac{\begin{array}{ccc}\underbrace{\Delta_1\,,\ \overline{\exists!a}^{(i)}\,,\ \overline{\Phi^x_a}^{(i)}} & & \underbrace{\Delta_3\,,\ \overline{a\in t}^{(i)}} \\ & \Delta_2 & \\ \Pi_1 & \Pi_2 & \Pi_3 \\ a\in t & \exists!t & \Phi^x_a\end{array}}{t=\{x\mid\Phi\}}\,(i)\quad,\quad \dfrac{\begin{array}{ccc} & \Gamma_1 & \Gamma_2 \\ & \Sigma_1 & \Sigma_2 \\ t=\{x\mid\Phi\} & \exists!v & \Phi^x_v\end{array}}{v\in t}\,\{\ \}\mathrm{E}_1\right]
$$

$$
=\left[\begin{array}{c}\Gamma_1\\ \Sigma_1\,,\\ \exists!v\end{array}\left[\begin{array}{c}\Gamma_2\\ \Sigma_2\,,\\ \Phi^x_v\end{array}\begin{array}{c}\underbrace{\Delta_1\,,\ \exists!v\,,\ \Phi^x_v}\\ \Pi_1{}^a_v\\ v\in t\end{array}\right]\right]
$$

$$
\left[\dfrac{\begin{array}{ccc}\underbrace{\Delta_1\,,\ \overline{\exists!a}^{(i)}\,,\ \overline{\Phi^x_a}^{(i)}} & & \underbrace{\Delta_3\,,\ \overline{a\in t}^{(i)}} \\ & \Delta_2 & \\ \Pi_1 & \Pi_2 & \Pi_3 \\ a\in t & \exists!t & \Phi^x_a\end{array}}{t=\{x\mid\Phi\}}\,(i)\quad,\quad \dfrac{t=\{x\mid\Phi\}}{\exists!t}\,\{\ \}\mathrm{E}_2\right]=\begin{array}{c}\Delta_2\\ \Pi_2\\ \exists!t\end{array}
$$

$$
\left[\dfrac{\begin{array}{ccc}\underbrace{\Delta_1\,,\ \overline{\exists!a}^{(i)}\,,\ \overline{\Phi^x_a}^{(i)}} & & \underbrace{\Delta_3\,,\ \overline{a\in t}^{(i)}} \\ & \Delta_2 & \\ \Pi_1 & \Pi_2 & \Pi_3 \\ a\in t & \exists!t & \Phi^x_a\end{array}}{t=\{x\mid\Phi\}}\,(i)\quad,\quad \dfrac{\begin{array}{ccc} & & \underbrace{\Gamma_2\ ,\ \overline{\Phi^x_v}^{(i)}} \\ & \Gamma_1 & \\ & \Sigma_1 & \Sigma_2 \\ t=\{x\mid\Phi\} & v\in t & \theta\end{array}}{\theta}\,\{\ \}\mathrm{E}_3\right]
$$

$$
=\left[\left[\begin{array}{c}\Gamma_1\\ \Sigma_1\ ,\\ v\in t\end{array}\begin{array}{c}\underbrace{\Delta_3\,,\ v\in t}\\ \Pi_3{}^a_v\\ \Phi^x_v\end{array}\right],\ \begin{array}{c}\underbrace{\Gamma_2\,,\ \Phi^x_v}\\ \Sigma_2\\ \theta\end{array}\right]
$$

[10] Note that in Core Logic the reduction procedures are used only in proving the *admissibility* of ‘Cut with potential epistemic gain’. All core proofs are in normal form. Reductions therefore do not eliminate maximal occurrences from core proofs, because there aren’t any such occurrences in core proofs. Reductions come into the picture only when core proofs are ‘strung together’, with the conclusion of one core proof occurring as a premise of another. The applicable reductions, when carried out by the core logician, will then furnish a core proof of some subsequent of the ‘target sequent’ that the follower of Gentzen would be happy to prove by stringing proofs together and repeatedly applying his structural rule of CUT. For the core logician, CUT is not a rule *of* or *in* the system. Nor is THINNING.

A degenerate application of { }I ensures that everything is the set of its members:

Theorem 4.1 *If t exists, then t is the set of all things bearing ∈ to t.*

Proof

$$\dfrac{\overset{(1)}{\overline{a \in t}} \quad \exists! t \quad \overline{a \in t}^{\,(1)}}{t = \{x \mid x \in t\}}\ (1)\ \{\ \}\mathrm{I}$$

□

So one needs to bear in mind that, with { }I in its present form, the universe of discourse is presumed to consist only of sets and to have no *Urelemente*. This is why the title of this subsection indicates that our rules are for theorizing about (hereditarily) *pure* sets.[11]

4.2 Natural-deduction rules for impure sets

If *Urelemente are* to be countenanced — meaning that one has to allow for the possibility of (hereditarily) *impure* sets — then the second premise (currently $\exists!t$) of the rule { }I will have to take the form St, to be interpreted as 't is a set'. By the Rule of Atomic Denotation this will also secure $\exists!t$, since S is an atomic predicate.

The S-modified introduction rule for { } is as follows.

$$\{\ \}\mathrm{I} \quad \dfrac{\begin{matrix} \underbrace{\overset{(i)}{\overline{\exists! a}}\ ,\ \overline{\Phi^x_a}^{\,(i)}} \\ \vdots \\ a \in t \end{matrix} \qquad St \qquad \begin{matrix} \overline{a \in t}^{\,(i)} \\ \vdots \\ \Phi^x_a \end{matrix}}{t = \{x \mid \Phi\}}\ (i)$$

, where a is parametric.

As a result of this modification, a corresponding change is needed only to $\{\ \}\mathrm{E}_2$ among the elimination rules:

$$\{\ \}\mathrm{E}_2 \quad \dfrac{t = \{x \mid \Phi\}}{St}\ .$$

Note that the S-modified $\{\ \}\mathrm{E}_2$ is *not* an instance of the Rule of Atomic Denotation. With S-modification the rule $\{\ \}\mathrm{E}_2$ has, as it were, 'come into its own' as an elimination rule making its own distinctive contribution.

The S-modified introduction rule for { } is in harmony with its corresponding elimination rules.

4.3 Single-barreled vs. double-barreled abstraction

The kind of introduction rule being considered here (with or without S-modification) will be called a *single-barreled* rule, because of the single occurrence of the set-

[11] The introduction and elimination rules just stated were first given in Tennant (1978).

abstraction operator in the rule's conclusion, dominant on the right-hand side. The intended contrast is with a *double-barreled* rule, such as Frege's ill-fated[12]

$$\text{(Va)} \quad \frac{\forall x(\Phi x \leftrightarrow \Psi x)}{\{x \mid \Phi x\} = \{x \mid \Psi x\}},$$

or even its 'free-logical' modification

$$\text{(fVa)} \quad \frac{\forall x(\Phi x \leftrightarrow \Psi x) \qquad \exists!\{x \mid \Phi x\}}{\{x \mid \Phi x\} = \{x \mid \Psi x\}}.$$

Such rules venture to 'introduce' (in the conclusion) *two* occurrences of the set-abstraction operator, one on each side of the identity sign. This represents a prima-facie limitation on the intended range of identifications afforded by the rule, since it does not involve the more general placeholder t that would be replaceable in licit applications by names or parameters in addition to set-abstraction terms.

If one is countenancing the possibility of *Urelemente*, and accordingly using the predicate S for '. . .is a set', then Theorem 4.1 will read 'If t is a set, then t is the set of all things bearing $\in$ to t.' And in the proof of this one will simply substitute St for $\exists!t$ in the proof of Theorem 4.1. This is worth stating as a separate theorem.

Theorem 4.2 (for the S-modified rule { }I) $\dfrac{St}{t = \{x \mid x \in t\}}$.

Proof

$$\frac{\overset{(1)}{\overline{a \in t}} \qquad St \qquad \overline{a \in t}^{(1)}}{t = \{x \mid x \in t\}} \text{(1) \{ \}I, } S\text{-modified form.}$$ □

The introduction and elimination rules are *ontologically neutral* — they characterize only the *logic* of one's talk about set abstraction, membership, and predication, not one's theory about what sets actually exist. Depending on a mutually agreed decision to confine one's theorizing to pure sets, or, alternatively, to allow for impure sets, *Frege would have conceded the validity of the corresponding natural-deduction rules — especially* the validity of the *elimination* rules for the set-term forming operator.

[12] The rules stated here as Va (on this page) and Vb (on page 102) are respectively the inferential equivalents of Frege's own Va and Vb on p. 69 of the *Grundgesetze*, in §53. From our free-logical perspective it is Va that is disastrous, in permitting easy derivation of Russell's paradox. Ironically, Frege himself, in his *Nachwort*, presented a regimentation of Russell's reasoning in the formalism of the *Grundgesetze*, which ended up laying the blame for Russell's paradox on Vb. (I am grateful here to Peter, for drawing this to my attention.) I rather suspect, though, that if one were to regiment *in natural deduction* Frege's own *Nachwort* reconstruction, within his own formal system, of Russell's reasoning, we would find that Frege was blaming the 'wrong half' of Basic Law V. Vindicating this suspicion, however, is beyond the scope of the present paper.

5 Results provable by the Fregean or by the natural-deduction theorist

In what follows, the *Theorems* stated (at least, up to and including Theorem 8.3) are results to the effect that such-and-such rules of the natural-deduction theorist allow one to derive so-and-so principle of the Fregean; and the *Lemmas* (at least, up to and including Lemma 8.9) are to the effect that so-and-so principles of the Fregean allow one to derive such-and-such rule of the natural-deduction theorist. The convenient abbreviation

$$R_1, \ldots, R_n \Rightarrow R$$

will be used to state these results. The rules $R_1, \ldots, R_n$ will be primitive for one of the theorists, and the rule R will be primitive or derivable for the other. Any other rules not mentioned, but which are used in the derivation, will be ones that are primitive for both of them (such as, for example, the Rule of Atomic Denotation). The aim, in the first instance, is to see whether the two theoretical approaches (roughly: Fregean vs. natural-deduction) are essentially equivalent.

The reader should be aware that we shall apply { }E$_3$ in its serial form (see footnote 9) rather than its parallelized form whenever it is convenient to do so. A good example of this is at the final step of the formal proof that follows, of Theorem 5.1.

Theorem 5.1 { }E$_3$ ⇒ F2.

Proof

$$\dfrac{\dfrac{\dfrac{t \in \{x \mid \Phi(x)\}}{\exists!\{x \mid \Phi(x)\}}\,\text{RAD}}{\{x \mid \Phi(x)\} = \{x \mid \Phi(x)\}}\,\text{Ref=} \qquad t \in \{x \mid \Phi(x)\}}{\Phi(t)}\,\{\ \}\text{E}_3$$

□

If Frege had been instructed on how to construct proofs using rules of inference in the manner employed here, he would have derived the rule

$$\{\ \}\text{E}_1 \quad \dfrac{t = \{x \mid \Phi\} \qquad \exists!v \qquad \Phi^x_v}{v \in t}$$

in the following even stronger form (by not availing himself of the premise $\exists!v$).

Lemma 5.2 F1 ⇒ { }E$_1$.

Proof

$$\dfrac{\dfrac{\Phi^x_v}{v \in \{x \mid \Phi\}}\,\text{F1} \qquad t = \{x \mid \Phi\}}{v \in t}$$

□

Bear in mind: one is talking in this instance of the Frege who is committed to each singular term's enjoying a denotation. That is why he would have eschewed the free logician's needed extra premise $\exists!v$ in the rule { }E$_1$.

Frege would also have been able to derive the rule { }E$_3$:

$$\{\ \}\text{E}_3 \quad \dfrac{t = \{x \mid \Phi\} \qquad v \in t}{\Phi^x_v}$$

Lemma 5.3 F2 ⇒ { }E3.

Proof

$$\dfrac{\dfrac{v \in t \qquad t = \{x \mid \Phi\}}{v \in \{x \mid \Phi\}}}{\Phi^x_v}\ \text{F2}$$

□

Suppose, for the sake of some imaginary and counterfactual speculation, that Frege could have been induced to consider the possibility (in advance of Russell's paradox) that certain kinds of singular terms might not always be secured denotations. He could have been invited to consider the possibility that the extensions of certain concepts were impossible to comprehend as individual, completed entities. Using Kripke's metaphor: such an 'extension' (because of some peculiarity of the concept whose extension it would erroneously be supposed to be) might resist being captured within the limited embrace of any 'intellectual lasso' trying to draw together all its members.

This speculative Frege, one presumes, would have recognized the free-logical validity of the rule { }E1, and would have remained content — with the free logician's concurrence — with the derivation which uses the rule F2 and which was supplied on his behalf in Lemma 5.3 for the rule { }E3. But he would have modified his erstwhile rule F1 to become fF1 ('fF' here for 'free-logical Frege'), furnished with the two existential presuppositions that are needed in the free-logical context:

$$\text{fF1} \quad \dfrac{\Phi(t) \qquad \exists!t \qquad \exists!\{x \mid \Phi(x)\}}{t \in \{x \mid \Phi(x)\}}$$

Lemma 5.4 fF1 ⇒ { }E1.

Proof

$$\dfrac{\dfrac{\Phi(v) \qquad \exists!v \qquad \dfrac{t = \{x \mid \Phi\}}{\exists!\{x \mid \Phi\}}}{v \in \{x \mid \Phi\}}\ \text{fF1} \qquad t = \{x \mid \Phi\}}{v \in t}$$

□

The natural-deduction theorist can return the favor, with the following converse.

Theorem 5.5 { }E1 ⇒ fF1.

Proof

$$\dfrac{\begin{array}{c}\exists!\{x \mid \Phi(x)\},\ \text{i.e.}\\ \exists y\, y = \{x \mid \Phi(x)\}\end{array} \qquad \dfrac{\dfrac{\overline{a = \{x \mid \Phi(x)\}}^{(1)} \qquad \exists!t \qquad \Phi(t)}{t \in a}\ \text{E}_1 \qquad \overline{a = \{x \mid \Phi(x)\}}^{(1)}}{t \in \{x \mid \Phi(x)\}}}{t \in \{x \mid \Phi(x)\}}\ (1)$$

□

6 Russell's paradox

Here we shall be scrupulous in using the 'official', parallelized form of the rule { }E3. This is in order to ensure the correctness of the claim that the Core logician can show that the Russell set does not exist.

