2 On-Shell Techniques for Tree-Level Amplitudes

Abstract

In this chapter we focus on the pole structure of tree-level amplitudes. We argue that amplitudes factorise on these poles into lower-point amplitudes. Moreover, universal factorisation structures emerge when two momenta become collinear as well as in the limit of low energy of a single particle—the soft limit. These factorisation properties are the basis of an efficient technique for computing tree-level scattering amplitudes in gauge theories and gravity recursively—without ever referring to Feynman rules or even a Lagrangian. These recursion relations use as input lower-point amplitudes, so that the gauge redundancy, which is partly responsible for the complexity of conventional Feynman graph calculations, is absent in this entirely on-shell based formalism. We then show the invariance of scattering amplitudes under Poincaré transformations, and introduce the conformal symmetry of gauge-theory tree-level amplitudes. Finally, we highlight a surprising double-copy relation between gluon and graviton amplitudes.

2.1 Factorisation Properties of Tree-Level Amplitudes

Important insights and constraints on tree-level scattering amplitudes may be gained by thinking about them as analytic functions of the external momenta. In this section we will restrict ourselves to the case of massless particles. As we already argued in Chap. 1 with Fig. 1.2, tree-level amplitudes have simple poles in multi-particle channels. This can be seen from the Feynman diagrammatic expansion. Take all diagrams which have a propagator $\sim 1/P_{ij}^2$, where $P_{ij} = p_i + \ldots + p_j$ is a sum of external momenta (which will be adjacent for colour-ordered amplitudes, or an arbitrary subset in gravity). We call P_{ij} the *region momentum*, as it is the total momentum associated to the region of momenta $\{p_i, \ldots, p_j\}$. As P_{ij} goes on shell, $P_{ij}^2 \to 0$, these singular diagrams will collect into a product of two

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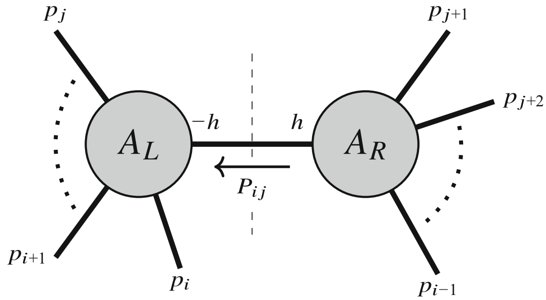


Fig. 2.1 The factorisation of a colour-ordered amplitude on the multi-particle pole $P_{ij} = p_i + p_{i+1} + \ldots + p_j$ when $P_{ij}^2 \to 0$. Here h denotes the helicity of the particles crossing the pole

on-shell amplitudes, to the left and right hand sides of the divergent propagator, in a mechanism known as *factorisation*. This process is illustrated in Fig. 2.1. We can understand the details of the procedure by studying the Feynman rules for the propagator that goes on-shell directly, each of which has the generic form

$$\frac{\mathrm{i}\, N(P_{ij})}{P_{ij}^2}\,, \tag{2.1}$$

where the numerator $N(P_{ij})$ depends on the type of particle. For example, for the specific case of gluons in the axial gauge ($n^\mu A_\mu = 0$) we would have $N(P) \to N_g^{\mu\nu}(P,n) = -\eta^{\mu\nu} + (P^\mu n^\nu + P^\nu n^\mu)/(P\cdot n)$. In the limit $P_{ij}^2 \to 0$ the numerator N can be rewritten in terms of the spin sum over external polarisation vectors or wave-functions. Again for the case of gluons, following the results of Exercise 1.7, we have that

$$N_g^{\mu\nu}(P,n) \overset{P^2\to 0}{\to} \sum_{h=\pm} \epsilon_h^\mu(P)\epsilon_h^{*\nu}(P) = \sum_{h=\pm} \epsilon_h^\mu(P)\epsilon_{-h}^\nu(-P)\,. \tag{2.2}$$

The polarisation vectors combine with the Feynman diagram components on either side of the divergent propagator to form on-shell amplitudes. A schematic form for a general particle type can be written as

$$A_n^{\text{tree}}(1,\ldots,n) \xrightarrow{P_{ij}^2\to 0} \sum_{s\in s_{\mathrm{P}}} A_L^{\text{tree}}\big(i,i+1,\ldots,j,-P_{ij}^{\bar{s}}\big)\frac{n_{\mathrm{P}}}{P_{ij}^2} A_R^{\text{tree}}\big(P_{ij}^s,j+1,\ldots,i-1\big), \tag{2.3}$$

where P indicates the particle type of the propagator with momentum P_{ij}, s_{P} are its possible spin states and n_{P} is a particle-dependent constant. As we see from the discussion above, for gluons $s_{\text{gluon}} = \pm$ are the two helicities and $n_{\text{gluon}} = \mathrm{i}$, for spin-$\frac{1}{2}$ fermions one can show that $s_{\text{fermion}} = \pm 1/2$ and $n_{\text{fermion}} = 1$, while spin 0 has $s_{\text{scalar}} = 0$ and $n_{\text{scalar}} = \mathrm{i}$. Other particle types are simple to determine following the same argument. Factorisation will also occur in the case of massive propagators where $P_{ij}^2 \to m_{ij}^2$ where m_{ij} is the mass of propagating particle.

2.1.1 Collinear Limits

A special case of factorisation is the two-particle pole also known as *collinear* singularity. Without loss of generality we take the two collinear particles to be 1 and 2. We then have $(p_1 + p_2)^2 = 0$, which implies $p_1 \cdot p_2 = 0$ or *collinearity* $p_1 \parallel p_2$. We again concentrate on the massless case. In fact, since the factorisation now involves a three-particle amplitude, such a pole can only occur for collinear external momenta. We already know from the discussion in the previous chapter that three-point amplitudes are subtle. In the strict collinear limit $p_1 \parallel p_2$ we may parametrise the collinear momenta p_1 and p_2 as

$$p_1^\mu = x\, P^\mu\,, \qquad\qquad p_2^\mu = (1-x)\, P^\mu\,, \tag{2.4}$$

with the total collinear momentum $P = p_1 + p_2$, and x parametrising the amount of P distribution over p_1 and p_2. Tree-level amplitudes have a universal (singular) behaviour in the collinear limit governed by the *splitting functions*,

$$A_n^{\text{tree}}\big(1^{h_1}, 2^{h_2}, \ldots\big) \xrightarrow{1\|2} \sum_{h=\pm} \text{Split}_h^{\text{tree}}\big(x, 1^{h_1}, 2^{h_2}\big)\, A_{n-1}^{\text{tree}}\big(P^{-h}, \ldots\big)\,, \tag{2.5}$$

as a special case of Eq. (2.3). In fact, the splitting functions are related to the three-particle amplitudes as

$$\text{Split}_h^{\text{tree}}\big(x, 1^{h_1}, 2^{h_2}\big) = \lim_{P^2 \to 0} A_3^{\text{tree}}\big(1^{h_1}, 2^{h_2}, -P^h\big)\, \frac{\mathrm{i}}{P^2}\,. \tag{2.6}$$

The splitting functions depend on the helicities of the collinear gluons but are independent of the helicities of the other legs $\{3, \ldots, n\}$ not participating in the collinear limit. This is known as the universality of the splitting functions.

Gluon Splitting Functions

For collinear gluons the splitting functions are given by

$$\begin{aligned}
\text{Split}_+^{\text{tree}}(x, a^+, b^+) &= 0 \\
\text{Split}_-^{\text{tree}}(x, a^+, b^+) &= \frac{g}{\sqrt{x(1-x)}\langle ab\rangle} \\
\text{Split}_+^{\text{tree}}(x, a^+, b^-) &= -\frac{(1-x)^2\, g}{\sqrt{x(1-x)}\langle ab\rangle} \\
\text{Split}_-^{\text{tree}}(x, a^+, b^-) &= -\frac{x^2\, g}{\sqrt{x(1-x)}[ab]}\,.
\end{aligned} \tag{2.7}$$

(continued)

The remaining splitting functions may be obtained by parity

$$\mathrm{Split}_h^{\mathrm{tree}}(x, a^{-\lambda_a}, b^{-\lambda_b}) = \mathrm{Split}_{-h}^{\mathrm{tree}}(x, a^{\lambda_a}, b^{\lambda_b})\Big|_{\langle ab\rangle \leftrightarrow [ba]} . \tag{2.8}$$

We shall now derive these expressions from our knowledge of the three-point MHV amplitude (1.170 and (1.171). As the collinear kinematics is subtle, it is advantageous to systematically approach the collinear configuration as

$$\begin{aligned} |1\rangle &= \cos\phi\,|P\rangle - z\,\sin\phi\,|r\rangle\,, \qquad & |1] &= \cos\phi\,|P] - z\,\sin\phi\,|r]\,, \\ |2\rangle &= \sin\phi\,|P\rangle + z\,\cos\phi\,|r\rangle\,, \qquad & |2] &= \sin\phi\,|P] + z\,\cos\phi\,|r] \end{aligned} \tag{2.9}$$

in the limit $z \to 0$. Here $P^\mu = p_1^\mu + p_2^\mu + \mathcal{O}(z^2)$ is the limiting collinear momentum vector, and r^μ is a null reference momentum not parallel to P^μ. Moreover, the collinear parameter above is $x = \cos^2\phi$. This parametrises the four-momenta of the collinear particles as

$$\begin{aligned} p_1 &= \cos^2\phi\; P - z\,\cos\phi\sin\phi\,\big(|P\rangle[r| + |r\rangle[P|\big) + z^2\,\sin^2\phi\, r\,, \\ p_2 &= \sin^2\phi\; P + z\,\cos\phi\sin\phi\,\big(|P\rangle[r| + |r\rangle[P|\big) + z^2\,\cos^2\phi\, r\,, \end{aligned} \tag{2.10}$$

implying that $p_1 + p_2 = P + z^2 r$, as claimed. One then has

$$\begin{aligned} \langle 12\rangle &= z\langle Pr\rangle\,, \qquad \langle 1P\rangle = z\sin\phi\,\langle Pr\rangle\,, \qquad \langle 2P\rangle = -z\cos\phi\,\langle Pr\rangle\,, \\ [12] &= z[Pr]\,, \qquad [1P] = z\sin\phi\,[Pr]\,, \qquad [2P] = -z\cos\phi\,[Pr]\,, \\ (p_1+p_2)^2 &= z^2\langle Pr\rangle[rP]\,. \end{aligned} \tag{2.11}$$

The splitting functions (2.7) follow immediately from Eq. (2.6) and the two MHV three-point amplitudes (1.170) and (1.171) upon using the above identities. The vanishing of $\mathrm{Split}_+^{\mathrm{tree}}(x, 1^+, 2^+)$ follows from the vanishing all-plus amplitude. Using the MHV_3 amplitude of Eq. (1.170)[1] we find

$$\mathrm{Split}_-^{\mathrm{tree}}(x, 1^+, 2^-) = \frac{-\mathrm{i}g\langle 2(-P)\rangle^3}{\langle 21\rangle\,\langle 1(-P)\rangle}\,\frac{\mathrm{i}}{(p_1+p_2)^2} = -\frac{\cos\phi^3 g}{z\sin\phi[Pr]}\,, \tag{2.12}$$

[1] Recall our convention (1.113) that $|-P\rangle = \mathrm{i}|P\rangle$ and $|-P] = \mathrm{i}|P]$.

as well as using the $\overline{\text{MHV}}_3$ amplitude of Eq. (1.171),

$$\text{Split}_{-}^{\text{tree}}(x,1^+,2^+) = \frac{\text{i}g[12]^3}{[1(-P)]\,[(-P)2]}\,\frac{\text{i}}{(p_1+p_2)^2} = \frac{g}{z\cos\phi\sin\phi\langle Pr\rangle}\,,$$

$$\text{Split}_{+}^{\text{tree}}(x,1^+,2^-) = \frac{\text{i}g[1(-P)]^3}{[12]\,[2(-P)]}\,\frac{\text{i}}{(p_1+p_2)^2} = -\frac{\sin\phi^3 g}{z\cos\phi\langle Pr\rangle}\,,$$

which prove the relations (2.7).

