7 Nonlinear Realization of Symmetry

In Parts I and II of the book, I reviewed the basic physics of *spontaneous symmetry breaking* (SSB) and the corresponding *Nambu–Goldstone* (NG) *bosons*. In order to keep the discussion simple, I relied mostly on general but elementary field-theoretic arguments. Where explicit results were necessary or useful, I resorted to simple models. However, one of the main virtues of SSB is universality: the low-energy physics is largely independent of the microscopic, short-distance details. This agrees with the general spirit of *effective field theory* (EFT); cf. Sect. 1.1. What SSB does for us is ensure a separation of resolution scales, which makes it possible to build a low-energy EFT solely using the NG degrees of freedom.

I will now set out on the main quest of this book: to develop an EFT framework for SSB that is based on symmetry alone. Thanks to the universality of SSB, the predictions of such a framework are guaranteed to match those of any microscopic theory with the same symmetry-breaking pattern. The construction of the EFT requires two main steps. The first step is to realize the action of the symmetry solely in terms of a set of NG fields. This problem is dealt with in the present chapter. The results will be used in the next chapter to construct effective actions for NG bosons.

The celebrated Wigner theorem of quantum mechanics states that any symmetry of a quantum system can be realized by a linear unitary or antilinear antiunitary operator on its Hilbert space (see Chap. 2 of [1] for a detailed discussion). As a consequence, states in the spectrum of a quantum system can be organized into multiplets, corresponding to irreducible representations of its symmetry group. In quantum field theory, the concept of linear realization of symmetry is naturally lifted from states to fields. In a perturbative setting, one usually has a one-to-one correspondence between one-particle states and (elementary) fields. Moreover, the fields span a linear space. One then expects the symmetry to act on fields linearly via some representation just like it does on states in the Hilbert space.

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Example 7.1

A field theory of a single complex scalar field ϕ may possess a U(1) symmetry under which the field transforms as $\phi \to e^{i\epsilon}\phi$, where ϵ is a real parameter. In this case, the field ϕ belongs to a complex one-dimensional representation of U(1). Similarly, the SO(n) "linear sigma model" includes an n-plet of real scalar fields ϕ^i, subject to symmetry transformations $\phi^i \to R^i{}_j\phi^j$, where $R \in \mathrm{SO}(n)$. In this case, the fields ϕ^i belong to the vector representation of SO(n).

As we saw in Sect. 5.3, however, a symmetry that is spontaneously broken is not necessarily realized by a set of unitary operators on the Hilbert space of states. Likewise, as illustrated in Chap. 2, it may be convenient to use field variables that do not belong to a linear representation of the symmetry. This may even become a necessity in the low-energy EFT where only the NG bosons of the broken symmetry are present; we simply do not have enough degrees of freedom to fill complete multiplets of the symmetry group.

Example 7.2

The complex scalar field of Example 7.1 can be represented by its real and imaginary parts, $\phi = \phi^1 + i\phi^2$. These span the vector representation of $\mathrm{SO}(2) \simeq \mathrm{U}(1)$. When the U(1) symmetry is spontaneously broken, it may however be more convenient to use the exponential parameterization of the field, $\phi = \varrho e^{i\theta}$, in terms of its modulus ϱ and phase θ. In the low-energy EFT, the modulus field is integrated out and the only remaining degree of freedom is the NG field θ. The latter transforms under $e^{i\epsilon} \in \mathrm{U}(1)$ as $\theta \to \theta + \epsilon$, which is not a linear representation. This is not merely a matter of bad choice of parameterization; the EFT contains a single real field, yet the group U(1) does not have any nontrivial real one-dimensional representations.

With the above observations in mind, I will develop in this chapter a formalism for nonlinear realization of symmetries. Mathematically, this amounts to generalizing the concept of a linear representation of a symmetry group to that of an *action* of the group. The space on which the group acts need not be linear itself; we can think of it as some manifold. Section 7.1 introduces the necessary mathematical terminology. The main argument, leading to a classification of possible nonlinear realizations of symmetry, is presented in Sect. 7.2. A reader interested mainly in ready-made results of the formalism may want to proceed directly to Sect. 7.3; this collects a number of practically useful formulas that I will refer to in the following chapters. Finally, Sect. 7.4 offers an alternative geometric viewpoint which illuminates some of the mathematical structure used to realize a symmetry nonlinearly. While most of the chapter is phrased in a rather elementary language, this last section relies on some concepts of differential geometry in an extent covered in Appendix A.

7.1 Group Action on Manifolds

To motivate the mathematical language that I need to introduce, suppose that we are given a theory of a set of fields, ψ^i, that possesses a symmetry group, G.[1] We would like to understand how to realize the symmetry group in terms of a set of transformations of the fields,

$$T_g : \psi^i \to \psi'^i \equiv \mathbb{F}^i(\psi, g) , \qquad g \in G , \tag{7.1}$$

where the functions $\mathbb{F}^i$ are assumed to be smooth in both of their arguments. The set of transformations T_g is constrained by the requirement that it respects the group structure of G. Thus, the unit element $e \in G$ must be represented by the identity map id, $\mathbb{F}^i(\psi, e) = \psi^i$. Consistency with group multiplication requires that

$$\mathbb{F}^i(\psi, g_1 g_2) = \mathbb{F}^i(\mathbb{F}(\psi, g_2), g_1) , \qquad g_1, g_2 \in G . \tag{7.2}$$

Finally, the transformation induced by the inverse of an element $g \in G$ has to satisfy

$$\mathbb{F}^i(\mathbb{F}(\psi, g), g^{-1}) = \psi^i , \qquad g \in G . \tag{7.3}$$

In the terminology introduced in Chap. 4, (7.1) is an example of a *point transformation* [2, 3]. The class of point transformations is clearly much broader than that of mere linear transformations, induced by a representation of G on the fields. It is therefore worthwhile to recall that even point transformations do not exhaust all conceivable, and physically relevant, realizations of symmetry. First, the field transformation may in principle depend explicitly on the spacetime coordinates. This feature can be included under the umbrella of point symmetries by treating fields and coordinates on the same footing; this will become relevant later when we talk about spacetime symmetries. Moreover, it is perfectly possible that the transformation of the fields also depends on their derivatives; this was dubbed *generalized local transformation* in Sect. 4.1. Such generalized symmetries play a minor role in this book, yet we will see some concrete examples in Chap. 10.

The restriction to point symmetries of the type (7.1), which I will make from now on unless explicitly stated otherwise, is a matter of practical convenience. Namely, it will allow us to disregard the fact that ψ^i actually are fields, that is functions of spacetime coordinates. Instead, I will treat them as independent variables that the group G acts upon. With this important technical assumption, we can now reformulate the problem of finding all possible realizations of a given symmetry group G on the fields ψ^i in geometric terms.

[1] Many of the concepts introduced below can be applied without change to any, even finite, group. However, I will always have implicitly in mind a connected Lie group, or the component of a Lie group connected to the unit element.

Consider a manifold $\mathcal{M}$ such that each point $x \in \mathcal{M}$ is uniquely specified by a set of values ψ^i. We can think of ψ^i as a set of (possibly only locally defined) coordinates on $\mathcal{M}$. An *action* of the group G on $\mathcal{M}$ is a set of smooth invertible maps $T_g : \mathcal{M} \to \mathcal{M}$ that satisfy the group constraints

$$T_e = \mathrm{id}\ , \qquad T_{g_1 g_2} = T_{g_1} \circ T_{g_2}\ , \qquad T_{g^{-1}} = (T_g)^{-1}\ , \qquad g, g_1, g_2 \in G\ . \tag{7.4}$$

In somewhat more abstract terms, the action of the group on the manifold is defined by a homomorphism from G to the group of diffeomorphisms on $\mathcal{M}$.

It is important to distinguish actual symmetry transformations on $\mathcal{M}$ from a mere change of coordinates. The action of the group G on $\mathcal{M}$ is defined geometrically by the maps T_g without reference to a particular set of coordinates ψ^i. Depending on the choice of coordinates, the same map T_g may correspond to different functions $\mathbb{F}^i$ as defined by (7.1). The freedom to choose coordinates on $\mathcal{M}$ mirrors the freedom to choose field variables in a given field theory. On the one hand, the independence of geometric properties of manifolds on the choice of local coordinates is a cornerstone of the language of differential geometry. On the other hand, it is an important result of quantum field theory that physical observables such as the S-matrix are invariant under (nearly) arbitrary field redefinitions [4, 5]. See [6] for a recent pedagogical discussion of this issue.

Let us now introduce some further terminology. For a given action of the group G on the manifold $\mathcal{M}$, the *orbit* of a point $x \in \mathcal{M}$ is the set of all points on $\mathcal{M}$ that can be reached from x by the action of some group element,

$$\mathcal{O}_x \equiv \{T_g x \mid g \in G\}\ . \tag{7.5}$$

The relation $y \sim x$ if and only if there is a $g \in G$ such that $y = T_g x$ is an equivalence. Orbits of the group G on the manifold $\mathcal{M}$ are the equivalence classes of this relation. As a consequence, $\mathcal{M}$ is a disjoint union of a (possibly infinite) set of orbits.

For a given point $x \in \mathcal{M}$, one defines its *isotropy group* (also called the *stabilizer* or the *little group* of x) as the subgroup of G that maps x to itself,

$$H_x \equiv \{h \in G \mid T_h x = x\}\ . \tag{7.6}$$

Two points lying on the same orbit of G have isomorphic isotropy groups. Indeed, if $y = T_g x$, then for any $h \in H_x$ we have $T_{ghg^{-1}} y = T_g T_h T_{g^{-1}} y = T_g T_h x = T_g x = y$. Conversely, it is easy to check that $T_h y = y$ implies $g^{-1} h g \in H_x$. Thus, H_x and

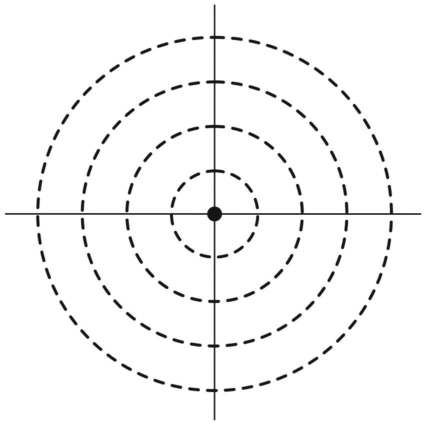

Fig. 7.1 Illustration of the action of the rotation group SO(2) on the Euclidean plane. Some orbits of SO(2) for which the isotropy group is trivial are displayed using *dashed lines*. The *black dot* indicates the single orbit whose isotropy group is the whole of SO(2)

H_y are conjugate as subgroups,

$$H_{T_g x} = g H_x g^{-1} \ . \tag{7.7}$$

Example 7.3

The rotation group SO(2) acts on the Euclidean plane by rotations around the origin, see Fig. 7.1 for an illustration. All points of the plane away from the origin have a trivial isotropy group. The corresponding orbits of SO(2) are circles centered at the origin. The origin itself forms an orbit with the isotropy group SO(2).