Let us use the abbreviation r for the set-term $\{x \mid \neg x \in x\}$. Consider the following proof

$$\begin{array}{c}\exists! r\\ \Sigma\\ \neg r \in r\end{array} \quad : \quad \dfrac{\dfrac{\dfrac{\exists! r}{r = \{x \mid \neg x \in x\}}\,\text{Ref.=} \qquad \dfrac{}{r \in r}\,(1) \qquad \dfrac{(2)\,\dfrac{}{\neg r \in r} \qquad \dfrac{}{r \in r}\,(1)}{\bot}}{\bot}\,\{\ \}E_3}{\neg r \in r}\,(1)$$

Now use Σ to construct the proof

$$\begin{array}{c}\exists! r\\ \Xi\\ r \in r\end{array} \quad : \quad \dfrac{\dfrac{\exists! r}{r = \{x \mid \neg x \in x\}} \qquad \exists! r \qquad \begin{array}{c}\exists! r\\ \Sigma\\ \neg r \in r\end{array}}{r \in r}\,\{\ \}E_1$$

Finally, embed Ξ twice as follows, to form the (dis)proof

$$\begin{array}{c}\exists! r\\ \Pi\\ \bot\end{array} \quad : \quad \dfrac{\dfrac{\exists! r}{r = \{x \mid \neg x \in x\}}\,\text{Ref.=} \qquad \begin{array}{c}\exists! r\\ \Xi\\ r \in r\end{array} \qquad \dfrac{(1)\,\dfrac{}{\neg r \in r} \qquad \begin{array}{c}\exists! r\\ \Xi\\ r \in r\end{array}}{\bot}}{\bot}\,(1)\ \{\ \}E_3$$

The disproof Π is in normal form. It avails itself of only the following rules:

1. Rule of Atomic Denotation,
2. Rule of Reflexivity of Identity,
3. $\neg$-Introduction,
4. $\neg$-Elimination,
5. $\{\ \}E_1$,
6. $\{\ \}E_3$,

and all of these would be acceptable to Frege. Conspicuously absent is any appeal to Basic Law V (or, more accurately, (Va)). So Frege was in error in concluding

> The error can only lie in our Law (Vb) which must therefore be false.
> (Der Fehler kann allein in unserm Gesetze (Vb) liegen, das also falsch sein muss.)

(See the *Nachwort* in Frege (1903), at p. 257.) To be sure, as will be seen by the end of this study, one can use (Va) to get into Russellian trouble; but Russell's paradox can (as just seen) be derived from much more basic logical materials, in a manner whose strict formalization makes no use at all of the beknighted (Va). Moreover, Russell's result, in this free-logical setting, is not a paradox at all. Rather, Π is a normal-form disproof of the claim $\exists! r$, that is, of $\exists y\, y = \{x \mid \neg x \in x\}$. By the rule $\neg$I it straightforwardly yields the negative existential theorem $\neg\exists y\, y = \{x \mid \neg x \in x\}$ in the logic of sets.

Constructivism and intuitionism in logic and mathematics, especially as formalisms, came well after Volume 2 of Frege's *Grundgesetze*. So Frege cannot be expected to have sought a constructive reductio, such as Π above, of the assumption that the Russell set exists. But it is worth pointing out that Π is indeed constructive. There is no application within Π of any strictly classical rule for negation. Frege was barking up a wrong tree when he informally invoked the Law of Excluded Middle in his informal presentation (in his *Nachwort* to Volume 2 of the *Grundgesetze*) of the Russellian reasoning. As he pondered the right revisionary response to Russell's paradox, he considered the possibility that one might have to abandon the Law of Excluded Middle. He seriously posed the question

> Should we assume the law of excluded middle fails for classes?
> (Sollen wir annehmen, das Gesetz vom ausgeschlossenem Dritten gelte von den Klassen nicht?)

(See the *Nachwort* in Frege (1903), at p. 254.) We can see now, however, that placing the blame on Excluded Middle, and abandoning it as a logical law, would have been futile. *For the Russell Paradox is a problem for the constructivist*, not just the classicist. Frege could have performed the formal reasoning in Π (even suppressing the middle premise of each application of { }E_1 therein), to reduce the assumption $\exists!\{x \mid \neg x \in x\}$ to absurdity, without any appeal to Excluded Middle.

Frege's ultimate mistake, on this analysis of the Russell Paradox, is his assumption that every grammatically well-formed singular term must denote something. It is ironic that he required this of any 'logically perfect' language. The real folly of Basic Law V is how it visits Fregean 'logical perfection' on set-abstraction terms, *even in the context of an explicitly free logic.* The folly can be seen at work in the direction given by (Va), from coextensiveness of concepts to identity of their extensions. Recall that (Va) can be expressed as a rule as follows:

$$\frac{\forall x(\Phi x \leftrightarrow \Psi x)}{\{x \mid \Phi x\} = \{x \mid \Psi x\}}$$

Take Φ for Ψ. Then *in free logic* we have the proof

$$\cfrac{\cfrac{\begin{array}{c}\vdots\ \text{Logic}\\ \forall x(\Phi x \leftrightarrow \Phi x)\end{array}}{\{x \mid \Phi x\} = \{x \mid \Phi x\}}\ \text{Va}}{\exists!\{x \mid \Phi x\}}\ \text{RAD}$$

So (Va) commits one to the existence of the set $\{x \mid \Phi x\}$ for *every* concept (or predicate) Φ.

It is frequently remarked that Frege's error was to believe in the *Axiom Schema of Naive Comprehension*:

$$\forall\Phi\exists X\forall y(y \in X \leftrightarrow \Phi y).$$

This is indeed derivable — even in free logic — by appeal to (Va). It is worth pointing out that the derivation makes use only of the *elimination* rules { }E_1 and { }E_3 for the set-abstraction operator. No use is made either of { }E_2 or of the introduction rule. This in turn means that what the following proof reveals is invariant across the 'pure' vs. 'impure' divide.[13] Nothing turns, that is, on whether the middle premise of the rule { }I takes the form $\exists!t$ or St. (We shall have occasion to remark once again on 'pure vs. impure invariance' in due course.)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\vdots_{\text{Logic}}}{\forall x(Fx \leftrightarrow Fx)}}{\{x \mid Fx\}=\{x \mid Fx\}}\,(\text{Va})}{\exists!\{x \mid Fx\}}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots_{\text{Logic}}}{\forall x(Fx \leftrightarrow Fx)}}{\{x \mid Fx\}=\{x \mid Fx\}}\,(\text{Va}) \quad \overline{a \in \{x \mid Fx\}}^{\,(1)}}{Fa}\,\{\,\}\mathrm{E}_3
\qquad
\dfrac{\dfrac{\dfrac{\vdots_{\text{Logic}}}{\forall x(Fx \leftrightarrow Fx)}}{\{x \mid Fx\}=\{x \mid Fx\}}\,(\text{Va}) \quad \overline{\exists!a}^{\,(2)} \quad \overline{Fa}^{\,(1)}}{a \in \{x \mid Fx\}}\,\{\,\}\mathrm{E}_1}{a \in \{x \mid Fx\} \leftrightarrow Fa}\,(1)}{\forall y(y \in \{x \mid Fx\} \leftrightarrow Fy)}\,(2)}
{\dfrac{\exists X \forall y(y \in X \leftrightarrow Fy)}{\forall \Phi \exists X \forall y(y \in X \leftrightarrow \Phi y)}}
$$

7 Extensionality

One might wonder whether the introduction rule for the set-term forming operator plays any philosophically important role. The answer is that it makes a crucial contribution in proving the *extensionality* of sets — that two sets are identical if they have exactly the same members. The proof of this result (Theorem 7.1 below) invokes Theorem 4.1 — whose own proof involved a degenerate application of the rule { }I — and then makes use of a further, *non*-degenerate application of { }I. The natural-deduction theorist proves Theorem 7.1 using the rules for pure set theory (favoring $\exists!t$ over St as the second premise of { }I). The reader can be left the exercise of modifying the proof so that with the rules involving St the result can be established in a suitably 'S-restricted' form.[14]

Theorem 7.1 *Sets are identical if they have exactly the same members.*

Sentential version: $\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y)$.

Inferential version:

$$
\dfrac{\exists!t \qquad \exists!u \qquad \begin{array}{c}\overline{a \in t}^{\,(i)}\\ \vdots \\ a \in u\end{array} \qquad \begin{array}{c}\overline{a \in u}^{\,(i)}\\ \vdots \\ a \in t\end{array}}{t = u}\,(i)
$$

[13] This remark is about only the sufficiency, and not necessarily the necessity, of eschewal of $\{\,\}\mathrm{E}_1$ and $\{\,\}\mathrm{E}_3$ for the invariance in question.

[14] The S-modified result to be proved in the sentential version in Theorem 7.1 is

$$\forall x(Sx \rightarrow \forall y(Sy \rightarrow \forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y))).$$

Its inferential version is

$$
\dfrac{St \qquad Su \qquad \begin{array}{c}\overline{a \in t}^{\,(i)}\\ \vdots \\ a \in u\end{array} \qquad \begin{array}{c}\overline{a \in u}^{\,(i)}\\ \vdots \\ a \in t\end{array}}{t = u}\,(i)\,.
$$

Proof Sentential version:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{a \in d}^{\,(1)}}{\exists! a} \quad \overline{\forall z(z \in c \leftrightarrow z \in d)}^{\,(2)}}{a \in c \leftrightarrow a \in d} \quad \overline{a \in d}^{\,(1)}}{a \in c} \quad \overline{\exists! c}^{\,(4)} \quad \dfrac{\dfrac{\dfrac{\overline{a \in c}^{\,(1)}}{\exists! a} \quad \overline{\forall z(z \in c \leftrightarrow z \in d)}^{\,(2)}}{a \in c \leftrightarrow a \in d} \quad \overline{a \in c}^{\,(1)}}{a \in d}}{c = \{x \mid x \in d\}}{\scriptstyle (1)\,\{\,\}\mathrm{I}} \quad \dfrac{\overline{\exists! d}^{\,(3)}}{d = \{x \mid x \in d\}}{\scriptstyle (\mathrm{Th}.4.1)}}{\dfrac{c = d}{\forall z(z \in c \leftrightarrow z \in d) \rightarrow c = d}{\scriptstyle (2)}}}{\dfrac{\forall y(\forall z(z \in c \leftrightarrow z \in y) \rightarrow c = y)}{\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y)}{\scriptstyle (4)}}{\scriptstyle (3)}
$$

Inferential version:

$$
\dfrac{\dfrac{\begin{matrix}\overline{a \in u}^{\,(i)} \\ \vdots \\ a \in t\end{matrix} \qquad \exists! t \qquad \begin{matrix}\overline{a \in t}^{\,(i)} \\ \vdots \\ a \in u\end{matrix}}{t = \{x \mid x \in u\}}\,{\scriptstyle (1)\,\{\,\}\mathrm{I}} \qquad \dfrac{\exists! u}{u = \{x \mid x \in u\}}\,{\scriptstyle (\mathrm{Th}.4.1)}}{t = u}
$$

□

Note that Extensionality as a *derived* result here — in either its sentential or its inferential version — does not itself contain any occurrences of the set-abstraction operator. In conventional (first-order) set theory, in the usual stripped-down language with ∈ as a predicate but *without* the set-abstraction operator primitive, one would need of course to follow Zermelo in *postulating* Extensionality as an axiom. One of the virtues of the natural-deduction rules essayed here is that Extensionality is 'built in' to the resulting conception of set, whether one is theorizing about pure or about impure sets.

8 Natural-deduction for set abstraction vs. Fregean abstraction

The question now arises: just how radical a departure from the (incoherent) Fregean conception of class (or set) is represented by the conception captured by the natural deduction rules for the set-abstraction operator, in a free logic? What is the relationship between the latter rules, and Frege's Basic Law V (whose half called Vb is what Frege — mistakenly, in our free-logical view — blamed for Russell's paradox)?

The following completely formal derivations will reveal the answer.

The innocent half of Basic Law V (call it Vb):

$$
(\mathrm{Vb}) \qquad \dfrac{\{x \mid Fx\} = \{x \mid Gx\}}{\forall x(Fx \leftrightarrow Gx)}
$$

can be derived by the natural-deduction theorist (see Theorem 8.1) *and by the Fregean* (see Lemma 8.2). This latter result is interesting; it stems from just the rules F1 and F2 adopted by the Fregean who assumes a 'logically perfect' language. So the

Frege who would venture to adopt both F1 and F2 as logically correct and primitive inferences need not have bothered to state Vb as a basic law, or as a conjunctive part of any other basic law (such as the relevant half, in one direction, of the biconditional Basic Law V).

Theorem 8.1 { }E$_1$, { }E$_3$ ⇒ Vb.

Proof Note the symmetry in the proof (to be expected with '=' and '↔'), and the use of the two main elimination rules for { }, but not of the introduction rule. This is another instance of 'pure vs. impure invariance'.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\{x|Fx\}=\{x|Gx\}}{\exists!\{x|Fx\}}}{\{x|Fx\}=\{x|Fx\}} \quad \dfrac{\{x|Fx\}=\{x|Gx\} \quad \overline{\exists!a}^{(2)} \quad \overline{Ga}^{(1)}}{a\in\{x|Fx\}}\,\{\,\}\mathrm{E}_1}{Fa}\,\{\,\}\mathrm{E}_3 \qquad \dfrac{\dfrac{\dfrac{\{x|Fx\}=\{x|Gx\}}{\exists!\{x|Gx\}}}{\{x|Gx\}=\{x|Gx\}} \quad \dfrac{\{x|Fx\}=\{x|Gx\} \quad \overline{\exists!a}^{(2)} \quad \overline{Fa}^{(1)}}{a\in\{x|Gx\}}\,\{\,\}\mathrm{E}_1}{Ga}\,\{\,\}\mathrm{E}_3}{Fa\leftrightarrow Ga}\,(1)}{\forall x(Fx\leftrightarrow Gx)}\,(2) \quad \square$$

Lemma 8.2 F1, F2 ⇒ Vb.

Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{Ga}^{(1)}}{a\in\{x\mid Gx\}}\,\mathrm{F1} \quad \{x\mid Fx\}=\{x\mid Gx\}}{a\in\{x\mid Fx\}}}{Fa}\,\mathrm{F2} \qquad \dfrac{\dfrac{\dfrac{\overline{Fa}^{(1)}}{a\in\{x\mid Fx\}}\,\mathrm{F1} \quad \{x\mid Fx\}=\{x\mid Gx\}}{a\in\{x\mid Gx\}}}{Ga}\,\mathrm{F2}}{Fa\leftrightarrow Ga}\,(1)}{\forall x(Fx\leftrightarrow Gx)}\,(2) \quad \square$$

Can Lemma 8.2 be strengthened by using the 'free logical' Fregean rule fF1 in place of F1? The answer is affirmative. See Lemma 8.9.

The natural-deduction theorist can prove not the converse of Vb — which would be the inconsistent Va — but a slight weakening of it, by adding an existential presupposition. The inference to be established is

$$\mathrm{fVa} \quad \dfrac{\forall x(Fx\leftrightarrow Gx) \qquad \exists!\{x\mid Fx\}}{\{x\mid Fx\}=\{x\mid Gx\}}.$$

Theorem 8.3 { }E$_1$, { }E$_3$, { }I ⇒ fVa.