Example: Collinear Limits of the Five-Point MHV Amplitude

Let us use this result to test our conjectured five-point MHV amplitude (1.192) from Chap. 1 for consistency under collinear limits. We set $g = 1$ for convenience. We have

$$\begin{aligned}\text{i}A_5^{\text{tree}}(1^-,2^-,3^+,4^+,5^+) &= \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\\ &\xrightarrow{4\|5} \frac{1}{\sqrt{x(1-x)}\langle 45\rangle}\times\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 3P\rangle\langle P1\rangle}\\ &= \text{Split}_{-}^{\text{tree}}(x,4^+,5^+)\times \text{i}A_4^{\text{tree}}(1^-,2^-,3^+,P^+)\,,\end{aligned} \tag{2.13}$$

where we parametrised the collinear limit by $|4\rangle = \sqrt{x}|P\rangle$ and $|5\rangle = \sqrt{1-x}|P\rangle$. Indeed we find agreement with the second function in Eq. (2.7). Next, we take the collinear limit in a $(-+)$ channel. We have

$$\begin{aligned}\text{i}A_5^{\text{tree}}(1^-,2^-,3^+,4^+,5^+) &\xrightarrow{2\|3} \frac{x^2}{\sqrt{x(1-x)}}\,\frac{1}{\langle 23\rangle}\,\frac{\langle 1P\rangle^4}{\langle 1P\rangle\langle P4\rangle\langle 45\rangle\langle 51\rangle}\\ &= \text{Split}_{+}^{\text{tree}}(z,2^-,3^+)\times \text{i}A_4^{\text{tree}}(1^-,P^-,4^+,5^+)\,,\end{aligned} \tag{2.14}$$

from which we deduce

$$\text{Split}_{+}^{\text{tree}}(x,a^-,b^+) = \frac{x^2}{\sqrt{x(1-x)}\,\langle ab\rangle}\,, \tag{2.15}$$

in agreement with the third expression in Eq. (2.7) whilst swapping a and b (and $x \to 1-x$). In order to check the fourth function we consider the collinear limit in a $(+-)$-channel,

$$A_5^{\text{tree}}(1^-,2^-,3^+,4^+,5^+) \xrightarrow{5\|1} \underbrace{\frac{(1-x)^2}{\sqrt{x(1-x)}\,\langle 51\rangle}}_{\text{Split}_+^{\text{tree}}(x,5^+,1^-)} A_4^{\text{tree}}(P^-,2^-,3^+,4^+)\,, \tag{2.16}$$

yielding the desired result via parity (2.8). In order to check the vanishing of the uniform helicity splitting function in Eq. (2.7) we have study the collinear factorisation of the 6-point MHV amplitude with the helicity distributions $A_6^{\text{tree}}(1^-,2^-,3^+,4^+,5^+,6^+)$ along the legs 5 and 6.

Exercise 2.1 (The Vanishing Splitting Function $\text{Split}_+^{\text{tree}}(x,a^+,b^+)$**)** Show that $\text{Split}_+^{\text{tree}}(x,a^+,b^+) = 0$ by studying the factorisation properties of the six-gluon MHV tree amplitude $A_6^{\text{tree}}(1^-,2^-,3^+,4^+,5^+,6^+)$ from Eq. (1.192) in the collinear limit $5 \parallel 6$. For the solution see Chap. 5.

Absence of Multi-Particle Poles in MHV Amplitudes
General n-gluon scattering amplitudes have multi-particle poles when region momenta go on-shell. However, MHV tree-amplitudes are special, and in fact *only* have two-particle poles or collinear singularities. The reason is the following. Consider the general factorisation formula (2.3). In a factorisation of an MHV amplitude there are only three negative-helicity legs (corresponding to the two external negative helicities, and one for the internal on-shell propagator) that are distributed over two partial amplitudes. However, we saw in the previous chapter that $A_n(1^\pm,2^+,\ldots,n^+) = 0$ (for $n > 3$). Therefore, either A_L or A_R in Eq. (2.3) is always zero unless one partial amplitude is a three-particle amplitude. The latter case corresponds to a two-particle pole or collinear singularity, as discussed above.

2.1.2 Soft Theorems

We continue our quest in the factorisations of scattering amplitudes with a kinematical limit subject to just a *single* leg: the soft limit. Here one particle involved in the scattering process has a very low energy—it is soft. Specifically, it refers to the limit where the four-momentum of the particle goes to zero. Again the tree-amplitude displays a universal factorisation property for photon, gluon and graviton amplitudes into a lower-point amplitude and a soft function. In order to take the limit, we parametrise the soft momentum of leg s as $p_s^\mu = \delta\, q^\mu$ and take $\delta \to 0$ (do

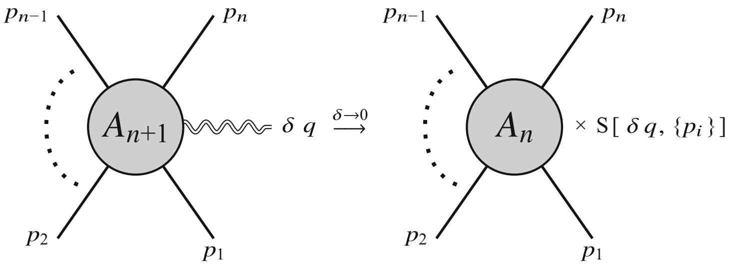


Fig. 2.2 The soft factorisation of a generic $(n+1)$-particle amplitude. The soft function $\mathcal{S}[\delta\, q, \{p_i\}]$ only depends on the momenta (not the polarisations) of the hard legs

not confuse $\delta\, q^\mu$ with a variation). The soft theorems state that

$$A_{n+1}^{\text{tree}}(\delta\, q, 1, \ldots, n) \xrightarrow{\delta \to 0} S[\delta\, q, \{p_1, \ldots, p_n\}]\, A_n^{\text{tree}}(1, \ldots, n)\,, \tag{2.17}$$

which is illustrated in Fig. 2.2. The factorised soft function $S[\delta\, q, \{p_1, \ldots, p_n\}]$ depends on the momentum $\delta\, q$ and helicity of the soft particle, as well as the momenta of the remaining hard legs $\{p_a\}$. It is however independent of the helicities and even particle types of the remaining hard legs, which may be massless or massive. The soft function diverges as $1/\delta$ at leading order, and also contains universal sub-leading pieces:

$$S[\delta\, q, \{p_a\}] = \frac{1}{\delta}\mathcal{S}^{[0]}(q) + \mathcal{S}^{[1]}(q) + \delta\, \mathcal{S}^{[2]}(q) + \mathcal{O}\big(\delta^2\big)\,. \tag{2.18}$$

It takes a universal form for photons [1], gluons and gravity [2], remarkably not only to leading order, but also to sub-leading order $\mathcal{S}^{[1]}(q)$ for gauge theories, and even to sub-sub-leading order $\mathcal{S}^{[2]}(q)$ for gravity [3].

Leading Soft Theorems
The leading soft divergences take the universal form

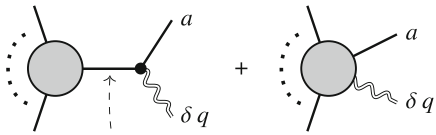
$$\frac{1}{\delta}\mathcal{S}^{[0]}(\delta q) = \begin{cases} \dfrac{1}{\delta}\mathcal{S}^{[0]}_{\text{EM}} = \dfrac{1}{\delta}\displaystyle\sum_{a=1}^{n} e_a \frac{\epsilon \cdot p_a}{p_a \cdot q}\,, & \text{photon,} \\ \dfrac{1}{\delta}\mathcal{S}^{[0]}_{\text{YM}} = \dfrac{g}{\delta\sqrt{2}}\left(\dfrac{\epsilon \cdot p_1}{p_1 \cdot q} - \dfrac{\epsilon \cdot p_n}{p_n \cdot q}\right), & \text{colour-ordered gluon,} \\ \dfrac{1}{\delta}\mathcal{S}^{[0]}_{\text{GR}} = \dfrac{\kappa}{\delta}\displaystyle\sum_{a=1}^{n} \frac{\epsilon_{\mu\nu}\, p_a^\mu\, p_a^\nu}{p_a \cdot q}\,, & \text{graviton.} \end{cases} \tag{2.19}$$

(continued)

Here ϵ denotes the polarisation vector (tensor) of the soft particle, e_a denotes the U(1) (electromagnetic) charge of the hard particle a.

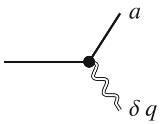
These results can be made plausible through the following argument based on Feynman diagrams. For a tree-level amplitude the soft leg $\delta\, q$ is attached either via a three-point coupling to an outgoing hard leg a, or to the "bulk" of the remaining amplitude:

$$A_n^{\text{tree}}(\delta\, q, \{p_a\}) = \sum_{a=1}^{n}$$



$$\frac{1}{(p_a + \delta\, q)^2 - m_a^2} \overset{\delta\to 0}{=} \frac{1}{2\delta}\,\frac{1}{p_a\cdot q} \tag{2.20}$$

We see that the divergence of the leading order soft function solely arises from the three-point coupling of the gauge or graviton field to the hard leg a. In the case of an amplitude with n-scalars and a soft photon or graviton these couplings take the form



$$\propto \begin{cases} e_a\, \epsilon\cdot p_a\,, & \text{photon,} \\ \kappa\, \epsilon_{\mu\nu}\, p_a^{\mu}\, p_a^{\nu}\,, & \text{graviton,} \end{cases} \tag{2.21}$$

which fixes the soft functions in Eq. (2.19) up to an overall factor. In the case of a colour-ordered pure gluon amplitude with one soft gluon leg, $A_n^{\text{tree}}(1, \ldots, n, \delta\, q)$, the soft leg can couple either to the hard gluon leg 1 or n due to colour ordering. A detailed look at the Feynman rules (1.149) then reveals the coupling $\pm g\; p_a \cdot \epsilon$ with $a = 1, n$ at leading order in δ with a relative sign factor reproducing the form of Eq. (2.19).

