Paradoxes, Intuitionism, and Proof-Theoretic Semantics

Reinhard Kahle and Paulo Guilherme Santos

> Wenn wir einen mathematischen Beweis erst am Resultate auf seine Zulässigkeit prüfen können, so brauchen wir überhaupt keinen Beweis.
>
> David Hilbert[1]

Abstract In this note, we review paradoxes like Russell's, the Liar, and Curry's in the context of intuitionistic logic. One may observe that one cannot blame the underlying logic for the paradoxes, but has to take into account the particular concept formations. For proof-theoretic semantics, however, this comes with the challenge to block some forms of direct axiomatizations of the Liar. A proper answer to this challenge might be given by Schroeder-Heister's *definitional freedom*.

Key words: paradoxes, liar, intuitionism, proof-theoretic semantics, definitional freedom

1 Weyl on the Grelling–Nelson paradox

Kurt Grelling presented in 1908, in a paper published jointly with Leonard Nelson (Grelling and Nelsen, 1908), the now well-known paradox concerning whether or not the adjective "heterologic" is heterologic.

Reinhard Kahle
Carl Friedrich von Weizsäcker Center, University of Tübingen, Germany, and CMA, FCT, Universidade Nova de Lisboa, Portugal, e-mail: reinhard.kahle@uni-tuebingen.de

Paulo Guilherme Santos
Centro de Matemática e Aplicações, NOVA School of Science and Technology, Universidade Nova de Lisboa, Portugal, e-mail: pgd.santos@campus.fct.unl.pt

[1] Hilbert (1917, p. 135). English translation: "If we can verify the admissibility of a mathematical proof only at the result, we do not need any proof at all."

T. Piecha and K. F. Wehmeier (eds.), *Peter Schroeder-Heister on Proof-Theoretic Semantics*, Outstanding Contributions to Logic 29,
https://doi.org/10.1007/978-3-031-50981-0_12

Hermann Weyl discusses this paradox in some length in the opening section of his "predicative manifesto", *Das Kontinuum* (Weyl, 1918):[2]

> But anyone who forgets that a proposition with such a structure can be meaningless is in danger of becoming trapped in absurdity — as a famous "paradox," essentially due to Russell, shows. Let a word which signifies a property be called *autological* if this word itself possesses the property which it signifies; if it does not possess that property, let it be called *heterological*. For example, the German word "kurz" (meaning "short") is itself *kurz* (i.e., is itself short — for a word in the German language which consists of only four letters will without question have to be described as a short one); hence "kurz" is autological. The word "long," on the other hand, is not itself long and, so, is heterological. Now what about the word "heterological" itself? If it is autological, then it has the property which it expresses and, so, is heterological. If, on the other hand, it is heterological, then it does not have this property and, so, is autological. Formalism regards this as an insoluble contradiction; but in reality this is a matter of scholasticism of the worst sort: for the slightest consideration shows that absolutely no sense can be attached to the question of whether the word "heterological" is itself auto- or heterological.

Weyl's warning, that propositions can be meaningless, can be taken as an indication that one would have to renounce the *tertium non datur* for such propositions. In other words, a logic which admits the formulation of such propositions cannot be classical; one will have to allow "truth values" — if they still can be called this way — beyond or between "true" and "false".

Historically, the natural choice for such a logic appears to be the intuitionistic one, which is distinguished for leaving out the *tertium non datur* from classical logic.[3] In the following we see, however, that intuitionistic logic is not much help regarding the paradoxes.

2 Russell's paradox in an intuitionistic setting

Russell's paradox had a profound impact on the development of modern logic. On the one hand, it forced set theory to reconsider its formal base, resulting eventually in Zermelo's axiomatization which can be taken as the standard set-theoretical foundation today. On the other hand, it questioned the mathematical concept formation, including the very notion of mathematical proof, prompting Hilbert to conceive proof theory as a tool to investigate the foundations of mathematics.

Interestingly, a simple inspection of Russell's paradox shows that it does not depend on the *tertium non datur*; in other words, the proof that the allowance to define the "Russell set" $\{X \mid X \notin X\}$ leads to a contradiction is carried out by logical reasoning which is intuitionistically valid.[4] As far as we know, it was first put on

---

[2] The English citation is from Weyl (1987, p. 6f).