Proof

$$\dfrac{\dfrac{\dfrac{\exists!\{x|Fx\}}{\{x|Fx\}=\{x|Fx\}} \quad \overline{\exists!a}^{(1)} \quad \dfrac{\dfrac{\overline{\exists!a}^{(1)} \quad \forall x(Fx\leftrightarrow Gx)}{Fa\leftrightarrow Ga} \quad \overline{Ga}^{(1)}}{Fa}}{a\in\{x|Fx\}}\,\{\,\}\mathrm{E}_1 \quad \exists!\{x|Fx\} \quad \dfrac{\dfrac{\dfrac{\exists!\{x|Fx\}}{\{x|Fx\}=\{x|Fx\}} \quad \overline{a\in\{x|Fx\}}^{(1)}}{Fa}\,\{\,\}\mathrm{E}_3 \quad \dfrac{\dfrac{\overline{a\in\{x|Fx\}}^{(1)}}{\exists!a} \quad \forall x(Fx\leftrightarrow Gx)}{Fa\leftrightarrow Ga}}{Ga}}{\{x|Fx\}=\{x|Gx\}}\,(1)\,\{\,\}\mathrm{I} \quad \square$$

Note that easy re-lettering will enable one to use the premise $\exists!\{x \mid Gx\}$ instead of $\exists!\{x \mid Fx\}$. The proof of Theorem 8.3 uses only the two main elimination rules for { }, along with its introduction rule.

The natural-deduction theorist's proof of Theorem 8.3 just given can be adapted so as to employ the rule { }I in its S-modified form. The adapted proof follows. Note that it appeals to Observation 8.4 in its middle immediate subproof. The natural-deduction theorist's proof of Observation 8.4 is presumed to be available here, and will be found on p. 106. It involves only two primitive steps.

$$\dfrac{\dfrac{\dfrac{\exists!\{x|Fx\}}{\{x|Fx\}=\{x|Fx\}}\quad \overline{\exists!a}^{\,(1)}\quad \dfrac{\dfrac{\overline{\exists!a}^{\,(1)}\quad \forall x(Fx\leftrightarrow Gx)}{Fa\leftrightarrow Ga}\quad \overline{Ga}^{\,(1)}}{Fa}}{a\in\{x|Fx\}}\,\{\,\}\mathrm{E}_1\qquad \dfrac{\exists!\{x|Fx\}}{S\{x|Fx\}}\,\text{Obs. 1}\qquad \dfrac{\dfrac{\dfrac{\exists!\{x|Fx\}}{\{x|Fx\}=\{x|Fx\}}\quad \overline{a\in\{x|Fx\}}^{\,(1)}}{Fa}\,\{\,\}\mathrm{E}_3\quad \dfrac{\dfrac{\overline{a\in\{x|Fx\}}^{\,(1)}}{\exists!a}\quad \forall x(Fx\leftrightarrow Gx)}{Fa\leftrightarrow Ga}}{Ga}}{\{x|Fx\}=\{x|Gx\}}\,(1)\ \{\,\}\mathrm{I},\ S\text{-modified form}$$

So we see that Basic Law V, suitably conditioned in its problematic direction with a much-needed existential premise, is derivable in our free logic for the set-term forming operator { }. It is fVa that is the (modified) Fregean way to express the fact that sets are extensional; and in deriving it, the natural-deduction theorist needs to use { }I.

Is the natural-deduction theorist's logic of sets tantamount to nothing more than Basic Law V thus modified? The answer would presumably be affirmative if, but only if, by using the inferences

$$\dfrac{\{x\mid Fx\}=\{x\mid Gx\}}{\forall x(Fx\leftrightarrow Gx)}\qquad\qquad \dfrac{\forall x(Fx\leftrightarrow Gx)\quad \exists!\{x\mid Fx\}}{\{x\mid Fx\}=\{x\mid Gx\}}$$

the free-logical Fregean could derive (in free logic) the introduction and elimination rules that have been stated for { }. The rules $\{\,\}\mathrm{E}_1$ and $\{\,\}\mathrm{E}_3$ have already been furnished with derivations of the requisite kind (see Theorems 5.2 and 5.3). The reader will recall that (in the case of pure set theory) $\{\,\}\mathrm{E}_2$ is a special case of the rule RAD of free logic. And in the case of a set theory countenancing *Urelemente*, the S-modified rule

$$\{\,\}\mathrm{E}_2\quad \dfrac{t=\{x\mid\Phi\}}{St}$$

will in due course be adopted by (us, on behalf of) the Fregean as a primitive rule, to be called F3 (see below). The remaining task, then, for the Fregean, is to derive { }I. Can this be done?

The answer is a cautious affirmative. The caution is occasioned by the residual need *on the part of the Fregean* to supply the inferential transition occurring twice in the following proof, as indicated by the descending dots:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\Delta_2\\ \Pi_2\\ \exists!t\\ \vdots\\ t=\{x\mid x\in t\}\end{array}}{\exists!\{x\mid x\in t\}}\qquad \dfrac{\dfrac{\begin{array}{c}\underbrace{{}^{(1)}\overline{\varphi a},\ \overline{\exists!a}^{\,(2)},\ \Delta_1}\\ \Pi_1\\ a\in t\end{array}\qquad \begin{array}{c}\underbrace{{}^{(1)}\overline{a\in t},\ \Delta_3}\\ \Pi_3\\ \varphi a\end{array}}{a\in t\leftrightarrow\varphi a}\,(1)}{\forall x(x\in t\leftrightarrow\varphi x)}\,(2)}{\{x\mid x\in t\}=\{x\mid\varphi x\}}\,\mathrm{fVa}\qquad \begin{array}{c}\Delta_2\\ \Pi_2\\ \exists!t\\ \vdots\\ t=\{x\mid x\in t\}\end{array}}{t=\{x\mid\varphi x\}}$$

That inferential transition is of course guaranteed by Theorem 4.1, to which the free-logical, natural-deduction theorist about sets is entitled. But would either the original Frege, or the speculatively 'free-logicized' Frege, be thus entitled?

Let us explore some conceptual and logical possibilities on Frege's behalf. We know that he conceived of his *Begriffsschrift* as a language for the pursuit of truths not just about the abstract realm, but also about concrete reality. So he would not have been satisfied with being confined to talk only of sets. 'Pure-set' theorizing would have been too restrictive, expressively, for Frege. So it is reasonable to infer that he would have been prepared to adopt a primitive predicate like S, so as to be able to express the informal idea, concerning any supplied argument (*Gegenstand*), that it is a set — that is, (for Frege) the completed extension of some concept. For any *Urelement*, of course, what is thus expressed is false.

Countenancing the possibility of *Urelemente*, Frege would have refused — correctly — to adopt the inferential principle

$$\frac{\exists!t}{St}\,.$$

But he would have been happy — rightly — with its converse:

$$\frac{St}{\exists!t}\,.$$

This rule says that every set exists. And, since St is an atomic predication, the rule is a special case of the Rule of Atomic Denotation.

The distinction was drawn earlier between *pure set* theorizing and *impure set* theorizing. Frege's theorizing was of the latter kind, since he regarded the universe of discourse as truly universal, and therefore containing all concrete objects (*Urelemente*, such as Julius Caesar) along with sets formed from them (such as the singleton {Julius Caesar}), *and* along with sets that happened to be hereditarily pure (such as $\{y \mid y = \{x \mid \neg x = x\}\}$). Suppose, however, that Frege had been asked to theorize in a more focused way about the (sub-)universe (for him) of hereditarily pure sets. It would have been both obvious and natural for him to give expression to this expressive focus by adopting the following rule π (for 'purity'):

$$\pi \quad \frac{\exists!t}{St}$$

where the intended reading of 'St' is (as always here) 't is a set'. It will follow from this that t is *hereditarily pure* as a set, since every member of t exists, hence (by rule π) is a set. Since no *Urelement* is a set:

$$\frac{Ut \qquad St}{\bot}$$

the membership pedigree of t does not contain any *Urelemente*, and t is accordingly pure. A little bit of sethood, given mere existence, goes a long way. The foregoing rule of contrariety is primitive for both the Fregean and the natural-deduction theorist when they allow for *Urelemente*.

What would it take, then, on the part of some *Gegenstand* t, to earn the honorific sortal S from Frege? Surely it would be enough that t really *be* the extension of some concept Φ. That is, the following inferential principle would be logico-analytically valid for the Fregean, and self-evident:

$$\text{F3} \quad \frac{t = \{x \mid \Phi(x)\}}{St}$$

This rules says that if t is the set of all Φs, then t is a set. It is primitive for the ND-theorist (for it is the set-abstraction elimination rule $\{\ \}\text{E}_2$), and the Fregean has every right to adopt it as a primitive rule too.

Observation 8.4 $\dfrac{\exists!\{x \mid \Phi(x)\}}{S\{x \mid \Phi(x)\}}$.

Proof (by the Fregean)

$$\cfrac{\exists!\{x \mid \Phi(x)\} \qquad \cfrac{\cfrac{\overline{a = \{x \mid \Phi(x)\}}^{\,(1)}}{Sa}\,\text{F3} \qquad \overline{a = \{x \mid \Phi(x)\}}^{\,(1)}}{S\{x \mid \Phi(x)\}}}{S\{x \mid \Phi(x)\}}\,(1)$$

□

Proof (by the ND-theorist)

$$\cfrac{\cfrac{\exists!\{x \mid \Phi(x)\}}{\{x \mid \Phi(x)\} = \{x \mid \Phi(x)\}}\,\text{Ref.=}}{S\{x \mid \Phi(x)\}}\,\{\ \}\text{E}_2$$

□

Observation 8.4 tells us that even though, as pointed out above, the inference

$$\frac{\exists!t}{St}$$

does not hold in general, it *does* hold when, more specifically, we have a set-abstraction term in place of t.

We have been considering a Frege who distinguishes between sets and *Urelemente*. So for fair comparison of his system with that of the free-logical natural-deduction theorist, the latter must give up commitment to theorizing only about sets. This means that the rule $\{\ \}$I must have its second premise in the form St; and, correspondingly, the elimination rule $\{\ \}\text{E}_2$ will be

$$\frac{t = \{x \mid \Phi\}}{St}$$

The sought derivation of the S-modified rule $\{\ \}$I using Fregean principles will eventually be found. See Lemma 8.8 below.

Now consider the prospect of having some sort of converse of principle F3. Suppose one is given just the premise St. Then one knows that t is the extension of *some* concept or other. And what might be the most general — indeed canonical — concept that would fit this bill? Why,

'... is a member of t',

of course. So the following inferential principle would be logico-analytically valid for the Fregean:

$$\text{F4} \quad \frac{St}{t = \{x \mid x \in t\}}$$

Note that the primitive rule F4 here being accepted by (us, on behalf of) the Fregean is Theorem 4.2 of the natural-deduction theorist. Both theorists have it *as part of their logic of sets*. The only difference is that for the Fregean it is primitive — because it has to be — whereas for the natural-deduction theorist it is derived.

Why do we say that F4 *has to be* primitive for the Fregean? The answer is that inspection of all of the Fregean's other primitive rules — RAD, Sub=, Ref=, fF1, F2, F3, fVa, and π — reveals that one cannot use them to derive the conclusion $t = \{x \mid x \in t\}$ from the premise St.

The 'pure-set' Fregean, by adopting the rules π and F4, is able to mimic the natural-deduction theorist's Theorem 4.1 as follows.

Lemma 8.5 $\dfrac{\exists ! t}{t = \{x \mid x \in t\}}$.

Proof
$$\dfrac{\dfrac{\exists ! t}{St}\,\pi}{t = \{x \mid x \in t\}}\,\text{F4}$$
□

The question arises: can the Fregeans prove Theorem 7.1 using their own rules thus far, so as to parallel and emulate what the natural-deduction theorist did? The answer is affirmative. We shall deal with Extensionality in its inferential form.

Lemma 8.6 F4, π, fVa $\Rightarrow$

$$\dfrac{\exists ! t \quad \exists ! u \quad \begin{array}{c}\overline{a \in t}^{\,(i)} \\ \vdots \\ a \in u\end{array} \quad \begin{array}{c}\overline{a \in u}^{\,(i)} \\ \vdots \\ a \in t\end{array}}{t = u}\,(i) \, .$$

Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\overline{a \in t}^{\,(1)} \\ \vdots \\ a \in u\end{array} \quad \begin{array}{c}\overline{a \in u}^{\,(1)} \\ \vdots \\ a \in t\end{array}}{a \in t \leftrightarrow a \in u}\,(1)}{\forall x (x \in t \leftrightarrow x \in u)} \quad \exists ! \{x \mid x \in t\}}{\{x \mid x \in t\} = \{x \mid x \in u\}}\,\text{fVa} \quad \dfrac{\exists ! t}{t = \{x \mid x \in t\}}\,\text{L8.5}}{t = \{x \mid x \in u\}} \quad \dfrac{\exists ! u}{u = \{x \mid x \in u\}}\,\text{L8.5}}{t = u}$$
□

Fregean rule F4 is potent in another important regard. Teaming up with fVa, it yields the natural-deduction theorist's introduction rule { }I — both in its original version (ensuring purity of sets) and in its S-modified version (allowing for *Urelemente*) as shown, respectively, by Lemma 8.7 and Lemma 8.8.

Lemma 8.7 F4, fVa, $\pi \Rightarrow \{\ \}$I (pure-set version).

Proof Recall that the proof of Lemma 8.5 used F4.

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\Delta_2\\ \Pi_2\\ \exists! t\end{array}}{t = \{x \mid x \in t\}}\,\text{L8.5}}{\exists!\{x \mid x \in t\}} \qquad \dfrac{\dfrac{\begin{array}{c}\underbrace{\overset{(1)}{\overline{\varphi a}}\,,\overset{(2)}{\overline{\exists! a}}\,,\Delta_1}\\ \Pi_1\\ a \in t\end{array} \qquad \begin{array}{c}\underbrace{\overset{(1)}{\overline{a \in t}}\,,\ \Delta_3}\\ \Pi_3\\ \varphi a\end{array}}{a \in t \leftrightarrow \varphi a}\,(1)}{\forall x(x \in t \leftrightarrow \varphi x)}\,(2)}{\{x \mid x \in t\} = \{x \mid \varphi x\}}\,\text{fVa}$$

$$\dfrac{\{x \mid x \in t\} = \{x \mid \varphi x\} \qquad \dfrac{\begin{array}{c}\Delta_2\\ \Pi_2\\ \exists! t\end{array}}{t = \{x \mid x \in t\}}\,\text{L8.5}}{t = \{x \mid \varphi x\}}$$

□

Lemma 8.8 F4, fVa $\Rightarrow \{\ \}$I (S-modified version).