It is instructive to study the gauge invariance of the leading soft functions (2.19). Gauge invariance requires the full amplitude $A_{n+1}(\delta\, q, p_1, \ldots, p_n)$ to vanish under the transformation $\epsilon^{\mu} \to q^{\mu}$ (in the graviton case we again take $\epsilon_{\mu\nu} = \epsilon_{\mu}\epsilon_{\nu}$). Hence, by consistency, the soft functions in Eq. (2.19) need to vanish:

$$\mathcal{S}^{[0]}(\delta\, q)\Big|_{\epsilon\to q} = \begin{cases} \displaystyle\sum_{a=1}^{n} e_a = 0\,, & \text{photon,} \\ \dfrac{g}{\sqrt{2}}(1-1) = 0\,, & \text{gluon,} \\ \kappa\, q_\mu \displaystyle\sum_{a=1}^{n} p_a^\mu = 0\,, & \text{graviton.} \end{cases} \tag{2.22}$$

They indeed vanish due to total charge conservation in the electromagnetic or total momentum conservation in the gravitational case (provided the gravitational coupling is universal to all matter). Hence these soft theorems are intimately connected to fundamental symmetries of space-time-matter.

2.1.3 Spinor-Helicity Formulation of Soft Theorems

It is instructive to translate the leading soft theorems to spinor-helicity language for the colour-ordered gluon and graviton cases. The leading order soft factorisation states that

$$A_n^{\text{tree}}(\ldots, a, \delta q^{\pm}, b, \ldots) \xrightarrow{\delta\to 0} \mathcal{S}^{[0]}(a, q^{\pm}, b)\, A_{n-1}^{\text{tree}}(\ldots, a, b, \ldots)\,. \tag{2.23}$$

The factorised soft function depends on the momenta and helicities of the soft gluon and the momenta of the colour-ordered neighbours a and b. It is independent, however, of the helicities and particle types of the neighbouring legs. From considering the soft limit of an MHV amplitude one easily establishes that

$$\mathcal{S}_{\text{YM}}^{[0]}(a, q^{+}, b) = g\, \frac{\langle ab\rangle}{\langle aq\rangle\langle qb\rangle}\,. \tag{2.24}$$

Via parity, in analogy to Eq. (2.8), we find

$$\mathcal{S}_{\text{YM}}^{[0]}(a, q^{-}, b) = -g\, \frac{[ab]}{[aq][qb]}\,. \tag{2.25}$$

Both results directly follow from Eq. (2.19) as well. In the graviton case we deduce

$$\mathcal{S}_{\text{GR}}^{[0]}(q^{++}, 1, \ldots, n) = \kappa \sum_{a=1}^{n} \frac{\langle xa\rangle\langle ya\rangle[qa]}{\langle xq\rangle\langle yq\rangle\langle aq\rangle}\,, \tag{2.26}$$

where x and y are arbitrary reference spinors associated to the polarisation vectors of the soft leg.

Exercise 2.2 (Soft Functions in the Spinor-Helicity Formalism) Starting from Eq. (2.19), derive the leading soft functions for a colour-ordered gluon and a graviton in the spinor-helicity formalism. For the solution see Chap. 5.

2.1.4 Subleading Soft Theorems

Remarkably, the universal soft factorisation survives to subleading order ($\mathcal{O}(\delta^0)$) in the gauge theory, and even sub-sub-leading order ($\mathcal{O}(\delta)$) in the gravitational case, cf. Eq. (2.18). Here we just state the results. They may again be derived from a careful study of gauge invariance and consistency arguments [4, 5]. The novelty is that the sub-leading soft functions are (necessarily) no longer true functions but rather *differential operators* in the external kinematical data. These then act on the factorised hard scattering-amplitude. Explicitly, one has the following sub-leading soft operator for a soft photon q^μ with polarisation ϵ^μ,

$$\mathcal{S}_{\text{EM}}^{[1]}(q) = \sum_{a=1}^{n} e_a \frac{\epsilon_\mu q_\nu J_a^{\mu\nu}}{p_a \cdot q} , \tag{2.27}$$

with the local angular momentum operator

$$J_a^{\mu\nu} = 2\, p_a^{[\mu} \frac{\partial}{\partial p_{a,\,\nu]}} + 2\, \epsilon_a^{[\mu} \frac{\partial}{\partial \epsilon_{a,\,\nu]}} . \tag{2.28}$$

In the above $[\mu\nu]$ denotes anti-symmetrisation with unit weight, see Appendix A. Similarly, for a colour-ordered soft gluon we have

$$\mathcal{S}_{\text{YM}}^{[1]}(n, q, 1) = -\mathrm{i}\frac{g}{\sqrt{2}} \left(\frac{\epsilon_\mu q_\nu J_1^{\mu\nu}}{p_1 \cdot q} - \frac{\epsilon_\mu q_\nu J_n^{\mu\nu}}{p_n \cdot q} \right) , \tag{2.29}$$

while for a soft gravitons we have a sub-leading and sub-sub-leading soft operators of the forms

$$\mathcal{S}_{\text{GR}}^{[1]}(q) = -\mathrm{i}\kappa \sum_{a=1}^{n} \frac{\epsilon \cdot p_a\, \epsilon_\mu q_\nu J_a^{\mu\nu}}{p_a \cdot q} , \qquad \mathcal{S}_{\text{GR}}^{[2]}(q) = -\frac{\kappa}{2} \sum_{a=1}^{n} \frac{\left(\epsilon_\mu q_\nu J_a^{\mu\nu}\right)^2}{p_a \cdot q} . \tag{2.30}$$

Again, one may show that these sub-leading soft operators are consistent with gauge invariance. While this is trivial for the gauge-theory operators by virtue of the anti-symmetry of $J^{\mu\nu}$, the gauge invariance of the gravitational $\mathcal{S}_{\text{EM}}^{[1]}(q)$ leads us to total angular momentum conservation,

$$\mathcal{S}^{[1]}_{\mathrm{GR}}(q)\Big|_{\epsilon\to q} = -\mathrm{i}\kappa\,\epsilon_\mu q_\nu \sum_{a=1}^{n} J_a^{\mu\nu} = 0\,, \tag{2.31}$$

nicely teaming up with the total momentum conservation in the leading case. Hence, the gravitational soft theorems are directly connected to the Poincaré invariance of scattering amplitudes.

Expressed in spinor-helicity variables the sub-leading soft operators take the form

$$\begin{aligned}\mathcal{S}^{[1]}_{\mathrm{YM}}(n, q^+, 1) &= g\left(\frac{[q\partial_n]}{\langle qn\rangle} - \frac{[q\partial_1]}{\langle q1\rangle}\right),\\ \mathcal{S}^{[1]}_{\mathrm{GR}}(q^+, 1, \ldots, n) &= \frac{\kappa}{2}\sum_{a=1}^{n}\frac{[aq]}{\langle aq\rangle}\left(\frac{\langle ax\rangle}{\langle qx\rangle} + \frac{\langle ay\rangle}{\langle qy\rangle}\right)[q\partial_a]\,,\end{aligned} \tag{2.32}$$

where as before x and y are arbitrary reference spinors. The sub-sub-leading gravity operator of Eq. (2.30) in spinor-helicity variables simply reads

$$\mathcal{S}^{[2]}_{\mathrm{GR}}(q^+, 1, \ldots, n) = \frac{\kappa}{2}\sum_{a=1}^{n}\frac{[aq]}{\langle aq\rangle}[q\partial_a]^2\,, \tag{2.33}$$

and was found in [3].

An important subtlety for the sub-leading soft theorems lies in balancing the total momentum conservation of the $(n+1)$-leg amplitude with the soft leg and the factorised n-point hard amplitude. The soft factorisation (2.17) is really a distributional identity involving delta-functions:

$$\begin{aligned}&\delta^{(D)}(\delta\, q + P)\,\mathcal{A}^{\mathrm{tree}}_{n+1}(\delta\, q, \{p_a\}) \xrightarrow{\delta\to 0}\\ &\qquad\qquad S[\delta\, q, \{p_a\}]\,\delta^{(D)}(P)\,\mathcal{A}^{\mathrm{tree}}_{n}(1, \ldots, n) + O\big(\delta^j\big)\,,\end{aligned} \tag{2.34}$$

with $P = p_1 + \ldots + p_n$ and $j = 1$ or 2 for gauge theory or gravity, respectively. This in fact implies that

$$S[\delta\, q, \{p_a\}]\,\delta^{(D)}(P) = \delta^{(D)}(\delta\, q + P)\,S[\delta\, q, \{p_a\}]\,, \tag{2.35}$$

from which one may strongly constrain the δ expansion of the soft function $S[\delta\, q, \{p_a\}]$ using Eq. (2.18). It necessitates the sub-leading terms to be differential operators, and even the functional forms of Eq. (2.32) are fixed by the knowledge of the leading soft functions [4].

Exercise 2.3 (A $\bar{q}qggg$ Amplitude from Collinear and Soft Limits) In Chap. 1 we established the following colour-ordered $\bar{q}qgg$ amplitudes involving a massless quark and anti-quark using colour-ordered Feynman rules:

$$A^{\text{tree}}_{\bar{q}qgg}(1^-_{\bar{q}}, 2^+_q, 3^+, 4^+) = 0\,, \tag{2.36}$$

$$A^{\text{tree}}_{\bar{q}qgg}(1^-_{\bar{q}}, 2^+_q, 3^-, 4^+) = -\mathrm{i}g^2 \frac{\langle 13\rangle^3 \langle 23\rangle}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle}\,. \tag{2.37}$$

Use these and the discussed splitting and soft factorisation properties for gluonic legs to make a guess for the five-point single quark-line tree amplitude $A^{\text{tree}}_{\bar{q}qggg}(1^-_{\bar{q}}, 2^+_q, 3^-, 4^+, 5^+)$. Check your guess against all known factorisation properties. Can you generalise your guess to the n-particle partial amplitudes $A^{\text{tree}}_{\bar{q}qg\ldots g}(1^-_{\bar{q}}, 2^+_q, 3^+, \ldots, n^+)$ and $A^{\text{tree}}_{\bar{q}qg\ldots g}(1^-_{\bar{q}}, 2^+_q, 3^-, 4^+, \ldots, n^+)$? For the solution see Chap. 5.