[3] Weyl joined Brouwer's intuitionism only shortly after the publication of *Das Kontinuum*. But his slight shift from predicativism to intuitionism does not affect the criticism stated in the paragraph above; rather to the contrary.

[4] See, for instance, Irvine and Deutsch (2016, §4). The first author learned this observation from Robert Lubarsky.

record by Neil Tennant (1982), see also footnote 6. He refers to Prawitz, who gives a proof which works in minimal logic (Prawitz, 1965, p. 95). At this point, however, Prawitz did not comment on the (weak) logical framework he is using.[5]

In consequence, this paradox (and, as we will see, others) cannot be resolved by just replacing the underlying classical logic by intuitionistic logic.[6]

3 The Liar and Curry's paradox

The Liar paradox — "This sentence is false." — is surely the oldest paradox in scientific history, and it appears to be obvious that it just contradicts the law of bivalance. In this case, the standard argument uses indeed classical logic, arguing by a case distinction on assuming that it is true or that it is false, both leading to contradictions.

Curry (1942) presented a paradox, now named after him, to simplify the *Kleene–Rosser paradox* which showed the inconsistency of a "λ-calculus logic" of Church. Curry points out that Kleene and Rosser had used the *Richard paradox*, while his argument is based on Russell or the Liar. In essence, Curry's paradox is based in the definability of sentences saying "This sentence implies φ" for any sentence φ. Only requiring some very simple rules for implication one can obtain, from the defined sentence, φ. As, in this way, every formula of the system is derivable, the system is inconsistent.

Apparently, this paradox does not even involve negation — but, we will argue in a minute that this is only apparent —, and therefore, it was taken to question our very intuition about implication. In particular, the reasoning used by Curry is clearly intuitionistically valid.

One can replace φ in Curry's sentence by the *falsum*, $\bot$, a propositional constant for a false sentence. The single instances follow then with the intuitionistically valid principle *ex falso quodlibet*. But taking into account that intuitionistic[7] negation $\neg\varphi$

[5] A more recent discussion of Prawitz and Tennant's treatment of Russell's paradox can be found in Schroeder-Heister and Tranchini (2017).

[6] See Tennant (1982, p. 268f.): "I shall always try to present my proof-theoretic considerations within *intuitionistic* logic. This should at least allay the suspicion that bivalence or excluded middle or some jaundiced relative is the source of contagion."

[7] See Troelstra and Schwichtenberg (2000, p. 3). Of course, in principle, also classical negation can be defined this way; but while, in intuitionistic logic, negation is directly *reduced* to $\bot$, in classical logic, we would have merely a syntactic variation, still requiring axioms or rules involving negated formulas as such — see, for instance, the difference of the absurdity rules $\bot_i$ and $\bot_c$ in Troelstra and Schwichtenberg (2000, p. 37).

can be defined as $\varphi \to \bot$, one observes that Curry's paradox (using $\bot$)[8] is essentially the Liar in intuitionistic terms.

"At the referee's suggestion" Löb (1955, Fn. 4) remarks that Curry's "paradox is derived without the word 'not' " (Löb, 1955, p. 117). Löb was carefully enough to speak only about the *word* "not". Beall and Murzi (2013, p. 144) rephrase it by saying that Curry's paradox "arises even in negation-free languages". But, as one can see from the definition of negation in an intuitionistic language without a primitive symbol for negation, the absence of the word does not imply necessarily the absence of the concept of negation. Even if Curry's paradox is still applicable in languages which would not even allow a definition by use of *falsum*, an implicit representation of negation will be present in any case.[9]

At the end — and as for Russell — revoking the bivalence is not enough to ban Liar-like paradoxes; rather, it appears that the paradoxes are independent of the underlying logic.

Concerning the given reading of intuitionistic negation we like to recall that a similar perspective was already given by Bernays (1979, p. 4) as a distinctive feature in comparison with classical logic:[10]

> As one knows, the use of the "tertium non datur" in relation to infinite sets, in particular in Arithmetic, was disputed by L. E. J. Brouwer, namely in the form or an opposition of the traditional logical principle of the excluded middle. Against this opposition is to say that it is just based on a reinterpretation of the negation. Brouwer avoids the usual negation non-A, and takes instead "A is absurd". It is then obvious that the general alternative "Every sentence A is true or absurd" is not justified.