This example has a straightforward generalization to the action of SO(n) on the Euclidean space $\mathbb{R}^n$. There, the origin has the isotropy group SO(n). All other points $x \in \mathbb{R}^n$ have the isotropy group SO($n-1$), corresponding to ($n-1$)-dimensional rotations that leave the line connecting x to the origin fixed. The corresponding orbits are ($n-1$)-dimensional spheres centered at the origin.

In somewhat loose terms, we can say that all points lying on the same orbit of G on the manifold $\mathcal{M}$ have "the same" properties, since they can be related by a symmetry transformation. We thus expect manifolds consisting of a single orbit to have particularly simple geometric properties. In this case, the action of the group G on the manifold $\mathcal{M}$ is called *transitive*; any point on the manifold can be reached from any other point by the action of a suitable group element. A manifold equipped with a transitive action of a group is referred to as a *homogeneous space*.

Homogeneous spaces will play a central role throughout the rest of this book. One of the special properties of a homogeneous space is that its structure is completely determined by the group G and its subgroup H that specifies the isotropy group of the homogeneous space.[2] To see why, let us introduce a relation between two elements of G: $g_2 \sim g_1$ if and only if there is an $h \in H$ such that $g_2 = g_1 h$. This is an equivalence relation. As a consequence, the group G is partitioned into a disjoint union of the corresponding equivalence classes. The equivalence class

$$gH \equiv \{gh \,|\, h \in H\} , \qquad g \in G , \tag{7.8}$$

is called the *left coset* (or simply *coset*) of H in G. Incidentally, the cosets themselves can be viewed as orbits of a group action if we treat the group G as the manifold on which the subgroup H acts by multiplication from the right.

The quotient set G/H, called the *coset space*, is the set of all cosets of H in G. Despite the different appearance, this is just another mathematical realization of a homogeneous space. Indeed, consider a homogeneous space $\mathcal{M}$ along with a single point $x \in \mathcal{M}$ and its isotropy group H. It is easy to see that for any two elements $g_1, g_2 \in G$, $T_{g_1}x = T_{g_2}x$ if and only if $g_1 \sim g_2$, that is if g_1 and g_2 belong to the same coset. Hence there is a one-to-one correspondence between the elements of the coset space G/H and the points of the manifold $\mathcal{M}$.

Example 7.4

The Euclidean group ISO(n) consists of proper rotations and translations in $\mathbb{R}^n$ and their combinations. It obviously acts transitively on $\mathbb{R}^n$ for the simple reason that any point in $\mathbb{R}^n$ can be reached from any other point by a suitable translation. Each point in $\mathbb{R}^n$ has a its stabilizer a particular subgroup of ISO(n), consisting of SO(n) rotations around that point. Hence $\mathbb{R}^n$ equipped with the action of ISO(n) is a homogeneous space, equivalent to the coset space ISO(n)/SO(n). This view of the Euclidean space in terms of its symmetry group follows the "Erlangen program," put forward by Felix Klein in 1872.

Example 7.5

The rotation group SO($n+1$) acts naturally on the n-dimensional unit sphere S^n if one thinks of the latter as embedded in $\mathbb{R}^{n+1}$. We can however think of the sphere S^n in itself as our manifold $\mathcal{M}$. In that case, the action of SO($n+1$) becomes transitive; any point on the sphere can be reached from any other point by a suitable rotation. The isotropy group of any point on the sphere is a particular SO(n) subgroup of SO($n+1$). Hence, the sphere S^n equipped with the action of SO($n+1$) is a homogeneous space, equivalent to the coset space SO($n+1$)/SO(n).

[2] Since the isotropy group is the same up to conjugation for all points on the same orbit, and hence all points of the homogeneous space, we can drop the subscript x.

Having set up the necessary mathematical background, I can now give a concise formulation of the main goal of this chapter. In order to understand how a given symmetry group G can act on a given set of fields, we need to classify all possible actions of G on a given manifold $\mathcal{M}$. This will be accomplished in the following section. A reader seeking further details on the mathematical background reviewed above is recommended to consult Chap. 13 of [7].

7.2 Classification of Nonlinear Realizations

At first sight, the task to classify all actions of an arbitrary group G on an arbitrary manifold $\mathcal{M}$ seems hopeless. We would only know what to do in the case of a linear action on a vector space, where the problem boils down to the good old representation theory of Lie groups. How can we make use of this?

In this section, I will answer the above question, following the classic work of Coleman, Wess and Zumino [8]. The key step is to observe that the action of a group on a manifold can be *partially* linearized by a suitable choice of coordinates on the manifold. Let us choose a fixed point $x_0 \in \mathcal{M}$. We can always introduce a set of coordinates ψ^i in the neighborhood of x_0 such that x_0 maps to the origin, $\psi^i = 0$. Now any linear transformation leaves the origin intact. On the contrary, only the isotropy group H_{x_0} keeps x_0 fixed; all the other elements of G translate x_0 to some other point on the manifold. Thus, the best we can hope for is that we find a set of coordinates in which the action of H_{x_0}, not of the whole group G, becomes linear.

We will see that provided the isotropy group H_{x_0} is compact, one can indeed construct a set of local coordinates on $\mathcal{M}$ in the neighborhood of x_0 in which the action of H_{x_0} is linear. The good news is that this is in fact sufficient to classify all nonlinear realizations of the whole group G. The details of the argument are the subject of the following two subsections.

While this whole chapter is intended to develop a mathematical formalism that will prove invaluable later, it might be helpful for the reader already now to keep in mind the corresponding concepts pertinent to SSB, introduced in Chap. 5. The fixed point $x_0 \in \mathcal{M}$ corresponds to the selected vacuum state, or the associated order parameter. The subgroup H_{x_0} that leaves x_0 intact is analogous to the subgroup of unbroken symmetries. Finally, the coset space G/H_{x_0}, which will turn out to span a submanifold of $\mathcal{M}$, is analogous to the vacuum manifold. For the reader's convenience, this correspondence is summarized in Table 7.1.

Table 7.1 Correspondence between the mathematical terminology used in this chapter and the physics terminology introduced in Chap. 5

Mathematics terminology	Physics terminology
"Origin" $x_0 \in \mathcal{M}$	Vacuum state (order parameter)
Isotropy group H_{x_0}	Unbroken subgroup H
Coset space G/H_{x_0}	Vacuum manifold

7.2.1 Linearization of Group Action

To simplify the notation, I will from now on denote the functions $\mathbb{F}^i$ defined in (7.1) directly as ψ'^i whenever possible. Suppose that we have a set of local coordinates ψ^i on $\mathcal{M}$ in which the chosen point x_0 maps to zero. In these coordinates, the isotropy group H_{x_0} will be represented by some nonlinear functions of ψ^i whose Taylor expansion in ψ^i starts at the linear order,

$$\psi'^i(\psi, h) = D(h)^i{}_j \psi^j + \mathcal{O}(\psi^2) \ , \tag{7.9}$$

where $h \in H_{x_0}$ and $D(h)^i{}_j$ is a set of matrix coefficients. The conditions (7.2) and (7.3) require that the matrices $D(h)$ form a representation of H_{x_0}. We can then change the coordinates in the vicinity of x_0 to

$$\Psi^i(\psi) \equiv \int_{H_{x_0}} \mathrm{d}h\, D(h^{-1})^i{}_j \psi'^j(\psi, h) \ , \tag{7.10}$$

where dh is an invariant measure on H_{x_0} normalized so that the total volume of H_{x_0} is one.[3] The normalization ensures that near the origin, $\Psi^i = \psi^i + \mathcal{O}(\psi^2)$. Hence, Ψ^i are a well-defined set of coordinates in some neighborhood of the origin. In these new coordinates, the action of the isotropy group is linear. This can be seen upon a short manipulation,

$$\begin{aligned}\Psi'^i(\Psi, h') &= \int_{H_{x_0}} \mathrm{d}h\, D(h^{-1})^i{}_j \mathbb{F}^j(\mathbb{F}(\psi, h'), h) = \int_{H_{x_0}} \mathrm{d}h\, D(h^{-1})^i{}_j \mathbb{F}^j(\psi, hh') \\ &= D(h')^i{}_j \int_{H_{x_0}} \mathrm{d}(hh')\, [D(hh')^{-1}]^j{}_k \psi'^k(\psi, hh') = D(h')^i{}_j \Psi^j \ ,\end{aligned} \tag{7.11}$$

for any $h' \in H_{x_0}$, where I used respectively the group composition law (7.2), the invariance of the integration measure, the representation property of the matrices $D(h)$, and the definition (7.10) of the new coordinates. This concludes the proof of the statement that the action of the isotropy group H_{x_0} can be linearized by a suitable choice of coordinates.

In defining the new coordinates (7.10), I tacitly assumed that H_{x_0} has a finite volume so that the invariant measure on H_{x_0} in fact *can* be normalized to unity. This is a key step in the proof, which requires that the isotropy group H_{x_0} be

(continued)

[3] A reader not familiar with group integration may find more information in Chap. 3 of [9]. I will not dwell on details, since we shall not need group integration again in this book.

compact. The technique of nonlinear realization of symmetry developed in this chapter is often introduced straight away under the assumption that the whole group G is compact. That is, however, not necessary.

