# Counterfactual Assumptions and Counterfactual Implications

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**Abstract** We define intuitionistic subatomic natural deduction systems for reasoning with elementary *would*-counterfactuals and causal *since*-subordinator sentences. The former kind of sentence is analysed in terms of counterfactual implication, the latter in terms of factual implication. Derivations in these modal proof systems make use of modes of assumptions which are sensitive to the factuality status of the formula that is to be assumed. This status is determined by means of the reference proof system on top of which a modal proof system is defined. The introduction and elimination rules for counterfactual (resp. factual) implication draw on this status. It is shown that derivations in the systems normalize and that normal derivations have the subexpression/subformula property. An intuitionistically acceptable proof-theoretic semantics is formulated in terms of canonical derivations. The systems are applied to so-called counterpossibles and to related constructions.

**Key words:** assumption, conditional logic, counterfactuals, counterpossibles, intuitionistic logic, natural deduction, proof-theoretic semantics

## 1 Introduction

The notion of assumption is essential to reasoning insofar as reasoning can be characterized as the activity of drawing conclusions from assumptions. It is the purpose of natural deduction systems (Gentzen, 1934; Jaśkowski, 1934) to depict this inferential activity as closely as possible and to lay it down formally in terms of inference rules. In his study of the notion of assumption in proof systems Schroeder-Heister (2004), Peter Schroeder-Heister stresses, using a tree-style format, the following asymmetry between assumptions and assertions (i.e., conclusions) in natural deduction:

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there is only an unspecific way of introducing assumptions, but there is both an unspecific and a specific way of introducing assertions. In order to introduce an *assumption* of formula $A$, one only has to state $A$ as an assumption. Schroeder-Heister calls this way of introducing an assumption unspecific, because "the form of $A$ is not specified" (Schroeder-Heister, 2004, p. 27). In order to introduce an assertion of $A$ in an unspecific way, one has to proceed like in the case of introducing an assumption of $A$, thereby making $A$ dependent on itself. To introduce an assertion of $A$ in a specific way, one has to derive $A$ as a conclusion using an inference rule (where $A$ is an axiom, in case the inference rule considered has no premisses). Schroeder-Heister argues that this asymmetry is responsible for limitations of expressive power and considers proof systems which, like the sequent calculus in Gentzen (1934), on his preferred reading, allow for both an unspecific and a specific way of introducing assumptions.

In what follows, we take up Schroeder-Heister's call to contribute to the exploration of the notion of assumption and of its significance to philosophical logic (cf. Schroeder-Heister, 2004, p. 45). However, we shall remain within the confines of natural deduction and suggest a way to widen its conception of assumption so as to put natural deduction in a position to "capture and codify reasoning" (Schroeder-Heister, 2004, p. 28) with *would-counterfactuals*, i.e., constructions of the form

(1) If $A$ were the case, $B$ would be the case.

More precisely, the aim of this contribution is to outline an intuitionistically (or constructively; cf. Dalen, 2002) acceptable formal approach to counterfactual reasoning and to the semantics of *would*-counterfactuals in terms of *modal proof systems* which are motivated directly by the practice of counterfactual inference making. Specifically, the systems use different *modes of making assumptions*.

Modes of assumptions, as we shall understand them in what follows, are dependent on the *factuality status* of the formula that is to be assumed. In a modal proof system this status is determined by means of a *reference proof system* on top of which the modal system is defined. In a nutshell, we assign factual status to a formula $A$, in case $A$ has been derived *canonically* (i.e., by means of an application of an introduction rule in the last inference step; cf. Dummett, 1991; Prawitz 2006; 2012) in the reference proof system $\mathsf{S}$. We shall distinguish three modes of making assumptions in modal proof systems. Intuitively, in order to assume $A$ in the *factual* mode, we need to make sure that a canonical derivation of $A$ in $\mathsf{S}$ has been constructed. To assume $A$ in the *counterfactual* mode, we need to make sure that a canonical derivation of $A$ in $\mathsf{S}$ has not been constructed. Finally, to assume $A$ in the *independent* mode, we just assume $A$ (without any proviso).

Whereas sentences of the form (1) are most adequately used in case $A$ does not count as a fact, *causal since-subordinator sentences* of the form

(2) Since $A$ is the case, it is the case that $B$.

are most adequately used in case $A$ does count as a fact (e.g., Dancygier and Sweetser, 2000, p. 126). Due to their sensitivity to the factuality status of the formula that is to be assumed, our modal natural deduction systems will be also in a position to deal with such *since*-constructions. Our analysis will focus exclusively on such intuitive uses. More specifically, constructions of the first form will be analysed as counterfactual implications, and those of the second form as factual implications. The meaning of the former will be explained by appeal to counterfactual assumptions, that of the latter by appeal to factual assumptions. In order to outline the main idea more clearly, the analysis will be confined to very elementary instances of (1) and (2).

The idea of using different ways of making assumptions for the purpose of a proof-theoretic analysis of counterfactual reasoning goes back at least to Thomason's (1970) Fitch-style natural deduction system FCS for Stalnaker's (1970) preferred conditional logic CS. However, not only the details but also the motivation of modal natural deduction systems differs from that underlying Thomason's FCS. We do not aim to formulate a structural proof system which is equivalent with a specific Hilbert-style axiom system for some specific conditional logic whose semantics is given non-inferentially. Rather, we shall develop our systems without any Hilbert-system in mind. One reason for this is that the axioms of such systems ultimately depend for their intelligibility on specific model-theoretic conditions. As a result, axioms may inherit undesirable features from these conditions. Recall, for example, that D. Lewis felt he should apologize for the "long and obscure" axiom $(\phi \mathrel{\Box\!\!\rightarrow} {\sim}\psi) \vee (((\phi \;\&\; \psi) \mathrel{\Box\!\!\rightarrow} \chi) \equiv (\phi \mathrel{\Box\!\!\rightarrow} (\psi \supset \chi)))$ of his simplest Hilbert-style system for VC (cf. Lewis, 2011, p. 133). It is for this reason that he preferred an equivalent axiomatization of VC in terms of his notion of "comparative possibility" of possible worlds (cf. Lewis, 2011, §2.5) rather than in terms of his *would*-counterfactual $\Box\!\!\rightarrow$. Furthermore, Hilbert-systems are not indispensable, if our aim is to formulate inferentially intuitive natural deduction systems which have good structural properties (e.g., normalization, subformula property) and which admit a *proof-theoretic semantics* (see Francez, 2015; Kahle and Schroeder-Heister, 2006; Piecha and Schroeder-Heister, 2016; Schroeder-Heister, 2018; Wansing, 2001) that is acceptable from an intuitionistic point of view (see Dummett, 1991, Prawitz 2006; 2012; for model-theoretic semantical considerations on Hilbert-systems for intuitionistic conditional logic see, e.g., Ciardelli and Liu, 2020 and Weiss, 2018).

Specifically, the intended proof-theoretic semantics is to be *semantically autarkic*, i.e., not defined in terms of a structural proof system that is itself defined by appeal to a formal semantics of a different kind (cf. Więckowski, 2021a). Since our approach to counterfactual reasoning is intended to be acceptable from an *intuitionistic* perspective, it will support a verification oriented conception of truth (cf. Dummett, 1991, Prawitz 2006; 2012).

The way in which this contribution is organized reflects the architecture of a modal natural deduction system. Section 2 defines the kind of proof system that will be used as reference proof system of such a system. Modal natural deduction systems for reasoning with factual and counterfactual implications will be then defined in Section 3. This section also contains the main results of this contribution (i.e., normalization and the subexpression/subformula property for the intended modal systems) and presents a proof-theoretic semantics with the desired properties. In Section 4, the modal proof systems will be used in an analysis of so-called counterpossibles (see, e.g., Berto, French, Priest, and Ripley, 2018 and Williamson, 2007) and related constructions. Section 5 makes some concluding remarks.

## 2 Reference proof systems

We shall now define the kind of reference proof system on top of which modal natural deduction systems will be defined in Section 3. We first specify the language and then formulate, in three steps, the intended kind of reference proof system. We choose subatomic natural deduction systems (cf. Więckowski 2011; 2016; 2021b; 2021a; 2023) as our reference proof systems, since we need, as mentioned above, a way to obtain canonical derivations of atomic sentences, if $A$ in (1) (resp. (2)) is atomic and if we want to assume $A$ in the counterfactual (factual) mode. Unlike standard natural deduction systems, subatomic natural deduction systems maintain introduction and elimination rules also for atomic sentences.

### 2.1 Subatomic systems

We first define the language $L0$ that we shall use in the formulation of reference proof systems.

**Definition 2.1** $L0$ is a first-order language which is defined in the usual inductive way. $C$ and $\mathcal{P}$ are the sets of individual (or nominal) constants (metavariables: $\alpha, \alpha_i$) and $n$-ary predicate constants (metavariables: $\varphi^n, \varphi^n_i$), respectively. $L0$-formulae are atomic formulae (form: $\varphi^n o_1 \ldots o_n$), absurdity ($\bot$), conjunctions ($A \& B$), disjunctions ($A \vee B$), implications ($A \supset B$), universal quantifications ($\forall x A$), and existential quantifications ($\exists x A$). In addition to defined operators for negation and bi-implication, $L0$ contains also a special non-primitive identity predicate:

1. $\neg A =_{def} A \supset \bot$
2. $A \leftrightarrow B =_{def} (A \supset B) \& (B \supset A)$
3. Let $\varphi^n$ be an $n$-ary predicate constant.

$$K^n_{\varphi^n}(o_1, o_2) =_{def} \forall z_1 \ldots \forall z_{n-1} \forall z_n ((\varphi^n o_1 z_2 \ldots z_n \leftrightarrow \varphi^n o_2 z_2 \ldots z_n) \\ \& (\varphi^n z_1 o_1 \ldots z_n \leftrightarrow \varphi^n z_1 o_2 \ldots z_n) \\ \& \ldots \& (\varphi^n z_1 \ldots z_{n-1} o_1 \leftrightarrow \varphi^n z_1 \ldots z_{n-1} o_2)).$$

Let $\varphi_1^{k_1}, \ldots, \varphi_m^{k_m}$ be all the predicate constants in $\mathcal{P}$, where $\varphi_i$ is $k_i$-ary.

$$o_1 \ddot{=} o_2 =_{def} K^{k_1}_{\varphi_1}(o_1, o_2) \& \ldots \& K^{k_m}_{\varphi_m}(o_1, o_2).$$

$Atm$ is the set of atomic sentences. $Atm(\alpha) =_{def} \{A \in Atm : A$ contains at least one occurrence of $\alpha \in C\}$ and $Atm(\varphi^n) =_{def} \{A \in Atm : A$ contains an occurrence of $\varphi^n \in \mathcal{P}\}$. Due to the presence of $\doteq$ in $L0$ we take $\mathcal{P}$ to be finite.