Proof

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\Delta_2\\ \Pi_2\\ St\end{array}}{t = \{x \mid x \in t\}}\,\text{F4}}{\exists!\{x \mid x \in t\}} \qquad \dfrac{\dfrac{\begin{array}{c}\underbrace{\overset{(1)}{\overline{\varphi a}}\,,\overset{(2)}{\overline{\exists! a}}\,,\Delta_1}\\ \Pi_1\\ a \in t\end{array} \qquad \begin{array}{c}\underbrace{\overset{(1)}{\overline{a \in t}}\,,\ \Delta_3}\\ \Pi_3\\ \varphi a\end{array}}{a \in t \leftrightarrow \varphi a}\,(1)}{\forall x(x \in t \leftrightarrow \varphi x)}\,(2)}{\{x \mid x \in t\} = \{x \mid \varphi x\}}\,\text{fVa}$$

$$\dfrac{\{x \mid x \in t\} = \{x \mid \varphi x\} \qquad \dfrac{\begin{array}{c}\Delta_2\\ \Pi_2\\ St\end{array}}{t = \{x \mid x \in t\}}\,\text{F4}}{t = \{x \mid \varphi x\}}$$

□

By Lemma 8.7, the Fregean rules F4, fVa, and π suffice for proof of { }I in its original form (ensuring purity); and by Lemma 8.8, F4 and fVa suffice for proof of { }I in its S-modified form (allowing for *Urelemente*). No matter which form of it is used, { }I in turn suffices for proof of extensionality in the relevant form (with unrestricted or S-restricted quantifiers, as seen from Theorem 7.1 and the comment thereon in footnote 14).

Recall Theorem 8.1:

$$\{\ \}\text{E}_1, \{\ \}\text{E}_3 \Rightarrow \text{Vb},$$

Lemma 5.4:

$$\text{fF1} \Rightarrow \{\ \}\text{E}_1,$$

and Lemma 5.3:

$$\text{F2} \Rightarrow \{\ \}\text{E}_3.$$

It follows by 'rule transitivity' that

$$\text{fF1}, \text{F2} \Rightarrow \text{Vb}.$$

Here is a more direct proof of this last result. It can be obtained by accumulating the proofs of Theorem 8.1, Lemma 5.4, and Lemma 5.3, and applying to that accumulation the two 'shrinking reductions' that the reader will find are obviously called for.

Lemma 8.9 fF1, F2 $\Rightarrow$ Vb.

Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{Ga}^{\,(1)} \quad \overline{\exists! a}^{\,(2)} \quad \dfrac{\{x|Fx\}=\{x|Gx\}}{\exists!\{x|Gx\}}}{a \in \{x|Gx\}}\,\text{fF1} \quad \{x|Fx\}=\{x|Gx\}}{a \in \{x|Fx\}}}{Fa}\,\text{F2} \qquad \dfrac{\dfrac{\dfrac{\overline{Fa}^{\,(1)} \quad \overline{\exists! a}^{\,(2)} \quad \dfrac{\{x|Fx\}=\{x|Gx\}}{\exists!\{x|Fx\}}}{a \in \{x|Fx\}}\,\text{fF1} \quad \{x|Fx\}=\{x|Gx\}}{a \in \{x|Gx\}}}{Ga}\,\text{F2}}{Fa \leftrightarrow Ga}\,(1)}{\forall x(Fx \leftrightarrow Gx)}\,(2)$$

□

So Vb is redundant for the (modified) Fregean. Note how no use is made of Vb in the (modified) Fregean's derivations of the ND-rules.

9 Taking stock

Let us take stock of the progress made thus far, in our comparison of the 'free-logical' Fregean with the (likewise 'free-logical') natural-deduction set theorist. Let us henceforth call each of them simply *free*, rather than 'free-logical'.

Bear in mind the following three crucial points of methodological agreement between them.

1. Both of these theorists are countenancing the possibility of *Urelemente*. So they both employ the atomic predicate S, for '. . . is a set'.
2. Both of them have the same conception of the truth conditions of atomic statements (including, of course, statements of identity). An atomic statement is false if any of its immediate constituent terms fails to denote.
3. Both of them use the usual introduction and elimination rules for free first-order logic with identity, based on $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\forall$, $\exists$, and $=$. Included among these rules are the Rule of Atomic Denotation, and the modified Rule of Reflexivity of Identity. The quantifier rules are also embellished with existential presuppositions in the usual well-understood way. (See Tennant, 1978, Ch. 7.)

Here are the basic inferential principles *for sets* espoused by these two theorists.

The free Fregean's basic inferential principles for sets:

fF1 $\quad \dfrac{\Phi(t) \qquad \exists!\{x \mid \Phi(x)\} \qquad \exists! t}{t \in \{x \mid \Phi(x)\}}$ $\qquad$ F4 $\quad \dfrac{St}{t = \{x \mid x \in t\}}$

F2 $\dfrac{t \in \{x \mid \Phi(x)\}}{\Phi(t)}$

F3 $\dfrac{t = \{x \mid \Phi(x)\}}{St}$

fVa $\dfrac{\forall x(\Phi x \leftrightarrow \Psi x) \qquad \exists!\{x \mid \Phi x\}}{\{x \mid \Phi x\} = \{x \mid \Psi x\}}$

π $\dfrac{\exists!t}{St}$

The free natural-deduction theorist's basic inferential principles for sets:

{ }I
$$\dfrac{\begin{array}{c}\underbrace{\overline{\exists!a}^{(i)}\ ,\ \overline{\Phi^x_a}^{(i)}}\\ \vdots \\ a \in t\end{array} \qquad \exists!t \qquad \begin{array}{c}\overline{a \in t}^{(i)} \\ \vdots \\ \Phi^x_a\end{array}}{t = \{x \mid \Phi\}}\,(i)$$
, where a is parametric.

{ }E$_1$ $\dfrac{t = \{x \mid \Phi\} \qquad \exists!v \qquad \Phi^x_v}{v \in t}$

{ }E$_2$ $\dfrac{t = \{x \mid \Phi\}}{\exists!t}$

{ }E$_3$ serial form:
$$\dfrac{t = \{x \mid \Phi\} \qquad v \in t}{\Phi^x_v}$$

parallelized form:
$$\dfrac{t = \{x \mid \Phi\} \qquad v \in t \qquad \begin{array}{c}\overline{\Phi^x_v}^{(i)} \\ \vdots \\ \theta\end{array}}{\theta}\,(i)$$

A remarkable contrast strikes the eye. The (free) Fregean states rules that make no provision for discharge of assumptions, whether in sentential form or in rule form. The natural-deduction theorist, however, states { }I so as to allow discharge of certain assumptions. It is also a single-barreled rule. These are crucial reasons why the latter's rules, overall, provide a more succinct and unified account of the interrelations among the concepts involved. This is the case even when both theorists are employing free logic and are achieving pure vs. impure invariance in their theorizing about sets. We have learned the lesson that the Gentzenian approach, allowing for rules that effect discharge of assumptions, is an essential advance over the Fregean one, and frees the set-logician to do more with less.

Summary of the Fregean's results: Lemma 5.2: F1 $\Rightarrow$ { }E$_1$.
Lemma 5.3: F2 $\Rightarrow$ { }E$_3$.
Lemma 5.4: fF1 $\Rightarrow$ { }E$_1$.
F3 is, hence $\Rightarrow$ { }E$_2$.
Lemma 8.2: F1, F2 $\Rightarrow$ Vb.
Lemma 8.5: $\dfrac{\exists!t}{t = \{x \mid x \in t\}}$.

Lemma 8.6: F4, π, fVa $\Rightarrow$
$$\dfrac{\exists!t \qquad \exists!u \qquad \begin{array}{c}\overline{a \in t}^{(i)} \\ \vdots \\ a \in u\end{array} \qquad \begin{array}{c}\overline{a \in u}^{(i)} \\ \vdots \\ a \in t\end{array}}{t = u}\,(i) \ .$$

Lemma 8.7: F4, fVa, $\pi \Rightarrow \{\ \}$I in its original form (ensuring purity).
Lemma 8.8: F4, fVa $\Rightarrow \{\ \}$I in its S-modified form (allowing *Urelemente*).
Lemma 8.9: fF1, F2 $\Rightarrow$ Vb.

Summary of the ND-theorist's results: Theorem 4.1: $\dfrac{\exists!t}{t=\{x \mid x \in t\}}$.

Theorem 4.2: $\{\ \}$I $\Rightarrow$ F4.
Theorem 5.1: $\{\ \}E_3 \Rightarrow$ F2.
Theorem 5.5: $\{\ \}E_1 \Rightarrow$ fF1.
$\{\ \}E_2$ is, hence $\Rightarrow$ F3.

Theorem 7.1: $\{\ \}$I $\Rightarrow$ $$\dfrac{\exists!t \quad \exists!u \quad \begin{matrix}\overline{a \in t}^{\,(i)} \\ \vdots \\ a \in u\end{matrix} \quad \begin{matrix}\overline{a \in u}^{\,(i)} \\ \vdots \\ a \in t\end{matrix}}{t=u}\,(i)$$.

Theorem 8.1: $\{\ \}E_1, \{\ \}E_3 \Rightarrow$ Vb.
Theorem 8.3: $\{\ \}E_1, \{\ \}E_3, \{\ \}$I $\Rightarrow$ fVa.

10 Summary of our comparison of the free Fregean approach with the free ND-approach

Clearly, the free Fregean approach is equivalent to the free ND-approach. Each primitive rule of the one theorist is either primitive for, or derivable by, the other theorist. The free Fregean, however, adopts as primitive the rule F4, which the ND-theorist easily but non-trivially derives. Furthermore, comparison of their respective proofs of the non-trivially derivable result of Extensionality reveals that the ND-proof is more succinct than the Fregean one. They tie, however, in proving Vb — taking ten primitive steps each.

There is a satisfying unity to the ND-approach that is lacking in the Fregean approach. Having harmoniously balanced introduction and elimination rules for set-abstraction is a definite plus. These rules require only minor tweaks to toggle between the pure and the impure conceptions of sets. The Fregean, by contrast, resorts to adopting two new primitive rules—F4 and π—to ensure restriction to pure sets.

The extension of Gentzenian methods from the usual logical operators so as to include also the operator for set-abstraction appears to yield a methodological advantage. If the methods of natural deduction had been available to Frege, the tradition could arguably have delivered an ontologically non-committal *logical foundation* for abstraction, membership, sethood and predication. On that foundation Zermelo and his successors could then have built further, by supplying axioms and axiom schemata for outright existence (e.g., the empty set) and conditional existence (e.g., the pair set of any two things).

The original sin revealed by Russell's paradox can be viewed, through this new lens, as Frege's insistence that every well-formed singular term denotes. The fateful Va codified that insistence as it concerned set-abstraction terms in particular. If Frege's erroneous conception of 'logical perfection' could have been eliminated earlier, the route would have been cleared to his acceptance of all the principles of modern free logic. He could have had a logic of sets, but without any sets as logical objects. That the set-term $\{x \mid \neg x \in x\}$ does not denote would then have been no more disastrous a discovery than that the definite descriptive term $\iota x(\Phi x \wedge \neg\Phi x)$ does not either.

Foundationalists know, as practicing mathematicians themselves, that all our mathematical reasoning is (intuitively) *relevant*, and therefore should be able to be regimented in a formal logic devoid of the paradoxes of irrelevance. There is foundational dispute over whether mathematical reasoning should be *constructive*; and we know now, even from a constructive standpoint, that the classical extensions of constructive theories are consistent if the latter are. The lesson that emerges here is that adopting a *free* logic for the foundations of mathematics appears to be crucial for the *constructivist* to ensure consistency.

This brings us to the end of our comparison of the Fregean and the natural-deduction theorist's approaches to the primitive rules for the *logic* of sets.

We turn now to explain how the natural-deduction theorist can pursue the project of framing introduction and elimination rules even further, so as to capture all the familiar 'compiled' concepts of set theory. The main philosophical lesson of the remaining part of our investigation is that one can capture the meanings of set-theoretic predicates and operators without incurring any ontological commitments. The latter commitments result only from subsequent existential postulations, either outright or conditional. But the concepts embedded in such postulates are *already understood*, thanks to the ontologically non-committal rules of introduction and elimination that govern them in a *free* logic. So the main 'takeaway' is that the postulates of set theory do *not* serve 'implicitly to define' the concepts involved. Rather, those concepts are already available to be grasped *before* any existential postulation is undertaken. Such is the power of harmoniously balanced introduction and elimination rules.

11 An inferentialist treatment of set-theoretic pasigraphs

At present our official list of primitive expressions contains only the logical operators $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\exists$, and $\forall$, the identity predicate $=$, the membership predicate $\in$, and the (singular) term-forming variable-binding set-abstraction operator $\{x \mid \ldots x \ldots\}$.

Set theory, however, is not laid out in such austerely primitive vocabulary. Set theorists and ordinary mathematicians making use of set-theoretic ideas employ a host of already familiar-looking *defined* expressions (such as '$\subseteq$' for 'is a subset of', and '$\mathscr{P}$' for 'the power set of'). These defined expressions are indispensable for communicating in a conveniently condensed fashion what would otherwise be extremely cumbrously expressed set-theoretical thoughts. The inferentialist seeks to frame rules of introduction and elimination for these defined expressions, so that they can be understood as being employed as 'local primitives' in mathematical discourse of the normal explicit texture.

When these defined function-operators and predicates are supplied with rules of inference, we call those operators and predicates *pasigraphs*, because they are so readily recognizable as pieces of notation in use, despite being absent from the official list of primitive expressions. In some cases we shall invent new pasigraphs, in order to express in yet more succinct symbolic form what practicing set theorists often render only in 'logician's English'. This enables one to be uniform and thorough in rigorously regimenting informal set-theoretical proofs as formal, logical proofs.

Every defined notion in set theory (as in any other branch of mathematics) stands at the apex of its own 'pyramid of preceding definitions'. The official primitive expressions form its base. There are of course only finitely many preceding definitions in any such pyramid; the process of constructing new concepts is always well-founded. Those definitions 'lower down' in the pyramid will be of pasigraphs that can be employed in the statement of introduction and elimination rules for the new notion at the apex. Not all of them need be thus employed; but they are eligible to be. Typically, the new notion at the apex will have its rules framed by using earlier notions just a layer or two down. But this is not necessarily the case in general.

We recall the special rule of free logic called the *Rule of Atomic Denotation*:

$$\frac{A(\ldots, t, \ldots)}{\exists! t}$$

We saw also that in free logic for languages containing function signs as primitives, there is the *Rule of Functional Denotation*:

$$\frac{\exists! f(\ldots, t, \ldots)}{\exists! t}$$

We shall adopt the following constraint on the formulation of any new concepts represented by our pasigraphs:

1. predicate-pasigraphs are to qualify as instances of A in the Rule of Atomic Denotation;
2. operator-pasigraphs are to qualify as instances of f in the Rule of Functional Denotation.