Equation (7.10) can serve as a useful practical tool to find explicitly the coordinates that linearize the action of the isotropy group. Let us have a look at a simple example, following [10].

Example 7.6

The action of SO(2) on the Euclidean plane, introduced in Example 7.3, can be recast in terms of an action of $\mathrm{U}(1) \simeq \mathrm{SO}(2)$ on the complex plane $\mathbb{C}$. Thus, under $\mathrm{e}^{\mathrm{i}\epsilon} \in \mathrm{U}(1)$, the complex coordinate z transforms as $z \to z' \equiv \mathrm{e}^{\mathrm{i}\epsilon} z$. This action is linear. One can however tweak it by changing the coordinate z to w such that $z = f(w)$, where f is a function analytic in the neighborhood of the origin of $\mathbb{C}$ such that $f(w) = w + \mathcal{O}(w^2)$. In the new coordinate w, the U(1) group acts via

$$w \to w' = f^{-1}(z') = f^{-1}(\mathrm{e}^{\mathrm{i}\epsilon} f(w)) \ . \tag{7.12}$$

In general, w' will be a nonlinear function of w. The original coordinate z, in which the action of U(1) is linear, can be reconstructed using (7.10). In case of U(1), the group integration in (7.10) amounts to averaging over the phase ϵ of the U(1) rotation. It can be written as an integral over a unit circle in the complex plane,

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\epsilon\, \mathrm{e}^{-\mathrm{i}\epsilon} f^{-1}(\mathrm{e}^{\mathrm{i}\epsilon} f(w)) = -\frac{\mathrm{i}}{2\pi}\oint \frac{\mathrm{d}c}{c^2} f^{-1}(cf(w)) \ , \tag{7.13}$$

where I introduced a new complex integration variable $c \equiv \mathrm{e}^{\mathrm{i}\epsilon}$. Given the assumptions I made on f, the function $f^{-1}(cf(w))/c^2$ of the complex variable c has a simple pole at the origin with the residue $f(w)$. It then follows at once from the residue theorem that the integral in (7.13) evaluates to $f(w) = z$, as expected.

Note that the coordinate z is not uniquely specified by the requirement that the action of U(1) is linear. We can for instance introduce a new variable w via

$$w = zf(z\bar{z}) \ , \tag{7.14}$$

where f is a smooth real function such that near the origin, $f(z\bar{z}) = 1 + \mathcal{O}(z\bar{z})$. Then w is a well-defined coordinate in some neighborhood of the origin of $\mathbb{C}$, upon which $\mathrm{e}^{\mathrm{i}\epsilon} \in \mathrm{U}(1)$ acts linearly as $w \to \mathrm{e}^{\mathrm{i}\epsilon} w$.

The above example suggests that the local coordinates Ψ^i in which the isotropy group H_{x_0} acts linearly are generally ambiguous. Namely, under the action of H_{x_0}, the manifold $\mathcal{M}$ splits into a disjoint union of orbits. The action of H_{x_0} will remain linear if we rescale the coordinates Ψ^i by an arbitrary H_{x_0}-invariant function on $\mathcal{M}$, that is a function which takes a constant value on any orbit of H_{x_0}. The only constraint is that such a rescaling leads to a well-defined set of new coordinates. The following example provides a nontrivial illustration of this ambiguity.

Example 7.7

Consider the action of $G \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$ on $\mathcal{M} \simeq \mathrm{SU}(2)$. As already hinted in Example 5.5 and explained in detail in Sect. 9.1, this is important for a low-energy EFT description of hadron physics. For a given $\mathcal{U} \in \mathcal{M}$ and a given element $(g_\mathrm{L}, g_\mathrm{R}) \in G$, the action is defined by

$$\mathcal{U} \to g_\mathrm{L} \mathcal{U} g_\mathrm{R}^{-1} \,. \tag{7.15}$$

The isotropy group of $\mathcal{U}_0 \equiv \mathbb{1}$ is the "diagonal" subgroup of G, $H_{\mathcal{U}_0} \simeq \mathrm{SU}(2)$, consisting of elements of the type (g, g), that is $g_\mathrm{L} = g_\mathrm{R} = g$. It is easy to guess a triplet of coordinates ψ^i, parameterizing $\mathcal{M}$ in the vicinity of $\mathcal{U}_0$, on which $H_{\mathcal{U}_0}$ acts linearly. Some common choices are

$$\mathcal{U} = \mathrm{e}^{\mathrm{i}\boldsymbol{\psi}\cdot\boldsymbol{\tau}} \,, \qquad \mathcal{U} = \frac{\mathbb{1} + \frac{\mathrm{i}}{2}\boldsymbol{\psi}\cdot\boldsymbol{\tau}}{\mathbb{1} - \frac{\mathrm{i}}{2}\boldsymbol{\psi}\cdot\boldsymbol{\tau}} \,, \qquad \mathcal{U} = \mathbb{1}\sqrt{1 - \boldsymbol{\psi}^2} + \mathrm{i}\boldsymbol{\psi}\cdot\boldsymbol{\tau} \,, \tag{7.16}$$

where $\boldsymbol{\tau}$ is the vector of Pauli matrices. All these choices coincide to linear order when expanded in powers of ψ^i, $\mathcal{U} = \mathbb{1} + \mathrm{i}\boldsymbol{\psi}\cdot\boldsymbol{\tau} + \mathcal{O}(\psi^2)$. All of them are mutually connected by coordinate redefinitions of the type $\psi'^i = \psi^i f(\boldsymbol{\psi}^2)$, where f is a suitably chosen function. The isotropy group $H_{\mathcal{U}_0}$ acts on $\mathcal{U}$ in all parameterizations shown in (7.16) via rotations of $\boldsymbol{\psi}$. Hence $\boldsymbol{\psi}^2$ is invariant under the action of $H_{\mathcal{U}_0}$ and any function $f(\boldsymbol{\psi}^2)$ is constant on the orbits of $H_{\mathcal{U}_0}$, as expected.

7.2.2 From Linear Representation to Nonlinear Realization

The ambiguity in the choice of coordinates that linearize the action of H_{x_0} around a chosen point $x_0 \in \mathcal{M}$ can be used to complete the classification of group actions. First of all, note that the set $\{T_g x_0 \,|\, g \in G\}$ defines a submanifold of $\mathcal{M}$. On this submanifold, G acts transitively; it is thus equivalent to the coset space G/H_{x_0}. We can choose a set of coordinates π^a, $a = 1, \dots, \dim G/H_{x_0}$, on it. Then we add another set of coordinates χ^ϱ, $\varrho = 1, \dots, \dim \mathcal{M} - \dim G/H_{x_0}$, so that (π^a, χ^ϱ) together is a well-defined coordinate system on $\mathcal{M}$ in the vicinity of x_0. In this coordinate system, the coset space G/H_{x_0} is embedded in $\mathcal{M}$ as the set of points $(\pi^a, 0)$.

As the next step, we subject the coordinates (π^a, χ^ϱ) to the linearization (7.10); with a slight abuse of notation, I will use the same symbols (π^a, χ^ϱ) for the resulting new coordinates. Importantly, the condition $\chi^\varrho = 0$ is preserved by the procedure. We thus end up with a set of coordinates (π^a, χ^ϱ) in which the isotropy group H_{x_0} is represented by linear transformations, and moreover the subset $(\pi^a, 0)$, parameterizing G/H_{x_0}, is invariant under the action of G. The latter implies that the representation of H_{x_0} on the coordinates (π^a, χ^ϱ) has an invariant subspace, that is, it is reducible. I now use once again the assumption that H_{x_0} is compact. This ensures that the representation of H_{x_0} is *completely* reducible. The action of H_{x_0} can then be brought to a block-diagonal form,

$$T_h(\pi, \chi) \equiv (\pi'^a, \chi'^\varrho) = (D^{(\pi)}(h)^a{}_b \pi^b, D^{(\chi)}(h)^\varrho{}_\sigma \chi^\sigma) , \qquad h \in H_{x_0} , \tag{7.17}$$

by orthogonalization that leaves the subspace $(\pi^a, 0)$ intact. The latter feature ensures that we can still use the coordinates π^a to parameterize the submanifold G/H_{x_0}.

A given point $x \in G/H_{x_0}$ with coordinates π^a can be reached from x_0 by the action of any element of G that lies in the coset of x. To proceed, we need to choose a concrete representative element of the coset. In line with the notation common in theoretical physics, I will denote this coset representative as $U(\pi)$. The concrete choice of the representative can be made fairly arbitrarily. There are however some natural requirements that will make our life easier:

- $U(\pi)$ should be a smooth function of π^a near the origin x_0.
- The origin x_0 itself, i.e. the trivial coset eH_{x_0}, should be represented by $U(0) = e$.
- The choice of $U(\pi)$ should reflect the linearity of the action of H_{x_0} in the coordinates π^a.

The first two constraints can always be satisfied. As to the third, the linearity of the action of H_{x_0} requires that $T_h x = T_h T_{U(\pi)} x_0 = T_{U(\pi')} x_0$ where π'^a are linear in π^a. Given that $T_h x = T_h T_{U(\pi)} T_{h^{-1}} x_0 = T_{hU(\pi)h^{-1}} x_0$, it is natural to pick $U(\pi)$ so that

$$U(\pi') = hU(\pi)h^{-1} . \tag{7.18}$$

We need to make sure, however, that this can be done consistently.

Consider the Lie algebra $\mathfrak{g}$ of G, which carries an adjoint action of H_{x_0},

$$Q \to hQh^{-1} , \qquad Q \in \mathfrak{g} , \quad h \in H_{x_0} . \tag{7.19}$$

This defines a linear representation of H_{x_0} which has an invariant subspace, namely the Lie algebra $\mathfrak{h}$ of H_{x_0}. Thus, the representation is reducible. Suppose now that it is, in fact, completely reducible. This is certainly the case when H_{x_0} is compact. More generally, coset spaces G/H_{x_0} for which the Lie algebra $\mathfrak{g}$ can be split as $\mathfrak{g} \simeq \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ where both $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are invariant subspaces under the adjoint action of H_{x_0} (7.19), are called *reductive*. For a reductive coset space, we can always choose $U(\pi)$ so that the linear action of H_{x_0} on G/H_{x_0} is realized by (7.18). We can for instance set $U(\pi) = \exp(\mathrm{i}\pi^a Q_a)$, where $Q_{a,b,\ldots}$ is a basis of $\mathfrak{g}/\mathfrak{h}$. This is however by far not the only choice, as illustrated by Example 7.7.