The first step of the definition of the intended kind of reference proof system consists in the definition of a subatomic system. In such systems we may introduce and eliminate atomic sentences using term assumptions for non-logical constants.

**Definition 2.2** A *subatomic system* $\mathcal{S}$ is a pair $\langle \mathcal{I}, \mathcal{R} \rangle$, where $\mathcal{I}$ is a *subatomic base* and $\mathcal{R}$ is a set of *introduction and elimination rules for atomic sentences*. $\mathcal{I}$ is a 3-tuple $\langle C, \mathcal{P}, v \rangle$, where $C$ and $\mathcal{P}$ are as above, and where $v$ is such that:

1. For any $\alpha \in C$, $v : C \to \wp(Atm)$, where $v(\alpha) \subseteq Atm(\alpha)$.
2. For any $\varphi^n \in \mathcal{P}$, $v : \mathcal{P} \to \wp(Atm)$, where $v(\varphi^n) \subseteq Atm(\varphi^n)$.

We let $\tau\Gamma =_{def} v(\tau)$ for any $\tau \in C \cup \mathcal{P}$, and call $\tau\Gamma$ the set of *term assumptions* for $\tau$. $\mathcal{R}$ contains I/E-rules of the following form:

$$\frac{\begin{array}{c}\mathcal{D}_0\\ \varphi_0^n\Gamma\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ \alpha_1\Gamma\end{array} \quad \dots \quad \begin{array}{c}\mathcal{D}_n\\ \alpha_n\Gamma\end{array}}{\varphi_0^n\alpha_1 \dots \alpha_n}\,(as\mathrm{I}) \qquad\qquad \frac{\begin{array}{c}\mathcal{D}_1\\ \varphi_0^n\alpha_1 \dots \alpha_n\end{array}}{\tau_i\Gamma}\,(as\mathrm{E}_i)$$

where $\varphi_0^n\alpha_1 \dots \alpha_n \in \varphi_0^n\Gamma \cap \alpha_1\Gamma \cap \dots \cap \alpha_n\Gamma$ (for $as$I); where $i \in \{0, \dots, n\}$ and $\tau_i \in \{\varphi_0^n, \alpha_1, \dots, \alpha_n\}$ (for $as$E$_i$)

Intuitively, a term assumption stores the elementary information which is associated with a non-logical constant and the $as$I-rule allows us to establish the truth of an atomic sentence on the basis of this information.

**Definition 2.3** *Derivations in $\mathcal{S}$-systems*.

*Basic step*. Any term assumption $\tau\Gamma$ and any atomic sentence $A$ (i.e., a derivation from the open assumption of $A$) is an $\mathcal{S}$-derivation.

*Induction step*. If $\mathcal{D}_i$, for $i \in \{0, \dots, n\}$, are $\mathcal{S}$-derivations, then an $\mathcal{S}$-derivation can be constructed by means of the $as$I/E-rules displayed above.

*Example 2.4* Let the $\mathcal{S}$-system contain only two predicates (i.e., $F, R$) and two nominal constants (i.e., $a$, $b$), and let the term assumptions be as follows: $F\Gamma = \{Fa, Fb\}$, $R\Gamma = \{Rab, Rba\}$, $a\Gamma = \{Fa, Rab, Rba\}$, and $b\Gamma = \{Fb, Rab, Rba\}$.

$$\text{(3)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{R\Gamma \quad a\Gamma \quad b\Gamma}{Rab}\,(as\mathrm{I})}{R\Gamma}\,(as\mathrm{E}_0) \qquad b\Gamma \qquad \dfrac{Fa}{a\Gamma}\,(as\mathrm{E}_1)}{Rba}\,(as\mathrm{I})}{a\Gamma}\,(as\mathrm{E}_2)$$

Derivation (3) contains two detours and is, therefore, not in normal form (or normal).

**Definition 2.5** *Detour conversion for as*:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_0\\ \varphi_0^n\Gamma\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ \alpha_1\Gamma\end{array} \quad \dots \quad \begin{array}{c}\mathcal{D}_n\\ \alpha_n\Gamma\end{array}}{\varphi_0^n\alpha_1 \dots \alpha_n}\,(as\mathrm{I})}{\tau_i\Gamma}\,(as\mathrm{E}_i) \quad \text{conv} \quad \begin{array}{c}\mathcal{D}_i\\ \tau_i\Gamma\end{array}$$

**Theorem 2.6** *Any derivation $\mathcal{D}$ in an $\mathcal{S}$-system can be transformed into a normal $\mathcal{S}$-derivation.*

***Proof*** Immediate. □

**Definition 2.7** Let $\mathcal{D}$ be a derivation in an $\mathcal{S}$-system.

1. An $\mathcal{S}$*-unit* in $\mathcal{D}$ is either an occurrence of (i) an atomic sentence or (ii) a term assumption $\tau\Gamma$ in $\mathcal{D}$. We use $U_{\mathcal{S}}, U'_{\mathcal{S}}$ (possibly with subscripts) for $\mathcal{S}$-units.
2. In case $U_{\mathcal{S}}$ is a term assumption $\tau\Gamma$ in $\mathcal{D}$, $\tau$ is *the expression in* $U_{\mathcal{S}}$.

**Theorem 2.8** *If $\mathcal{D}$ is a normal $\mathcal{S}$-derivation of an $\mathcal{S}$-unit $U_{\mathcal{S}}$ from a set of $\mathcal{S}$-units* $\Gamma$*, then each $\mathcal{S}$-unit in $\mathcal{D}$ is a subexpression of an expression in* $\Gamma \cup \{U_{\mathcal{S}}\}$.

***Proof*** Immediate. □

## 2.2 Subatomic identity systems

The next step of the definition of the intended kind of reference proof system consists in the extension of subatomic systems to subatomic identity systems by adding I/E-rules for non-primitive identity sentences. Roughly, two nominal constants are identical in this sense if they are indistinguishable with respect to the elementary information associated with them (cf. Definition 2.2).

**Definition 2.9** Atomic sentences $\varphi(\alpha_1)$ and $\varphi(\alpha_2)$ are *mirror atomic sentences* if and only if they are exactly alike except that the former contains occurrences of $\alpha_1$ at all the places at which the latter contains occurrences of $\alpha_2$, and vice versa.

**Definition 2.10** A *subatomic identity system* $\mathcal{S}^{\ddot{=}}$ is a 3-tuple $\langle \mathcal{I}, \mathcal{R}, \mathcal{R}^{\ddot{=}} \rangle$ which extends a subatomic system with a set $\mathcal{R}^{\ddot{=}}$ of *I/E-rules for* $\ddot{=}$*-sentences*:

$$\dfrac{\begin{matrix}[\varphi_1(\alpha_1)]^{(1_1)} \\ \mathcal{D}_{1_1} \\ \varphi_1(\alpha_2)\end{matrix} \quad \begin{matrix}[\varphi_1(\alpha_2)]^{(1_2)} \\ \mathcal{D}_{1_2} \\ \varphi_1(\alpha_1)\end{matrix} \quad \ldots \quad \begin{matrix}[\varphi_k(\alpha_1)]^{(k_1)} \\ \mathcal{D}_{k_1} \\ \varphi_k(\alpha_2)\end{matrix} \quad \begin{matrix}[\varphi_k(\alpha_2)]^{(k_2)} \\ \mathcal{D}_{k_2} \\ \varphi_k(\alpha_1)\end{matrix}}{\alpha_1 \ddot{=} \alpha_2}\,(\ddot{=}\mathrm{I}), 1_1, 1_2, \ldots, k_1, k_2$$

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ \alpha_1 \ddot{=} \alpha_2\end{matrix} \quad \begin{matrix}\mathcal{D}_{i_2} \\ \varphi_i(\alpha_1)\end{matrix}}{\varphi_i(\alpha_2)}\,(\ddot{=}\mathrm{E}_i 1) \qquad \dfrac{\begin{matrix}\mathcal{D}_1 \\ \alpha_1 \ddot{=} \alpha_2\end{matrix} \quad \begin{matrix}\mathcal{D}_{i_1} \\ \varphi_i(\alpha_2)\end{matrix}}{\varphi_i(\alpha_1)}\,(\ddot{=}\mathrm{E}_i 2)$$

where $i \in \{1, \ldots, k\}$, and $\varphi_i(\alpha_1)$ and $\varphi_i(\alpha_2)$ are mirror atomic sentences.

*Remark 2.11* In the $\ddot{=}$I/E-rules the operators figuring in the definiens of $\ddot{=}$ (Definition 2.1) have been absorbed, so to speak, into the metalanguage.

**Definition 2.12** *Derivations in* $\mathcal{S}^{\ddot{=}}$*-systems.*

*Basic step*. Any derivation in an $\mathcal{S}$-system and any identity sentence $\alpha_i \doteq \alpha_j$ (i.e., a derivation from the open assumption of $\alpha_i \doteq \alpha_j$), where possibly $i = j$, is an $\mathcal{S}^{\doteq}$-derivation.