2.2 BCFW Recursion for Gluon Amplitudes

The Britto-Cachazo-Feng-Witten (BCFW) recursion relation [6, 7] is an efficient way to compute higher-point tree-level amplitudes from lower-point ones. It does not make use of Feynman rules but builds upon unitarity by artfully exploiting the factorisation property of scattering amplitudes (2.3) when region momenta go on-shell. As we will see, our knowledge of the gluon (and graviton) three-point amplitudes of Eqs. (1.170) and (1.171) allows for the construction of *all* higher-point tree-level amplitudes in a recursive fashion. To begin with, let us concentrate on the colour-ordered case and leave the discussion of gravitons for later. The central idea of the recursion is to consider a deformation in a *single* complex variable z of two adjacent momenta in a colour-ordered amplitude that maps the singularities of the amplitude into poles in $z \in \mathbb{C}$. For the tree-level n-gluon amplitude $A_n(p_1, \ldots p_n)$ we introduce the following complex shift of the helicity spinors of two arbitrary adjacent particles, taken to be 1 and n without loss of generality:

$$\begin{aligned} \lambda_1 &\to \hat{\lambda}_1(z) = \lambda_1 - z\lambda_n\,, & \tilde{\lambda}_1 &\to \tilde{\lambda}_1\,, \\ \lambda_n &\to \lambda_n\,, & \tilde{\lambda}_n &\to \hat{\tilde{\lambda}}_n(z) = \tilde{\lambda}_n + z\tilde{\lambda}_1\,. \end{aligned} \tag{2.38}$$

We denote the shifted, z-dependent quantities by a hat. This shift is often termed an $[n\,1\rangle$ shift. It results in a deformation of the momenta,

$$p_1^{\dot{\alpha}\alpha} \to \hat{p}_1^{\dot{\alpha}\alpha}(z) = \tilde{\lambda}_1^{\dot{\alpha}}\,(\lambda_1 - z\lambda_n)^{\alpha}\,, \qquad p_n^{\dot{\alpha}\alpha} \to \hat{p}_n^{\dot{\alpha}\alpha}(z) = (\tilde{\lambda}_n + z\tilde{\lambda}_1)^{\dot{\alpha}}\,\lambda_n^{\alpha}\,. \tag{2.39}$$

Importantly, the shift preserves both overall momentum conservation and the on-shell conditions:

$$\hat{p}_1(z) + \hat{p}_n(z) = p_1 + p_n\,, \qquad \hat{p}_1^2(z) = 0\,, \qquad \hat{p}_n^2(z) = 0\,. \tag{2.40}$$

The $[n\,1\rangle$ shift generates a one-parameter family of amplitudes,

$$A_n(z) \coloneqq A_n\big(\hat{p}_1(z), p_2, \ldots, p_{n-1}, \hat{p}_n(z)\big)\,. \tag{2.41}$$

Note that $\hat{p}_1$ and $\hat{p}_n$ in Eq. (2.39) are now complex, as the underlying helicity spinors $\lambda_{1,n}$ and $\tilde{\lambda}_{1,n}$ are no-longer complex conjugates of each other. This makes the three-point amplitudes involving these states of Eqs. (1.170) and (1.171) non-vanishing. They will become the seeds of the recursion relation. What are the analytic properties of the deformed amplitude $A_n(z)$? Factorisation implies that the deformed amplitude $A_n(z)$ has precisely $n-3$ *simple* poles in z. Using the region momenta $P_i \coloneqq \sum_{j=1}^{i-1} p_j$, these $n-3$ poles take the form

$$\frac{\mathrm{i}}{\hat{P}_i^2(z)} \coloneqq \frac{\mathrm{i}}{P_i^2 - z\,\langle n|P_i|1]} = -\frac{1}{\langle n|P_i|1]}\,\frac{\mathrm{i}}{z - z_{P_i}}\,, \tag{2.42}$$

where $\hat{P}_i(z) = \hat{p}_1(z) + p_2 + \cdots p_{i-1}$, and

$$z_{P_i} = \frac{P_i^2}{\langle n|P_i|1]}\,, \qquad \forall i \in \{3, \ldots, n-1\}\,. \tag{2.43}$$

Note that any region momentum containing *both* $\hat{p}_1(z)$ and $\hat{p}_n(z)$ is independent of z by virtue of Eq. (2.40), and hence cannot contribute to a z-pole. This is why we find $n-3$ poles. It follows that, as $z \to z_{P_i}$, the amplitude $A_n(z)$ factorises as

$$A_n(z) \xrightarrow{z \to z_{P_i}}$$

$$\frac{\mathrm{i}}{\hat{P}_i^2(z)} \sum_h A_L\big[\hat{1}(z_{P_i}), 2, \ldots, i-1, -\hat{P}^{-h}(z_{P_i})\big] A_R\big[\hat{P}^h(z_{P_i}), i, \ldots, n-1, \hat{n}(z_{P_i})\big], \tag{2.44}$$

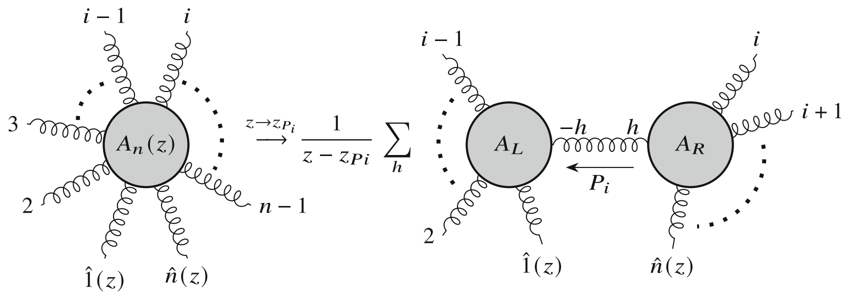


Fig. 2.3 Factorisation of the z-deformed amplitude $A_n(z)$

as depicted in Fig. 2.3. The sum on h runs over all possible helicity states h propagating between A_L and A_R, and depends on the field content of the theory considered. For gluons it is a sum over $h = \{+1, -1\}$.

In the end we are only interested in the undeformed amplitude, i.e. $A_n(z{=}0)$, and we can use complex analysis to construct it from the knowledge of the residues of $A_n(z)$:

$$
\begin{aligned}
A_n(z{=}0) &= \frac{1}{2\pi \mathrm{i}} \oint_{C_0} \frac{\mathrm{d}z}{z}\, A_n(z) \\
&= \sum_{i=2}^{n-1} \sum_{h=\pm} A_L^{-h}(z_{P_i}) \frac{\mathrm{i}}{P_i^2} A_R^{h}(z_{P_i}) + \mathrm{Res}(z=\infty)\,.
\end{aligned}
\tag{2.45}
$$

Here C_0 is a small circle around $z{=}0$ that only contains the pole around the origin. To obtain Eq. (2.45) we have deformed this into a large circle at infinity, now encircling all the poles z_{P_i} in the complex plane but with an opposite orientation. See Fig. 2.4. If $A_n(z) \to 0$ as $z \to \infty$ we can drop the boundary term $\mathrm{Res}(z = \infty)$. As we shall argue in a moment, this is the case for gauge theories under certain conditions.

BCFW Recursion Relation for Gluon Amplitudes
With this assumption, we arrive at the BCFW recursion relation [7]:

$$
\begin{aligned}
A_n(1, \ldots, n) = &\sum_{i=3}^{n-1} \sum_{h=\pm} A_i\Big(\hat{1}(z_{P_i}), 2, \ldots, -\hat{P}_i^{-h}(z_{P_i})\Big) \\
&\frac{\mathrm{i}}{P_i^2}\, A_{n+2-i}\Big(\hat{P}_i^{h}(z_{P_i}), i, \ldots, n-1, \hat{n}(z_{P_i})\Big)\,,
\end{aligned}
\tag{2.46}
$$

(continued)

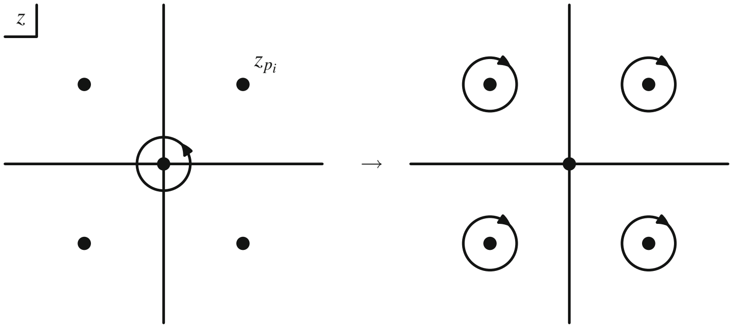


Fig. 2.4 Using Cauchy's theorem to obtain Eq. (2.45) we may pull the initial circle C_0 off to infinity thereby encircling the other poles clock-wise

with z_{P_i} defined in (2.43), and $P_i = p_1 + p_2 + \ldots + p_{i-1}$. This relation is constructive: the amplitudes appearing on the right-hand side have lower multiplicity than the initial A_n. Hence, with the seed three-gluon amplitudes (1.170) and (1.171) we can bootstrap this relation to construct *all* n-gluon trees without using Feynman diagrams at all! Even more, our nuclei—the three-point amplitudes—were obtained from helicity scaling arguments alone, as discussed in Sect. 1.11.3. In this derivation we chose to shift two neighbouring legs $\hat{1}$ and $\hat{n}$. In fact, one can also shift non-neighbouring legs or even more than two legs to obtain alternative recursion relations, see e.g. [8, 9].

An open issue is the vanishing of the boundary term in (2.45). For this we need to have that

$$\frac{1}{2\pi \mathrm{i}} \oint_{\infty} \frac{dz}{z} \, A_n(z) = 0 \,, \tag{2.47}$$

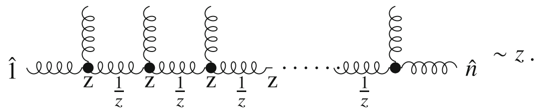
which in turns requires a large-z falloff of the amplitude as $A_n(z) \sim z^{-1}$. In fact, the large-z behaviour depends on the helicities of the shifted legs, and one can show that

$$A\big(\hat{1}^+, \hat{n}^-\big) \overset{z\to\infty}{\sim} \frac{1}{z} \,, \qquad A\big(\hat{1}^+, \hat{n}^+\big) \overset{z\to\infty}{\sim} \frac{1}{z} \,, \qquad A\big(\hat{1}^-, \hat{n}^-\big) \overset{z\to\infty}{\sim} \frac{1}{z} \,, \tag{2.48}$$

yet $A\big(\hat{1}^-, \hat{n}^+\big) \overset{z\to\infty}{\sim} z^3$, which is then a forbidden $[n1\rangle$ shift. In the following we show the first relation; the other scalings are more technical to derive, and may be found in [10].