Example 7.8

There are many examples of reductive coset spaces for which the isotropy group H is not compact. One can for instance start with a compact H, for which the reductive property is guaranteed, and then switch to a related noncompact Lie group. For a concrete example, take as G the Poincaré group of isometries of D-dimensional Minkowski spacetime and as H its subgroup, the Lorentz group $\mathrm{SO}(d, 1)$. The subspace $\mathfrak{g}/\mathfrak{h}$ can be spanned on the generators of spacetime translations, which carry the vector representation of the Lorentz group. This coset space is a cousin of the Euclidean space $\mathbb{R}^D \simeq \mathrm{ISO}(D)/\mathrm{SO}(D)$, for which $H \simeq \mathrm{SO}(D)$ is compact.

It is easy to promote the action (7.18) of H_{x_0} on G/H_{x_0} to an action of the whole group G. Indeed, the action of any $g \in G$ on an element $x = T_{U(\pi)}x_0$ of G/H_{x_0} is completely characterized by the product $gU(\pi)$. The latter can be, at least in the vicinity of the unit element, uniquely decomposed as $U(\pi')h(\pi, g)$, where π'^a is defined by $T_g x = T_{U(\pi')}x_0$ and the factor $h(\pi, g) \in H_{x_0}$ ensures that the correct representative of the coset $T_g x$ is used.

In order to lift the action of G from the submanifold G/H_{x_0} to the whole of $\mathcal{M}$, we now make one last change of coordinates. Namely, we define new coordinates $\tilde{\pi}^a$ and $\tilde{\chi}^\varrho$ by the requirement that the point $(\pi^a, \chi^\varrho) \in \mathcal{M}$ can be expressed as

$$(\pi^a, \chi^\varrho) = T_{U(\tilde{\pi})}(0, \tilde{\chi}) \; . \tag{7.20}$$

This notation can be intuitively thought of as defining a slicing of $\mathcal{M}$ by orbits of G, starting from the subset of points $(0, \tilde{\chi}^\varrho)$. The submanifold G/H_{x_0} is a special case corresponding to $\tilde{\chi}^\varrho = 0$. In order to see that $\tilde{\pi}^a, \tilde{\chi}^\varrho$ in fact are well-defined coordinates on $\mathcal{M}$, note that (7.20) defines uniquely π^a, χ^ϱ for given $\tilde{\pi}^a, \tilde{\chi}^\varrho$. It follows from the fact that $U(0) = e$ that if $\tilde{\pi}^a = 0$, then $\pi^a = 0$ and $\chi^\varrho = \tilde{\chi}^\varrho$. Likewise, it follows from the definition of coordinates on the submanifold G/H_{x_0} that if $\tilde{\chi}^\varrho = 0$, then $\chi^\varrho = 0$ and $\pi^a = \tilde{\pi}^a$. Hence the Jacobian matrix $\partial(\pi, \chi)/\partial(\tilde{\pi}, \tilde{\chi})$ equals $\mathbb{1}$ at the origin, and (7.20) defines a valid coordinate system in some neighborhood of x_0.

The linearity of the action of H_{x_0} is preserved in the new coordinates since

$$T_h T_{U(\tilde{\pi})}(0,\tilde{\chi}) = T_{hU(\tilde{\pi})h^{-1}} T_h(0,\tilde{\chi}) = T_{hU(\tilde{\pi})h^{-1}}(0, D^{(\chi)}(h)\tilde{\chi}) \tag{7.21}$$

for any $h \in H_{x_0}$. It is now a matter of a short manipulation to see that the action of an arbitrary $g \in G$ on (7.20) is already completely fixed,

$$\begin{aligned} T_g T_{U(\tilde{\pi})}(0,\tilde{\chi}) &= T_{gU(\tilde{\pi})}(0,\tilde{\chi}) = T_{U(\tilde{\pi}')h(\tilde{\pi},g)}(0,\tilde{\chi}) = T_{U(\tilde{\pi}')} T_{h(\tilde{\pi},g)}(0,\tilde{\chi}) \\ &= T_{U(\tilde{\pi}')}(0, D^{(\chi)}(h(\tilde{\pi},g))\tilde{\chi}) \ . \end{aligned} \tag{7.22}$$

This is the end of the line. Equation (7.22) shows that in some neighborhood of a chosen point $x_0 \in \mathcal{M}$, the action of any Lie group G can by a change of coordinates be brought to a "standard form" such that (dropping the tildes)

$$U(\pi) \to U(\pi') = gU(\pi)h(\pi,g)^{-1} \ , \quad \chi^\varrho \to \chi'^\varrho = D^{(\chi)}(h(\pi,g))^\varrho{}_\sigma \chi^\sigma \ , \tag{7.23}$$

where $h(\pi, g) \in H_{x_0}$ and $D^{(\chi)}$ is a matrix representation of H_{x_0}. The only technical assumption that was required in the proof was that the isotropy group H_{x_0} is compact. This is the main result of the chapter. We can now harvest the fruits of our labors.

7.3 Standard Realization of Symmetry

Having in mind that the reader might have skipped the last, somewhat technical section, let me give here a brief but self-contained summary of its main result. Consider a manifold $\mathcal{M}$ equipped with an action of a group G, and let us choose a fixed point $x_0 \in \mathcal{M}$. Suppose that the isotropy group H_{x_0} of x_0 is compact. Then it is always possible to redefine coordinates in a neighborhood of x_0 so that the new, "standard" coordinates (π^a, χ^ϱ) have the following properties. First, the point x_0 itself corresponds to the origin (0, 0). The subset $(\pi^a, 0)$ spans a submanifold of $\mathcal{M}$, equivalent to the coset space G/H_{x_0}. Every point on the coset space can be uniquely characterized by a choice of a representative element $U(\pi)$ of the corresponding coset. The representative $U(\pi)$ can be chosen so that $U(0) = e$, and that the adjoint action of $h \in H_{x_0}$, $U(\pi) \to hU(\pi)h^{-1}$, defines a linear transformation of the coordinates π^a. The group G acts on the coset space via left multiplication, which defines implicitly an element $h(\pi, g)$ of H_{x_0} through

$$gU(\pi) = U(\pi'(\pi,g))h(\pi,g) \ , \qquad g \in G \ . \tag{7.24}$$

The last two properties of $U(\pi)$ imply that $h(\pi, g) = g$ for any $g \in H_{x_0}$.

The action of an element $g \in G$ on the whole manifold $\mathcal{M}$ is now defined in terms of the standard coordinates (π^a, χ^ϱ) as

$$\begin{aligned} U(\pi) &\xrightarrow{g} U(\pi'(\pi, g)) = gU(\pi)h(\pi, g)^{-1} , \\ \chi^\varrho &\xrightarrow{g} \chi'^\varrho(\chi, \pi, g) = D(h(\pi, g))^\varrho{}_\sigma \chi^\sigma , \end{aligned} \tag{7.25}$$

where the matrices $D(h)$ define some linear representation of H_{x_0}. Altogether, the action of G is fully specified by the choice of coset representative $U(\pi)$, which fixes the first line of (7.25), and the choice of representation $D(h)$ of H_{x_0}, which fixes the second line thereof.

The above construction of the standard realization of group action goes through without change even if H_{x_0} is noncompact provided the coset space G/H_{x_0} is reductive. In that case, however, the line of argument in Sect. 7.2 fails and it is no longer guaranteed that the standard realization is unique up to a coordinate redefinition. There may then be more mutually inequivalent nonlinear realizations of the group G on the manifold $\mathcal{M}$, of which the standard realization is but one example.

Finally, one may try to follow the same steps of the construction of the standard realization even when the coset space G/H_{x_0} is not reductive. Then, however, many of the simple features of the standard realization are lost. A concrete example of a nonreductive coset space is worked out in [11].

In the standard nonlinear realization (7.25), the coordinates π^a transform under G on their own, independently of χ^ϱ. The transformation of the latter, on the other hand, is nonlinear in π^a but linear in χ^ϱ themselves. It is therefore possible to set the χ^ϱs to zero consistently. This is not surprising. In the field theory language, the coordinates π^a correspond to NG bosons and their universal presence therefore mirrors the Goldstone theorem as reviewed in Chap. 6. The remaining coordinates on $\mathcal{M}$, χ^ϱ, represent other degrees of freedom that are not of NG nature. In the jargon of EFT, they are usually called *matter fields*. Among all the nonlinear realizations of the given symmetry group G with the given subgroup H, there is therefore a “minimal” nonlinear realization, defined on the coset space G/H, which only includes the NG degrees of freedom. On a general manifold $\mathcal{M}$, additional matter fields may be present. Throughout this book, I will focus mostly on minimal nonlinear realizations due to their significance for low-energy EFT description of broken symmetries.

Example 7.9

One can gain insight into the standard realization (7.25) of the action of G by looking at some special choices of the isotropy group. If there is a point $x_0 \in \mathcal{M}$ such that $H_{x_0} \simeq G$, then in its vicinity, we do not have any coordinates π^a. The coset space G/H_{x_0} consists of a single point that we can represent with $U = e$, which is consistent with the first line of (7.25) if $h(g) = g$ for all $g \in G$. All coordinates in the neighborhood of x_0 are of the χ^ϱ type. The second line of (7.25) guarantees that the action of G can be completely linearized, $\chi'^\varrho(\chi, g) = D(g)^\varrho{}_\sigma \chi^\sigma$. This is the usual textbook realization of symmetry via a linear representation.