*Induction step*. If $\mathcal{D}_1$, $\mathcal{D}_{1_1}$, $\mathcal{D}_{1_2}$, . . ., $\mathcal{D}_{k_1}$, $\mathcal{D}_{k_2}$, $\mathcal{D}_{i_1}$, and $\mathcal{D}_{i_2}$ are $\mathcal{S}$-derivations, then an $\mathcal{S}^{\doteq}$-derivation can be constructed using the $\doteq$I/E-rules listed above.

*Example 2.13* For simplicity, let the $\mathcal{S}^{\doteq}$-system contain only one predicate (i.e., $F$) and two nominal constants (i.e., $a$ and $b$). And let the term assumptions be as follows: $F\Gamma = \{Fa, Fb\}$, $a\Gamma = \{Fa\}$, and $b\Gamma = \{Fb\}$.

(4)

$$\dfrac{\dfrac{\dfrac{[Fa]^{(1_1)}}{F\Gamma}\,(as\mathrm{E}_0) \qquad b\Gamma}{Fb}\,(as\mathrm{I}) \qquad \dfrac{\dfrac{[Fb]^{(1_2)}}{F\Gamma}\,(as\mathrm{E}_0) \qquad a\Gamma}{Fa}\,(as\mathrm{I})}{a \doteq b}\,(\doteq\mathrm{I}),\ 1_1,\ 1_2$$

(5)

$$\dfrac{\dfrac{\dfrac{[Fa]^{(1_1)}}{F\Gamma}\,(as\mathrm{E}_0) \qquad \dfrac{[Fa]^{(1_1)}}{a\Gamma}\,(as\mathrm{E}_1)}{Fa}\,(as\mathrm{I}) \qquad \dfrac{\dfrac{[Fa]^{(1_2)}}{F\Gamma}\,(as\mathrm{E}_0) \qquad \dfrac{[Fa]^{(1_2)}}{a\Gamma}\,(as\mathrm{E}_1)}{Fa}\,(as\mathrm{I})}{a \doteq a}\,(\doteq\mathrm{I}),\ 1_1,\ 1_2$$

*Example 2.14* Let $\varphi_1(\alpha), \ldots, \varphi_k(\alpha) \in Atm(\alpha)$ for any $\alpha \in C$. The following is a derivation in any $\mathcal{S}^{\doteq}$-system:
(6)

$$\dfrac{[\varphi_1(\alpha)]^{(1_1)} \quad [\varphi_1(\alpha)]^{(1_2)} \quad \ldots \quad [\varphi_k(\alpha)]^{(k_1)} \quad [\varphi_k(\alpha)]^{(k_2)}}{\alpha \doteq \alpha}\,(\doteq\mathrm{I}),\ 1_1, 1_2, \ldots, k_1, k_2$$

*Remark 2.15* According to Example 2.14, $\alpha \doteq \alpha$ does not need to be postulated as an axiom. In particular, it is not declared, as it is usually the case, a conclusion of a zero premiss I-rule. Rather it is inferred, on a non-empty basis, by appeal to mirror formulae.

**Definition 2.16** *Detour conversions for* $\doteq$:

$$\dfrac{\dfrac{\begin{matrix}[\varphi_1(\alpha_1)]^{(1_1)}\\ \mathcal{D}_{1_1}\\ \varphi_1(\alpha_2)\end{matrix} \quad \begin{matrix}[\varphi_1(\alpha_2)]^{(1_2)}\\ \mathcal{D}_{1_2}\\ \varphi_1(\alpha_1)\end{matrix} \quad \ldots \quad \begin{matrix}[\varphi_k(\alpha_1)]^{(k_1)}\\ \mathcal{D}_{k_1}\\ \varphi_k(\alpha_2)\end{matrix} \quad \begin{matrix}[\varphi_k(\alpha_2)]^{(k_2)}\\ \mathcal{D}_{k_2}\\ \varphi_k(\alpha_1)\end{matrix}}{\alpha_1 \doteq \alpha_2}\,(\doteq\mathrm{I}) \qquad \begin{matrix}\mathcal{D}_{i_2}\\ \varphi_i(\alpha_1)\end{matrix}}{\varphi_i(\alpha_2)}\,(\doteq\mathrm{E}_i 1) \qquad \text{conv} \qquad \begin{matrix}\mathcal{D}_{i_2}\\ [\varphi_i(\alpha_1)]\\ \mathcal{D}_{i_1}\\ \varphi_i(\alpha_2)\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}[\varphi_1(\alpha_1)]^{(1_1)}\\ \mathcal{D}_{1_1}\\ \varphi_1(\alpha_2)\end{matrix} \quad \begin{matrix}[\varphi_1(\alpha_2)]^{(1_2)}\\ \mathcal{D}_{1_2}\\ \varphi_1(\alpha_1)\end{matrix} \quad \ldots \quad \begin{matrix}[\varphi_k(\alpha_1)]^{(k_1)}\\ \mathcal{D}_{k_1}\\ \varphi_k(\alpha_2)\end{matrix} \quad \begin{matrix}[\varphi_k(\alpha_2)]^{(k_2)}\\ \mathcal{D}_{k_2}\\ \varphi_k(\alpha_1)\end{matrix}}{\alpha_1 \doteq \alpha_2}\,(\doteq\mathrm{I}) \qquad \begin{matrix}\mathcal{D}_{i_1}\\ \varphi_i(\alpha_2)\end{matrix}}{\varphi_i(\alpha_1)}\,(\doteq\mathrm{E}_i 2) \qquad \text{conv} \qquad \begin{matrix}\mathcal{D}_{i_1}\\ [\varphi_i(\alpha_2)]\\ \mathcal{D}_{i_2}\\ \varphi_i(\alpha_1)\end{matrix}$$

**Theorem 2.17** *Any derivation* $\mathcal{D}$ *in an* $\mathcal{S}^{\doteq}$*-system can be transformed into a normal* $\mathcal{S}^{\doteq}$*-derivation.*

***Proof*** Cf. Więckowski (2016). □

**Definition 2.18** Let $\mathcal{D}$ be a derivation in an $\mathcal{S}^{\doteq}$-system.

1. An $\mathcal{S}^{\doteq}$-*unit* in $\mathcal{D}$ is either an occurrence of (i) an atomic sentence, (ii) an identity sentence, or (iii) a term assumption $\tau\Gamma$ in $\mathcal{D}$. We use $U_{\mathcal{S}^{\doteq}}, U'_{\mathcal{S}^{\doteq}}$ (possibly with subscripts) for $\mathcal{S}^{\doteq}$-units.
2. In case $U_{\mathcal{S}^{\doteq}}$ is a term assumption $\tau\Gamma$ in $\mathcal{D}$, $\tau$ is *the expression in* $U_{\mathcal{S}^{\doteq}}$.

**Theorem 2.19** *If $\mathcal{D}$ is a normal $\mathcal{S}^{\doteq}$-derivation of an $\mathcal{S}^{\doteq}$-unit $U_{\mathcal{S}^{\doteq}}$ from a set of $\mathcal{S}^{\doteq}$-units $\Gamma$, then each $\mathcal{S}^{\doteq}$-unit in $\mathcal{D}$ is a subexpression of an expression in* $\Gamma \cup \{U_{\mathcal{S}^{\doteq}}\}$.

***Proof*** Cf. Więckowski (2016). □

## 2.3 Subatomic natural deduction systems

We now complete the definition of the intended kind of reference proof system. In order to reduce complexity and to focus on the main idea underlying modal proof systems, we define these reference systems, $\mathbf{I}(\mathcal{S}^{\doteq})$-systems, only for a fragment of $L0$.

**Definition 2.20** *The language* $L0'$. $L0'$ is the fragment of $L0$ which comprises only $\perp$, atomic, $\doteq$-, and $\supset$-formulae.

**Definition 2.21** *Derivations in* $\boldsymbol{I}(\mathcal{S}^{\doteq})$*-systems.*

*Basic step*. Any derivation in an $\mathcal{S}^{\doteq}$-system and any $L0'$-formula $A$ (i.e., a derivation from the open assumption of $A$) is an $\mathbf{I}(\mathcal{S}^{\doteq})$-derivation.

*Induction step*. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are $\mathbf{I}(\mathcal{S}^{\doteq})$-derivations, then an $\mathbf{I}(\mathcal{S}^{\doteq})$-derivation can be constructed by means of the following rules:

$$\dfrac{\begin{matrix}[A]^{(u)}\\ \mathcal{D}_1\\ B\end{matrix}}{A \supset B}\,(\supset\mathrm{I}), u \qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ A \supset B\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ A\end{matrix}}{B}\,(\supset\mathrm{E}) \qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ \perp\end{matrix}}{A}\,(\perp\mathrm{i})$$

**Definition 2.22** 1. A derivation $\mathcal{D}$ of a formula $A$ in an $\mathbf{I}(\mathcal{S}^{\doteq})$-system is a *canonical derivation* iff it derives $A$ by means of an application of an I-rule in the last step of $\mathcal{D}$.

2. A canonical derivation $\mathcal{D}$ of $A$ in an $\mathbf{I}(\mathcal{S}^{\doteq})$-system is a *canonical proof* of $A$ in that system iff there are no applications of $as$-rules and no undischarged assumptions in $\mathcal{D}$.
3. The conclusions of canonical $\mathbf{I}(\mathcal{S}^{\doteq})$-derivations are $\mathbf{I}(\mathcal{S}^{\doteq})$-*theses* and the conclusions of $\mathbf{I}(\mathcal{S}^{\doteq})$-proofs are also $\mathbf{I}(\mathcal{S}^{\doteq})$-*theorems*.

*Example 2.23* Let the $\mathbf{I}(\mathcal{S}^{\doteq})$-system maintain only one predicate (i.e., $F$) and two nominal constants (i.e., $a$, $b$). Let the term assumptions be like in Example 2.13: $F\Gamma = \{Fa, Fb\}$, $a\Gamma = \{Fa\}$, and $b\Gamma = \{Fb\}$.