2.2.1 Large z Falloff

In order to estimate the large z behaviour of generic tree-level amplitudes we perform an analysis based on Feynman graphs. There are three sources for z dependence in a generic colour-ordered amplitude: the propagators, the interaction vertices, and the polarisation vectors. Consider a generic graph contributing to the tree-level n-gluon amplitude ($\hat{1}$ and $\hat{n}$ are assumed to be neighbours). The z dependence occurs only along the path from $\hat{1}$ to $\hat{n}$, see Fig. 2.5. Along this path, each three-gluon vertex, being linear in the momenta, maximally contributes a factor of z, while four-gluon vertices do not contribute, and all propagators along the path contribute a factor of $1/z$. We may derive an upper bound for the z-scaling by considering the diagrams with maximal powers of z. This happens when the path from $\hat{1}$ to $\hat{n}$ contains *only* three-valent vertices. In that case it is easy to see that the graph scales as z:



$\sim z\,.$ (2.49)

Finally, there is an additional z dependence arising from the polarisation vectors at legs 1 and n:

$$\epsilon_1^{+\alpha\dot{\alpha}} = -\sqrt{2}\,\frac{\tilde{\lambda}_1^{\dot{\alpha}}\mu_1^{\alpha}}{\langle\hat{\lambda}_1(z)\mu_1\rangle} \sim \frac{1}{z}\,, \qquad \epsilon_1^{-\alpha\dot{\alpha}} = \sqrt{2}\,\frac{\hat{\lambda}_1^{\alpha}(z)\tilde{\mu}_1^{\dot{\alpha}}}{[\lambda_1\mu_1]} \sim z\,, \tag{2.50}$$

$$\epsilon_n^{+\alpha\dot{\alpha}} = -\sqrt{2}\,\frac{\hat{\tilde{\lambda}}_n^{\dot{\alpha}}(z)\mu_n^{\alpha}}{\langle\lambda_n\mu_n\rangle} \sim z\,, \qquad \epsilon_n^{-\alpha\dot{\alpha}} = \sqrt{2}\,\frac{\lambda_n^{\alpha}\tilde{\mu}_n^{\dot{\alpha}}}{[\hat{\lambda}_n(z)\mu_n]} \sim \frac{1}{z}\,. \tag{2.51}$$

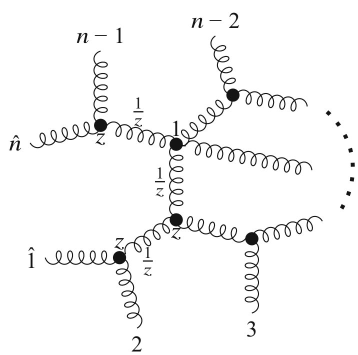


Fig. 2.5 The z scaling of a generic graph: along the path from $\hat{1}$ to $\hat{n}$ the propagators scale as $1/z$, the three-point vertices as z, while four-point vertices do not scale. This sample graph scales as 1. However, if we would replace the four-point vertex by a three-point one, it would scale as z. On top of that we have to consider the z scaling of the polarisation vectors

Therefore, taking all sources of z dependence into account, we conclude that individual graphs scale at worst as

$$A\big(\hat{1}^+, \hat{n}^-\big) \overset{z\to\infty}{\sim} \frac{1}{z}\,, \qquad A\big(\hat{1}^+, \hat{n}^+\big) \overset{z\to\infty}{\sim} z\,,$$
$$A\big(\hat{1}^-, \hat{n}^-\big) \overset{z\to\infty}{\sim} z\,, \qquad A\big(\hat{1}^-, \hat{n}^+\big) \overset{z\to\infty}{\sim} z^3\,. \tag{2.52}$$

This shows that the $[-+\rangle$-shift has the desired falloff properties that allow us to drop the boundary term at infinity in the BCFW formula (2.46). By cyclicity, it is always possible to find a $\{\hat{1}^+, \hat{n}^-\}$ pair. In fact, the above bound is too conservative. It was shown in [10] that the $[++\rangle$ and $[--\rangle$-shifts also lead to an overall $1/z$ scaling once the sum over all Feynman graphs is performed, as the terms scaling as z or 1 cancel out. Only the $[+-\rangle$-shift gives a non-vanishing $\mathrm{Res}(z = \infty)$ in general, and may not be used as basis for a BCFW recursion.

2.3 BCFW Recursion for Gravity and Other Theories

Can we generalise the BCFW recursion to other *massless* quantum field theories? If we analyse its derivation in Sect. 2.2, we see that only two ingredients were needed to establish it:

1. Tree-level amplitudes factorise on simple poles whenever the square of the sum of a subset of external momenta vanishes. While for colour-ordered amplitudes we only need to consider adjacent channels, this is not essential for the derivation of the BCFW recursion: factorisation is a completely general property, and that is all that is needed.
2. The deformed amplitude $A_n(z)$ falls off as $1/z$ at infinity. This depends on the theory and is related to its ultraviolet behaviour.

Therefore, in order to construct tree-level amplitudes recursively without colour ordering from their factorisation properties, we need to consider all multi-particle channels that may occur. We thus generalise the region momenta to include any subset I of the momenta $\{p_1, \ldots, p_n\}$,

$$P_I^\mu := \sum_{i\in I} p_i^\mu\,. \tag{2.53}$$

Whenever $P_I^2 = 0$ we have a pole, and, if a two-particle BCFW shift is used, the set I must contain only one of the shifted momenta so that P_I^2 becomes z-dependent. Concretely, the BCFW recursion for a shift of legs 1 and n as in Eq. (2.38) in gravity takes the form [11, 12]

$$M_n = \sum_Q \sum_{h=\pm\pm} M_L\Big(\hat{1}(z_{P_Q}), Q, -\hat{P}_Q^{-h}(z_{P_Q})\Big)\frac{\mathrm{i}}{P_Q^2}\, M_R\Big(\hat{P}_Q^h(z_{P_Q}), \bar{Q}, \hat{n}(z_{P_Q})\Big)\,, \tag{2.54}$$

where the first sum runs over *all* subsets Q of momenta in $\{p_2, \dots, p_{n-1}\}$, $\bar{Q}$ is the complement of Q, and $P_Q = p_1 + \sum_{i\in Q} P_i$. Again, the recursion is only valid for the $[n1\rangle$ shifts

$$|\hat{1}\rangle = |1\rangle - z|n\rangle\,, \qquad |\hat{n}] = |n] + z|1]\,, \tag{2.55}$$

of the types $[-+\rangle$, $[++\rangle$, and $[--\rangle$. For a derivation see [10].

Finally, we note that the BCFW recursion can be generalised to massive theories [13, 14] to be discussed in Sect. 2.5, to the rational parts of one-loop amplitudes in QCD and gravity [15–19] and form factors [20, 21]. In supersymmetric Yang-Mills theory a supersymmetric version of the BCFW recursion may be formulated [22, 23]. In fact this recursion could be solved analytically [24].

2.4 MHV Amplitudes from the BCFW Recursion Relation

2.4.1 Proof of the Parke-Taylor Formula

As an application of the colour-ordered BCFW recursion (2.46), we now derive the Parke-Taylor formula (1.192) for MHV amplitudes. We already know from Sect. 1.11 that it is true for $n = 3$ and $n = 4$ through an explicit computation. Therefore we shall prove by induction that the formula is correct. We focus here on the case where particles n and 1 have negative helicity, and perform the $[n1\rangle$ shifts of Eq. (2.38). The MHV amplitude has no multi-particle factorisation, as was discussed in Sect. 2.1.1. Hence, only the two BCFW diagrams of Fig. 2.6 contribute to the BCFW recursion of Eq. (2.46). We recall the $[n1\rangle$ shift,

$$|\hat{1}\rangle = |1\rangle - z|n\rangle\,, \quad |\hat{n}] = |n] + z|1]\,, \quad \hat{P} = P - z\,|n\rangle[1|\,, \tag{2.56}$$

whereas $|\hat{1}] = |1]$ and $|\hat{n}\rangle = |n\rangle$ are left inert.

In fact, diagram (II) does not contribute. Here, the right diagram is of $\overline{\text{MHV}}_3$ type. Its numerator reads $[\hat{P}|n-1]^3$, cf. Eq. (1.171), which vanishes:

$$\begin{aligned}[n-1|\hat{P}] &= \frac{[n-1|\hat{P}|n\rangle}{\langle \hat{P}\, n\rangle} = \frac{[n-1|(P - z\,|1]\,\langle n|)|n\rangle}{\langle \hat{P}\, n\rangle} \\ &= \frac{[n-1|P|n\rangle}{\langle \hat{P}\, n\rangle} \overset{P=-p_{n-1}-p_n}{=} 0\,. \end{aligned} \tag{2.57}$$

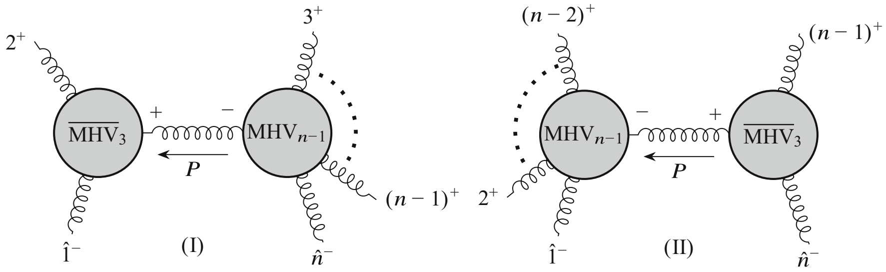


Fig. 2.6 The two BCFW diagrams contributing to the MHV_n amplitude. In fact, diagram (II) does not contribute

In fact, this vanishing is consistent with the observation that the $\overline{\mathrm{MHV}}_3$ kinematical assumption of collinear left-handed spinors, i.e. $\langle(n-1)|\hat{n}\rangle = \langle(n-1)|n\rangle = 0$, forces the two momenta p_{n-1} and p_n to be collinear, $p_{n-1} \parallel p_n$. This is an inconsistent assumption on the n-particle kinematics. This is not a problem for diagram (I), where the analogue condition reads $\langle 2|\hat{1}\rangle = 0$, which does not imply collinearity of p_1 and p_2 as $|\hat{1}\rangle \neq |1\rangle$.