4 Intuitionism

The fact that changing from classical to intuitionistic logic does not resolve the paradoxes, neither Russell's nor the Liar, leads to the conclusion that one cannot hold

[8] The original version of Curry, allowing arbitrary formulas in the consequence is, in some sense, more general, as it involves as a particular case "Löb's Theorem" (which, however, is not really a paradox any longer, at least not in the sense that it leads to a contradiction). The relation between Curry's paradox and Löb's Theorem is an interesting issue in itself; see, for instance, Ruitenburg (1991).

[9] Therefore, it seems to be misleading to say, as Benthem (1978, p. 49), that "Curry's paradox shows that negation is not essential in this connection"; it is just the negation *symbol* which appears not to be essential.

[10] German original: "Wie man weiß, ist die Verwendung des ‚tertium non datur' in bezug auf unendliche Gesamtheiten, insbesondere schon in der Arithmetik, von L. E. J. Brouwer angefochten worden, und zwar in der Form einer Opposition gegen das traditionelle logische Prinzip vom ausgeschlossenen Dritten. Gegenüber dieser Opposition ist zu bemerken, daß sie ja auf einer Umdeutung der Negation beruht. Brouwer vermeidet die übliche Negation nicht-A, und nimmt stattdessen ‚A ist absurd'. Es ist dann klar, daß eine allgemeine Alternative ‚Jede Aussage A ist wahr oder ist absurd' nicht berechtigt ist."

the logic responsible for them.[11] Thus, one should look for the concept formations involved in the paradoxes when searching for solutions.

In standard semantics, one takes care of concept formations by careful choices of interpretations. This involves either a staunch platonistic insight in the interpretation or, at least, a firm confidence in set-theoretic constructions for them.[12]

Zermelo (1908) gave the classical example of *concept formation* when he axiomatized set theory with the explicit aim to ban the paradoxes:

> Under these circumstances there is this point nothing left for us to do but [. . .] to seek out the principles requried for establishing the foundations of this mathematical discipline. In solving the problem we must, on the one hand, restrict these principles sufficiently to exclude all contradictions and, on the other, take them sufficiently wide to retain all that is valuable in this theory.[13]

The principles of set theory, of course, serve as (implicit) definitions of the set-theoretical concepts. In practice, Zermelo's set theory fully satisfies the needs of the mathematicians. But Poincaré was not convinced (cited according to Gray, 2013, p. 540): "But even though he has closed his sheepfold carefully, I am not sure that he has not set the wolf to mind the sheep." Thus, without a consistency proof for the axiomatized set theory, the situation remains unsatisfactory from a philosophical point of view.

Interestingly, also Brouwer can cope with the problem, insofar one puts the concept formation ahead of the logic. This is in line with his idea that mathematics goes ahead of logic.[14] For Russell's paradox, one may note that Brouwer clearly rejected Cantorian set theory as such and abstract set formation principles are plainly anti-intuitionistic. In the same way, formalizations of the Liar and Curry's paradox depend on self-referential features of formal languages — to be implemented by some kind of Gödelization. But such formal languages are not the subject of intuitionism. In this perspective, the paradoxes may even support Brouwer's anti-logical convictions.

This perspective also vindicates Weyl (1987, p. 5), who was using his criticism of (the scholasticism around) the Grelling–Nelson paradox, not to advocate a many-valued logic, but rather to demand a careful delimitation of the "categories" to which a meaningful proposition is affiliated.

Here is not the place to evaluate the success of intuitionism to provide convincing techniques for concept formation. Brouwer coined the name with reference to the

[11] Here, we are not going into the attempts to mutilate further the logical framework (as, for instance, by questioning *modus ponens*); nor do we discuss informal notions of provability (Weaver, 2012) or validity versions of the Liar (Beall and Murzi, 2013) which are sometimes used to clarify the situation.

[12] See, for instance, Feferman (2000, p. 72).