The opposite extreme, $H_{x_0} \simeq \{e\}$, is more interesting. Here (7.25) reduces to

$$U(\pi) \xrightarrow{g} gU(\pi) , \qquad \chi^\varrho \xrightarrow{g} \chi^\varrho . \tag{7.26}$$

The coset space G/H_{x_0} corresponds to the group manifold G and carries an action of G defined by simple left multiplication. Whatever other coordinates χ^ϱ on $\mathcal{M}$ are present can always be chosen to be invariant under G. This is quite surprising. We are used to working with fields spanning linear multiplets of G; it is not obvious that the same physical content can be encoded in a set of fields that do not transform under G at all. The resolution of this apparent paradox lies in the freedom to choose coordinates at will. Namely, if we start with a set of fields Ψ^ϱ transforming under G as $\Psi^\varrho \xrightarrow{g} D(g)^\varrho{}_\sigma \Psi^\sigma$, we can make the redefinition $\chi^\varrho \equiv D(U(\pi)^{-1})^\varrho{}_\sigma \Psi^\sigma$. The new variables χ^ϱ are invariant under G in accord with (7.26).

7.3.1 Nonlinear Realization on Coset Spaces

Can we be more explicit about the way that the coordinates (π^a, χ^ϱ) transform under the action of G? The standard realization (7.25) of the group action requires the knowledge of the nonlinear functions $\pi'^a(\pi, g)$ and $h(\pi, g)$. We would like to be able to compute these, at least for small transformations, that is for $g \in G$ infinitesimally close to unity.[4] To that end, it is sufficient to consider the action of G on the coset space G/H; from now on will I drop the subscript x_0 on H unless it is needed to explicitly distinguish the isotropy groups of different points on the coset space. Once the minimal realization of G on G/H is found, it can be extended to any other manifold by specifying the linear representation $D(h)$ of H under which the additional coordinates χ^ϱ transform.

[4] I will only introduce the concept of a metric on a group, and more generally on a homogeneous space, in Sect. 7.4. Statements about infinitesimal distance of group elements should therefore be interpreted within a faithful matrix representation of the group using some standard matrix norm. The same remark applies whenever a sum or difference of group elements is considered below.

The general algorithm for calculation of the desired functions $\pi'^a(\pi, g)$ and $h(\pi, g)$ is as follows. Take the first line of (7.25), multiply it with $U(\pi)^{-1}$, and subtract $U(\pi)^{-1}U(\pi) = e$. This gives the master relation

$$U(\pi)^{-1}\delta U(\pi) = U(\pi)^{-1}gU(\pi)h(\pi, g)^{-1} - e , \tag{7.27}$$

where $\delta U(\pi) \equiv U(\pi') - U(\pi)$. For $g \in G$ that is infinitesimally close to unity, both $U(\pi)^{-1}gU(\pi)$ and $h(\pi, g)$ are infinitesimally close to unity as well. By a systematic comparison of the left- and right-hand sides of (7.27), one can then determine $\delta\pi^a(\pi, g) \equiv \pi'^a(\pi, g) - \pi^a$ as well as $h(\pi, g)$.

To make further progress, we first have to establish some notation. In order to be able to discuss different symmetry transformations from G separately, we choose a basis $Q_{A,B,\dots}$ of $\mathfrak{g}$. A subset of these spans a basis of the Lie subalgebra $\mathfrak{h}$ and will be denoted with Greek indices, $Q_{\alpha,\beta,\dots}$. The rest of the generators spans the subspace $\mathfrak{g}/\mathfrak{h}$ and will be denoted as $Q_{a,b,\dots}$. The structure constants of the Lie algebra $\mathfrak{g}$ will be called f^C_{AB} with a conventional factor of i in the commutation relations, that is, $[Q_A, Q_B] = \mathrm{i} f^C_{AB} Q_C$. The structure constant does not have to be fully antisymmetric in its three indices even if G is compact. It does have to be antisymmetric under the exchange of its lower two indices though. It also has to satisfy the Jacobi identity,

$$f^E_{AB} f^D_{EC} + f^E_{BC} f^D_{EA} + f^E_{CA} f^D_{EB} = 0 . \tag{7.28}$$

The basic commutator of the Lie algebra $\mathfrak{g}$, $[Q_A, Q_B] = \mathrm{i} f^C_{AB} Q_C$, can be unfolded into three separate conditions on the subsets of generators $Q_{\alpha,\beta,\dots}$ and $Q_{a,b,\dots}$,

$$\begin{aligned}
[Q_\alpha, Q_\beta] &= \mathrm{i} f^\gamma_{\alpha\beta} Q_\gamma , \\
[Q_\alpha, Q_b] &= \mathrm{i} f^c_{\alpha b} Q_c , \\
[Q_a, Q_b] &= \mathrm{i}(f^\gamma_{ab} Q_\gamma + f^c_{ab} Q_c) .
\end{aligned} \tag{7.29}$$

The first of these encodes the requirement that the generators $Q_{\alpha,\beta,\dots}$ span a closed Lie algebra (that is $f^c_{\alpha\beta} = 0$). The second of these likewise expresses the assumption that the coset space G/H is reductive (that is $f^\gamma_{\alpha b} = 0$).

Next, we are going to need two simple statements from linear algebra, which I reproduce here for the sake of completeness. The first of these is usually known under the name *Hadamard lemma*,

$$\mathrm{e}^A B \mathrm{e}^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots , \tag{7.30}$$

where A, B are arbitrary (square) matrices. The second identity, which to the best of my knowledge does not have an established name, reads

$$e^{-A}de^{A} = \int_0^1 dt\, e^{-tA}(dA)e^{tA} \; . \tag{7.31}$$

Here the symbol d acting on A and e^A can be thought of as a differential, but also as a derivative with respect to whatever parameter A might depend on. Once multiplied from the left with e^A, we can think of (7.31) as a continuous version of the Leibniz (product) rule, applied to e^A. It follows from (7.31) combined with the Hadamard lemma (7.30) that whenever A is a function with values in a Lie algebra, $e^{-A}de^{A}$ will take values in the same Lie algebra.

Equation (7.31) prepares the ground for the introduction of a concept of central importance for calculus on coset spaces: the *Maurer–Cartan* (MC) *form*,

$$\omega(\pi) \equiv -iU(\pi)^{-1}dU(\pi) \; . \tag{7.32}$$

For the time being, the reader may think of the d herein as an ordinary differential of a function. The true geometric significance of the MC form as a differential 1-form will become clear in Sect. 7.4. Recall that any element of a Lie group sufficiently close to unity, in our case $U(\pi)$, can be obtained as the exponential of an element of the corresponding Lie algebra. Equation (7.31) then implies that the MC form takes values in the Lie algebra $\mathfrak{g}$. We can split it into pieces that belong to $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$, and represent each of these in terms of their components in a chosen basis of generators,

$$\begin{aligned} &\omega \equiv \omega_{\parallel} + \omega_{\perp} \; , \\ &\omega \equiv \omega^A Q_A \; , \qquad \omega_{\parallel} \equiv \omega^{\alpha} Q_{\alpha} \; , \qquad \omega_{\perp} \equiv \omega^a Q_a \; . \end{aligned} \tag{7.33}$$

Finally, by writing $dU(\pi) = [\partial U(\pi)/\partial \pi^a]d\pi^a$, one can introduce explicit components of the MC form in a chosen set of local coordinates π^a on G/H, $\omega^A \equiv \omega^A_a d\pi^a$.

It is instructive to check how the MC form is affected by the action of G on the coset space. This follows directly from (7.25). It is a simple exercise to verify that for given $g \in G$,

$$\begin{aligned} &\omega_{\parallel}(\pi) \xrightarrow{g} \omega_{\parallel}(\pi'(\pi, g)) = h(\pi, g)\omega_{\parallel}(\pi)h(\pi, g)^{-1} - ih(\pi, g)dh(\pi, g)^{-1} \; , \\ &\omega_{\perp}(\pi) \xrightarrow{g} \omega_{\perp}(\pi'(\pi, g)) = h(\pi, g)\omega_{\perp}(\pi)h(\pi, g)^{-1} \; . \end{aligned} \tag{7.34}$$

Note how (7.31) guarantees that $\omega_{\parallel}(\pi')$ still takes values in the Lie algebra $\mathfrak{h}$. While not of direct relevance right here and now, the transformation rules (7.34) will help us understand the geometric meaning of the MC form in Sect. 7.4.

We still need a few last pieces of notation. Conjugation of elements of $\mathfrak{g}$ by $U(\pi)$ will be abbreviated as

$$U(\pi)^{-1}Q_A U(\pi) \equiv \nu_A^B(\pi)Q_B \;, \tag{7.35}$$

which defines a set of nonlinear functions $\nu_A^B(\pi)$ on the coset space. Finally, for the action of an element $g \in G$ infinitesimally close to unity, I will use the notation

$$\begin{aligned} g &\approx e + \mathrm{i}\epsilon^A Q_A \;, \\ \delta\pi^a(\pi, g) &\approx \epsilon^A \xi_A^a(\pi) \;, \qquad h(\pi, g) \approx e + \mathrm{i}\epsilon^A k_A^\alpha(\pi) Q_\alpha \;. \end{aligned} \tag{7.36}$$

The $\approx$ symbol indicates that I have expanded all the quantities to linear order in the transformation parameters ϵ^A, defined by the first line of (7.36). The second line thereof introduces notation for the infinitesimal versions of the functions $\pi'^a(\pi, g)$ and $h(\pi, g)$. Of particular interest are the functions $\xi_A^a(\pi)$ that realize the motion induced on the coset space G/H by the group G.

With all the notation at hand, we can now expand (7.27) to linear order in ϵ^A and compare coefficients of the various generators of $\mathfrak{g}$ on the left- and right-hand sides. This leads to the identities

$$\begin{aligned} \nu_A^\alpha(\pi) &= \xi_A^a(\pi)\omega_a^\alpha(\pi) + k_A^\alpha(\pi) \;, \\ \nu_A^a(\pi) &= \xi_A^b(\pi)\omega_b^a(\pi) \;. \end{aligned} \tag{7.37}$$

These are still valid for any choice of the coset representative $U(\pi)$. Once it is fixed, the functions $\omega_a^A(\pi)$ are determined by (7.32). Likewise, the functions $\nu_A^B(\pi)$ are fixed by (7.35). The identities (7.37) then constitute a set of linear equations for $\xi_A^a(\pi)$ and $k_A^\alpha(\pi)$. At the origin, $\nu_A^B(0) = \delta_A^B$ as a consequence of the fact that $U(0) = e$. The second line of (7.37) then implies that $\omega_b^a(0)$ is nonsingular. By continuity, it must remain nonsingular in some neighborhood of the origin. This guarantees that a solution of (7.37) for $\xi_A^a(\pi)$ and $k_A^\alpha(\pi)$ exists and it is unique.