Therefore only the BCFW diagram (I) contributes, where A_L is a three-point $\overline{\mathrm{MHV}}$ amplitude, and A_R is an $(n-1)$-point MHV amplitude. From Eq. (2.43), the position of the pole is

$$z_P = \frac{(p_1+p_2)^2}{\langle n|P|1]} = \frac{\langle 12\rangle[21]}{\langle n2\rangle[21]} = \frac{\langle 12\rangle}{\langle n2\rangle}\,. \tag{2.58}$$

The amplitudes A_L and A_R are then given by

$$A_L = A_3^{\overline{\mathrm{MHV}}}\left(\hat{1}^-, 2^+, -\hat{P}^+\right) = \mathrm{i}g\,\frac{[2(-\hat{P})]^3}{[\hat{1}2][(-\hat{P})\hat{1}]}\,,$$

$$A_R = A_{n-1}^{\mathrm{MHV}}\left(\hat{P}^-, 3^+, 4^+, \ldots (n-1)^+, \hat{n}^-\right) = -\mathrm{i}g^{n-3}\,\frac{\langle\hat{n}\hat{P}\rangle^3}{\langle\hat{P}3\rangle\langle 34\rangle\cdots\langle(n-1)\hat{n}\rangle}\,. \tag{2.59}$$

Using (1.113), the fact that λ_n and $\tilde{\lambda}_1$ are not shifted in our $[n1\rangle$ shift of Eq. (2.55), as well as

$$\langle\hat{n}\hat{P}\rangle[\hat{P}2] = \langle n\hat{1}\rangle\,[12] = \langle n1\rangle\,[12]\,, \qquad \langle 3\hat{P}\rangle\,[\hat{P}1] = \langle 32\rangle\,[2\hat{1}] = \langle 32\rangle\,[21]\,, \tag{2.60}$$

we find

$$A_L \frac{\mathrm{i}}{(p_1+p_2)^2} A_R = \mathrm{i}g^{n-2} \frac{\langle n1\rangle^3\,[12]^3}{[12][21]\langle 32\rangle[21]\,\langle 12\rangle\langle 34\rangle\cdots\langle (n-1)\,n\rangle}$$
$$= -\mathrm{i}g^{n-2}\frac{\langle n1\rangle^4}{\langle 12\rangle\cdots\langle n1\rangle}\,, \tag{2.61}$$

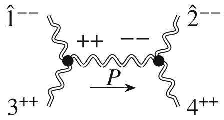
in agreement with the conjecture (1.192) for the chosen helicities. This proves the Parke-Taylor formula for adjacent negative-helicity states.

2.4.2 The Four-Graviton MHV Amplitude

As a second example using the BCFW recursion for non-colour-ordered amplitudes (2.54) we compute the four-graviton amplitude $M_4(1^{--}, 2^{--}, 3^{++}, 4^{++})$. For this we perform a $[2^{--}1^{--}\rangle$ shift,

$$|\hat{1}\rangle = |1\rangle - z|2\rangle\,, \quad |\hat{2}] = |2] + z|1]\,. \tag{2.62}$$

As $M_n(--, ++, \ldots, ++) = 0$ for $n > 3$, we again only find two channels in the BCFW recursion, as shown in Fig. 2.6 with the hatted leg $\hat{n}$ now replaced by $\hat{2}$, while the positive-helicity legs are summed over. Still, the same argument for the vanishing of the type-(II) diagrams applies. For the special case of the four-point graviton amplitude we therefore have

$$M_4(1^{--}, 2^{--}, 3^{++}, 4^{++}) =$$


$$+ (3 \leftrightarrow 4)\,, \tag{2.63}$$

with $P = -p_1 - p_3$. Inserting the three-graviton amplitudes (1.178) this becomes (setting $\kappa = 1$)

$$M_4(1^{--}, 2^{--}, 3^{++}, 4^{++}) = \left(\frac{[\hat{P}3]^3}{[31][1\hat{P}]}\right)^2 \frac{\mathrm{i}}{(p_1+p_3)^2}\left(\frac{\langle\hat{P}2\rangle^3}{\langle 24\rangle\langle 4\hat{P}\rangle}\right)^2 + (3 \leftrightarrow 4)$$
$$= \frac{\mathrm{i}}{s_{13}}\frac{\langle 2|\hat{P}|3]^6}{\langle 24\rangle^2[31]^2\langle 4|\hat{P}|1]^2} + (3 \leftrightarrow 4)\,.$$

We now use $\hat{P} = p_2 + p_4 + z|2\rangle[1|$ to find $\langle 2|\hat{P}|3] = \langle 24\rangle[43]$ and $\langle 4|\hat{P}|1] = \langle 42\rangle[21]$. Hence, the z dependance drops out! Inserting these two relations one finds

$$M_4(1^{--}, 2^{--}, 3^{++}, 4^{++}) = \mathrm{i}\frac{[34]^6}{[12]^2}\left(\frac{\langle 24\rangle^2}{\langle 13\rangle[31]^3} + \frac{\langle 23\rangle^2}{\langle 14\rangle[41]^3}\right), \tag{2.64}$$

where $s_{ij} = (p_i + p_j)^2$. While this is the final result, one may write it using the Mandelstam variables

$$s = (p_1 + p_2)^2\,, \quad t = (p_2 + p_3)^2\,, \quad u = (p_1 + p_3)^2\,. \tag{2.65}$$

Doing this one finally arrives at the compact result (reinstating the coupling)

$$M_4(1^{--}, 2^{--}, 3^{++}, 4^{++}) = \mathrm{i}\kappa^2\,\frac{\langle 12\rangle^4[34]^4}{stu}\,, \tag{2.66}$$

with a peculiar pole structure. The correct helicity scaling is easily checked. Similarly to the gluon case, a closed expression for the MHV n-graviton tree-level amplitude may also be conjectured and proven via BCFW recursion. Yet, it is more involved and may be found in [25].

Exercise 2.4 (The Six-Gluon Split-Helicity NMHV Amplitude) Determine the first non-trivial next-to-maximally-helicity-violating (NMHV) amplitude

$$A_6^{\mathrm{tree}}(1^+, 2^+, 3^+, 4^-, 5^-, 6^-)$$

from a BCFW recursion relation and our knowledge of the MHV amplitudes. Consider a shift of the two helicity states 1^+ and 6^-, and show that

$$\begin{aligned} A_6^{\mathrm{tree}}(1^+, 2^+, 3^+, 4^-,5^-, 6^-) = \mathrm{i}g^4\Bigg(&\frac{\langle 6|P_{12}|3]^3}{\langle 61\rangle\langle 12\rangle[34][45][5|P_{16}|2\rangle} \\ &\frac{1}{(p_6 + p_1 + p_2)^2} + \frac{\langle 4|P_{56}|1]^3}{\langle 23\rangle\langle 34\rangle[16][65][5|P_{16}|2\rangle} \\ &\frac{1}{(p_5 + p_6 + p_1)^2}\Bigg), \end{aligned} \tag{2.67}$$

where $P_{ij} = p_i + p_j$. For the solution see Chap. 5.

Exercise 2.5 (Soft Limit of the Six-Gluon Split-Helicity Amplitude) Check the consistency of the six-gluon split-helicity amplitude of Eq. (2.67) with the soft limit of leg 5. For the solution see Chap. 5.

2.5 BCFW Recursion with Massive Particles

So far we restricted our attention to amplitudes involving massless particles, i.e. gluons, gravitons, and massless fermions and scalars. Yet, scattering amplitudes involving massive particles are of great relevance in physics, and consequently on-shell recursions have been devised for this case as well. Let us focus here on colour-ordered gauge-theory amplitudes involving also massive coloured matter fields (for the colour-ordered amplitudes the representation of the matter field is irrelevant). Concretely, we consider amplitudes involving at least two massless gluons, which we take to be neighbours for simplicity of the discussion, say at positions 1 and n, together with $n-2$ massive states:

$$A_n(p_1, p_2, \dots, p_n)\,, \qquad p_1^2 = 0 = p_n^2\,, \qquad p_i^2 = m_i^2\,. \tag{2.68}$$

Let us now see how the BCFW on-shell recursion derived in Sect. 2.2 may be generalised to gauge theory amplitudes with massive particles. We closely follow reference [13] in our exposition.

As before in Eq. (2.38), we consider a complex shift of the null gluon momenta at positions 1 and n by a parameter $z \in \mathbb{C}$,

$$\hat{p}_1(z) = p_1 - z|n\rangle[1|\,, \qquad \hat{p}_n(z) = p_n + z|n\rangle[1|\,. \tag{2.69}$$

This entails a z-shift of the region momentum $P_i := p_1 + \dots p_{i-1}$,

$$\hat{P}_i(z) = P_i - z|n\rangle[1|\,, \qquad i \in \{3, \dots, n-1\}\,, \tag{2.70}$$

which also featured in the BCFW recursion relation of Eq. (2.42). Importantly, the on-shellness of the deformed legs $\hat{1}$ and $\hat{n}$ as well as total momentum conservation is preserved under the shift. As the arguments leading to the BCFW recursion relation are purely based on factorisation, they are applicable to a generic quantum field theory involving massive particles as well. The BCFW recursion was obtained by thinking about the deformed amplitude $A_n(z)$ as an analytic function in z. Its poles arise whenever an internal propagator associated to the z-shifted region momentum $\hat{P}_i(z)$ goes on-shell. This reasoning does not change in the massive case, i.e. we will have a pole whenever a z-shifted region momentum goes on-shell, i.e. $\hat{P}_i^2(z) = m^2$

with $i \in \{3, \ldots, n-1\}$. The pole then reads in generalisation of Eq. (2.42) as

$$\frac{1}{\hat{P}_i(z)^2 - m_{P_i}^2} = \frac{1}{(\hat{p}_1(z) + p_2 + \ldots p_{i-1})^2 - m_{P_i}^2} = \frac{1}{P_i^2 - m_{P_i}^2 - z\langle n|P_i|1]}\,, \tag{2.71}$$

where m_{P_i} is the mass of the associated intermediate particle going on-shell. Generalising Eq. (2.43), the location of the pole is shifted to

$$z_{P_i} = \frac{P_i^2 - m_{P_i}^2}{\langle n|P_i|1]}\,, \qquad \forall i \in \{3, \ldots, n-1\}\,. \tag{2.72}$$

BCFW Recursion Relation with Massive Particles
Using again the complex analysis arguments of Fig. 2.4, one immediately arrives at the on-shell recursion relation for amplitudes including massive particles:

$$A_n(1, \ldots, n) = \sum_{i=3}^{n-1} \sum_{s \in s_{\mathrm{P}}} A_L\big(\hat{1}(z_{P_i}), 2, \ldots, i-1, -\hat{P}^{\bar{s}}(z_{P_i})\big) \frac{\mathrm{i}\, n_{\mathrm{P}}}{P_i^2 - m_{P_i}^2} \times A_R\big(\hat{P}^s(z_{P_i}), i, \ldots, n-1, \hat{n}(z_{P_i})\big) + \mathrm{Res}(z=\infty)\,, \tag{2.73}$$

where the sum now is over the spins s_{P} of the intermediate particle P and n_{P} is the particle-dependent constant appearing in the factorisation as described below Eq. (2.3). We recall that the legs 1 and n are assumed to be massless.

Again, this formula is only of use if the residue at infinity, $\mathrm{Res}(z=\infty)$, vanishes. This turns out to be the case if the gluon helicities of the shifted legs are not of the $[n^+1^-\rangle$ type, just as in Eq. (2.52). Hence, the statement is

$$\mathrm{Res}(z=\infty) = 0 \quad \text{iff} \quad (h_1, h_n) = (+,-)\,,\ (+,+)\,,\ (-,-)\,. \tag{2.74}$$

See [13] for a derivation. This renders the massive BCFW recursion relation (2.73) very useful.