[13] Zermelo (1967, p. 200). German orginial (Zermelo, 1908, p. 261): "Unter diesen Umständen bleibt gegenwärtig nichts anderes übrig, als [. . .] die Prinzipien aufzusuchen, welche zur Begründung dieser mathematischen Disziplin erforderlich sind. Diese Aufgabe muß in der Weise gelöst werden, daß man die Prinzipien einmal eng genug einschränkt, um alle Widersprüche auszuschließen, gleichzeitig aber auch weit genug ausdehnt, um alles Wertvolle diese Lehre beizubehalten."

[14] See the third chapter of Brouwer's dissertation, reprinted in English translation in L. E. J. Brouwer (1975), which contains the theses: *Mathematics is independent of logic* and *Logic depends upon mathematics*.

intuition of mathematicians. The resulting risk of subjectivity was criticised by Lorenzen with respect to the rejection of the *tertium non datur*:[15] "Unfortunately, the explanation which Brouwer himself offers for this phenomenon is an esoteric issue: only one who listened the Master himself understands him." Anyhow, it is the supposed intuition which should save the intuitionist from contradictions, not the underlying logic.

To avoid misunderstandings, we have to stress that Brouwer's conception of intuitionism was by no means motivated by the paradoxes — quite contrary to Hilbert's motivations for his foundational research.[16] In fact, Brouwer did not comment on the paradoxes at all, except for a plain rejection of Cantorian or axiomatic set theory in his dissertation and in a paper of 1912; cf. L. E. J. Brouwer (1975, pp. 80ff. and 130ff.).

5 Proof-theoretic semantics

Proof-theoretic semantics "is based on the fundamental assumption that the central notion in terms of which meanings are assigned to certain expressions of our language, in particular to logical constants, is that of *proof* rather than *truth*. In this sense proof-theoretic semantics is *semantics in terms of proofs*" (Schroeder-Heister, 2016b). "Proof-theoretic semantics is intuitionistically biased" (Schroeder-Heister, 2016b, §3.5)[17]. As such, it is confronted with the paradoxes in the very same way as intuitionism itself; but, as we will see, there is an additional challenge.

In a first step, proof-theoretic semantics may follow the "solution" we attributed to Brouwer: turning to the particular concept formations. In the case of Russell's paradox, this means that one would have to provide a proof-theoretic semantics for the set formation principles (expecting that such a semantics blocks the possibility to introduce the "Russell set"). Such an approach was, in fact, already initiated by Hallnäs (2016). With respect to the Liar in its usual form, one needs a framework axiomatizing truth and providing some form of term representation of formulas (as *Gödelization*). There are plenty of truth theories around and giving them a proof-theoretic semantics can be subsumed under the "open problem" of *Proof-Theoretic Semantics Beyond Logic* addressed by Schroeder-Heister (2016a, §4).

However, there is another form of treating the liar in a formal theory, which constitute a genuine challenge to proof-theoretic semantics. One may axiomatize a self-contradicting atom R with $R \leftrightarrow \neg R$. Schroeder-Heister (2012a) introduced such an R in a sequent calculus by the following two rules:

[15] German original in Lorenzen (1960): "Unglücklicherweise ist die Erklärung, die Brouwer selbst für dieses Phänomen anbietet, eine esoterische Angelegenheit: nur, wer den Meister selber hörte, versteht ihn."

[16] For Hilbert's motivation, see Kahle (2006).

[17] "Most forms of proof-theoretic semantics are intuitionistic in spirit, which means in particular that principles of classical logic such as the law of excluded middle or the double negation law are rejected or at least considered problematic." (Schroeder-Heister, 2016b, §1.2)

Paradoxical rules

$$\frac{\Gamma, \neg R \ \vdash \ C}{\Gamma, R \ \vdash \ C}\,(R \vdash) \qquad\qquad \frac{\Gamma \ \vdash \ \neg R}{\Gamma \ \vdash \ R}\,(\vdash R)$$

In a fine analysis of the contradiction, which one can derive from these rules (together with the usual structural rules and rules for negation), Schroeder-Heister shows that the contradiction can be blocked by imposing some restriction on each of the structures rules of Identity, Contraction, or Cut — restrictions which would not harm " 'ordinary' mathematical reasoning" (Schroeder-Heister 2012a; 2016c). From the point of proof-theoretic semantics, however, we don't see that such restrictions — in fact, any switch to sub-structural logics — is justifiable, as one would not like to dismiss the usual structural rules in other contexts.