11.1 Pasigraphs for restricted quantifications

We shall start with a pasigraph that is neither an operator- nor a predicate-pasigraph. We are very familiar with the following form of generalization:

$$\forall x \in t\, \varphi(x).$$

But strictly speaking the universal quantifier $\forall$, as a logical primitive, is a *unary* quantifier; so the foregoing form is that of a binary-quantifier *pasigraph*, which needs either to be defined explicitly:

$$\forall x(x \in t \to \varphi(x))$$

or to have its meaning specified by rules that can govern it as an apparent primitive:

$$\forall\text{I} \quad \dfrac{\begin{array}{c}\overline{a \in t}^{\,(i)} \\ \vdots \\ \varphi^x_a\end{array}}{\forall x \in t\, \varphi}\,(i) \qquad\qquad \forall\text{E} \quad \dfrac{\forall x \in t\, \varphi \qquad u \in t}{\varphi^x_u}$$

We are very familiar also with the following form of generalization:

$$\exists x \in t\, \varphi(x).$$

But strictly speaking the existential quantifier $\exists$, as a logical primitive, is a *unary* quantifier; so the foregoing form is that of a binary-quantifier *pasigraph*, which needs either to be defined explicitly:

$$\exists x(x \in t \wedge \varphi(x))$$

or to have its meaning specified by rules that can govern it as an apparent primitive:

$$\exists\text{I} \quad \dfrac{u \in t \qquad \varphi^x_u}{\exists x \in t\, \varphi} \qquad\qquad \exists\text{E} \quad \dfrac{\exists x \in t\, \varphi \qquad \begin{array}{c}\underbrace{{}^{(i)}\,\overline{a \in t}\;,\;\overline{\varphi^x_a}\,{}^{(i)}} \\ \vdots \\ \psi\end{array}}{\psi}\,(i)$$

11.2 Pasigraph $\emptyset$ for empty set

In primitive set-theoretic vocabulary the Axiom of Empty Set is as follows:

$$\exists!\{x \mid \neg x = x\}.$$

We shall now introduce the *constant pasigraph* $\emptyset$ for the empty set, governed by the following (existentially non-committal) rules.

$$\emptyset\text{I} \quad \dfrac{\exists! t \qquad \begin{array}{c}\overline{a \in t}^{\,(i)} \\ \vdots \\ \bot\end{array}}{t = \emptyset}\,(i) \qquad\qquad \emptyset\text{E} \quad \dfrac{t = \emptyset}{\exists! t} \qquad \dfrac{t = \emptyset \qquad u \in t}{\bot}$$

The Axiom of Empty Set can now take the form $\exists!\emptyset$. This is an outright existence postulate. Its inferential form is the zero-premise rule

$$\overline{\exists!\emptyset}\,.$$

11.3 Pasigraph for separated sets

Suppose φ has x free. Then

$$\{x \in t \mid \varphi\} =_{df} \{x \mid x \in t \wedge \varphi\}$$

The Axiom Scheme of Separation can now take the rule form

$$\frac{\exists! t}{\exists! \{x \in t \mid \varphi\}}$$

Instead of stipulating that $\{x \in t \mid \varphi\}$ is an abbreviation of $\{x \mid x \in t \wedge \varphi\}$, we could adopt as a grammatical primitive the term-formation operation on a term t and a formula φ with x free, that produces $\{x \in t \mid \varphi\}$ as a genuine term of the language, from the two constituents mentioned. We could then furnish the operation with its own introduction and elimination rules as follows.

$$\frac{\begin{array}{c}\underbrace{{}^{(i)}\,\overline{\varphi^x_a}\ ,\ \overline{a \in v}\,{}^{(i)}}\\ \vdots \\ a \in t\end{array} \qquad \exists! t \qquad \begin{array}{c}\overline{a \in t}\,{}^{(i)}\\ \vdots \\ a \in v\end{array} \qquad \begin{array}{c}\overline{a \in t}\,{}^{(i)}\\ \vdots \\ \varphi^x_a\end{array}}{t = \{x \in v \mid \varphi\}}\,{}^{(i)}$$

$$\frac{t = \{x \in v \mid \varphi\} \qquad \varphi^x_u \qquad u \in v}{u \in t} \qquad\qquad \frac{t = \{x \in v \mid \varphi\}}{\exists! t}$$

$$\frac{t = \{x \in v \mid \varphi\} \qquad u \in t}{u \in v} \qquad\qquad \frac{t = \{x \in v \mid \varphi\} \qquad u \in t}{\varphi^x_u}$$

By virtue of the foregoing Introduction and Elimination Rules for the Separation Pasigraph, along with the Introduction and Elimination Rules for the Set-Abstraction Operator, we have

$$t = \{x \in v \mid \varphi\} \vdash t = \{x \mid x \in v \wedge \varphi\}$$

and its converse

$$t = \{x \mid x \in v \wedge \varphi\} \vdash t = \{x \in v \mid \varphi\}.$$

11.4 Pasigraph for pair-sets

In primitive set-theoretic vocabulary the Axiom of Pairing is as follows:

$$\forall x \forall y \exists! \{z \mid z = x \vee z = y\}$$

We shall now introduce a (binary) *operator pasigraph.*

We shall write $\mathbb{P}(t, u)$ for the pair set $\{z \mid z = t \vee z = u\}$. Note that the ‘pair’ set $\mathbb{P}(t, u)$ is a *singleton* if $t = u$ (whence both t and u exist).

The Axiom of Pairing can now also be expressed as follows:

$$\forall x \forall y \exists! \mathbb{P}(x, y).$$

The inferentialist working in free logic requires the interdeducibility

$$t = \mathbb{P}(u, v) \dashv\vdash t = \{z \mid z = u \vee z = v\}$$

rather than the provability of the identity

$$\mathbb{P}(u, v) = \{z \mid z = u \vee z = v\},$$

The latter identity commits one to the existence of the pair set of u and v:

$$\exists! \mathbb{P}(u, v).$$

The interdeducibility, however, does not carry such existential commitment. It allows one to pin down the meaning of $\mathbb{P}$ as an operator on sets without committing one to its being everywhere (or indeed: *any*where) defined. One can grasp what $\mathbb{P}$ means without yet adopting the Axiom of Pair Sets. And when we *do* adopt that axiom, it does not serve implicitly to define the meaning of $\mathbb{P}$. For that meaning will already have been defined by the Introduction and Elimination rules that we frame for $\mathbb{P}$.

We propose the following introduction and elimination rules for the pairing operator $\mathbb{P}$.

$$\mathbb{P}\text{-I} \qquad \dfrac{u \in t \qquad v \in t \qquad \begin{array}{c} \overline{a \in t}^{\,(i)} \\ \vdots \\ a = u \vee a = v \end{array}}{t = \mathbb{P}(u, v)}\,(i)$$

$$\mathbb{P}\text{-E}_1 \quad \frac{t = \mathbb{P}(u, v)}{u \in t} \qquad \mathbb{P}\text{-E}_2 \quad \frac{t = \mathbb{P}(u, v)}{v \in t} \qquad \mathbb{P}\text{-E}_3 \quad \frac{t = \mathbb{P}(u, v) \qquad w \in t}{w = u \vee w = v}$$

Lemma 11.1 *The operator pasigraph* $\mathbb{P}$ *obeys the Rule of Functional Denotation; that is, the following are provable:*

$$\frac{\exists! \mathbb{P}(u, v)}{\exists! u} \qquad \frac{\exists! \mathbb{P}(u, v)}{\exists! v}.$$

Proof

$$\dfrac{\begin{array}{c}\exists! \mathbb{P}(u, v),\ \text{i.e.,} \\ \exists x\, x = \mathbb{P}(u, v)\end{array} \qquad \dfrac{\dfrac{\overline{a = \mathbb{P}(u, v)}^{\,(1)}}{u \in a}\,\mathbb{P}\text{-E}_1}{\exists! u}\,\text{RAD}}{\exists! u}\,(1) \qquad \dfrac{\begin{array}{c}\exists! \mathbb{P}(u, v),\ \text{i.e.,} \\ \exists x\, x = \mathbb{P}(u, v)\end{array} \qquad \dfrac{\dfrac{\overline{a = \mathbb{P}(u, v)}^{\,(1)}}{v \in a}\,\mathbb{P}\text{-E}_1}{\exists! u}\,\text{RAD}}{\exists! u}\,(1)$$

□

Lemma 11.2 $t = \mathbb{P}(u, v) \vdash t = \{z \mid z = u \vee z = v\}$.

Proof

$$\dfrac{\dfrac{\overline{a = u \vee a = v}\ ^{(1)} \qquad \dfrac{\overline{a=u}\ ^{(2)} \quad \dfrac{t=\mathbb{P}(u,v)}{u\in t}\ \mathbb{P}\text{-E}_1}{a\in t} \qquad \dfrac{\overline{a=v}\ ^{(2)} \quad \dfrac{t=\mathbb{P}(u,v)}{v\in t}\ \mathbb{P}\text{-E}_2}{a\in t}}{a\in t}\ (2) \qquad \dfrac{t=\mathbb{P}(u,v)}{\exists! t} \qquad \dfrac{t=\mathbb{P}(u,v) \quad \overline{a\in t}\ ^{(1)}}{a=u\vee a=v}\ \mathbb{P}\text{-E}_3}{t=\{z \mid z=u\vee z=v\}}\ (1)\ \{\,\}\text{-I}$$

□

Lemma 11.3 $t = \{z \mid z=u \vee z=v\},\ \exists! u,\ \exists! v \vdash t = \mathbb{P}(u, v)$.

Proof Abbreviate $z=u \vee z=v$ as Φz, where convenient, to reduce sideways spread.

$$\dfrac{\dfrac{t=\{z \mid \Phi z\} \qquad \dfrac{\dfrac{\exists! u}{u=u}}{\Phi u}}{u \in t}\ \{\,\}\text{-E}_{PM} \qquad \dfrac{t=\{z \mid \Phi z\} \qquad \dfrac{\dfrac{\exists! v}{v=v}}{\Phi v}}{v \in t}\ \{\,\}\text{-E}_{PM} \qquad \dfrac{t=\{z \mid \Phi z\} \qquad \overline{a \in t}\ ^{(1)}}{a=u \vee a=v}\ \{\,\}\text{-E}_{MP}}{t = \mathbb{P}(u, v)}\ (1)\ \mathbb{P}\text{-I}$$

□

11.5 Pasigraph for singletons

A special case of pairs $\mathbb{P}(u, v)$ arises when $u = v$. Here, the talk is of *singletons*. $\mathbb{P}(u, u)$ is often abbreviated as $\{u\}$. We, however, shall introduce a special single operator σ, and write σu for $\{u\}$.

The introduction and elimination rules for σ arise from the obvious simple modifications of the rules for $\mathbb{P}$.

$$\sigma\text{-I} \quad \dfrac{u \in t \qquad \begin{matrix}\overline{a \in t}\ ^{(i)} \\ \vdots \\ a=u\end{matrix}}{t=\sigma u}\ (i) \qquad\qquad \sigma\text{-E}_1 \quad \dfrac{t=\sigma u}{u \in t} \qquad\qquad \sigma\text{-E}_2 \quad \dfrac{t=\sigma u \qquad w \in t}{w=u}$$

The next two Lemmas answer a query from Ethan Brauer: "Do the rules enable one to prove that $\mathbb{P}(u, u)$ is σu, given the existence of either one of them?".

Lemma 11.4 $\dfrac{\exists! \mathbb{P}(u, u)}{\mathbb{P}(u, u) = \sigma u}$.

Proof

$$\dfrac{\dfrac{\dfrac{\exists! \mathbb{P}(u, u)}{\mathbb{P}(u, u) = \mathbb{P}(u, u)}\ \text{Ref=}}{u \in \mathbb{P}(u, u)}\ \mathbb{P}\text{E}_1 \qquad \dfrac{\dfrac{\dfrac{\exists! \mathbb{P}(u, u)}{\mathbb{P}(u, u) = \mathbb{P}(u, u)}\ \text{Ref=} \qquad \overline{a \in \mathbb{P}(u, u)}\ ^{(2)}}{a = u \vee a = u}\ \mathbb{P}\text{E}_3 \qquad \overline{a = u}\ ^{(1)} \qquad \overline{a = u}\ ^{(1)}}{a = u}\ (1)}{\mathbb{P}(u, u) = \sigma u}\ (2)\ \sigma\text{I}$$

□

Lemma 11.5 $\dfrac{\exists! \sigma u}{\sigma u = \mathbb{P}(u, u)}$.

Proof

$$\dfrac{\dfrac{\dfrac{\exists!\sigma u}{\sigma u = \sigma u}\,\text{Ref=}}{u \in \sigma u}\,\sigma\text{E}_1 \qquad \dfrac{\dfrac{\exists!\sigma u}{\sigma u = \sigma u}\,\text{Ref=}}{u \in \sigma u}\,\sigma\text{E}_1 \qquad \dfrac{\dfrac{\dfrac{\exists!\sigma u}{\sigma u = \sigma u}\,\text{Ref=} \quad \overline{a \in \sigma u}^{\,(1)}}{a = u}\,\sigma\text{E}_2}{a = u \vee a = u}}{\sigma u = \mathbb{P}(u, u)}\,(1)\,\mathbb{P}\text{I}$$

□

11.6 The binary predicate pasigraph $\in^2$ for 'is a member of a member of'

Now consider the simple notion that *t is a member of a member of u*. Let us use for this the (binary) *predicate* pasigraph

$$t \in^2 u.$$

Here are the introduction and elimination rules for this new pasigraph:

$$\in^2\text{-I} \quad \dfrac{t \in v \qquad v \in u}{t \in^2 u} \qquad\qquad \in^2\text{-E} \quad \dfrac{t \in^2 u \qquad \begin{matrix} \underbrace{{}^{(i)}\,\overline{t \in a}\;,\;\overline{a \in u}\,{}^{(i)}} \\ \vdots \\ \psi \end{matrix}}{\psi}\,(i)$$

Lemma 11.6 $\dfrac{t \in^2 u}{\exists! t}, \quad \dfrac{t \in^2 u}{\exists! u}.$

Proof

$$\dfrac{t \in^2 u \qquad \dfrac{\overline{t \in a}^{\,(1)}}{\exists! t}\,\text{RAD}}{\exists! t}\,(1)\in^2\text{-E} \qquad\qquad \dfrac{a \in u \qquad \dfrac{\overline{t \in a}^{\,(1)}}{\exists! u}\,\text{RAD}}{\exists! u}\,(1)\in^2\text{-E}$$

□

Lemma 11.7 $t \in^2 u \dashv\vdash \exists x(t \in x \wedge x \in u)$.