(7)
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[a \doteqdot a]^{(1)} \quad [Fa]^{(2_1)}}{Fa}\,(\doteqdot\mathrm{E})}{F\Gamma}\,(as\mathrm{E}_0) \qquad b\Gamma}{Fb}\,(as\mathrm{I}) \qquad \dfrac{\dfrac{[Fb]^{(2_2)}}{F\Gamma}\,(as\mathrm{E}_0) \qquad a\Gamma}{Fa}\,(as\mathrm{I})}{a \doteqdot b}\,(\doteqdot\mathrm{I}),\, 2_1,\, 2_2}{a \doteqdot a \supset a \doteqdot b}\,(\supset\mathrm{I}),\, 1$$

(8)
$$\dfrac{[a \doteqdot b]^{(1)}}{a \doteqdot b \supset a \doteqdot b}\,(\supset\mathrm{I}),\, 1$$

It can be readily verified, largely relying on standard methods (cf. Prawitz, 1965; Troelstra and Schwichtenberg, 2000), that derivations in $\mathbf{I}(\mathcal{S}^{\doteqdot})$-systems can be transformed into normal derivations and that normal derivations possess the subexpression/subformula property.

**Definition 2.24** *Detour conversions in $\boldsymbol{I}(\mathcal{S}^{\doteqdot})$-systems*. The detour conversions for *as* and $\doteqdot$ are like those in Definitions 2.5 and 2.16. These are supplemented with the detour conversion for $\supset$:

$$\dfrac{\dfrac{\begin{array}{c}[A]^{(u)}\\ \mathcal{D}_1\\ B\end{array}}{A \supset B}\,(\supset\mathrm{I}),\, u \qquad \begin{array}{c}\mathcal{D}_2\\ A\end{array}}{B}\,(\supset\mathrm{E}) \qquad \text{conv} \qquad \begin{array}{c}\mathcal{D}_2\\ [A]\\ \mathcal{D}_1\\ B\end{array}$$

**Theorem 2.25** *Normalization for $\boldsymbol{I}(\mathcal{S}^{\doteqdot})$-systems: Any derivation $\mathcal{D}$ in an $\boldsymbol{I}(\mathcal{S}^{\doteqdot})$-system can be transformed into a normal $\boldsymbol{I}(\mathcal{S}^{\doteqdot})$-derivation.*

***Proof*** A consequence of the normalization proof in Więckowski (2016). □

*Remark 2.26* (3) is neither normal nor canonical, but it can be transformed into a normal $\mathbf{I}(\mathcal{S}^{\doteqdot})$-derivation (that is not canonical). (4)-(5) and (7)-(8) are normal canonical $\mathbf{I}(\mathcal{S}^{\doteqdot})$-derivations. (6) and (8) have the form of normal canonical $\mathbf{I}(\mathcal{S}^{\doteqdot})$-proofs.

**Definition 2.27** Let $\mathcal{D}$ be a derivation in an $\mathbf{I}(\mathcal{S}^{\doteqdot})$-system.

1. A *unit* in $\mathcal{D}$ is either (i) a formula or (ii) the occurrence of an $\mathcal{S}^{\doteqdot}$-unit in $\mathcal{D}$. We use $U, U'$ (possibly with subscripts) for units.
2. In case $U$ is a term assumption $\tau\Gamma$ in $\mathcal{D}$, $\tau$ is *the expression in* $U$.

**Theorem 2.28** *Subexpression property for $\boldsymbol{I}(\mathcal{S}^{\doteqdot})$-systems: If $\mathcal{D}$ is a normal $\boldsymbol{I}(\mathcal{S}^{\doteqdot})$-derivation of a unit $U$ from a set of units $\Gamma$, then each unit in $\mathcal{D}$ is a subexpression of an expression in $\Gamma \cup \{U\}$.*

***Proof*** A consequence of the corresponding proof in Więckowski (2016). □

**Corollary 2.29** *Subformula property for $\boldsymbol{I}(\mathcal{S}^{\doteqdot})$-systems: If $\mathcal{D}$ is a normal $\boldsymbol{I}(\mathcal{S}^{\doteqdot})$-derivation of formula $A$ from a set of formulae $\Gamma$, then each formula in $\mathcal{D}$ is a subformula of a formula in $\Gamma \cup \{A\}$.*

*Remark 2.30* (4)-(8) possess the subexpression property, and so does (3) after its transformation into a normal derivation. Concerning the subformula property, we have to bear in mind that $\doteqdot$-formulae are abbreviations according to Definition 2.1.

# 3 Modal proof systems

Modal proof systems, as we shall understand them, are proof systems which, given a reference proof system that serves to determine what counts as a fact, distinguish between various modes of making assumptions. Section 3.1 defines modal proof systems for reasoning with elementary *would*-counterfactuals and causal *since*-subordinator sentences. Section 3.2 formulates a proof-theoretic semantics for such constructions.

## 3.1 IFC-systems

The natural deduction systems to be defined in this section maintain three modes of assumption. Derivations in these systems reflect, as it were, how the modes of assumptions are related to the moods of implications. The systems are defined for the language $L1$.

**Definition 3.1** *The language* $L1$. The notion of a formula of $L1$ is inductively defined by the following clauses:

1. Any formula of $L0'$ is a formula of $L1$.
2. If $A, B$ are formulae of $L1$, then $A \supset_f B$ (factual implication), $A \supset_c B$ (counterfactual implication), and $A \supset B$ (mode-sensitive implication) are formulae of $L1$.

Note that we do not use a different symbol for mode-sensitive implication. Let $\circ \in \{\supset_f, \supset_c, \supset\}$. Call the $\circ$-operators *implication-operators* and formulae with principal $\circ$ *implication-formulae*. *Defined operators of* $L1$: $\neg_f A =_{def} A \supset_f \bot$ (factual negation), $\neg_c A =_{def} A \supset_c \bot$ (counterfactual negation), $\neg A =_{def} A \supset \bot$ (mode-sensitive negation).

For instance, we symbolize sentences of the form (1) by $A \supset_c B$ and sentences of the form (2) by $A \supset_f B$.

**Definition 3.2** Let $Fml_0$ be the set of formulae of $L0'$ and let $Fml_1$ be the set of formulae of $L1$. An $L1$-formula $A$ is a *modal formula* in case $A \in Fml_1 \setminus Fml_0$.

We now define the intended modal proof systems.

**Definition 3.3** An IFC-*system* is a modal natural deduction system for intuitionistic factual, counterfactual, and mode-sensitive implication which, given a reference proof system (Definition 3.4), distinguishes three modes of making assumptions (Definition 3.5): factual, counterfactual, and independent. Derivations in IFC-systems are defined on the basis of these modes (Definition 3.7).

**Definition 3.4** *Reference proof system* S. Let S be an $\mathbf{I}(\mathcal{S}^{\doteqdot})$-system. Let an *established thesis of* S be an $L0'$-formula for which a canonical S-derivation (Definition 2.22(1)) has been constructed. And let $\Theta_{\mathsf{S}}$ be the set of established theses (or facts).

**Definition 3.5** *Modes of assumptions (IFC-systems)*. There are three modes of making assumptions in IFC-systems:

1. $|A|$ indicates that $A$ is assumed in the *factual mode*, given that $A \in Fml_0$ and $A \in \Theta_{\mathsf{S}}$.
2. $\wr A \wr$ indicates that $A$ is assumed in the *counterfactual mode*, given that $A \in Fml_0$ and $A \in \Theta^c_{\mathsf{S}}$, where $\Theta^c_{\mathsf{S}} =_{def} Fml_0 \setminus \Theta_{\mathsf{S}}$.
3. $A$ (no markers) indicates that $A$ is assumed in the *independent mode*, where $A \in Fml_1$. Specifically, in case $A$ is also an $L0'$-formula, $A$ is assumed independently of whether it is contained in $\Theta_{\mathsf{S}}$ or $\Theta^c_{\mathsf{S}}$.

We write $/A/$ to indicate that $A$ is assumed in one of the three modes.

*Remark 3.6* A consequence of Definition 3.5 is that modal $L1$-formulae can be assumed only in the independent mode.

**Definition 3.7** *Derivations in* IFC*-systems*.

*Basic step*. Any derivation in the reference system S of an IFC-system, any $L0'$-formula $A$ assumed in the factual (resp. counterfactual) mode $|A|$ ($\wr A \wr$), i.e., a derivation from the open factual (counterfactual) assumption of $A$, and any $L1$-formula $A$ assumed in the independent mode, i.e., a derivation from the open independent assumption of $A$, is a derivation in that IFC-system.

*Induction step*. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are IFC-derivations, then an IFC-derivation can be constructed by means of I/E-rules for *as* and $\doteqdot$, which now also take the modes of assumptions into account, and the following rules:

$$\dfrac{\begin{matrix}[|A|]^{(u)}\\ \mathcal{D}_1\\ B\end{matrix}}{A \supset_f B}\,(\supset_f\text{I}),\, u \qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ A \supset_f B\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ A\end{matrix}}{B}\,(\supset_f\text{E}) \qquad \dfrac{\begin{matrix}[\wr A \wr]^{(u)}\\ \mathcal{D}_1\\ B\end{matrix}}{A \supset_c B}\,(\supset_c\text{I}),\, u \qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ A \supset_c B\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ A\end{matrix}}{B}\,(\supset_c\text{E})$$

$$\dfrac{\begin{matrix}[/A/]^{(u)}\\ \mathcal{D}_1\\ B\end{matrix}}{A \supset B}\,(\supset\text{I}),\, u \qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ A \supset B\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ A\end{matrix}}{B}\,(\supset\text{E}) \qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ \bot\end{matrix}}{A}\,(\bot\text{i})$$

1. *Side conditions*:

   SC1. $\supset_f$I: No empty discharge; and no empty discharge contained in $\mathcal{D}_1$.

   SC2. $\supset_f$E: The minor premiss $A$ has *factual status*, i.e.: $A$ depends on no counterfactual assumption in $\mathcal{D}_2$; and either $A$ depends on at least one factual assumption in $\mathcal{D}_2$, or $\mathcal{D}_2$ contains at least one term assumption, or $\mathcal{D}_2$ is a derivation in S.