Substructural logics (Došen and Schroeder-Heister, 1993) do not intended to mutilate logics (see footnote 11), but they rather serve as tactical modifications of standard logic to obtain a fine-grained analysis of the interplay of different strutural operations. Some of the substructural logics turned out to be of interest in special applications, as the Lambek calculus (Lambek, 1958) in linguistics and Linear Logic (Girard, 1987) for resource aware reasoning. The "sub"structural character of the latter is manifest in the possibility to reestablish classical reasoning by use of the bang operator. Intuitionistic logic, however, when considered as a substructural logic in the form of mono-succedent sequent calculus has further claims. Brouwer and Weyl were aiming, indeed, to replace classical reasoning in Mathematics. And they invoked sofisticated philosophical arguments — although these arguments were dismissed (or ignored) by the mathematical community at large. But they neither addressed the paradoxes nor were concerned with technical properties of calculi.

If, thus, classical reasoning should not be dismissed in general, proof-theoretic semantics should provide an argument to invalidate directly the mentioned *paradoxical rules*. We are facing here a "*tonk*-like phenomenon",[18] and as such it is discussed in Tranchini (2016). As for *tonk*, the sheer definition of $(R \vdash)$ and $(\vdash R)$ would spoil our calculus. To deal with *tonk*, *proof-theoretic harmony* was conceived as a possible solution.[19] But the two paradoxical rules appear to be in perfect harmony; and this was already observed by Read (2010, §7).

Next to harmony — as the "first principle" of proof-theoretic semantics —, we can consider a "second principle": *normalizability of proofs*. In fact, Tennant (1982) used

[18] The "tonk" connective was introduced by Prior (1960) by the following two rules:

$$\frac{A}{A \text{ tonk } B}\,(\text{tonk-I}) \qquad\qquad \frac{A \text{ tonk } B}{B}\,(\text{tonk-E})$$

It is widely discussed in the literature and, for our context, we may refer to Read (2010) or Tranchini (2016) for further information.

[19] This concept is due to Dummett (1981; 1991); for a recent discussion see, for instance, Tranchini (2021).

normalizability[20] exactly to block the paradoxical rules under discussion.[21] This is today one of the main directions proof-theoretic semantics is following and Tranchini (2015; 2016) proposed, quite convincingly, a combined treatment of *tonk* and the paradoxical rules in this vein by combining harmony and normalizability.

The problem with the "second principle" is that it is *global* and not any longer *local*; i.e., we cannot assign a proof-theoretic meaning to the connectives by solely inspect the given rules, but we have to prove properties of derivability in general.[22] In the last consequence, we are confronted with Hilbert's concern as expressed in the citation at the beginning of the paper: the admissibility of a proof can only be verified *a posteriori*.[23]

But even using normalizability as a proof-theoretic principle, we have no superordinate philosophical argument that this blocks *all* potential contradictions; we just verified empirically that it will block the liar.[24] There is an important lesson to learn from Martin-Löf's first type theory (Martin-Löf, 1971). It was conceived in a way that it was not subject to a direct liar-like contradiction; only a much more subtle reasoning, expressed in *Girard's paradox* showed its inconsistency (Coquand, 1986; Hurkens, 1995).[25] Thus, global proof-theoretic conditions which block "one or another" paradox might be far from being sufficient to convince one from the consistency of a system as a whole.

With reference to Hallnäs (1991; 2006), Schroeder-Heister proposes a possible solution: *Definitional Freedom* (Schroeder-Heister 2012b, §2 and 2016a, p. 276). Under this freedom, one does not forbid any rules, but has to single out the "well-behaved" ones by (*a posteriori*) mathematical arguments. Qualitatively, this was already done by Gödel, when he formalized the sentence "I am not provable in T" in an arithmetical (consistent) theory T.[26] The definition is perfectly fine, but the

[20] Ekman (2016, p. 212) observed that non-normalizability can be related to some form of "overloading" of (the use of) propositions: "A *self-contradictory argument* is, informally, an argument [. . .] in which there is a proposition which is used in two or more ways such that not all of the ways of using the proposition are compatible."