2.5.1 Four-Point Amplitudes with Gluons and Massive Scalars

Let us construct an explicit example. We consider a theory of a massive complex scalar field coupled to gauge theory. Concretely, we want to evaluate the four-point amplitude involving two neighbouring gluons of positive helicity and two scalars,

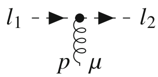
$$A_4(1^+, 2_\phi, 3_{\bar{\phi}}, 4^+)\,. \tag{2.75}$$

The scalars have mass m^2. This amplitude vanishes in the massless limit $m = 0$, similarly to the vanishing of the above amplitude when the scalars are replaced by massless fermions, as was shown in Eq. (1.164). In fact, amplitudes of the above type are of interest even in massless theories at the one-loop level. There, the need to regulate divergences leads one to consider internal particles propagating in $D = 4 - 2\epsilon$ dimensions which may be modelled using masses, as we shall discuss in detail in the next chapter.

Returning to our concrete example we employ the massive on-shell recursion of Eq. (2.73). Only the scalar channel contributes,

$$A_4(1^+, 2_\phi, 3_{\bar{\phi}}, 4^+) = A_3\big(\hat{1}^+, 2_\phi, -\hat{P}_{\bar{\phi}}\big)\,\frac{\mathrm{i}}{P^2 - m^2}\,A_3\big(\hat{P}_\phi, 3_{\bar{\phi}}, \hat{4}^+\big)\,, \tag{2.76}$$

as an amplitude with a single scalar vanishes, $A_3(\hat{1}^+, 2_\phi, 3^\pm) = 0$. All that is needed are the $(\phi g \bar{\phi})$-amplitudes. These follow from the colour-ordered Feynman vertices of two charged scalars and a gluon of Eq. (1.151),

$$V_3(l_1, p^\mu, l_2) =$$ 

$$= \mathrm{i}\frac{g}{\sqrt{2}}\,(l_2^\mu - l_1^\mu)\,, \tag{2.77}$$

where 1 (2) denotes a $\bar{\phi}$ (ϕ) leg, respectively. Contracting this with the positive-helicity gluon polarisation of Eq. (1.124) one obtains the on-shell three-point amplitudes (setting $g = 1$)

$$A_3(l_{1\,\bar{\phi}}, p^+, l_{2\,\phi}) = -\mathrm{i}\frac{\langle r|l_1|p]}{\langle rp\rangle} = A_3(l_{1\,\phi}, p^+, l_{2\,\bar{\phi}})\,, \tag{2.78}$$

where the last relation follows by reflection. Note that here r is the arbitrary null reference momentum of the gluon leg related to the local gauge invariance of the theory. By similar arguments one establishes the three-point amplitudes involving a negative helicity gluon:

$$A_3(l_{1\,\bar{\phi}}, p^-, l_{2\,\phi}) = -\mathrm{i}\frac{\langle p|l_1|r]}{[pr]} = A_3(l_{1\,\phi}, p^-, l_{2\,\bar{\phi}})\,. \tag{2.79}$$

Before moving on with the recursion, let us address a seemingly dramatic problem: the amplitudes of Eqs. (2.78) and (2.79) apparently depend on the reference momentum r—how can that be? It turns out that, despite their representation, the amplitudes in Eqs. (2.78) and (2.79) are actually *independent* of the choice of r. Taking the initial reference spinor $|r\rangle$ and $|p\rangle$ as a basis in Weyl spinor space, we may parametrise an arbitrary reference spinor different from $|r\rangle$ as $|r'\rangle = \alpha|r\rangle + \beta|p\rangle$. Clearly, Eq. (2.78) is invariant under rescaling of the reference spinor $|r\rangle \to \Lambda\,|r\rangle$, thus without loss of generality we may parameterise $|r'\rangle = |r\rangle + \gamma|p\rangle$, or infinitesimally write $\delta_r|r\rangle \propto |p\rangle$. This entails that the amplitude Eq. (2.78) changes under a variation of the reference spinor $|r\rangle$ as

$$\delta_r A_3(l_1^+, p^+, l_2^-) \propto \frac{\langle p|l_1|p]}{\langle rp\rangle} = 0\,, \tag{2.80}$$

where the vanishing follows from $\langle p|l_1|p] = 2p\cdot l_1 = 0$, which is a consequence of the three-point kinematics:

$$(l_1+p)^2 = l_2^2 \quad \to \quad l_1\cdot p = 0 \quad \text{as} \quad l_1^2 = l_2^2 = m^2\,,\ p^2 = 0\,. \tag{2.81}$$

A similar argument applies to Eq. (2.79). Again we see the subtleties in three-point amplitudes: the expressions in Eqs. (2.78) and (2.79) are actually independent of r.

Coming back to the recursive construction of the amplitude $A_4(1^+, 2_\phi, 3_{\bar\phi}, 4^+)$, we have (cf. Fig. 2.7)

$$\begin{aligned} A_4(1^+, 2_\phi, 3_{\bar\phi}, 4^+) &= A_3\big(-\hat P_{\bar\phi}, \hat 1^+, 2_\phi\big)\,\frac{\mathrm{i}}{P^2-m^2}\,A_3\big(3_{\bar\phi}, \hat 4^+, \hat P_\phi\big) \\ &= \left(\mathrm{i}\,\frac{\langle r_1|\hat P|\hat 1]}{\langle r_1\hat 1\rangle}\right)\frac{\mathrm{i}}{P^2-m^2}\left(-\mathrm{i}\,\frac{\langle r_4|p_3|\hat 4]}{\langle r_4\hat 4\rangle}\right), \end{aligned} \tag{2.82}$$

with $P = p_1 + p_2$. Here $r_{1/4}$ denote the reference momenta of the gluon legs 1 and 4. Things are simplified considerably with the gauge choice

$$r_1 = \hat p_4\,, \qquad r_4 = \hat p_1\,. \tag{2.83}$$

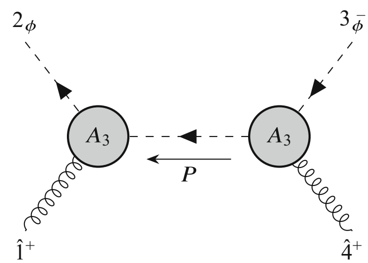


Fig. 2.7 On-shell recursion for the massive $A_4(1^+, 2_\phi, 3_{\bar\phi}, 4^+)$ amplitude. All external momenta are outgoing, P runs from right to left

Noting that $|\hat{1}] = |1]$ and $|\hat{4}\rangle = |4\rangle$, we then have

$$\langle r_1|\hat{P}|\hat{1}] = \langle 4|\hat{P}|1] = \langle 4|P|1] = -\langle 4|p_3|1]\,, \qquad \langle r_1\hat{1}\rangle = \langle 4\hat{1}\rangle = \langle 41\rangle\,,$$
$$\langle r_4|p_3|\hat{4}] = \langle \hat{1}|p_3|\hat{4}]\,, \qquad \langle r_4\hat{4}\rangle = \langle \hat{1}4\rangle = \langle 14\rangle\,. \tag{2.84}$$

Plugging these into the above we find

$$A_4 = \mathrm{i}\,\frac{\langle 4|p_3|1]\,\langle\hat{1}|p_3|\hat{4}]}{\langle 14\rangle^2\,[(p_1+p_2)^2-m^2]}\,. \tag{2.85}$$

The numerator may be simplified with a trace identity (see Eq. (1.29)) to

$$\begin{aligned}\langle 4|p_3|1]\,\langle\hat{1}|p_3|\hat{4}] &= \frac{1}{2}\,\mathrm{Tr}\left(\hat{\not{p}}_4\,\not{p}_3\,\hat{\not{p}}_1\,\not{p}_3\right)\\ &= 2\left(2(p_3\cdot\hat{p}_4)\,(p_3\cdot\hat{p}_1) - p_3^2\,(\hat{p}_1\cdot\hat{p}_4)\right).\end{aligned} \tag{2.86}$$

In fact $p_3\cdot\hat{p}_4 = 0$, which follows from momentum conservation:

$$\hat{P}^2 = (p_3+\hat{p}_4)^2 \quad\Rightarrow\quad m^2 = 2p_3\cdot\hat{p}_4 + p_3^2 \quad\Rightarrow\quad p_3\cdot\hat{p}_4 = 0\,. \tag{2.87}$$

Moreover, we find $2\hat{p}_1\cdot\hat{p}_4 = \langle\hat{1}4\rangle[\hat{4}1] = \langle 14\rangle[41]$. Putting everything together we arrive at the compact expression for the four-point amplitude

$$A_4(1^+,2_\phi,3_{\bar{\phi}},4^+) = \mathrm{i}\,\frac{[14]\,m^2}{\langle 14\rangle\,[(p_1+p_2)^2-m^2]}\,, \tag{2.88}$$

which indeed vanishes in the massless limit, as claimed.

Exercise 2.6 (Mixed-Helicity Four-Point Scalar-Gluon Amplitude) Compute the four-point massive-scalar-gluon amplitude with one positive and one negative gluon,

$$A_4(1^+,2_\phi,3_{\bar{\phi}},4^-) = \mathrm{i}\,\frac{\langle 4|p_3|1]^2}{(p_1+p_4)^2\,[(p_1+p_2)^2-m^2]}\,, \tag{2.89}$$

using the above recursive techniques. For the solution see Chap. 5.

2.6 Symmetries of Scattering Amplitudes

We now turn to a more conceptual yet very important subject: the question of how the space-time symmetries of the Poincaré group (and beyond) that we discussed in Sect. 1.1 manifest themselves at the level of scattering amplitudes. This has proven to be a very rich subject in particular at tree level. The space-time symmetries of scattering amplitudes may be grouped into obvious and less obvious symmetries.

The obvious symmetries are the Poincaré transformations of Sect. 1.1, under which scattering amplitudes should be invariant. Sticking to massless amplitudes, working in the spinor-helicity formulation of momentum space is highly advantageous. Here the momentum generator $p^{\alpha\dot{\alpha}}$ is represented by a multiplicative operator,

$$p^{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \lambda_i^{\alpha}\, \tilde{\lambda}_i^{\dot{\alpha}}\,, \tag{2.90}$$

and any amplitude $\mathcal{A}_n$ must obey

$$p^{\alpha\dot{\alpha}}\, \mathcal{A}_n(\{\lambda_i, \tilde{\lambda}_i\}) = 0\,. \tag{2.91}$$

This is in fact true in the distributional sense of the relation $p\,\delta(p) = 0$, thanks to the total momentum conserving delta-function present in each amplitude (that we often drop):

$$\mathcal{A}_n(\lambda_i, \tilde{\lambda}_i) = \delta^{(4)}\Big(\sum_i p_i\Big)\, A_n\left(\lambda_i, \tilde{\lambda}_i\right)\,. \tag{2.92}$$

In our notation we distinguish the full amplitude $\mathcal{A}_n$ and the delta-function "stripped" amplitude A_n.