Proof

$$\dfrac{t \in^2 u \qquad \dfrac{\dfrac{{}^{(1)}\,\overline{a \in u}}{\exists! a} \qquad \dfrac{\overline{t \in a}^{\,(1)} \qquad \overline{a \in u}^{\,(1)}}{t \in a \wedge a \in u}}{\exists x(t \in x \wedge x \in u)}}{\exists x(t \in x \wedge x \in u)}\,(1)$$

$$\dfrac{\exists x(t \in x \wedge x \in u) \qquad \dfrac{{}^{(2)}\,\overline{t \in a \wedge a \in u} \qquad \dfrac{{}^{(1)}\,\overline{t \in a} \qquad \overline{a \in u}^{\,(1)}}{t \in^2 u}}{t \in^2 u}}{t \in^2 u}\,(2)$$

□

11.7 Pasigraph for unions

We shall next choose the Axiom of Unions to illustrate further the inferentialist's method of treating *operator* pasigraphs. The pasigraph $\bigcup$, for the Union of a given set, is *unary* (unlike the pasigraph $\mathbb{P}$ for Pairing, which, as we have seen, is binary). Set-theorists handle the pasigraph $\bigcup$ with ease, as though it were a familiar primitive expression of their language.

The genuinely primitive form of the Axiom of Unions:

$$\forall x \exists ! \{y \mid \exists z(y \in z \wedge z \in x)\}$$

can be re-written

$$\forall x \exists ! \bigcup x$$

provided only that $\bigcup$ is furnished with Introduction and Elimination rules so that

$$t = \bigcup v \dashv\vdash t = \{y \mid \exists z(y \in z \wedge z \in v)\}.$$

We choose to require the interdeducibility

$$t = \bigcup v \dashv\vdash t = \{y \mid \exists z(y \in z \wedge z \in v)\}$$

rather than require the provability of the identity

$$\bigcup v = \{y \mid \exists z(y \in z \wedge z \in v)\}$$

because we are working in a *free* logic. The latter identity commits one to the existence of the union of v:

$$\exists ! \bigcup v.$$

The interdeducibility, however, does not. It allows one to pin down the meaning of $\bigcup$ as an operator on sets without committing one to its being everywhere (or indeed: *any*where) defined. One can grasp what $\bigcup$ means without yet adopting the Axiom of Unions. And when we *do* adopt that axiom, it does not serve implicitly to define the meaning of $\bigcup$. For that meaning will already have been defined by the following Introduction and Elimination rules for $\bigcup$.

$$\bigcup\text{-I} \quad \dfrac{\exists ! t \quad \exists ! v \quad \begin{matrix} \overline{a \in^2 v}^{\,(i)} \\ \vdots \\ a \in t \end{matrix} \quad \begin{matrix} \overline{a \in t}^{\,(i)} \\ \vdots \\ a \in^2 v \end{matrix}}{t = \bigcup v}\,(i)$$

$$\bigcup\text{-E}_1 \quad \dfrac{t = \bigcup v \quad w \in^2 v}{w \in t} \qquad \bigcup\text{-E}_2 \quad \dfrac{t = \bigcup v \quad w \in t}{w \in^2 v}$$

$$\bigcup\text{-E}_3 \quad \dfrac{t = \bigcup v}{\exists ! t} \qquad \bigcup\text{-E}_4 \quad \dfrac{t = \bigcup v}{\exists ! v}$$

Lemma 11.8 *The operator pasigraph $\bigcup$ obeys the Rule of Functional Denotation; that is, the following is provable:*

$$\frac{\exists!\bigcup u}{\exists! u}.$$

Proof

$$\dfrac{\begin{array}{c}\exists!\bigcup u,\ \text{i.e.,}\\ \exists x\, x=\bigcup u\end{array} \qquad \dfrac{\overline{a=\bigcup u}^{\,(1)}}{\exists! u}\ \bigcup\text{-E}_1}{\exists! u}\ (1)$$

□

Lemma 11.9 $t = \bigcup v \vdash t = \{y \mid \exists z(y \in z \wedge z \in v)\}$.

Proof

$$\dfrac{\dfrac{t=\bigcup v \qquad \dfrac{{}^{(1)}\overline{a\in b\wedge b\in v} \qquad \dfrac{{}^{(2)}\overline{a\in b}\quad \overline{b\in v}^{\,(2)}}{a\in^2 v}\ \in^2\text{-I}}{a\in^2 v}\ (1)}{a\in t}\ \bigcup\text{-E}_1 \qquad \dfrac{t=\bigcup v}{\exists! t} \qquad \dfrac{\dfrac{t=\bigcup v \quad \overline{a\in t}^{\,(4)}}{a\in^2 v} \qquad \dfrac{\dfrac{{}^{(3)}\overline{a\in c}}{\exists! c} \quad \dfrac{\overline{a\in c}^{\,(3)} \quad \overline{c\in v}^{\,(3)}}{a\in c\wedge c\in v}}{\exists z(a\in z\wedge z\in v)}}{\exists z(a\in z\wedge z\in v)}\ (3)\ \in^2\text{-E}}{t=\{y \mid \exists z(y\in z\wedge z\in v)\}}\ (4)\ \{\}\text{-I}$$

□

Lemma 11.10 $t = \{y \mid \exists z(y \in z \wedge z \in v)\},\ \exists! v \vdash t = \bigcup v$.

Proof Abbreviate $\exists z(y \in z \wedge z \in v)$ as Φy, where convenient, to reduce sideways spread. Let Π_1 be the following fragment of the final proof that we shall construct. Note that the two discharge strokes labeled (2) are being put in place in advance of the eventual step labeled (2) (in the final proof) that will effect that discharge.

$$\dfrac{t=\{y \mid \Phi y\} \qquad \dfrac{{}^{(2)}\overline{a\in^2 v} \qquad \dfrac{\dfrac{{}^{(4)}\overline{a\in c}}{\exists! c} \quad \dfrac{\overline{a\in c}^{\,(4)} \quad \overline{c\in v}^{\,(4)}}{a\in c\wedge c\in v}}{\exists z(a\in z\wedge z\in v)}}{\exists z(a\in z\wedge z\in v)}\ (4)\ \in^2\text{-E} \qquad \dfrac{\overline{a\in^2 v}^{\,(2)}}{\exists! a}\ \text{L11.6}}{a\in t}\ \{\}\text{-E}_{\text{PM}}$$

Now let Π_2 be the following fragment of the final proof that we shall construct. Note once again that the discharge stroke labeled (2) has already been put in place, in advance of the eventual step labeled (2) (in the final proof) that will effect that discharge.

$$\dfrac{\dfrac{t=\{y \mid \Phi y\} \quad \overline{a\in t}^{\,(2)}}{\Phi a}\ \{\}\text{-E}_{\text{MP}} \qquad \dfrac{{}^{(3)}\overline{a\in b\wedge b\in v} \qquad \dfrac{{}^{(1)}\overline{a\in b} \quad \overline{b\in v}^{\,(1)}}{a\in^2 v}\ \in^2\text{-I}}{a\in^2 v}\ (1)}{a\in^2 v}\ (3)\ \exists\text{-E}$$

Now we can form the final proof

$$\dfrac{\dfrac{t=\{y \mid \Phi y\}}{\exists! t} \qquad \exists! v \qquad \begin{matrix}\Pi_1 \\ a \in t\end{matrix} \qquad \begin{matrix}\Pi_2 \\ a \in^2 v\end{matrix}}{t = \bigcup v} \; (2) \; \bigcup\text{-I}$$

□

11.8 Pasigraph for inclusion, or subset

The notion ⊆ of inclusion is one of the most familiar and frequently used binary, but *ancillary*, or *defined*, relations in set theory. The usual reading is 't is a subset of u'. The usual definition in primitive vocabulary would be

$$t \subseteq u \equiv_{df} \forall x(x \in t \rightarrow x \in u).$$

The inferentialist, however, working in free logic, lays down instead the following introduction and elimination rules for this pasigraph:

$$\subseteq\text{I} \quad \dfrac{\exists! t \qquad \exists! u \qquad \begin{matrix}\overline{a \in t}^{\,(i)} \\ \vdots \\ a \in u\end{matrix}}{t \subseteq u} \; (i)$$

$$\subseteq\text{E}_1 \quad \dfrac{t \subseteq u}{\exists! t} \qquad \subseteq\text{E}_2 \quad \dfrac{t \subseteq u}{\exists! u} \qquad \subseteq\text{E}_3 \quad \dfrac{t \subseteq u \qquad v \in t}{v \in u}$$

That the predicate pasigraph ⊆ obeys the Rule of Atomic Denotation is obvious from $\subseteq\text{E}_1$ and $\subseteq\text{E}_2$.

11.9 The unary predicate pasigraph 'trans'

A *transitive* set is one that contains as members all members of its members. That is, every member of a transitive set is a subset of it. Thus we have the following introduction rule:

$$\text{trans-I} \quad \dfrac{\exists! t \qquad \begin{matrix}\overline{a \in t}^{\,(i)} \\ \vdots \\ a \subseteq t\end{matrix}}{\text{trans}(t)} \; (i)$$

matched by these elimination rules:

$$\text{trans-E} \quad \dfrac{\text{trans}(t)}{\exists! t} \qquad \dfrac{\text{trans}(t) \qquad u \in t}{u \subseteq t}$$

11.10 Pasigraph for power sets

We can now turn our attention to the Axiom of Power Sets to illustrate further the inferentialist's method for set theory.

We shall apply our earlier method to the unary operator-pasigraph $\mathscr{P}$ which set-theorists handle with ease, as though it were a familiar primitive expression of their language. The genuinely primitive form of the Axiom of Power Sets:

$$\forall x \exists! \{y \mid \forall z(z \in y \rightarrow z \in x)\}$$

i.e.,

$$\forall x \exists! \{y \mid y \subseteq x\}$$

can be re-written

$$\forall x \exists! \mathscr{P} x$$

provided only that $\mathscr{P}$ is furnished with Introduction and Elimination rules so that

$$t = \mathscr{P} v \dashv\vdash t = \{y \mid \forall z(z \in y \rightarrow z \in x)\}.$$

As with Unions, we choose with Power Sets to require the 'general term t'-involving interdeducibility

$$t = \mathscr{P} v \dashv\vdash t = \{y \mid \forall z(z \in y \rightarrow z \in v)\}$$

rather than require the provability of the identity

$$\mathscr{P} v = \{y \mid \forall z(z \in y \rightarrow z \in v)\},$$

because we are working in a *free* logic. The latter identity commits one to the existence of the power set of v:

$$\exists! \mathscr{P} v.$$

The interdeducibility, however, does not. It allows one to pin down the meaning of $\mathscr{P}$ as an operator on sets without committing one to its being everywhere (or indeed: *any*where) defined. One can grasp what $\mathscr{P}$ means without yet adopting the Axiom of Power Sets. And when we *do* adopt that axiom, it does not serve implicitly to define the meaning of $\mathscr{P}$. For that meaning will already have been defined by the following Introduction and Elimination rules for $\mathscr{P}$.

$$\mathscr{P}\text{-I} \qquad \frac{\exists! t \qquad \exists! v \qquad \begin{array}{c} {}^{(i)}\overline{\;a \subseteq v\;} \\ \vdots \\ a \in t \end{array} \qquad \begin{array}{c} {}^{(i)}\overline{\;a \in t\;} \\ \vdots \\ a \subseteq v \end{array}}{t = \mathscr{P} v}\,(i)$$

$$\mathscr{P}\text{-E}_1 \quad \frac{t = \mathscr{P} v}{\exists! t} \qquad\qquad \mathscr{P}\text{-E}_2 \quad \frac{t = \mathscr{P} v}{\exists! v}$$

$$\mathscr{P}\text{-E}_3\quad \dfrac{t=\mathscr{P}v \qquad u\subseteq v}{u\in t} \qquad\qquad \mathscr{P}\text{-E}_4\quad \dfrac{t=\mathscr{P}v \qquad u\in t}{u\subseteq v}$$

Lemma 11.11 $\dfrac{u\in\mathscr{P}v}{\exists! v}$.

Proof

$$\dfrac{\exists!\mathscr{P}v,\ \text{i.e.,}\ \exists x\, x=\mathscr{P}v \qquad \dfrac{\overline{a=\mathscr{P}v}^{\,(1)}}{\exists! v}\,\mathscr{P}\text{-E}_2}{\exists! v}\,(1)$$

□

Lemma 11.12 $\dfrac{\exists!\mathscr{P}v \qquad u\subseteq v}{u\in\mathscr{P}v}$.

Proof

$$\dfrac{\exists!\mathscr{P}v,\ \text{i.e.,}\ \exists x\, x=\mathscr{P}v \qquad \dfrac{\overline{a=\mathscr{P}v}^{\,(1)} \qquad u\subseteq v}{u\in\mathscr{P}v}\,\mathscr{P}\text{-E}_3}{u\in\mathscr{P}v}\,(1)$$

□

Lemma 11.13 $\dfrac{u\in\mathscr{P}v}{u\subseteq v}$.

Proof

$$\dfrac{\dfrac{u\in\mathscr{P}v}{\exists!\mathscr{P}v,\ \text{i.e.,}\ \exists x\, x=\mathscr{P}v} \qquad \dfrac{\overline{a=\mathscr{P}v}^{\,(1)} \qquad u\in\mathscr{P}v}{u\subseteq v}\,\mathscr{P}\text{-E}_4}{u\subseteq v}\,(1)$$

□

Lemma 11.14 $t=\mathscr{P}v\vdash t=\{y\mid\forall z(z\in y\rightarrow z\in v)\}$.

Proof

$$\dfrac{\dfrac{t=\mathscr{P}v \qquad \dfrac{{}^{(3)}\overline{\exists! a} \qquad \dfrac{t=\mathscr{P}v}{\exists! v} \qquad \dfrac{{}^{(1)}\overline{c\in a} \qquad \dfrac{{}^{(3)}\overline{\forall z(z\in a\rightarrow z\in v)} \qquad \dfrac{{}^{(1)}\overline{c\in a}}{\exists! c}}{c\in a\rightarrow c\in v}}{c\in v}}{a\subseteq v}\,(1)\subseteq\text{I}}{a\in t} \qquad \dfrac{t=\mathscr{P}v}{\exists! t} \qquad \dfrac{\dfrac{\dfrac{\dfrac{t=\mathscr{P}v \qquad \overline{a\in t}^{\,(3)}}{a\subseteq v}\,\mathscr{P}\text{E}_4 \qquad \overline{b\in a}^{\,(2)}}{b\in v}\,\subseteq\text{E}_3}{b\in a\rightarrow b\in v}\,(2)}{\forall z(z\in a\rightarrow z\in v)}}{t=\{y\mid\forall z(z\in y\rightarrow z\in v)\}}\,(3)\,\{\,\}\text{I}$$

□

Lemma 11.15 $\exists! v,\ t=\{y\mid\forall z(z\in y\rightarrow z\in v)\}\vdash t=\mathscr{P}v$.