[21] For critical evaluations of Tennant's approach see, for instance, Schroeder-Heister and Tranchini (2017) and Petrolo and Pistone (2019).

[22] See also *Local and Global Proof-Theoretic Semantics* (Schroeder-Heister, 2016a, §2.4).

[23] In principle, normalizability could be proven, for a given set of axioms, before performing the single proofs. But there are two problems: first, normalizability will be, in general, an undecidable property; second, proof-theoretic semantics would depend on such a (meta-)proof of normalizability, which is rather delicate with respect to the philosophical claim of proof-theoretic semantics.

[24] More exactly: the liar and some other known paradoxes; see Tennant (1982). Tennant is well aware of the limitations of his approach: "I fully realise how inadequate any supposedly final word on this matter [the paradoxes] would be." (Tennant, 1982, p. 278). Thus, we are still in need for a *philosophical* argument that normalizability is more than an "ad hoc reply" to the known paradoxes.

[25] Admittedly, Girard's paradox will not be detected by a "slightest consideration"; thus, the situation is far more complex than Weyl might have judged it in 1917.

[26] Gödel explicitely refers to the paradoxes as heuristic motivation: "The analogy between this result and Richard's antinomy leaps to the eye; there is also a close relationship with the 'liar' antinomy"; in a footnote he continues: "Every epistemological antinomy can likewise be used for a similar undecidability proof" Gödel (1931, p. 175, translated); see also Lethen (2021).

provability predicate of T turns out to be incomplete, i.e., not every sentence is either provable or refutable in terms of such a formal predicate.[27]

The situation is also exemplified in recursion theory, where one does not like to forbid the definition of partial functions, but rather likes to single out, *a posteriori*, the functions which are total (or the domain of a partial function).[28] For recursion theory, there are adequate formal systems to incorporate the reasoning about partiality, namely *free logics* (Bencivenga, 2002) or the *logic of partial terms* (Beeson, 1985). For logical calculi, there exist a largely forgotten attempt by Behmann (1959)[29], but a modern worked out formalism is still a desideratum. Incorporating the reasoning about the "well-behaviour" of definitions would, in fact, vindicate both, Weyl and Hilbert: for Weyl, the slightest (or not so slight) consideration about the sense of a definition would turn explicit; for Hilbert, the admissibility of (the concepts used in) a proof would be checked not only at the result, but — if not globally, which we cannot expect any longer in view of undecidability phenomena — at least locally, for every proof in advance.

In fact, it is one of the features of the paradoxes that they work with *locally correct reasoning*.[30] According to our analysis, the paradoxes are not phenomena of the underlying logic, but of the concept formations; therefore, a proper treatment has to take into account the reasoning about these concepts. For proof-theoretic semantics such a reasoning should be a part of the game, and we agree with Schroeder-Heister (2012b, p. 78) to allow such reasoning within the formal frameworks:

> We strongly propose definitional freedom in the sense that there should be one or several formats for definitions, but within this format one should be free. Whether a certain definition is well-behaved is a matter of (mathematical) 'observation', and not something to be guaranteed from the very beginning.

References

Beall, J. C. and J. Murzi (2013). Two flavors of Curry's paradox. *Journal of Philosophy* 110, 143–165.

Beeson, M. (1985). *Foundations of Constructive Mathematics*. Ergebnisse der Mathematik und ihrer Grenzgebiete; 3. Folge, Band 6. Springer.

[27] Apparently, it was Russell, who didn't get the point here, when he argues as if Gödel had shown an inconsistency in mathematics in a letter to Leon Henkin of 1 April 1963, cited in Dawson, Jr. (1988, p. 89ff).

[28] The common underlying reason for partiality in recursion theory and incompleteness in arithmetic is, of course, *diagonalization*. It is the source of the very most paradoxes, and *definitional freedom* requires to deal positively with it, rather than forbitting it. Discussions of such positive handling in Mathematics can be found, for instance, in Sommaruga-Rosolemos (1991) and Santos (2020).

[29] See also Thiel (2019).