SC3. $\supset_c$I: Like SC1.

SC4. $\supset_c$E: (a) The minor premiss $A$ has *counterfactual status*, i.e.: $A$ depends on at least one counterfactual assumption in $\mathcal{D}_2$. (b) In case $A$ and $B$ are distinct formulae, the conclusion of $\supset_c$E must not be the minor premiss of another application of $\supset_c$E or of $\supset$E (*break formula*, for short).

2. *Terminology*: The minor premiss $A$ of $\supset$E has *independent status* in case $A$ depends on no factual or counterfactual assumption in $\mathcal{D}_2$ and has not been derived by means of a term assumption in $\mathcal{D}_2$.

3. *Assumption Principles*: The following principles are respected by derivations in IFC-systems:

AP1. No formula is assumed in more than one mode in $\mathcal{D}$.

AP2. The mode in which an antecedent $A$ is assumed in $\circ$I-applications in $\mathcal{D}$ determines the *modal status* (factual, counterfactual, independent) of all antecedent $A$-nodes (i.e., minor premisses of $\circ$E-applications) in $\mathcal{D}$.

*Remark 3.8* 1. *Factual implication*: SC1 ensures that $A$ is indeed assumed factually and that factual implication behaves inferentially in an intuitively required non-monotonic manner. SC2 ensures that the minor premiss $A$ is rooted in facts and does not rest on the unestablished.

2. *Counterfactual implication*: SC3 ensures that $A$ is indeed assumed counterfactually and that counterfactual implication behaves non-monotonically. SC4a ensures that the minor premiss $A$ is neither based entirely on facts nor on assumptions made in the independent mode. SC4b excludes break formulae in order to block transitivity.

3. *(Mode-sensitive) implication*: There are no side conditions on the I/E-rules for $\supset$. The $\supset$-rules of S-systems are special cases of the $\supset$-rules of IFC-systems. We may regard these special cases, allowing only for independent assumptions, as governing $\supset$ in the usual (i.e., mode-less) sense of implication. As mentioned, if $A$ is a modal $L1$-formula (Definition 3.2), $[/A/]^{(u)}$ in an $\supset$I-application can be only of the form $[A]^{(u)}$.

4. We obtain a minimal system (abbr. MFC-system) from an IFC-system, if we remove the $\bot$i-rule from the latter.

**Definition 3.9** *Canonical derivation, canonical proof, thesis, and theorem (*IFC*-systems)*. Analogous to Definition 2.22.

*Example 3.10*

(9)
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\wr Rab \wr]^{(1)}}{R\Gamma}\,(asE_0) \quad \dfrac{[|Fc|]^{(2)}}{c\Gamma}\,(asE_1) \quad d\Gamma}{Rcd}\,(asI)}{Fc \supset_f Rcd}\,(\supset_f I),\,2}{Rab \supset_c (Fc \supset_f Rcd)}\,(\supset_c I),\,1 \quad \wr Rab \wr}{Fc \supset_f Rcd}\,(\supset_c E) \quad \dfrac{F\Gamma \quad c\Gamma}{Fc}\,(asI)}{\dfrac{Rcd}{c\Gamma}\,(asE_1)}\,(\supset_f E)$$

(10) (a)

$$\dfrac{\dfrac{\dfrac{[\neg_c A]^{(2)} \qquad [\wr A \wr]^{(1)}}{\bot}\,(\supset_c \mathrm{E})}{\neg\neg_c A}\,(\supset \mathrm{I}),\,2}{A \supset_c \neg\neg_c A}\,(\supset_c \mathrm{I}),\,1$$

(b)

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg_f B]^{(2)} \qquad \dfrac{[A \supset_f B]^{(1)} \qquad [|A|]^{(3)}}{B}\,(\supset_f \mathrm{E})}{\bot}\,(\supset_f \mathrm{E})}{\neg_f A}\,(\supset_f \mathrm{I}),\,3}{\neg_f B \supset \neg_f A}\,(\supset \mathrm{I}),\,2}{(A \supset_f B) \supset (\neg_f B \supset \neg_f A)}\,(\supset \mathrm{I}),\,1$$

The side conditions imposed on the $\supset_c$-rules guarantee that IFC-systems do justice to traditional *counterfactual fallacies* (cf. Stalnaker, 1968, pp. 48–49; see also Lewis, 2011, §1.8). The failures of transitivity and contraposition are reflected by the illegal (11a) and (11b), respectively. In both derivations $B$ is a break formula. (12a) and (12b) contain violations of weakening and are related to the fallacy of strengthening of the antecedent.

(11) (a)

$$\dfrac{\dfrac{B \supset_c C \qquad \dfrac{A \supset_c B \qquad [\wr A \wr]^{(1)}}{B}\,(\supset_c \mathrm{E})\ \textit{illeg.}}{C}\,(\supset_c \mathrm{E})}{A \supset_c C}\,(\supset_c \mathrm{I}),\,1$$

(b)

$$\dfrac{\dfrac{\dfrac{\dfrac{[\wr \neg B \wr]^{(2)} \qquad \dfrac{[A \supset_c B]^{(1)} \qquad [\wr A \wr]^{(3)}}{B}\,(\supset_c \mathrm{E})\ \textit{illeg.}}{\bot}\,(\supset \mathrm{E})}{\neg A}\,(\supset \mathrm{I}),\,3}{\neg B \supset_c \neg A}\,(\supset_c \mathrm{I}),\,2}{(A \supset_c B) \supset (\neg B \supset_c \neg A)}\,(\supset \mathrm{I}),\,1$$

(12) (a)

$$\dfrac{\dfrac{[\wr A \wr]^{(1)}}{B \supset_c A}\,(\supset_c \mathrm{I})\ \textit{illeg.}}{A \supset_c (B \supset_c A)}\,(\supset_c \mathrm{I}),\,1$$

(b)

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg(A \supset_c B)]^{(1)} \qquad \dfrac{[/B/]^{(2)}}{A \supset_c B}\,(\supset_c \mathrm{I})\ \textit{illeg.}}{\bot}}{\neg B}\,(\supset \mathrm{I}),\,2}{A \supset_c \neg B}\,(\supset_c \mathrm{I})\ \textit{illeg.}}{\neg(A \supset_c B) \supset (A \supset_c \neg B)}\,(\supset \mathrm{I}),\,1$$

Derivations in IFC-systems may contain detours (i.e., cut or maximum formulae). (9) is an example. Such derivations can be transformed into normal derivations.

**Definition 3.11** 1. The occurrence of a formula in a derivation $\mathcal{D}$ in and IFC-system is a *cut (or maximum) formula*, if it is the conclusion of an application of an

I-rule and the (major) premiss of an E-rule. A *maximal* cut formula in $\mathcal{D}$ is a cut formula with maximal rank $r$.
2. The *cut rank* of $\mathcal{D}$, $cr(\mathcal{D})$, is $\langle d, n\rangle$, where $d = max\{r(A) : A$ cut formula in $\mathcal{D}\}$, and where $n$ is the number of maximal cut formulae in $\mathcal{D}$. A derivation is *normal (or in normal form)*, if it contains no cut formulae.

**Definition 3.12** *Detour conversions in* IFC*-systems*. The detour conversions for $as$ and $\doteq$ are like those in Definitions 2.5 and 2.16, except that assumptions of $as$- and $\doteq$-formulae now occur in $//$. These are supplemented with detour conversions for the $\circ$-operators:

$$\dfrac{\dfrac{\begin{array}{c}[|A|]^{(u)}\\ \mathcal{D}_1\\ B\end{array}}{A \supset_f B}(\supset_f \mathrm{I}), u \qquad \begin{array}{c}\mathcal{D}_2\\ A\end{array}}{B}(\supset_f \mathrm{E}) \qquad \text{conv} \qquad \begin{array}{c}\mathcal{D}_2\\ [|A|]\\ \mathcal{D}_1\\ B\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}[\wr A\wr]^{(u)}\\ \mathcal{D}_1\\ B\end{array}}{A \supset_c B}(\supset_c \mathrm{I}), u \qquad \begin{array}{c}\mathcal{D}_2\\ A\end{array}}{B}(\supset_c \mathrm{E}) \qquad \text{conv} \qquad \begin{array}{c}\mathcal{D}_2\\ [\wr A\wr]\\ \mathcal{D}_1\\ B\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}[/A/]^{(u)}\\ \mathcal{D}_1\\ B\end{array}}{A \supset B}(\supset \mathrm{I}), u \qquad \begin{array}{c}\mathcal{D}_2\\ A\end{array}}{B}(\supset \mathrm{E}) \qquad \text{conv} \qquad \begin{array}{c}\mathcal{D}_2\\ [/A/]\\ \mathcal{D}_1\\ B\end{array}$$

*Remark 3.13* 1. The $\supset$-conversion of S-systems (Definition 3.12) is a special case of the $\supset$-conversion of IFC-systems.

2. Recall that, in virtue of AP1, a formula can be only assumed in exactly one mode in a derivation. Because of $[|A|]^{(u)}$, the minor premiss $A$ of the $\supset_f$E-application in the $\supset_f$-conversion has, by AP2, factual status. Similarly, since we have $[\wr A\wr]^{(u)}$ on the left-hand side of the $\supset_c$-conversion, the minor premiss $A$ of the $\supset_c$E-application in the $\supset_c$-conversion has counterfactual status. Finally, in the $\supset$-conversion, $A$ is assumed in exactly one of the three modes in $[/A/]^{(u)}$. By AP2, this mode determines the modal status of the two $A$-nodes in the derivation on the left-hand side of the conversion.

The following considerations supplement Remark 3.13(2).