I have now achieved the main goal of this subsection: to give an algorithm how, for a chosen set of coordinates π^a, to realize the action of the group G on the coset space G/H. None of the nonlinear functions involved—$\omega_a^A(\pi)$, $\nu_A^B(\pi)$, $\xi_A^a(\pi)$ and $k_A^\alpha(\pi)$—can however in general be evaluated in a closed form. For practical applications, it is useful to have explicit expressions for these functions, even if just as a series expansion in a specific set of coordinates π^a. One popular choice of parameterization for which this can easily be done is

$$U(\pi) = \exp(\mathrm{i}\pi^a Q_a) \;. \tag{7.38}$$

The Hadamard lemma (7.30) then tells us at once that

$$\nu_B^A(\pi) = \delta_B^A - f_{Ba}^A \pi^a + \frac{1}{2} f_{Ba}^C f_{Cb}^A \pi^a \pi^b + O(\pi^3) \;. \tag{7.39}$$

Likewise, it follows quickly from (7.31) that

$$\omega_a^A(\pi) = \delta_a^A - \frac{1}{2} f_{ab}^A \pi^b + \frac{1}{6} f_{ab}^B f_{Bc}^A \pi^b \pi^c + O(\pi^3) \; . \tag{7.40}$$

Finally, (7.37) can be solved iteratively for the remaining pieces,

$$\begin{aligned}
\xi_A^a(\pi) &= \delta_A^a - \left(f_{Ab}^a - \frac{1}{2} \delta_A^e f_{eb}^a \right) \pi^b \\
&\quad + \frac{1}{2} \left(f_{Ab}^\alpha f_{\alpha c}^a - \frac{1}{3} \delta_A^e f_{eb}^B f_{Bc}^a + \frac{1}{2} \delta_A^e f_{eb}^d f_{dc}^a \right) \pi^b \pi^c + O(\pi^3) \; , \\
k_A^\alpha(\pi) &= \delta_A^\alpha - \left(f_{Aa}^\alpha - \frac{1}{2} \delta_A^e f_{ea}^\alpha \right) \pi^a \\
&\quad + \frac{1}{2} \left(f_{Aa}^\beta f_{\beta b}^\alpha - \frac{1}{3} \delta_A^e f_{ea}^B f_{Bb}^\alpha + \frac{1}{2} \delta_A^e f_{ea}^d f_{db}^\alpha \right) \pi^a \pi^b + O(\pi^3) \; .
\end{aligned} \tag{7.41}$$

With these explicit expressions at hand, it is easy to illustrate some of the general properties of the standard nonlinear realization. For instance, the action of the isotropy group H reduces to the linear adjoint transformation of the coordinates, $\xi_\alpha^a(\pi) = -f_{\alpha b}^a \pi^b$. Likewise, we find that $k_\beta^\alpha(\pi) = \delta_\beta^\alpha$, which is an infinitesimal version of the relation $h(\pi, g) = g$ for any $g \in H$.

Example 7.10

All the structure introduced above takes a particularly simple form when the group G is Abelian. Then $\xi_A^a(\pi) = \delta_A^a$: the group G acts on the standard coordinates π^a by a mere set of shifts. The MC form reduces to $\omega(\pi) = \mathrm{d}\pi^a Q_a$. This suggests a simple interpretation of $\omega_\perp = \omega_b^a \mathrm{d}\pi^b Q_a$ in the general situation when G is non-Abelian. Namely, ω^a supplies us with a generalized derivative (or differential) of π^a which, as (7.34) shows, is covariant under the action of G.

7.3.2 Symmetric Coset Spaces

Up to some technical assumptions, we have already accomplished a complete classification of possible nonlinear realizations of symmetry on manifolds. In the language of field theory, this amounts to finding all possible point symmetry transformations of a given set of fields under a given symmetry group G. The price we had to pay for the generality of this result was the rather complicated transformation rule (7.25). This involves nonlinear functions on the coset space G/H that cannot be evaluated in a closed explicit form. There is, however, an important class of coset spaces for which we can do much better.

The coset space G/H is called *symmetric*, and the associated homogeneous space is called a *symmetric space*, if the Lie algebra $\mathfrak{g}$ admits an involutive automorphism under which $\mathfrak{h}$ is even and $\mathfrak{g}/\mathfrak{h}$ is odd. In other words, we require that there is a linear map $\mathcal{R} : \mathfrak{g} \to \mathfrak{g}$ such that

$$\begin{aligned} \mathcal{R}([Q_1, Q_2]) &= [\mathcal{R}(Q_1), \mathcal{R}(Q_2)] , & Q_1, Q_2 &\in \mathfrak{g} , \\ \mathcal{R}(Q) &= Q , & Q &\in \mathfrak{h} , \\ \mathcal{R}(Q) &= -Q , & Q &\in \mathfrak{g}/\mathfrak{h} . \end{aligned} \tag{7.42}$$

This property guarantees the vanishing of any structure constant with an odd number of $\mathfrak{g}/\mathfrak{h}$ indices. In particular, $f^{\gamma}_{\alpha b} = 0$: any symmetric coset space is automatically reductive. In addition, $f^{c}_{ab} = 0$, that is the last term in the commutation relations (7.29) is missing.

Example 7.11

The fundamental commutation relation of the SO(n) group reads

$$[J_{ij}, J_{kl}] = \mathrm{i}(\delta_{ik} J_{jl} + \delta_{jl} J_{ik} - \delta_{il} J_{jk} - \delta_{jk} J_{il}) , \tag{7.43}$$

where $i, j, k, l = 1, \dots, n$ and J_{ij} is the antisymmetric tensor of angular momentum in n spatial dimensions. The Lie algebra of SO(n) possesses an automorphism $\mathcal{R}$ under which $\mathcal{R}(J_{\alpha\beta}) = J_{\alpha\beta}$ and $\mathcal{R}(J_{\alpha n}) = -J_{\alpha n}$, where $\alpha, \beta = 1, \dots, n-1$. One can think of this automorphism geometrically as an inversion of the n-th coordinate axis. The coset space $\mathrm{SO}(n)/\mathrm{SO}(n-1) \simeq S^{n-1}$ is therefore symmetric.

In a similar vein, the Euclidean space $\mathbb{R}^n$ is a symmetric space. This can be seen most easily by recalling that $\mathbb{R}^n \simeq \mathrm{ISO}(n)/\mathrm{SO}(n)$. The desired automorphism $\mathcal{R}$ is the spatial inversion. Under this, all the generators of SO(n) (rotations) remain intact, whereas the remaining generators of ISO(n)/SO(n) (translations) change sign.

Example 7.12

Consider the "chiral" coset spaces of the type $G_{\mathrm{L}} \times G_{\mathrm{R}}/G_{\mathrm{V}}$, where all the three groups $G_{\mathrm{L}}, G_{\mathrm{R}}, G_{\mathrm{V}}$ are isomorphic to the same Lie group G; this is a generalization of Example 7.7 where $G \simeq \mathrm{SU}(2)$. The chiral group $G_{\mathrm{L}} \times G_{\mathrm{R}}$ consists of elements $(g_{\mathrm{L}}, g_{\mathrm{R}})$ where $g_{\mathrm{L}}, g_{\mathrm{R}} \in G$. The "vector" isotropy group G_{V} consists of elements of the type (g, g), that is $g_{\mathrm{L}} = g_{\mathrm{R}} = g$. The generators of the chiral group include two copies, $Q_{\mathrm{L},A}$ and $Q_{\mathrm{R},A}$, of the generators of G. The Lie algebra of the chiral group is defined in terms of the structure constants f^{C}_{AB} of G by

$$[Q_{\mathrm{L},A}, Q_{\mathrm{L},B}] = \mathrm{i} f^{C}_{AB} Q_{\mathrm{L},C} , \qquad [Q_{\mathrm{R},A}, Q_{\mathrm{R},B}] = \mathrm{i} f^{C}_{AB} Q_{\mathrm{R},C} , \\ [Q_{\mathrm{L},A}, Q_{\mathrm{R},B}] = 0 . \tag{7.44}$$

These commutation relations are invariant under the exchange of $Q_{\mathrm{L},A}$ and $Q_{\mathrm{R},A}$, which defines the desired automorphism,

$$\mathcal{R}(Q_{\mathrm{L},A}) = Q_{\mathrm{R},A} , \qquad \mathcal{R}(Q_{\mathrm{R},A}) = Q_{\mathrm{L},A} . \tag{7.45}$$

The generators of G_{V}, equal to $Q_{\mathrm{L},A} + Q_{\mathrm{R},A}$ up to overall normalization, are even under this automorphism. The generators of the complementary space $\mathfrak{g}/\mathfrak{h}$, which are to be odd under $\mathcal{R}$, can be taken as $Q_{\mathrm{L},A} - Q_{\mathrm{R},A}$ up to an overall factor.

The automorphism $\mathcal{R}$ of the Lie algebra $\mathfrak{g}$ can be lifted, at least locally near the unit element, to the Lie group G. It is then possible to choose the coset representative $U(\pi)$ so that $\mathcal{R}(U(\pi)) = U(\pi)^{-1}$; one can use for instance the exponential parameterization (7.38). We now take the first line of (7.25) and multiply it with the inverse of its image under $\mathcal{R}$. The result is a surprise: for symmetric coset spaces, there is a parameterization of G/H in which the whole group G is realized linearly,

$$\Sigma(\pi) \equiv U(\pi)^2 , \qquad \Sigma(\pi) \xrightarrow{g} \Sigma(\pi'(\pi, g)) = g\Sigma(\pi)\mathcal{R}(g)^{-1} . \tag{7.46}$$

This is of such utility that whenever one deals with a symmetric coset space, one almost always uses the linearly transforming variable Σ instead of working directly with the coordinates π^a. Note, however, that one may need the coordinates π^a if one wishes to extend the coset space G/H to a larger manifold $\mathcal{M}$. This is because the transformation of χ^{ϱ} in (7.25) requires the functions $h(\pi, g)$ that depend on π^a.