[30] Schroeder-Heister (2016a, §4.2) remarked this in the context of clausal definitions: "This connects the proof theory of clausal definitions with theories of paradoxes, which conceive paradoxes as based on locally correct reasoning."

Behmann, H. (1959). Der Prädikatenkalkül mit limitierten Variablen. Grundlegung einer natürlichen exakten Logik. *Journal of Symbolic Logic* 24, 112–140.

Bencivenga, E. (2002). Free logics. In: *Handbook of Philosophical Logic*. Ed. by D. M. Gabbay and F. Guenthner. Vol. 5. Dordrecht: Springer, 147–196.

Benthem, J. F. A. K. van (1978). Four paradoxes. *Journal of Philosophical Logic* 7, 49–72.

Bernays, P. (1979). Bemerkungen zu LORENZEN's Stellungnahme in der Philosophie der Mathematik. In: *Konstruktionen versus Positionen, Vol. 1*. Ed. by K. Lorenz. Berlin: de Gruyter, 3–16.

Coquand, T. (1986). An analysis of Girard's paradox. In: *Proceedings Symposium on Logic in Computer Science*. IEEE Computer Society Press, 227–236.

Curry, H. (1942). The inconsistency of certain formal logics. *Journal of Symbolic Logic* 7, 115–117.

Dawson, Jr., J. W. (1988). The reception of Gödel's incompleteness theorems. In: *Gödel's Theorem in focus*. Ed. by S. G. Shanker. Routledge, 74–95.

Došen, K. and P. Schroeder-Heister (1993). *Substructural Logics*. Vol. 2. Studies in Logic and Computation. Oxford University Press.

Dummett, M. (1981). *Frege. Philosophy language*. 2nd ed. Duckworth.

– (1991). *The Logical Basis of Metaphysics*. Duckworth.

Ekman, J. (2016). Self-contradictory reasoning. In: *Advances in Proof-Theoretic Semantics*. Ed. by T. Piecha and P. Schroeder-Heister. Springer, 211–229.

Feferman, S. (2000). Does reductive proof theory have a viable rationale? *Erkenntnis* 53, 63–96.

Girard, J.-Y. (1987). Linear logic. *Theoretical Computer Science* 50, 1–102.

Gödel, K. (1931). Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I. *Monatshefte für Mathematik und Physik* 38, 173–198.

Gray, J. (2013). *Henri Poincaré*. Princeton University Press.

Grelling, K. and L. Nelsen (1908). Bemerkungen zu den Paradoxien von Russell und Burali-Forti. *Abhandlungen der Fries'schen Schule II*. reprinted in: Nelson, Leonard. Gesammelte Schriften III. Die kritische Methode in ihrer Bedeutung für die Wissenschaften. Hamburg: Felix Meiner Verlag, 1974, pp. 95–127, 301–334.

Hallnäs, L. (1991). Partial inductive definitions. *Theoretical Computer Science* 87, 115–142.

– (2006). On the proof-theoretic foundation of general definition theory. *Synthese* 148, 589–602.

– (2016). On the proof-theoretic foundations of set theory. In: *Advances in Proof-Theoretic Semantics*. Ed. by T. Piecha and P. Schroeder-Heister. Springer, 161–171.

Hilbert, D. (1917). *Mengenlehre*. Vorlesung Sommersemester 1917, Ausarbeitung (Bibliothek des Mathematischen Instituts der Universität Göttingen).

Hurkens, A. J. C. (1995). A simplification of Girard's paradox. In: *Typed Lambda Calculi and Applications: Second International Conference on Typed Lambda Calculi and Applications, TLCA '95 Edinburgh, United Kingdom, April 10–12, 1995 Proceedings*. Ed. by M. Dezani-Ciancaglini and G. Plotkin. Springer, 266–278.

Irvine, A. D. and H. Deutsch (2016). Russell's paradox. In: *The Stanford Encyclopedia of Philosophy*. Ed. by E. N. Zalta. Winter 2016 edition. Metaphysics Research Lab, Stanford University.

Kahle, R. (2006). *David Hilbert über Paradoxien*. Pré-Publicações. Preprint Number 06-17. Departamento de Matemática, Universidade de Coimbra.