*Remark 3.14* Let $F$ be an implication-formula $A \circ B$. Let $\mathcal{D}^F$ be a derivation which derives $F$ by means of an $\circ$I-application in its last step, and let $\mathcal{D}^B$ be the subderivation of $\mathcal{D}^F$ which derives the premiss $B$ of that $\circ$I-application. Let $\mathcal{D}^A$ be a derivation which derives $A$. The tables below list the cases in which an $\circ$E-application can be used to construct a derivation $\mathcal{D}^*$ of $B$ (i.e., a detour derivation with $F$ being a detour formula) from derivations $\mathcal{D}^F$ and $\mathcal{D}^A$. (Since no mode-related side conditions are imposed on the I/E-rules for *as*- and $\doteq$-formulae, we consider only cases in which $B$ and $A$ have been obtained by means of $\circ$-rules.) The columns with the heading $\mathcal{D}^B$ (resp. $\mathcal{D}^A$) indicate the last rule applied in $\mathcal{D}^B$ ($\mathcal{D}^A$). '+' means that the construction of $\mathcal{D}^*$ is legal and that $\mathcal{D}^*$'s conversion is successful. '−' means that a conversion is precluded, since the construction is not legal. In case $\mathcal{D}^B$ ends with an $\circ$E-application there are two entries. The first [second] entry indicates the result for the case in which $F$ is introduced discharging an assumption used to derive the major [minor] premiss of that $\circ$E-application.

| T.1. | $\mathcal{D}^B$: | $\mathcal{D}^A$: | $\supset_f$-c | $\supset_c$-c | $\supset$-c |
|---|---|---|---|---|---|
| a. | $\supset_f$I | $\supset_f$I | − | − | + |
| b. | $\supset_f$I | $\supset_f$E | + | + | + |
| c. | $\supset_f$E | $\supset_f$I | $- \mid -$ | $- \mid -$ | $+ \mid +$ |
| d. | $\supset_f$E | $\supset_f$E | $+ \mid +$ | $+ \mid -$ | $+ \mid +$ |

| T.2. | $\mathcal{D}^B$: | $\mathcal{D}^A$: | $\supset_f$-c | $\supset_c$-c | $\supset$-c |
|---|---|---|---|---|---|
| a. | $\supset_f$I | $\supset_c$I | − | − | + |
| b. | $\supset_f$I | $\supset_c$E | − | − | − |
| c. | $\supset_f$E | $\supset_c$I | $- \mid -$ | $- \mid -$ | $+ \mid +$ |
| d. | $\supset_f$E | $\supset_c$E | $- \mid -$ | $- \mid -$ | $- \mid -$ |

| T.3. | $\mathcal{D}^B$: | $\mathcal{D}^A$: | $\supset_f$-c | $\supset_c$-c | $\supset$-c |
|---|---|---|---|---|---|
| a. | $\supset_f$I | $\supset$I | + | + | + |
| b. | $\supset_f$I | $\supset$E | + | + | + |
| c. | $\supset_f$E | $\supset$I | $+ \mid +$ | $+ \mid -$ | $+ \mid +$ |
| d. | $\supset_f$E | $\supset$E | $+ \mid +$ | $+ \mid -$ | $+ \mid +$ |

| T.4. | $\mathcal{D}^B$: | $\mathcal{D}^A$: | $\supset_f$-c | $\supset_c$-c | $\supset$-c |
|---|---|---|---|---|---|
| a. | $\supset_c$I | $\supset_f$I | − | − | + |
| b. | $\supset_c$I | $\supset_f$E | + | + | + |
| c. | $\supset_c$E | $\supset_f$I | $- \mid -$ | $- \mid -$ | $+ \mid +$ |
| d. | $\supset_c$E | $\supset_f$E | $+ \mid +$ | $+ \mid +$ | $+ \mid +$ |

| T.5. | $\mathcal{D}^B$: | $\mathcal{D}^A$: | $\supset_f$-c | $\supset_c$-c | $\supset$-c |
|---|---|---|---|---|---|
| a. | $\supset_c$I | $\supset_c$I | − | − | + |
| b. | $\supset_c$I | $\supset_c$E | − | − | − |
| c. | $\supset_c$E | $\supset_c$I | $- \mid -$ | $- \mid -$ | $+ \mid +$ |
| d. | $\supset_c$E | $\supset_c$E | $- \mid -$ | $- \mid -$ | $- \mid -$ |

| T.6. | $\mathcal{D}^B$: | $\mathcal{D}^A$: | $\supset_f$-c | $\supset_c$-c | $\supset$-c |
|---|---|---|---|---|---|
| a. | $\supset_c$I | $\supset$I | + | + | + |
| b. | $\supset_c$I | $\supset$E | + | + | + |
| c. | $\supset_c$E | $\supset$I | $+ \mid +$ | $+ \mid +$ | $+ \mid +$ |
| d. | $\supset_c$E | $\supset$E | $+ \mid +$ | $+ \mid +$ | $+ \mid +$ |

| T.7. | $\mathcal{D}^B$: | $\mathcal{D}^A$: | $\supset_f$-c | $\supset_c$-c | $\supset$-c |
|---|---|---|---|---|---|
| a. | $\supset$I | $\supset_f$I | − | − | + |
| b. | $\supset$I | $\supset_f$E | + | + | + |
| c. | $\supset$E | $\supset_f$I | $- \mid -$ | $- \mid -$ | $+ \mid +$ |
| d. | $\supset$E | $\supset_f$E | $+ \mid +$ | $+ \mid +$ | $+ \mid +$ |

| T.8. | $\mathcal{D}^B$: | $\mathcal{D}^A$: | $\supset_f$-c | $\supset_c$-c | $\supset$-c |
|---|---|---|---|---|---|
| a. | $\supset$I | $\supset_c$I | − | − | + |
| b. | $\supset$I | $\supset_c$E | − | − | − |
| c. | $\supset$E | $\supset_c$I | $- \mid -$ | $- \mid -$ | $+ \mid +$ |
| d. | $\supset$E | $\supset_c$E | $- \mid -$ | $- \mid -$ | $- \mid -$ |

| T.9. | $\mathcal{D}^B$: | $\mathcal{D}^A$: | $\supset_f$-c | $\supset_c$-c | $\supset$-c |
|---|---|---|---|---|---|
| a. | $\supset$I | $\supset$I | + | + | + |
| b. | $\supset$I | $\supset$E | + | + | + |
| c. | $\supset$E | $\supset$I | $+ \mid +$ | $+ \mid +$ | $+ \mid +$ |
| d. | $\supset$E | $\supset$E | $+ \mid +$ | $+ \mid +$ | $+ \mid +$ |

**Theorem 3.15** *Normalization for* IFC*-systems: Any derivation* $\mathcal{D}$ *in an* IFC*-system can be transformed into a normal* IFC*-derivation.*

***Proof*** We proceed in the familiar way by applying detour conversions (Definition 3.12) to $\mathcal{D}$, in order to arrive at $cr(\mathcal{D}) = \langle 0, 0\rangle$. □

(9), for instance, can be transformed into $c\Gamma$, a derivation in normal form. (10a) and (10b) are normal derivations.

Normal derivations have a simple structure. It can be shown, adapting standard methods (cf. Prawitz, 1965; Troelstra and Schwichtenberg, 2000), that they possess the subexpression property (of which the subformula property is a special case).

**Definition 3.16** Let $\mathcal{D}$ be a derivation in an IFC-system.

1. A *unit* in $\mathcal{D}$ is either (i) a formula or (ii) the occurrence of an $\mathcal{S}^{\doteqdot}$-unit in $\mathcal{D}$. We use $U$, $U'$ (possibly with subscripts) for units.
2. In case $U$ is a term assumption $\tau\Gamma$ in $\mathcal{D}$, $\tau$ is *the expression in* $U$.

**Definition 3.17** Let $\mathcal{D}$ be a normal derivation in an IFC-system, let $\circ \in \{\supset_f, \supset_c, \supset\}$. A sequence of unit occurrences $U_0, \ldots, U_n$ such that

1. $U_0$ is a top formula occurrence $A_0$ enclosed in // or a top occurrence of a term assumption $\tau\Gamma_0$ in $\mathcal{D}$, and
2. for $0 \leq i < n$, $U_{i+1}$ is immediately below $U_i$, and
3. $U_i$ is not the minor premiss of $\doteqdot$E or $\circ$E,

is a *track* of $\mathcal{D}$. A track of *order* 0 in $\mathcal{D}$ is a track ending in the conclusion of $\mathcal{D}$. A track of *order* $n + 1$ in $\mathcal{D}$ is a track ending in the minor premiss of an application of $\doteqdot$E or $\circ$E with the major premiss belonging to a track of order $n$.

**Theorem 3.18** *Let $\mathcal{D}$ be a normal derivation in an* IFC*-system and let $\pi$ be a track $U_0, \ldots, U_n$ in $\mathcal{D}$. Then there is a unit $U_i$ in $\pi$, the minimum part of the track, which divides $\pi$ into two (possibly empty) parts, an E-part $U_0, \ldots, U_{i-1}$ and an I-part $U_{i+1}, \ldots, U_n$. The E-part is constructed exclusively by E-rule applications. The I-part is constructed exclusively by I-rule applications. $U_i$ is the conclusion of an E-rule, and in case $i < n$, a premiss of an I-rule or of $\perp i$.*

***Proof*** By Theorem 3.15, a major premiss of an E-rule application cannot be a conclusion of an I-rule application. The result is a consequence of this insight. □

**Theorem 3.19** *Subexpression property for* IFC*-systems: If $\mathcal{D}$ is a normal* IFC*-derivation of a unit $U$ from a set of units $\Gamma$, then each unit in $\mathcal{D}$ is a subexpression of an expression in $\Gamma \cup \{U\}$.*

***Proof*** Making use of Theorem 3.18, the result is established by induction of the order of tracks $n$. □

**Corollary 3.20** *Subformula property for* IFC*-systems: If $\mathcal{D}$ is a normal* IFC*-derivation of formula $A$ from a set of formulae $\Gamma$, then each formula in $\mathcal{D}$ is a subformula of a formula in $\Gamma \cup \{A\}$.*

As a consequence of Theorem 3.19, full analyticity can be claimed for the systems. We may use, relying on Corollary 3.20, the following method (cf. Więckowski, 2021a) in order to show that a formula of the language of IFC-systems cannot be derived as a theorem in these systems:

**Definition 3.21** *Method of counter-derivations*. Construct a candidate for a normal canonical IFC-proof of formula $A$ by proceeding bottom-up using the rules for the operators ignoring the side conditions on them. In case (i) the construction has been successful, check whether the candidate violates a side condition. If this is the case, (ia) we obtain a counter-derivation for $A$, otherwise (ib) we obtain a normal IFC-proof of $A$. In case (ii) the construction of a candidate has not been successful, we may conclude that $A$ cannot be derived as a theorem. Consequently, we get a decision concerning the IFC-derivability of $A$ as a theorem. It is derivable as a theorem in case (ib), and underivable in cases (ia) and (ii).