With the automorphism $\mathcal{R}$ at hand, one may easily project out the $\omega_{\parallel}$ and $\omega_{\perp}$ components of the MC form,

$$\omega_{\parallel} = \frac{1}{2}[\omega + \mathcal{R}(\omega)] , \qquad \omega_{\perp} = \frac{1}{2}[\omega - \mathcal{R}(\omega)] . \tag{7.47}$$

The latter has a practically convenient expression in terms of Σ,

$$\omega_{\perp} = -\frac{\mathrm{i}}{2} U^{-1} \mathrm{d}\Sigma U^{-1} = \frac{\mathrm{i}}{2} U \mathrm{d}\Sigma^{-1} U . \tag{7.48}$$

Example 7.13

Let us see how the variable $\Sigma(\pi)$ is realized on the chiral coset spaces $G_L \times G_R/G_V$ discussed in Example 7.12. Here we can choose the coset representative as $U = (u, u^{-1})$ where $u \in G$; this satisfies the requirement that under the automorphism $\mathcal{R}(g_L, g_R) = (g_R, g_L)$ of the chiral group $G_L \times G_R$, U is turned into its inverse. The general transformation rule as given by the first line of (7.25) then translates to

$$(u, u^{-1}) \xrightarrow{(g_L, g_R)} (g_L, g_R)(u, u^{-1})(h^{-1}, h^{-1}) , \tag{7.49}$$

where $h \in G$. The linearly transforming variable (7.46) is given by $\Sigma = U^2 = (u^2, u^{-2})$, where $u^2 \equiv \mathcal{U}$ transforms under $G_L \times G_R$ as $\mathcal{U} \to g_L \mathcal{U} g_R^{-1}$.

It turns out that (7.46) can be further simplified in case $\mathcal{R}$ is an inner automorphism of G, that is, when there is an element $R \in G$ such that

$$\mathcal{R}(g) = R^{-1} g R , \qquad g \in G . \tag{7.50}$$

Then a slight modification of (7.46) leads to a variable that transforms linearly under the adjoint action of G,

$$N(\pi) \equiv \Sigma(\pi) R = U(\pi) R U(\pi)^{-1} , \quad N(\pi) \xrightarrow{g} N(\pi'(\pi, g)) = g N(\pi) g^{-1} . \tag{7.51}$$

An advantage of trading $\Sigma(\pi)$ for $N(\pi)$ is that $N(\pi)^2$ is a constant independent of π^a, which may be convenient in concrete applications. Indeed, since $\mathcal{R}(\mathcal{R}(g)) = g$ for any $g \in G$, it follows from (7.50) that R^2 belongs to the center of G. Consequently, $N(\pi)^2 = U(\pi) R^2 U(\pi)^{-1} = R^2$.

Example 7.14

Consider the coset space SU(2)/U(1), relevant for description of quantum systems with magnetic ordering such as (anti)ferromagnets. In the fundamental representation, the generators of SU(2) are $Q_A = \tau_A/2$, and the generator of the U(1) isotropy group can be taken as $\tau_3/2$. This coset space is symmetric thanks to an inner automorphism (7.50) with $R = \mathrm{i}\tau_3$. While the factor of i here is required to make R an element of SU(2), we do not need it for the definition of $N(\pi)$. Let us therefore set

$$N(\pi) \equiv U(\pi) \tau_3 U(\pi)^{-1} . \tag{7.52}$$

This matrix variable is unitary and Hermitian. Moreover, it is traceless and it squares to $\mathbb{1}$. It can thus be mapped on a unit vector variable $\boldsymbol{n}(\pi)$ such that

$$N(\pi) = \boldsymbol{\tau} \cdot \boldsymbol{n}(\pi) \ , \tag{7.53}$$

which belongs to the vector representation of SU(2). The fact that we ended up describing the coset space in terms of a unit vector is not a coincidence. Thanks to the local isomorphism of SU(2) and SO(3) we also have a local equivalence of coset spaces, $\mathrm{SU}(2)/\mathrm{U}(1) \simeq \mathrm{SO}(3)/\mathrm{SO}(2) \simeq S^2$.

7.4 Geometry of the Coset Space

In this final section of the chapter, I will show that some of the structure that enters the standard realization of symmetry on a coset space can be given a neat geometric interpretation. A reader not familiar with basics of differential geometry is advised to consult Appendix A before proceeding. Further information about the geometry of homogeneous spaces at an easily accessible level can be found in [12].

To start, recall that each point of a coset space G/H corresponds to an entire class of elements of the group G. The approach I have used so far was to parameterize each $x \in G/H$ with coordinates π^a in terms of a fixed coset representative $U(\pi)$. The choice of $U(\pi)$ is however arbitrary and can be changed locally by multiplication from the right; any $\tilde{U}(\pi) = U(\pi)h(\pi)$ with $h(\pi) \in H_{x_0}$ is equally good. One can then promote the basic transformation rule (7.25) for $U(\pi)$ to

$$U(\pi) \xrightarrow{g,h(\pi)} U(\pi'(\pi, g)) = gU(\pi)h(\pi)^{-1} \ , \tag{7.54}$$

where $h(\pi)$ is independent of g. This realizes an action of the product group $G \times H_{\text{gauge}}$. The group G acts on $U(\pi)$ by left multiplication as usual. The local group H_{gauge} isomorphic to H acts on $U(\pi)$ by right multiplication with $h(\pi)^{-1}$ and encodes the freedom to choose locally the coset representative. This is a typical example of a gauge redundancy; any geometrically or physically well-defined quantity must be independent of the arbitrary choice of coset representative. In theoretical physics, the approach that views the left action of G separately from the right action of the local group H_{gauge} is called "hidden local symmetry" [13].

The different geometric roles of G and its isotropy subgroup can be intuitively understood by looking at Fig. 7.2. One-parameter subgroups of G define a set of flows on G/H that in general translate a given point x to some other point. The isotropy group H_x maps x to itself. It does, however, act nontrivially in the vicinity of x. Based on the figure, we expect that the action of H_x projects to a set of linear maps on the tangent space $\mathcal{T}_x G/H$ at x. By induction, such linear H_x-transformations should exist for any tensor at x.

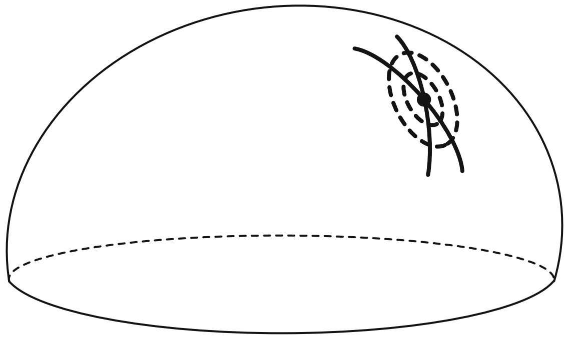

Fig. 7.2 Illustration of the action of G and H_x around a fixed point x (*black dot*) on the coset space. The thick *solid lines* indicate orbits of one-parameter subgroups of G passing through x; these correspond to generators from $\mathfrak{g}/\mathfrak{h}$ and act as translations on the coset space. The thick *dashed lines* indicate some orbits of H_x in the vicinity of x. The action of H_x naturally induces a set of linear transformations on the tangent space at x

The local group H_{gauge} gives us the freedom to do independent H-transformations of tangent vectors (and generally tensors) point by point. In order to be able to analyze the properties of vector (tensor) fields on G/H, we therefore need a way to relate tangent vectors (tensors) at different points. This resembles closely the motivation behind the construction of a local frame, or vielbein, and the corresponding connection on a manifold. The only difference to the formalism reviewed in Appendix A.5 is that here the structure group is not GL(dim G/H) but H itself. This restriction makes it possible to realize changes in the local frame in terms of a fixed matrix representation of H.

It turns out that we already have both the local frame and the connection: they are granted to us by the MC form (7.32). Specifically, the components ω^a of $\omega_\perp$ define a coframe on G/H. By the second line of (7.34), these indeed transform linearly under the local action of H_{gauge} as they should. Likewise, the $\mathfrak{h}$ part of the MC form, $\omega_\parallel$, provides the necessary H-connection. This is confirmed by the transformation rule on the first line of (7.34).

This observation stresses the significance of the MC form as a differential 1-form, taking values in the Lie algebra $\mathfrak{g}$. Its exterior derivative reflects the structure of G,

$$\begin{aligned} \mathrm{d}\omega &= -\mathrm{i}\mathrm{d}U^{-1} \wedge \mathrm{d}U = \mathrm{i}U^{-1}\mathrm{d}U \wedge U^{-1}\mathrm{d}U = -\mathrm{i}\omega \wedge \omega \\ &= -\frac{\mathrm{i}}{2}\omega^B \wedge \omega^C [Q_B, Q_C] = \frac{1}{2} f^A_{BC}\omega^B \wedge \omega^C Q_A \,. \end{aligned} \tag{7.55}$$

This can be split into a pair of equations, one for $\omega_\parallel$ and one for $\omega_\perp$,

$$\begin{aligned} \mathrm{d}\omega^\alpha &= \frac{1}{2} f^\alpha_{\beta\gamma}\omega^\beta \wedge \omega^\gamma + \frac{1}{2} f^\alpha_{bc}\omega^b \wedge \omega^c \ , \\ \mathrm{d}\omega^a &= f^a_{\beta c}\omega^\beta \wedge \omega^c + \frac{1}{2} f^a_{bc}\omega^b \wedge \omega^c \ , \end{aligned} \tag{7.56}$$

where I used the facts that $f^a_{\beta\gamma} = 0$ (the Lie algebra $\mathfrak{h}$ closes) and that $f^\alpha_{\beta c} = 0$ (the coset space is reductive). These so-called *Maurer–Cartan equations* provide a link between the algebraic and geometric properties of the coset space.