L. E. J. Brouwer (1975). *Collected Works*. Vol. 1: Philosophy and Foundations of Mathematics. Edited by A. Heyting. North-Holland.

Lambek, J. (1958). The mathematics of sentence structure. *American Mathematical Monthly* 65, 154–169.

Lethen, T. (2021). Kurt Gödel on logical, theological, and physical antinomies. *The Bulletin of Symbolic Logic* 27, 267–297.

Löb, M. (1955). Solution of a problem of Leon Henkin. *Journal of Symbolic Logic* 20, 115–118.

Martin-Löf, P. (1971). *A Theory of Types*. Tech. rep. Department of Mathematics, University of Stockholm.

Petrolo, M. and P. Pistone (2019). On Paradoxes in Normal Form. *Topoi* 38, 605–617.

Prawitz, D. (1965). *Natural Deduction*. Vol. 3. Acta Universitatis Stockholmiensis, Stockholm Studies in Philosophy. Almqvist & Wiksell.

Prior, A. N. (1960). The runabout inference-ticket. *Analysis* 21, 38–39.

Read, S. (2010). General-elimination harmony and the meaning of the logical constants. *Journal of Philosophical Logic* 39, 557–576.

Ruitenburg, W. (1991). Constructive logic and the paradoxes. *Modern Logic* 1, 271–301.

Santos, P. G. (2020). *Diagonalization in Formal Mathematics*. BestMasters. Springer.

Schroeder-Heister, P. (2012a). Paradoxes and structural rules. In: *Insolubles and Consequences: Essays in honour of Stephen Read*. Ed. by C. D. Novaes and O. T. Hjortland. College Publications, 203–211.

– (2012b). Proof-theoretic semantics, self-contradiction, and the format of deductive reasoning. *Topoi* 31, 77–85.

– (2016a). Open problems in proof-theoretic semantics. In: *Advances in Proof-Theoretic Semantics*. Ed. by T. Piecha and P. Schroeder-Heister. Springer, 253–283.

– (2016b). Proof-theoretic semantics. In: *The Stanford Encyclopedia of Philosophy*. Ed. by E. N. Zalta. Winter 2016 edition. Metaphysics Research Lab, Stanford University.

– (2016c). Restricting initial sequents: the trade-offs between identity, contraction and cut. In: *Advances in Proof Theory*. Ed. by R. Kahle, T. Strahm, and T. Studer. Springer, 339–351.

Sommaruga-Rosolemos, G. (1991). *Fixed point constructions in various theories of Mathematical Logic*. Bibliopolis.

Tennant, N. (1982). Proof and paradox. *Dialectica* 36, 265–296.

Thiel, C. (2019). Heinrich Behmanns Beitrag zur Grundlagendebatte. In: *Siegener Beiträge zur Geschichte und Philosophie der Mathematik*. Ed. by D. Koenig, G. Nickel, S. Shokrani, and R. Krömer. Vol. 11. Universitätsverlag Siegen, 191–202.

Tranchini, L. (2015). Harmonising harmony. *The Review of Symbolic Logic* 8, 41–423.

Tranchini, L. (2016). Proof-theoretic semantics, paradoxes and the distinction between sense and denotation. *Journal of Logic and Computation* 26, 495–512.

– (2021). Proof-theoretic harmony: towards an intensional account. *Synthese* 198 (Suppl 5), 1145–1176.

Troelstra, A. S. and H. Schwichtenberg (2000). *Basic Proof Theory*. 2nd ed. Cambridge University Press.

Weaver, N. (2012). Intuitionism and the liar paradox. *Annals of Pure and Applied Logic* 163, 1437–1445.

Weyl, H. (1918). *Das Kontinuum*. Veit.

– (1987). *The Continuum: A Critical Examination of the Foundation of Analysis*. Thomas Jefferson University Press. Corrected re-publication, Dover 1994. English translation of Weyl (1918).

Zermelo, E. (1908). Untersuchungen über die Grundlagen der Mathematik. I. *Mathematische Annalen*. English translation in Zermelo (1967), 261–281.

– (1967). Inverstigations in the foundations of set theory I. In: *From Frege to Gödel*. Ed. by J. van Heijenoort. Harvard University Press, 199–215.