*Remark 3.22* Some of the derivations in (11) and (12) can be seen as counter-derivations which show that their conclusions are not theorems.

*Remark 3.23* 1. As mentioned in Section 1, the idea of using different ways of making assumptions in the context of natural deduction for counterfactuals can be traced back at least to Thomason's (1970) FCS. Crucially, Thomason introduces the notion of a *strict derivation*. He takes it that in an ordinary derivation from an assumption $A$, we suppose that $A$ is the case in the actual situation. By contrast, in a strict derivation, we may "hold in abeyance certain portions of our knowledge about our actual situation, and envisage another situation in which something is supposed to hold" (Thomason, 1970, p. 398). Since it may happen that in the envisaged alternative situations not all our knowledge about the actual situation is available, Thomason imposes restrictions on the availability of that knowledge in strict derivations. Formally, this is achieved by introducing special *reiteration rules* which govern reiteration into strict derivations. One may infer a *would*-counterfactual (Thomason's notation: $A > B$), by means of an introduction rule on the basis of a strict derivation of $B$ from the, as it were, "strict" assumption of $A$. In particular, one may assume also known propositions in this counterfactual way. Thomason establishes the equivalence of FCS and CS. However, he does not discuss the proof-theoretic properties of FCS.

2. It seems possible to use factual implication for the formal analysis of those constructions of the form (2) in which "since" can be equivalently replaced by "because". For discussion of the relation between these two kinds of causal subordinator see, e.g., Dancygier and Sweetser (2000) and Guillaume (2013). For an outline of a formal system for reasoning with "because" see Schnieder (2011).

*Remark 3.24* In defining modal natural deduction systems there are several decisions to be made. These may concern, for instance, the choice of the reference proof system, the conception of an established fact, the modes, in which modal formulae can be legally assumed, the shape of the rules, or the side conditions that are to be imposed on them. Thus, the complexity that pertains to formal accounts of counterfactual reasoning is not moved to conditions on external (e.g., model-theoretic) structures, but rather enters the systems via such proof-theoretic design options.

### 3.2 A proof-theoretic semantics

On the basis of Theorem 3.15, we may formulate a proof-theoretic semantics for the non-logical constants, the atomic sentences, the identity sentences, and for the formulae composed of the operators of IFC-systems.

**Definition 3.25** *Meaning*: Let the modal proof system $M$ be an IFC-system.

1. The meaning of a *non-logical constant* $\tau$ is given by the term assumptions $\tau\Gamma$ for $\tau$ which are determined by the subatomic base of the $\mathcal{S}^{\doteq}$-system of the reference proof system S (Definition 2.2) of $M$.
2. The meaning of an $L1$-*formula* $A$ is given by the set of canonical derivations of $A$ in $M$ (Definition 3.9).

*Remark 3.26* 1. Definition 3.25 contains a proof-theoretic semantics for the non-logical constants and formulae of $L0'$, defined in terms of S-derivations, as a special case.

2. Since meaning is defined in terms of canonical derivations (cf. Dummett, 1991; Prawitz, 2006), the semantics specified above is acceptable from an *intuitionistic* point of view.

3. The proposed proof-theoretic semantics is *semantically autarkic*, since the modal natural deduction systems do not draw on a formal semantics of a different kind (e.g., a possible worlds similarity semantics; cf. Lewis, 2011; Nute and Cross, 2001; Stalnaker, 1968; Stalnaker and Thomason, 1970). For instance, labelled (e.g., Negri and Olivetti, 2015; Negri and Sbardolini, 2016; Poggiolesi, 2016) or internal (e.g., Lellmann and Pattinson, 2012; Olivetti and Pozzato, 2015) structural proof systems for standard counterfactual logics all of which are formulated in a classical context do not allow for an autarkic proof-theoretic semantics. Labelled proof systems incorporate model-theoretic structures in terms of which truth conditions are formulated into their rules by means of labels (for worlds) and labelled formulae (for similarity). A proof-theoretic semantics based on such a calculus (envisaged in Girlando, Negri, and Olivetti, 2018) would certainly not be autarkic. By contrast, internal proof systems for a given counterfactual logic can be characterized as not involving a syntax that cannot be defined in terms of the object language of that logic. As a result, the sequents of such a calculus do not wear their genesis on their sleeves. However, the internal systems mentioned above make use of structural operators and specific rules which directly imitate model-theoretic structures involved in the semantics. (Translations of internal into labelled systems and back are considered in Girlando, 2019; Girlando, Negri, and Olivetti, 2018.) From a foundational point of view—or seeing proof-theoretic semantics as an "alternative to truth-condition semantics" (Schroeder-Heister, 2018, p. 1)—neither an internalization of model-theoretic truth conditions nor an imitation of model-theoretic structures seems to be appealing.

**Definition 3.27** A [subatomic] proof system is *meaning-integral*, if a proof-theoretic semantics is available for it that is based on [term assumptions and] canonical derivations.

Given a meaning-integral proof system, we may define a notion of *derivation-based intuitionistic truth* (cf. Więckowski, 2023).

**Definition 3.28** Let $I$ be an IFC-system and call $\{A : \Gamma \vdash_c^I A\}$, i.e., the set of formulae which have been canonically derived in $I$ from a set of units $\Gamma$, the *canonical $I$-set*. The *$I$-truth* of $A$ *with respect to* $\Gamma$ is defined by: $\Gamma \Vdash^I A =_{def} A \in \{A : \Gamma \vdash_c^I A\}$. Special case: $A$ is a *logical $I$-truth* (i.e., $\Vdash^I A$) in case $A$ is the conclusion of a canonical $I$-proof (cf. Definition 3.9).

*Remark 3.29* $I$-truth may serve to single out certain atomic sentences, identity sentences, and logical compounds. And it is logical $I$-truth, rather than plain $I$-truth, that can be used to single out certain $I$-true identity sentences (i.e., self-identities) and logical compounds further. In general, the canonical derivations on which $I$-truths are based can be seen as formal verifications (for verificationism in the context of proof-theoretic semantics see, e.g., Dummett, 1991, Prawitz 2006; 2012).

## 4 A philosophical application: counterpossibles

We shall now apply the modal proof systems developed in the previous section to the following constructions (cf. Williamson, 2007, p. 174):

(13) If Hesperus had not been Phosphorus, Hesperus would not have been Hesperus.

(14) If Hesperus had not been Phosphorus, Hesperus would not have been Phosphorus.

Given certain philosophical presuppositions, in particular, the doctrine of the *necessity of identity* (NI, for short; cf. Kripke, 1980; Marcus, 1961), conditional sentences of this kind are sometimes called "counterpossibles" (see, e.g., Berto, French, Priest, and Ripley, 2018; Williamson, 2007), since their antecedents turn out to be impossible: If Hesperus is Phosphorus, then, given NI, this is so of necessity. Moreover, given the interdefinability of the operators for necessity and possibility (cf. Williamson, 2007, p. 295), guaranteed by *classical* modal logic, "their" distinctness is, therefore, impossible.

A further common presupposition in the discussion of counterpossibles is "orthodoxy" (cf. Berto, French, Priest, and Ripley, 2018, p. 694), that is, the aforementioned similarity semantics for counterfactuals. A semantics of this kind makes use of truth conditions and explains the formal meaning of counterfactuals in terms of subset relations on possible worlds. Roughly, a sentence of the form (1) is true at world $w$ exactly if $B$ is true in all the possible worlds in which $A$ is true that are most similar to $w$. On this account, there are no $A$-worlds, in case $A$ is impossible. This means that an instance of (1) with impossible $A$ is true, since, given orthodoxy, for any $B$, $B$ is true (vacuously) at all the most similar $A$-worlds. So-called *vacuists* (e.g., Williamson, 2007) accept this consequence for both (13) and (14). By contrast, *non-vacuists* argue, typically admitting also impossible worlds into the orthodox picture (see, e.g., Berto, French, Priest, and Ripley, 2018 and the references therein), that the first counterpossible is false and that the second is true. Our discussion of (13) and (14) below will presuppose neither NI, nor classicality, nor orthodoxy.