Proof Abbreviate $\forall z(z\in y\rightarrow z\in v)$ by Φy and $\forall z(z\in a\rightarrow z\in v)$ by Φa, where convenient, to reduce sideways spread. Moreover, let Π be

$$\dfrac{\dfrac{{}^{(3)}\overline{b\in a} \qquad \dfrac{\dfrac{{}^{(1)}\overline{a\in t} \qquad t=\{y\mid\Phi y\}}{\Phi a} \qquad \dfrac{\overline{b\in a}^{\,(3)}}{\exists! b}}{b\in a\rightarrow b\in v}}{b\in v}}{a\subseteq v}\,(3)\subseteq\text{I}$$

Then form the proof

$$\dfrac{\dfrac{t=\{y\,|\,\Phi y\}}{\exists! t}\qquad \exists! v\qquad \dfrac{t=\{y\,|\,\Phi y\}\qquad \dfrac{\dfrac{\dfrac{\overline{a\subseteq v}^{\,(1)}\qquad \overline{b\in a}^{\,(1)}}{b\in v}\,{\scriptstyle\subseteq\mathrm{E}_3}}{b\in a\rightarrow b\in v}\,{\scriptstyle(2)}}{\Phi a}\qquad \dfrac{\overline{a\subseteq v}^{\,(1)}}{\exists! a}}{a\in t}\,{\scriptstyle\{\,\}\mathrm{E_{PM}}}\qquad \Pi}{t=\mathscr{P}v}\,{\scriptstyle(1)}$$

□

Lemma 11.16 $\exists!\mathscr{P}(t),\ \mathsf{trans}(t)\vdash\mathsf{trans}(\mathscr{P}(t))$.

Proof

$$\dfrac{\dfrac{\dfrac{\exists!\mathscr{P}(t)\qquad \dfrac{\mathsf{trans}(t)\qquad \dfrac{{}^{(1)}\,\overline{c\in d}\qquad \dfrac{\overline{d\in\mathscr{P}(t)}^{\,(2)}}{d\subseteq t}\,{\scriptstyle\mathrm{L}11.13}}{c\in t}\,{\scriptstyle\subseteq\mathrm{E}_3}}{c\subseteq t}\,{\scriptstyle\textsf{trans-E}}}{c\in\mathscr{P}(t)}\,{\scriptstyle\mathrm{L}11.12}}{d\subseteq\mathscr{P}(t)}\,{\scriptstyle(1)\ \subseteq\mathrm{I}}}{\mathsf{trans}(\mathscr{P}(t))}\,{\scriptstyle(2)\ \textsf{trans-I}}$$

□

11.11 The binary predicate pasigraph $\rceil\!\lfloor$ for 'is disjoint from'

We now introduce a new binary-relation pasigraph: $t \rceil\!\lfloor u$ is to mean that t is disjoint from u — that is, they have no member in common. This pasigraph will be useful in the formulation of the Axiom (or Rule) of Regularity. The introduction and elimination rules for $\rceil\!\lfloor$ are as follows.

$$\rceil\!\lfloor\text{-I}\qquad \dfrac{\exists! t\qquad \exists! u\qquad \begin{array}{c}{}^{(i)}\,\underbrace{\overline{a\in t}\ ,\ \overline{a\in u}}\,{}^{(i)}\\ \vdots\\ \bot\end{array}}{t\rceil\!\lfloor u}\,{\scriptstyle(i)}$$

$$\rceil\!\lfloor\mathrm{E}\qquad \dfrac{t\rceil\!\lfloor u}{\exists! t}\qquad \dfrac{t\rceil\!\lfloor u}{\exists! u}\qquad \dfrac{t\rceil\!\lfloor u\qquad v\in t\qquad v\in u}{\bot}$$

Lemma 11.17 $t \rceil\!\lfloor u \dashv\vdash t\cap u=\emptyset$.

11.12 Pasigraph for ranges

Suppose φxy has the variables x, y free, and is functional from x to y, at least for x in t. The Replacement Pasigraph $\varphi_{xy}[t]$ can then be explicitly defined as follows.

$$\varphi_{xy}[t] =_{df} \{y \mid \exists x(x \in t \wedge \varphi)\}.$$

The subscripting with x and y registers the fact that this pasigraph binds those two variables.

[The Axiom Scheme of] Replacement, due to Fraenkel, can be formulated as the following 'conditional existence' rule (where '$\exists_1$' is the uniqueness quantifier):

$$\dfrac{\begin{array}{c}\overline{a \in t}^{\,(i)}\\ \vdots \\ \exists_1 y\varphi ay\end{array}}{\exists!\varphi_{xy}[t]}\,(i)$$

The pasigraph could also be taken as a grammatical primitive, furnished with the following rules of introduction and elimination.

$$\ldots[\ldots]\text{-Intro}\quad \dfrac{\begin{array}{c}{}^{(i)}\overline{a \in v}\\ \vdots \\ w \in t\end{array}\quad \begin{array}{c}{}^{(i)}\overline{a \in v}\\ \vdots \\ \varphi wa\end{array}\quad \begin{array}{c}\underbrace{{}^{(i)}\overline{c \in t}\ ,\ \overline{\varphi cb}^{\,(i)}}\\ \vdots \\ b \in v\end{array}\quad \begin{array}{c}\underbrace{{}^{(i)}\overline{c \in t}\ ,\overline{\varphi cb}^{\,(i)}\ ,\overline{\varphi cd}^{\,(i)}}\\ \vdots \\ b = d\end{array}}{v = \varphi_{xy}[t]}\,(i)$$

Note that the first two subproofs call for some same term w in their conclusions. This means that the elimination rule will have a part that corresponds to these two subproofs taken together. The final subproof ensures the functionality of φxy on t as its domain.

$$\ldots[\ldots]\text{-Elim}\quad \dfrac{v = \varphi_{xy}[t] \qquad u \in v \qquad \begin{array}{c}\underbrace{{}^{(i)}\overline{a \in t}\ ,\ \overline{\varphi au}^{\,(i)}}\\ \vdots \\ \theta\end{array}}{\theta}\,(i)$$

$$\dfrac{v = \varphi_{xy}[t] \qquad u \in t \qquad \varphi uw}{w \in v} \qquad \dfrac{v = \varphi_{xy}[t] \qquad u \in t \qquad \varphi uv \qquad \varphi uw}{v = w}$$

In set theory, the successor of a set u is defined as $u \cup \{u\}$ — or, in the notation we have thus far introduced, as $\bigcup \mathbb{P}(u, \sigma u)$. We shall now introduce a unary operator pasigraph s to represent successor, and furnish it with introduction and elimination rules that secure for it the same meaning.

$$\mathsf{sI}\quad \dfrac{u \in t \qquad \begin{array}{c}\overline{a \in u}^{\,(i)}\\ \vdots \\ a \in t\end{array} \qquad \begin{array}{c}\overline{b \in t}^{\,(i)}\\ \vdots \\ b \in u \vee b{=}u\end{array}}{t{=}\mathsf{s}u}\,(i)$$

$$\mathsf{sE}\quad \dfrac{t{=}\mathsf{s}u}{u \in t} \qquad \dfrac{t{=}\mathsf{s}u \qquad v \in u}{v \in t} \qquad \dfrac{t{=}\mathsf{s}u \qquad v \in t}{v \in u \vee v{=}u}$$

Lemma 11.18 $t = u \cup \{u\} \dashv\vdash t = \bigcup \mathbb{P}(u, \sigma u)$.

11.13 The unary predicate pasigraph 'comp'

$\mathsf{comp}(t, u)$ is to mean that t and u are comparable in terms of the membership relation:

$$\mathsf{comp}(t, u) \equiv_{df} t \in u \vee t = u \vee u \in t.$$

Introduction and elimination rules that secure this meaning directly are as follows.

$$\text{comp-I} \quad \frac{t \in u}{\mathsf{comp}(t, u)} \qquad \frac{t = u}{\mathsf{comp}(t, u)} \qquad \frac{u \in t}{\mathsf{comp}(t, u)}$$

$$\text{comp-E} \quad \dfrac{\mathsf{comp}(t, u) \quad \begin{array}{c} \overline{t \in u}^{\,(i)} \\ \vdots \\ \theta/\bot \end{array} \quad \begin{array}{c} \overline{t = u}^{\,(i)} \\ \vdots \\ \theta/\bot \end{array} \quad \begin{array}{c} \overline{u \in t}^{\,(i)} \\ \vdots \\ \theta/\bot \end{array}}{\theta/\bot}\,(i)$$

11.14 The unary predicate pasigraph 'conn'

$\mathsf{conn}(t)$ is to mean that t is connected by the membership relation:

$$\mathsf{conn}(t) \equiv_{df} \forall x \forall y ((x \in t \wedge y \in t) \rightarrow (x \in y \vee x = y \vee y \in x)).$$

Introduction and elimination rules that secure this meaning directly are as follows.

$$\text{conn-I} \quad \dfrac{\exists! t \quad \begin{array}{c} \underbrace{{}^{(i)}\overline{a \in t}\ ,\ \overline{b \in t}^{\,(i)}} \\ \vdots \\ \mathsf{comp}(a, b) \end{array}}{\mathsf{conn}(t)}\,(i)$$

$$\text{conn-E} \quad \frac{\mathsf{conn}(t)}{\exists! t} \qquad \frac{\mathsf{conn}(t) \quad u \in t \quad v \in t}{\mathsf{comp}(u, v)}$$

11.15 The unary predicate pasigraph 'O' (for 'is an ordinal')

$\mathsf{O}(t)$ is to mean that t is an ordinal. The introduction and elimination rules are as follows.

$$\text{O-I}\quad \frac{\mathsf{trans}(t) \qquad \mathsf{conn}(t)}{\mathsf{O}(t)} \qquad\qquad \text{O-E}\quad \frac{\mathsf{O}(t)}{\mathsf{trans}(t)} \qquad \frac{\mathsf{O}(t)}{\mathsf{conn}(t)}$$

11.16 The unary predicate pasigraph 'IoS' (for 'is an initial or successor ordinal')

$\mathsf{IoS}(t)$ is to mean that t is an *initial or successor* ordinal. The introduction and elimination rules are as follows.

$$\text{IoS-I}\quad \frac{t=\emptyset}{\mathsf{IoS}(t)} \qquad \frac{\mathsf{O}(u) \qquad t=\mathsf{s}u}{\mathsf{IoS}(t)} \qquad\qquad \text{IoS-E}\quad \frac{\mathsf{IoS}(t) \qquad \begin{matrix} \overline{t=\emptyset}^{\,(i)} \\ \vdots \\ \theta/\bot \end{matrix} \qquad \begin{matrix} \underbrace{\overline{\mathsf{O}(a)}^{\,(i)}\ ,\ \overline{t=\mathsf{s}a}^{\,(i)}} \\ \vdots \\ \theta/\bot \end{matrix}}{\theta/\bot}\,(i)$$

11.17 The unary predicate pasigraph 'fO' (for 'is a finite ordinal')

$\mathsf{fO}(t)$ is to mean that t is a *finite* ordinal. The introduction and elimination rules are as follows.

$$\text{fO-I}\quad \frac{\mathsf{O}(t) \qquad \mathsf{IoS}(t) \qquad \begin{matrix} \overline{a \in t}^{\,(i)} \\ \vdots \\ \mathsf{IoS}(a) \end{matrix}}{\mathsf{fO}(t)}\,(i) \qquad\qquad \text{fO-E}\quad \frac{\mathsf{fO}(t)}{\mathsf{O}(t)} \qquad \frac{\mathsf{fO}(t)}{\mathsf{IoS}(t)} \qquad \frac{\mathsf{fO}(t) \qquad u \in t}{\mathsf{IoS}(u)}$$

11.18 The constant pasigraph ω

The set ω of finite ordinals:

$$\omega =_{df} \{x \mid \mathsf{fO}(x)\}$$

is the canonical choice among set theorists of a *countably infinite* set. The Axiom of Infinity is usually formulated as the statement that ω exists:

$$\exists!\omega.$$

Theorem 11.19 $\exists!t \ \vdash \ t = \bigcup \mathscr{P}t.$

Proof The following proof uses the new rules for $\bigcup$, $\subseteq$, $\mathscr{P}$, and σ. It also appeals to the existence of singletons, and the existence of power sets.

$$
\dfrac{\exists!t \quad \dfrac{\exists!t}{\exists!\mathscr{P}t} \quad \dfrac{\dfrac{\dfrac{\dfrac{\exists!t}{\exists!\mathscr{P}t}}{\mathscr{P}t=\mathscr{P}t} \quad \overline{b\in\mathscr{P}t}^{\,(1)}}{b\subseteq t}\,{\scriptstyle\mathscr{P}\mathrm{E}_4} \quad \overline{a\in b}^{\,(1)}}{a\in t}\,{\scriptstyle\subseteq\mathrm{E}_3} \quad \dfrac{\dfrac{\dfrac{\exists!t}{\exists!\mathscr{P}t}}{\mathscr{P}t=\mathscr{P}t} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{c\in t}^{\,(1)}}{\exists!c}}{\exists!\sigma c}}{\sigma c=\sigma c} \quad \overline{d\in\sigma c}^{\,(2)}}{d=c}\,{\scriptstyle\sigma\mathrm{E}_2} \quad \overline{c\in t}^{\,(1)}}{d\in t}}{\sigma c\subseteq t}\,{\scriptstyle(2)\subseteq\mathrm{I}}}{\sigma c\in\mathscr{P}t}\,{\scriptstyle\mathscr{P}\mathrm{E}_3}}{t=\bigcup\mathscr{P}t}\,{\scriptstyle(1)\bigcup\mathrm{I}}
$$

□

We do *not* have the operator-commutation of Theorem 11.19, namely $\exists!t \vdash t = \mathscr{P}\bigcup t$. Here is a counterexample: *Take* $t = \{\{a\}\}$, *where* $\exists!a$, *whence* $\exists!t$. *Then* $\bigcup t = \{a\}$. *So* $\mathscr{P}\bigcup t = \{\emptyset, \{a\}\} \neq \{\{a\}\} = t$.

12 New rules for identity and existence in free logic

In our deployment of free first-order logic thus far, we have used the abbreviation $\exists!t$ to express the thought that t exists, or is defined. Moreover, $\exists!t$ has been taken as a mere abbreviation of the longer, 'official' sentence $\exists x\, x=t$.

We shall now provide new introduction and elmination rules for the *separate* expressions !, =, ∃ and ∀, which do a better job of capturing their logical roles.

The formal sentence $!t$ will now, officially, be a well-formed sentence produced by our logical grammar. $!t$ will *take over* the role formerly played by $\exists!t$. Since ! will be treated as a *primitive* expression, $!t$ will *not* be a definitional abbreviation.

$!t$ means "t exists", or, as mathematicians often put it, "t is defined".

12.1 Introduction and elimination rules for !

! has two parts to its introduction rule:

$$
\frac{A(\ldots t\ldots)}{!t}\,,\ \text{where } A \text{ is an atomic predicate;} \qquad \frac{!f(\ldots t\ldots)}{!t}\,.
$$

So: $!t$ is a consequence of any atomic fact involving (the denotation of) t; and is a consequence also of the existence of any function's yielding a value on arguments among which is (the denotation of) t.