7.4.1 Canonical and Torsion-Free Connection

The local basis of 1-forms ω^a transforms under the action of the structure group H as a tangent vector. It follows from (7.34) that upon an infinitesimal transformation by $h \approx e + \mathrm{i}\epsilon^\alpha Q_\alpha$, $\delta\omega^a = -\epsilon^\alpha f^a_{\alpha b}\omega^b$. This determines the matrix elements of the action of the H-connection $\omega_\parallel$ on tangent vectors, in the notation of Appendix A.5,

$$\Omega^a{}_b = -f^a_{\alpha b}\omega^\alpha \ . \tag{7.57}$$

The MC equations (7.56) encode information about both the torsion and the curvature of this so-called *canonical connection* on G/H. The torsion 2-form follows from the second line of (7.56),

$$T^a \equiv \mathrm{d}\omega^a + \Omega^a{}_b \wedge \omega^b = \frac{1}{2} f^a_{bc}\omega^b \wedge \omega^c \ . \tag{7.58}$$

Similarly, the curvature 2-form follows from the first line of (7.56),

$$R^a{}_b \equiv \mathrm{d}\Omega^a{}_b + \Omega^a{}_c \wedge \Omega^c{}_b = -\frac{1}{2} f^a_{\alpha b} f^\alpha_{cd}\omega^c \wedge \omega^d \ , \tag{7.59}$$

where I used the Jacobi identity (7.28) to simplify the result. In the language of field theory, this is nothing but the field-strength 2-form of $\omega_\parallel$.

The definition (7.57) of the canonical connection arises naturally from the splitting (7.33) of the MC form into the $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ subspaces. This is not the only possible connection on G/H though. In fact, there is an infinite class of them. Let us set

$${}^\lambda\Omega^a{}_b \equiv -f^a_{\alpha b}\omega^\alpha - \lambda f^a_{cb}\omega^c \ , \qquad \lambda \in \mathbb{R} \ . \tag{7.60}$$

This time, it takes some manipulation to derive the corresponding torsion and curvature 2-forms. The final result is

$$\begin{aligned} {}^{\lambda}T^a &= \left(\frac{1}{2}-\lambda\right) f^a_{bc}\omega^b \wedge \omega^c \ , \\ {}^{\lambda}R^a{}_b &= -\frac{1}{2}\left(f^a_{\alpha b} f^\alpha_{cd} + \lambda f^a_{eb} f^e_{cd} - 2\lambda^2 f^a_{ec} f^e_{bd}\right)\omega^c \wedge \omega^d \ , \end{aligned} \tag{7.61}$$

which generalizes (7.58) and (7.59) to any nonzero λ.

Within the class of connections (7.60), there is one with vanishing torsion, corresponding to $\lambda = 1/2$. One might expect that it should be possible to recover this connection from a suitable Riemannian metric on the coset space. This is indeed the case under some further technical assumptions, as I will explain below. For symmetric coset spaces, the whole class of connections (7.60) becomes degenerate, and is automatically torsion-free.

7.4.2 Riemannian Metric

Every Lie algebra $\mathfrak{g}$ possesses a bilinear form invariant under the adjoint action of the corresponding Lie group G. To see this, just take any faithful matrix representation of the generators Q_A and set

$$g_{AB} \equiv \text{tr}(Q_A Q_B) \ . \tag{7.62}$$

The invariance under $Q_A \to g Q_A g^{-1}$ for any $g \in G$ is manifest. The infinitesimal version of the invariance condition follows by setting $g \approx e + \text{i}\epsilon^C Q_C$ and expanding to linear order in ϵ^C,

$$f^D_{CA} g_{DB} + f^D_{CB} g_{AD} = 0 \ . \tag{7.63}$$

Since the components ω^a of the MC form define a basis of the space of 1-forms on G/H, any rank-2 covariant tensor can be constructed as a linear combination of $\omega^a \otimes \omega^b$. Given the invariant bilinear form g_{AB}, it is then natural to introduce the following metric on G/H,

$$g_{G/H} \equiv g_{ab}\omega^a \otimes \omega^b \ , \tag{7.64}$$

where g_{ab} is the restriction of g_{AB} to the $\mathfrak{g}/\mathfrak{h}$ subspace.

This construction is not guaranteed to work without further assumptions. A Riemannian metric should be positive-definite. A sufficient, though not necessary, condition for this is that g_{AB} itself is positive-definite, which is generally only true for compact semisimple Lie algebras. For what follows, I will only need the weaker assumption that g_{AB} is nondegenerate and that $g_{a\beta} = 0$, i.e. that the subspace $\mathfrak{g}/\mathfrak{h}$ can be chosen to be "orthogonal" to $\mathfrak{h}$. Interestingly, this alone already ensures that

G/H is a reductive coset space. Namely, a short manipulation using (7.63) gives

$$0 = f^{\delta}_{\alpha\beta} g_{\delta c} = f^{D}_{\alpha\beta} g_{Dc} = -f^{D}_{\alpha c} g_{\beta D} = -f^{\delta}_{\alpha c} g_{\beta\delta} \ , \tag{7.65}$$

which implies that $f^{\delta}_{\alpha c} = 0$ thanks to the fact that $g_{\alpha\beta}$ is nondegenerate. With the assumption that $g_{a\beta} = 0$, the invariance condition (7.63) also splits into two separate conditions on g_{ab},

$$f^{d}_{\gamma a} g_{db} + f^{d}_{\gamma b} g_{ad} = 0 \ , \qquad f^{d}_{ca} g_{db} + f^{d}_{cb} g_{ad} = 0 \ . \tag{7.66}$$

The metric (7.64) is invariant under the left action of G, since the MC form itself is. It is however also invariant under the right action of H_{gauge} as defined by (7.34). This follows from the first condition in (7.66). The G-invariance of the metric guarantees the existence of a set of Killing vector fields that realize infinitesimal group motions on the coset space. In the local coordinates π^a, these are nothing but the functions $\xi^a_A(\pi)$ introduced in (7.36). According to (7.37), we have $\xi^b_A(0)\omega^a_b(0) = \delta^a_A$ at the origin. This means that the Killing vectors $\boldsymbol{\xi}_a(0)$ corresponding to generators from $\mathfrak{g}/\mathfrak{h}$ define a local frame dual to $\omega^a(0)$, which further illuminates the geometric nature of the MC form. Away from the origin, the duality between $\omega_\perp$ and the subset of Killing vectors realizing infinitesimal translations on G/H is still expressed by the second line of (7.37). One just has to recall that the isotropy group of $x = T_{U(\pi)}x_0$ is $H_x = U(\pi)H_{x_0}U(\pi)^{-1}$. This conjugation is supplied by the matrix $\nu^B_A(\pi)$ on the left-hand side of (7.37). The local basis of 1-forms $\omega^a(\pi)$ is then dual to the local frame consisting of Killing vectors of the generators $U(\pi)Q_aU(\pi)^{-1}$.

Example 7.15

The metric (7.64) is particularly easy to evaluate explicitly on symmetric coset spaces. Using the expression (7.48) for $\omega_\perp$ in terms of the linearly transforming variable Σ, we find at once that

$$g_{G/H} = \frac{1}{4}\,\text{tr}(\mathrm{d}\Sigma \otimes \mathrm{d}\Sigma^{-1}) = \frac{1}{4}\,\text{tr}(\partial_a\Sigma\partial_b\Sigma^{-1})\mathrm{d}\pi^a \otimes \mathrm{d}\pi^b \ , \tag{7.67}$$

where I used the abbreviation $\partial_a \equiv \partial/\partial\pi^a$. As an illustration, consider the coset space $\mathrm{SU}(2)/\mathrm{U}(1) \simeq \mathrm{SO}(3)/\mathrm{SO}(2) \simeq S^2$ discussed in Example 7.14. Here we find that up to an overall factor,

$$g_{S^2} \propto \mathrm{d}\boldsymbol{n} \otimes \mathrm{d}\boldsymbol{n} = \mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta\,\mathrm{d}\varphi \otimes \mathrm{d}\varphi \ , \tag{7.68}$$

where I used standard spherical coordinates to parameterize the unit vector $\boldsymbol{n}$. This is just an elaborate way to show that the unique SO(3)-invariant Riemannian metric on S^2 is given by projecting (pulling back) the Euclidean metric on $\mathbb{R}^3$ to the sphere.

It remains to clarify the relationship between the metric (7.64) and the connections (7.60). The covariant derivative of the metric with respect to these connections, in the direction of an arbitrary vector field $\boldsymbol{v}$, is easily calculated in the local frame,

$$(\nabla_{\boldsymbol{v}} g_{G/H})_{ab} = \boldsymbol{v}[g_{ab}] - g_{cb}\,{}^{\lambda}\Omega^{c}{}_{a}(\boldsymbol{v}) - g_{ac}\,{}^{\lambda}\Omega^{c}{}_{b}(\boldsymbol{v})\ . \tag{7.69}$$

The first term vanishes since g_{ab} is merely a set of constants. The sum of the second and the third term vanishes for any λ as a consequence of the combination of (7.60) and (7.66). Thus, the whole class of connections (7.60) is compatible with the metric (7.64). In case the metric is (pseudo-)Riemannian, it is known that there is a unique metric connection without torsion, called the *Levi-Civita connection*. As shown in the previous subsection, this corresponds to the choice $\lambda = 1/2$.

The metric on the coset space is not uniquely fixed by the requirement of invariance under $G \times H_{\text{gauge}}$. Although it was natural to start the construction with the Cartan–Killing form (7.62) on the whole Lie algebra $\mathfrak{g}$ and then restrict it to the $\mathfrak{g}/\mathfrak{h}$ subspace, we could have as well started from the latter. One then finds that possible metrics on G/H invariant under $G \times H_{\text{gauge}}$ are classified by constant symmetric tensors g_{ab} invariant under the adjoint action of H_{gauge}, that is g_{ab} satisfying the first relation in (7.66). For g_{ab} that does not satisfy the second relation in (7.66), the connections (7.60) with $\lambda \neq 0$ are not metric-compatible. In particular ${}^{1/2}\Omega^{a}{}_{b}$ is not the Levi-Civita connection despite being torsion-free.

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