We symbolize (13) and (14) as $\neg a \doteq b \supset_c \neg a \doteq a$ and $\neg a \doteq b \supset_c \neg a \doteq b$, respectively. For simplicity, let the reference proof system S of the IFC-system contain only a single predicate constant (i.e., $F$) and two nominal constants (i.e., $a$, $b$). Moreover, let the term assumptions be, again, like in Example 2.13: $F\Gamma = \{Fa, Fb\}$, $a\Gamma = \{Fa\}$, and $b\Gamma = \{Fb\}$. (15) is a canonical derivation for (13):

(15)
$$\dfrac{\dfrac{\dfrac{[\wr\neg a \doteq b\wr]^{(1)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[a \doteq a]^{(2)} \quad [Fa]^{(3_1)}}{Fa}(\doteq\mathrm{E})}{F\Gamma}(as\mathrm{E}_0) \quad b\Gamma}{Fb}(as\mathrm{I}) \qquad \dfrac{\dfrac{[Fb]^{(3_2)}}{F\Gamma}(as\mathrm{E}_0) \quad a\Gamma}{Fa}(as\mathrm{I})}{a \doteq b}(\doteq\mathrm{I}), 3_1, 3_2}{\bot}(\supset\mathrm{E})}{\neg a \doteq a}(\supset\mathrm{I}), 2}{\neg a \doteq b \supset_c \neg a \doteq a}(\supset_c\mathrm{I}), 1$$

The conclusion of (15) is only a *thesis* of the specific IFC-system. (An alternative derivation would be, e.g., one in which the subderivation of $a \doteq b$ in (15) were replaced by $|a \doteq b|$, or one in which it were replaced, e.g., by the S-derivation (4).) Note that we would not be in a position to assume $\neg a \doteq b$ in the counterfactual mode, if we had established $\neg a \doteq b$ as a fact. If we had done so, we would not be in a position to arrive at the intended conclusion. (16) is a canonical derivation for (14). Its conclusion is a *theorem* of the IFC-system:

(16)
$$\dfrac{[\wr\neg a \doteq b\wr]^{(1)}}{\neg a \doteq b \supset_c \neg a \doteq b}(\supset_c\mathrm{I}), 1$$

*Comments*:

1. We may regard both (13) and (14) as true (cf. Definition 3.28). The latter can be taken to be also logically true, since its canonical derivation is a proof. Thus, the present assessment of these sentences as true seems to be closer to the vacuist one. Note, however, that no appeal to some notion of vacuity is being made in the explanation of their truth.

2. (15) shows how the *self-distinctness* of Hesperus can be inferred form the counterfactual assumption of the distinctness of Hesperus and Phosphorus. On the present semantics, (15) is one of those derivations which constitute the meaning of its conclusion.

3. By Definition 3.25, the meaning of $\neg a \doteq b \supset_c \neg a \doteq a$ does not coincide with that of $\neg a \doteq b \supset_c \neg a \doteq b$, nor do the meanings of any two theorems, or those of any two logically equivalent formulae. As a consequence, the present semantics is sensitive to *hyperintensional* distinctions (a recent collection on hyperintensionality is Duží and Jespersen, 2015; for discussion in the context of proof-theoretic semantics see, e.g., Pezlar, 2018).

4. Consider the structure of (4) and (6) which canonically derive $a \doteqdot b$ and $a \doteqdot a$ (an instance of $\alpha \doteqdot \alpha$), respectively. $a \doteqdot a$ can be derived as a theorem in any IFC-system, whereas $a \doteqdot b$ cannot be derived as a theorem in any such system. If we regard formulae that have been derived as theorems as *necessities*, and those that have been derived only as theses as *contingencies*, we may classify $a \doteqdot a$ as necessary and $a \doteqdot b$ as contingent.

5. In order to derive $a \doteqdot b$ canonically, we have to look to the term assumptions and to apply the $as$I-rule. However, in order to derive $a \doteqdot a$ canonically these steps are not required. If we regard, taking a derivation-oriented perspective, the conclusion of a canonical derivation as *a posteriori* in case the derivation requires an application of an $as$-rule, and if we regard the conclusion of a canonical derivation as *a priori* in case the derivation does not need to make use of such rules, we may classify $a \doteqdot b$ as *a posteriori* and $a \doteqdot a$ as *a priori*. Moreover, depending on whether the constants symbolize denoting or non-denoting names, we may also distinguish between a denotational (or referential) and a non-denotational kind of the *a posteriori*. Furthermore, taking this derivation-oriented perspective, we may also consider adding a distinction between an empirical and a non-empirical kind of the denotational *a posteriori*.

6. The above categorization of 'Hesperus is Hesperus' as necessary *a priori* and of 'Hesperus is Phosphorus' as contingent *a posteriori* is relatively old-fashioned in nature. It differs from Kripke's well-known proposal (cf. Kripke, 1980), according to which, given NI (and other prerequisites), 'Hesperus is Phosphorus' expresses an *a posteriori* necessity.

7. Analogous remarks apply to instances of (13) and (14) which feature *empty names* (e.g., let $a$ symbolize 'Superman' and $b$ 'Clark Kent'). For such instances, the idea that a proper name is a rigid designator (and so denotes the same object in every possible world) which lies at the hart of NI, does not seem to be appealing as, intuitively, there is nothing for such names to designate whether rigidly or not.

We shall next look at constructions related to (13) and (14), in order to obtain a sharper contrast. First, consider the following counterfactuals:

(17) If Hesperus were Phosphorus, Hesperus would be Hesperus.

(18) If Hesperus were Hesperus, Hesperus would be Phosphorus.

We symbolize (17) and (18) as $a \doteqdot b \supset_c a \doteqdot a$ and $a \doteqdot a \supset_c a \doteqdot b$, respectively. Let the IFC-system be like that in the analysis of (13) and (14), but with $F\Gamma = \{Fa\}$, $a\Gamma = \{Fa\}$, and $b\Gamma = \{Fb\}$.

(19)
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\wr a \doteqdot b \wr]^{(1)} \quad [Fa]^{(2_1)}}{Fb}\,(\doteqdot\mathrm{E})}{F\Gamma}\,(as\mathrm{E}_0) \qquad a\Gamma}{Fa}\,(as\mathrm{I}) \qquad \dfrac{\dfrac{[Fa]^{(2_2)}}{F\Gamma}\,(as\mathrm{E}_0) \qquad a\Gamma}{Fa}\,(as\mathrm{I})}{a \doteqdot a}\,(\doteqdot\mathrm{I}),\, 2_1,\, 2_2}{a \doteqdot b \supset_c a \doteqdot a}\,(\supset_c\mathrm{I}),\, 1$$

(20)

$$\dfrac{\dfrac{\dfrac{\dfrac{[\wr a \doteqdot a \wr]^{(1)}\,\mathit{illeg.} \qquad [Fa]^{(2_1)}}{Fa}\,(\doteqdot\mathrm{E})}{F\Gamma}\,(as\mathrm{E}_0) \qquad b\Gamma}{Fb}\,(as\mathrm{I})\ \mathit{illeg.} \qquad \dfrac{\dfrac{[Fb]^{(2_2)}}{F\Gamma}\,(as\mathrm{E}_0) \qquad a\Gamma}{Fa}\,(as\mathrm{I})}{\dfrac{a \doteqdot b}{a \doteqdot a \supset_c a \doteqdot b}\,(\supset_c\mathrm{I}),\ 1}\,(\doteqdot\mathrm{I}),\ 2_1,\ 2_2$$

In the present IFC-system, we can neither make a factual assumption of $a \doteqdot b$ nor can we derive that formula by means of a canonical $\mathcal{S}^{\doteqdot}$-derivation. In (20), $a \doteqdot a$ cannot be assumed in the counterfactual mode given (6), and $Fb$ cannot be derived by means of $as$I given that $Fb \notin F\Gamma$.

Now, consider the following two factuals:

(21) Since Hesperus is Hesperus, Hesperus is Phosphorus.

(22) Since Hesperus is Phosphorus, Hesperus is Hesperus.

We symbolize (21) as $a \doteqdot a \supset_f a \doteqdot b$ and (22) as $a \doteqdot b \supset_f a \doteqdot a$. Let the IFC-system be exactly like that in the discussion of (13) and (14).

(23)

$$\dfrac{\dfrac{\dfrac{\dfrac{[|a \doteqdot a|]^{(1)} \qquad [Fa]^{(2_1)}}{Fa}\,(\doteqdot\mathrm{E})}{F\Gamma}\,(as\mathrm{E}_0) \qquad b\Gamma}{Fb}\,(as\mathrm{I}) \qquad \dfrac{\dfrac{[Fb]^{(2_2)}}{F\Gamma}\,(as\mathrm{E}_0) \qquad a\Gamma}{Fa}\,(as\mathrm{I})}{\dfrac{a \doteqdot b}{a \doteqdot a \supset_f a \doteqdot b}\,(\supset_f\mathrm{I}),\ 1}\,(\doteqdot\mathrm{I}),\ 2_1,\ 2_2$$

(24)

$$\dfrac{\dfrac{\dfrac{\dfrac{[|a \doteqdot b|]^{(1)} \qquad [Fa]^{(2_1)}}{Fb}\,(\doteqdot\mathrm{E})}{F\Gamma}\,(as\mathrm{E}_0) \qquad a\Gamma}{Fa}\,(as\mathrm{I}) \qquad \dfrac{\dfrac{[Fa]^{(2_2)}}{F\Gamma}\,(as\mathrm{E}_0) \qquad a\Gamma}{Fa}\,(as\mathrm{I})}{\dfrac{a \doteqdot a}{a \doteqdot b \supset_f a \doteqdot a}\,(\supset_f\mathrm{I}),\ 1}\,(\doteqdot\mathrm{I}),\ 2_1,\ 2_2$$

None of the conclusions of (19), (20), (23), and (24) is a theorem.

## 5 Concluding remarks

We have defined rudimentary modal natural deduction systems for reasoning with relatively simple *would*-counterfactuals and causal *since*-subordinator sentences. The systems are motivated by inferential practice. They allow for different modes of making assumptions relative to their reference proof systems which serve to determine the factuality status of the formulae that are to be assumed.

Normalization and the subexpression/subformula property have been established for these systems along largely familiar lines. Due to the subexpression/subformula result, the systems are fully analytic. Due to the normalization result, the systems admit a proof-theoretic semantics. The proposed proof-theoretic semantics is acceptable from an intuitionistic point of view, since it is defined in terms of canonical derivations. Moreover, it is semantically autarkic, since the modal natural deduction systems do not draw on a formal semantics of a different kind (e.g., by internalizing a possible worlds similarity semantics).

Some aspects of the proposal are philosophically significant. Due to an absence of a semantic ontology (e.g., possible or impossible worlds), neither metaphysical nor epistemological considerations concerning such entities are triggered. Furthermore, an approach to formal epistemology is supported, according to which we arrive at knowledge of counterfactuals and factuals by means of constructive derivation and proof.

It is hoped that useful and less rudimentary modal proof systems can be obtained along the lines suggested in this outline.

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