When a proposition can be inferred from each of such a wide range of propositions, it must be extremely weak; and its own consequences will be at least as weak.

So when the question arises: *What might legitimately be inferred from* $!t$*?*, given its own two-part introduction rule, the answer must be: an atomic proposition, involving t as its only constituent term, that is bound to be true no matter what 'positive' atomic facts might obtain (involving the denotation of t), and no matter what mappings might be effected by what functions involving the denotation of t as an input. An excellent candidate for such an atomic proposition would be $t=t$. The Elimination Rule for ! is, accordingly:

$$\text{!E}\quad \frac{!t}{t=t}\,.$$

12.2 Introduction and elimination rules for =

$$=\!\text{I}\quad \dfrac{\begin{matrix}\overline{F(t)}^{\,(i)}\\ \vdots\\ F(u)\end{matrix}\qquad !t\qquad !u}{t=u}\,{\scriptstyle(i)}\ \text{, where } F \text{ is parametric;}\qquad =\!\text{E}\quad \dfrac{t=u\qquad \psi(t)}{\psi(u)}\quad \dfrac{t=u}{!t}\quad \dfrac{t=u}{!u}\,.$$

Note how the last two parts of =E are already covered by the rule !I — since identity statements are atomic.

Theorem 12.1 $!t\vdash t=t$.

Proof $$\dfrac{\overline{Ft}^{\,(i)}\qquad !t\qquad !t}{t=t}\,{\scriptstyle =\text{I}}$$ □

It is interesting that this derived result using =I simply *is* the rule !E.

$$\dfrac{\dfrac{\begin{matrix}\Delta\\ \Pi\\ A(t)\end{matrix}}{!t}\,{\scriptstyle !\text{I}}}{t=t}\,{\scriptstyle !\text{E}}\ \leadsto\ {\scriptstyle !1}\,\dfrac{\overline{Ft}\qquad \dfrac{\begin{matrix}\Delta\\ \Pi\\ A(t)\end{matrix}}{!t}\,{\scriptstyle !\text{I}}\qquad \dfrac{\begin{matrix}\Delta\\ \Pi\\ A(t)\end{matrix}}{!t}\,{\scriptstyle !\text{I}}}{t=t}\,{\scriptstyle (1)=\text{I}}\qquad \dfrac{\dfrac{\begin{matrix}\Delta\\ \Pi\\ !f(t)\end{matrix}}{!t}\,{\scriptstyle !\text{I}}}{t=t}\,{\scriptstyle !\text{E}}\ \leadsto\ {\scriptstyle !1}\,\dfrac{\overline{Ft}\qquad \dfrac{\begin{matrix}\Delta\\ \Pi\\ !f(t)\end{matrix}}{!t}\,{\scriptstyle !\text{I}}\qquad \dfrac{\begin{matrix}\Delta\\ \Pi\\ !f(t)\end{matrix}}{!t}\,{\scriptstyle !\text{I}}}{t=t}\,{\scriptstyle (1)=\text{I}}$$

We see, then, that the two inferences

$$\frac{A(t)}{t=t}\quad \text{and}\quad \frac{!f(t)}{t=t}$$

have normal proofs. We shall henceforth adopt them as primitive inferences, while mindful that they are actually derived rules.

With $!t$ as our preferred way of expressing the existence or definedness of t, our earlier proof of the sequent $\exists!t : t=\bigcup\mathscr{P}t$ can be rewritten as follows:

$$\dfrac{!t\qquad \dfrac{!t}{!\mathscr{P}t}\qquad \dfrac{\dfrac{\dfrac{\dfrac{!t}{!\mathscr{P}t}}{\mathscr{P}t=\mathscr{P}t}\qquad \overline{b\in\mathscr{P}t}^{\,(1)}}{b\subseteq t}\,{\scriptstyle \mathscr{P}\text{E}_4}\qquad \overline{a\in b}^{\,(1)}}{a\in t}\,{\scriptstyle \subseteq\text{E}_3}\qquad \dfrac{\dfrac{\dfrac{!t}{!\mathscr{P}t}}{\mathscr{P}t=\mathscr{P}t}\qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{c\in t}^{\,(1)}}{!c}}{!\sigma c}}{\sigma c=\sigma c}\qquad \overline{d\in\sigma c}^{\,(2)}}{d=c}\,{\scriptstyle \sigma\text{E}_2}\qquad \overline{c\in t}^{\,(1)}}{d\in t}}{\sigma c\subseteq t}\,{\scriptstyle (2)\subseteq\text{I}}}{\sigma c\in\mathscr{P}t}\,{\scriptstyle \mathscr{P}\text{E}_3}}{t=\bigcup\mathscr{P}t}\,{\scriptstyle (1)\bigcup\text{I}}$$

12.3 Some results for !, =, ∃, and ∀

Theorem 12.2 $!t \vdash \exists x\, x=t$.

Proof

$$\dfrac{!t \qquad \dfrac{!t}{t=t}\,!\mathrm{E}}{\exists x\, x=t}\,\exists\mathrm{I}$$

□

Theorem 12.3 $\exists x\, x=t \vdash\, !t$.

Proof

$$\dfrac{\exists x\, x=t \qquad \dfrac{\overline{a=t}^{\,(1)}}{!t}\,!\mathrm{I}}{!t}\,(1)\,\exists\mathrm{E}$$

□

Theorem 12.4 $\vdash \forall x\, !x$.

Proof

$$\dfrac{\overline{!a}^{\,(1)}}{\forall x\, !x}\,(1)\,\forall\mathrm{I}$$

□

Theorem 12.5 $\vdash \forall x\, x=x$.

Proof

$$\dfrac{\dfrac{\overline{!a}^{\,(1)}}{a=a}\,!\mathrm{E}}{\forall x\, x=x}\,(1)\,\forall\mathrm{I}$$

□

12.4 Using the new pasigraph ! to reformulate the introduction and elimination rules for set-theoretic pasigraphs

The rules for set-theoretical pasigraphs, rewritten: ∅.

$$\emptyset\mathrm{I} \qquad \dfrac{!t \qquad \begin{matrix}\overline{a\in t}^{\,(i)}\\ \vdots\\ \bot\end{matrix}}{t=\emptyset}\,(i) \qquad\qquad \emptyset\mathrm{E} \qquad \dfrac{t=\emptyset}{!t} \qquad \dfrac{t=\emptyset \qquad u\in t}{\bot}$$

The rules for set-theoretical pasigraphs, rewritten: ⊆.

$$\subseteq\mathrm{I} \qquad \dfrac{!t \qquad !u \qquad \begin{matrix}\overline{a\in t}^{\,(i)}\\ \vdots\\ a\in u\end{matrix}}{t\subseteq u}\,(i)$$

$$\subseteq\mathrm{E}_1 \qquad \dfrac{t\subseteq u}{!t}$$

$$\subseteq\mathrm{E}_2 \qquad \dfrac{t\subseteq u}{!u}$$

$$\subseteq\mathrm{E}_3 \qquad \dfrac{t\subseteq u \qquad v\in t}{v\in u}$$

The rules for set-theoretical pasigraphs, rewritten: ℙ. These rules are unchanged.

$$\mathbb{P}\text{-I} \qquad \dfrac{u\in t \qquad v\in t \qquad \begin{matrix}\overline{a\in t}^{\,(i)}\\ \vdots\\ a=u\vee a=v\end{matrix}}{t=\mathbb{P}(u,v)}\,(i)$$

$$\mathbb{P}\text{-E}_1 \quad \frac{t=\mathbb{P}(u,v)}{u \in t} \qquad \mathbb{P}\text{-E}_2 \quad \frac{t=\mathbb{P}(u,v)}{v \in t} \qquad \mathbb{P}\text{-E}_3 \quad \frac{t=\mathbb{P}(u,v) \qquad w \in t}{w=u \vee w=v}$$

The rules for set-theoretical pasigraphs, rewritten: $\boldsymbol{\sigma}$. These rules are unchanged.

$$\sigma\text{-I} \quad \frac{u \in t \qquad \begin{array}{c}\overline{a \in t}^{\,(i)} \\ \vdots \\ a \in u\end{array}}{t=\sigma u}\,(i) \qquad\qquad \sigma\text{-E}_1 \quad \frac{t=\sigma u}{u \in t} \qquad \sigma\text{-E}_2 \quad \frac{t=\sigma u \qquad w \in t}{w=u}$$

The rules for set-theoretical pasigraphs, rewritten: ⊆.

$$\bigcup\text{-I} \quad \frac{!t \quad !v \qquad \begin{array}{c}\underbrace{{}^{(i)}\overline{a \in b}\,,\ \overline{b \in v}^{\,(i)}} \\ \vdots \\ a \in t\end{array} \qquad \begin{array}{c}{}^{(i)}\overline{c \in t} \\ \vdots \\ c \in w\end{array} \qquad \begin{array}{c}{}^{(i)}\overline{c \in t} \\ \vdots \\ w \in v\end{array}}{t = \bigcup v}\,(i) \quad , a,b,c \text{ parametric}$$

$$\bigcup\text{-E}_1 \quad \frac{t = \bigcup v \qquad w \in u}{u \in v} \qquad\qquad \bigcup\text{-E}_3 \quad \frac{t = \bigcup v}{!t}$$

$$\bigcup\text{-E}_2 \quad \frac{t = \bigcup v \quad u \in t \qquad \begin{array}{c}\underbrace{{}^{(i)}\overline{u \in a}\,,\ \overline{a \in v}^{\,(i)}} \\ \vdots \\ \theta\end{array}}{\theta}\,(i) \quad , a \text{ parametric}; \qquad \bigcup\text{-E}_4 \quad \frac{t = \bigcup v}{!v}$$

The rules for set-theoretical pasigraphs, rewritten: Separation.

$$\text{Intro} \quad \frac{\begin{array}{c}\underbrace{{}^{(i)}\overline{\varphi^x_a}\,,\ \overline{a \in v}^{\,(i)}} \\ \vdots \\ a \in t\end{array} \qquad !t \qquad \begin{array}{c}{}^{(i)}\overline{a \in t} \\ \vdots \\ a \in v\end{array} \qquad \begin{array}{c}{}^{(i)}\overline{a \in t} \\ \vdots \\ \varphi^x_a\end{array}}{t = \{x \in v \mid \varphi\}}\,(i)$$

$$\text{Elim} \quad \frac{t = \{x \in v \mid \varphi\} \qquad \varphi^x_u \qquad u \in v}{u \in t} \qquad\qquad \frac{t = \{x \in v \mid \varphi\}}{!t}$$

$$\frac{t = \{x \in v \mid \varphi\} \qquad u \in t}{u \in v} \qquad\qquad \frac{t = \{x \in v \mid \varphi\} \qquad u \in t}{\varphi^x_u}$$

The rules for set-theoretical pasigraphs, rewritten: $\mathscr{P}$.

$$\mathscr{P}\text{-I} \quad \frac{!t \quad !v \qquad \begin{array}{c}{}^{(i)}\overline{a \subseteq v} \\ \vdots \\ a \in t\end{array} \qquad \begin{array}{c}{}^{(i)}\overline{a \in t} \\ \vdots \\ a \subseteq v\end{array}}{t = \mathscr{P} v}\,(i)$$

$$\mathscr{P}\text{-E}_1 \quad \frac{t = \mathscr{P}v}{!t} \qquad\qquad \mathscr{P}\text{-E}_2 \quad \frac{t = \mathscr{P}v}{!v}$$

$$\mathscr{P}\text{-E}_3 \quad \frac{t = \mathscr{P}v \quad u \subseteq v}{u \in t} \qquad\qquad \mathscr{P}\text{-E}_4 \quad \frac{t = \mathscr{P}v \quad u \in t}{u \subseteq v}$$

The rules for ontologially committal — and *classical* — set-theory, rewritten.

Extensionality

This is now derivable, given the I- and E-rules for $\{x \mid \ldots x \ldots\}$.

Existence of Empty Set

$$\frac{}{!\emptyset}$$

Separation Schema

$$\frac{!t}{!\{x \in t \mid \varphi\}}$$

Pairing

$$\frac{!t \quad !u}{!\mathbb{P}(t,u)}$$

Union

$$\frac{!t}{!\bigcup t}$$

Power Set

$$\frac{!t}{!\mathscr{P}t}$$

Regularity, or Foundation

$$\frac{u \in t \qquad \begin{matrix} \underbrace{\overline{a \in t}^{(i)} \ , \ \overline{t \urcorner a}^{(i)}} \\ \vdots \\ \theta \end{matrix}}{\theta}\,(i)$$

Replacement Schema

$$\frac{\begin{matrix} \overline{a \in t}^{(i)} \\ \vdots \\ \exists_1 y \varphi a y \end{matrix}}{!\varphi[t]}\,(i)$$

Infinity

$$\frac{}{!\omega}$$

Choice

$$\frac{(\forall x \in t)(\exists y \in u)R(x,y)}{(\exists \phi : t \mapsto u)(\forall x \in t)R(x, \phi x)}$$

13 Concluding remarks

It is important to remind the reader of the methodological underpinnings of this study. We have sought to illuminate the meanings of set-theoretical expressions in such a way as to secure agreement on those meanings from classicists, intuitionists, and constructivists alike. This we have done by laying down ontologically non-committal rules for set-theoretical expressions, no matter whether they are, conventionally, either primitive or defined. This captures the 'analytical core' of set-theoretical talk — what Quine once called 'virtual set theory'. Interestingly, all the proofs thus far involved in delivering this analytical core are proofs in Core Logic.

It is then a *further* question what sets actually exist — either outright or conditionally. Two simple examples, respectively, will illustrate this. That the empty set exists is an *outright* existential assertion. That, given any two sets, their pair set exists, is a *conditional* existential assertion. On such simple, finitistic assertions it is no surprise that no theorist from any of the competing camps — classical, intuitionistic, or constructivist — demurs. Disagreements arise only when one begins to deal with such matters as *completed infinities*; sets being specified by means of *effectively undecidable* formulae; sets being specified by means of *impredicative* formulae; and/or whether there can be sets answering to formulae whose extensions would be *too extensive*. Constructive set theorists also have to be vigilant about their choices of constructively distinguishable (i.e., non-equivalent) formulations of axioms or axiom-schemes that the classicist is able to regard as equivalent (possibly, *modulo* other, 'more basic', or 'secure' axioms already laid down). Such is the case with various possible forms of the Axiom of Choice; of the Axiom of Regularity, or Foundation; and (so this author contends) with the Axiom Scheme of Separation.[15]

In the conduct of the further investigations to which these latter considerations give rise, the present author offers the *common parlance* of the pasigraphs treated above. They provide the *lingua franca* within which classicists, intuitionists, and constructivists can subsequently disagree, or agree to differ, given their respective doctrinal grounds concerning the nature of mathematical existence, the bivalence of mathematical truth, whether such truth is epistemically constrained, etc.

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[15] See Tennant (2020; 2021).

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