The Lorentz generators in the helicity spinor basis come in two pairs of symmetric rank-two tensors $m_{\alpha\beta}$ and $\overline{m}_{\dot{\alpha}\dot{\beta}}$, originating from the projections $M^{\mu\nu}\,(\sigma_{\mu\nu})_{\alpha\beta} = m_{\alpha\beta}$ and $M^{\mu\nu}\,(\bar{\sigma}_{\mu\nu})_{\dot{\alpha}\dot{\beta}} = \overline{m}_{\dot{\alpha}\dot{\beta}}$. They are first-order differential operators in helicity spinor space,

$$m_{\alpha\beta} = \sum_{i=1}^{n} \lambda_{i\,(\alpha}\, \partial_{i\,\beta)}\,, \qquad \overline{m}_{\dot{\alpha}\dot{\beta}} = \sum_{i=1}^{n} \tilde{\lambda}_{i\,(\dot{\alpha}}\, \partial_{i\,\dot{\beta})}\,, \tag{2.93}$$

where $\partial_{i\alpha} := \frac{\partial}{\partial\lambda_i^{\alpha}}$, $\partial_{i\dot{\alpha}} := \frac{\partial}{\partial\tilde{\lambda}_i^{\dot{\alpha}}}$ and $r_{(\alpha\beta)} := (r_{\alpha\beta} + r_{\beta\alpha})/2$ denotes symmetrisation with unit weight, cf. Exercise 1.6. The invariance of $A_n(\lambda_i, \tilde{\lambda}_i)$ under Lorentz-transformations,

$$m_{\alpha\beta}\, A_n(\lambda_i, \tilde{\lambda}_i) = 0 = \overline{m}_{\dot{\alpha}\dot{\beta}}\, A_n(\lambda_i, \tilde{\lambda}_i)\,, \tag{2.94}$$

is manifest, as it is an immediate consequence of the proper contraction of all Weyl indices within A_n, i.e. the fact that the spinor brackets $\langle ij\rangle$ and $[ij]$ are invariant under $m_{\alpha\beta}$ and $\overline{m}_{\dot{\alpha}\dot{\beta}}$. See the solution of Exercise 1.6 for an explicit calculation.

Let us now discuss a set of less obvious symmetries of $\mathcal{A}_n(\lambda_i, \tilde{\lambda}_i)$ in the case of pure colour-ordered gluon amplitudes. Classical Yang-Mills theory is in fact invariant under a larger symmetry group than Poincaré: due to the absence of dimensionful parameters in the theory (the coupling g is dimensionless) pure Yang-Mills theory (as well as massless QCD or scalar QCD) is invariant under a scale transformation,

$$x^\mu \to \Lambda^{-1} x^\mu , \quad \text{respectively} \quad p^\mu \to \Lambda\, p^\mu . \tag{2.95}$$

The scale transformations of the momenta are generated by the dilatation operator d, whose representation in spinor-helicity variables acting on amplitudes is

$$d = \sum_{i=1}^{n} \left(\frac{1}{2}\lambda_i^\alpha\, \partial_{i\,\alpha} + \frac{1}{2}\tilde{\lambda}_i^{\dot{\alpha}}\, \partial_{i\,\dot{\alpha}} + d_0 \right) , \qquad d_0 \in \mathbb{R} , \tag{2.96}$$

reflecting the mass dimension 1/2 of the λ_i and $\tilde{\lambda}_i$ helicity spinors, i.e. $[d, \lambda_i] = \lambda_i/2$ and $[d, \tilde{\lambda}_i] = \tilde{\lambda}_i/2$. The constant d_0 is undetermined at this point. It may be fixed by requiring invariance of the MHV gluon amplitudes,

$$\mathcal{A}_n^{\text{MHV}} = \delta^{(4)}(p)\, \frac{\langle st\rangle^4}{\langle 12\rangle \ldots \langle n1\rangle} , \tag{2.97}$$

where $p = \sum_{i=1}^n p_i$. The dilatation operator d of Eq. (2.96) simply measures the weight in units of mass of the amplitude it acts on adding a factor of $n\, d_0$, namely

$$d\, \mathcal{A}_n = \big([\mathcal{A}_n] + n\, d_0\big)\, \mathcal{A}_n , \tag{2.98}$$

where $[\mathcal{O}]$ indicates the weight of $\mathcal{O}$ in dimensions of mass. We note the weights $[\delta^{(4)}(p)] = -4$, $[\langle st\rangle^4] = 4$ and $[\langle 12\rangle \ldots \langle n1\rangle] = n$, hence

$$d\, \mathcal{A}_n^{\text{MHV}} = (-4 + 4 - n + n\, d_0)\, \mathcal{A}_n^{\text{MHV}} , \tag{2.99}$$

which vanishes for the choice $d_0 = 1$. One easily checks the invariance under dilatations of the $q\bar{q}gg$-amplitude of Eq. (1.194) and of the $\overline{\text{MHV}}_n$ amplitudes as well.

The scaling symmetry comes with a further less obvious symmetry of vectorial nature known as special conformal transformations. Their generators, denoted by $k_{\alpha\dot{\alpha}}$, are realised in terms of a second-order differential operator in the spinor variables,

$$k_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \partial_{i\,\alpha}\, \partial_{i\,\dot{\alpha}}\,. \tag{2.100}$$

Together with the Poincaré and dilatation generators, the set $\{p_{\alpha\dot{\alpha}}, k_{\alpha\dot{\alpha}}, m_{\alpha\beta}, \overline{m}_{\dot{\alpha}\dot{\beta}}, d\}$ generates the *conformal group* in four dimensions, SO(2, 4).

Let us now prove the invariance of the MHV gluon amplitudes under special conformal transformations. As the only dependence of $\mathcal{A}_n^{\mathrm{MHV}}$ on the conjugate spinors $\tilde{\lambda}_i$ resides in the momentum conserving delta-function, we have

$$\begin{aligned} k_{\alpha\dot{\alpha}}\, \mathcal{A}_n^{\mathrm{MHV}} &= \sum_{i=1}^{n} \frac{\partial}{\partial \lambda_i^{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} \left(\delta^{(4)}(p)\, A_n^{\mathrm{MHV}}\right) \\ &= \sum_{i=1}^{n} \frac{\partial}{\partial \lambda_i^{\alpha}} \left[\frac{\partial p^{\beta\dot{\beta}}}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} \left(\frac{\partial}{\partial p^{\beta\dot{\beta}}}\, \delta^{(4)}(p)\right) A_n^{\mathrm{MHV}} \right] \\ &= \left[\left(n\, \frac{\partial}{\partial p^{\alpha\dot{\alpha}}} + p^{\beta\dot{\beta}}\, \frac{\partial}{\partial p^{\beta\dot{\alpha}}}\, \frac{\partial}{\partial p^{\alpha\dot{\beta}}} \right) \delta^{(4)}(p) \right] A_n^{\mathrm{MHV}} \\ &\quad + \left(\frac{\partial\, \delta^{(4)}(p)}{\partial p^{\beta\dot{\alpha}}} \right) \sum_{i=1}^{n} \lambda_i^{\beta}\, \frac{\partial}{\partial \lambda_i^{\alpha}}\, A_n^{\mathrm{MHV}}\,. \end{aligned} \tag{2.101}$$

The last term may be rewritten as follows. First, we note the relation

$$\sum_{i=1}^{n} \lambda_{i\,\alpha}\, \partial_{i\,\beta} = \sum_{i=1}^{n} \lambda_{i\,(\alpha}\, \partial_{i\,\beta)} + \frac{1}{2}\, \epsilon_{\alpha\beta} \sum_{i} \lambda_i^{\gamma}\, \partial_{i\,\gamma}\,, \tag{2.102}$$

which follows from decomposing the left-hand side in a symmetric and anti-symmetric piece. The first term on the right-hand side is the Lorentz generator $m_{\alpha\beta}$, which we already know annihilates A_n^{MHV}. The remaining term yields

$$\sum_{i=1}^{n} \lambda_i^{\beta}\, \frac{\partial}{\partial \lambda_i^{\alpha}}\, A_n^{\mathrm{MHV}} = \frac{1}{2}\, \delta_{\alpha}^{\beta} \sum_{i} \lambda_i^{\delta}\, \partial_{i\,\delta}\, A_n^{\mathrm{MHV}} = (4-n)\, A_n^{\mathrm{MHV}}\,. \tag{2.103}$$

Hence Eq. (2.101) turns into

$$k_{\alpha\dot{\alpha}}\, \mathcal{A}_n^{\mathrm{MHV}} = \left[\left(4\, \frac{\partial}{\partial p^{\alpha\dot{\alpha}}} + p^{\beta\dot{\beta}}\, \frac{\partial}{\partial p^{\beta\dot{\alpha}}}\, \frac{\partial}{\partial p^{\alpha\dot{\beta}}} \right) \delta^{(4)}(p) \right] A_n^{\mathrm{MHV}}\,. \tag{2.104}$$

Indeed in a distributional sense we have

$$p^{\beta\dot{\beta}}\,\frac{\partial}{\partial p^{\beta\dot{\alpha}}}\,\frac{\partial}{\partial p^{\alpha\dot{\beta}}}\,\delta^{(4)}(p) = -4\,\frac{\partial}{\partial p^{\alpha\dot{\alpha}}}\,\delta^{(4)}(p)\,, \tag{2.105}$$

which one may see by integrating the second derivative expression against a test function $F(p)$:

$$\begin{aligned}
&\int \mathrm{d}^4 p\, F(p)\, p^{\beta\dot{\beta}}\,\frac{\partial}{\partial p^{\beta\dot{\alpha}}}\,\frac{\partial}{\partial p^{\alpha\dot{\beta}}}\,\delta^{(4)}(p) = \\
&\qquad = \int \mathrm{d}^4 p\left[\left(\frac{\partial}{\partial p^{\beta\dot{\alpha}}}\,F(p)\right)2\,\delta_{\alpha}^{\beta} + \left(\frac{\partial}{\partial p^{\alpha\dot{\beta}}}\,F(p)\right)2\,\delta_{\dot{\alpha}}^{\dot{\beta}}\right] \\
&\qquad = 4\int \mathrm{d}^4 p\left(\frac{\partial}{\partial p^{\alpha\dot{\alpha}}}\,F(p)\right)\delta^{(4)}(p)\,.
\end{aligned} \tag{2.106}$$

This proves the vanishing of $k_{\alpha\dot{\alpha}}\,\mathcal{A}_n^{\mathrm{MHV}}$, as claimed.

Summarising, we have constructed a representation of the conformal group whose generators are represented by differential operators of degree one ($m_{\alpha\beta}$, $\overline{m}_{\dot{\alpha}\dot{\beta}}$, d), of degree two ($k_{\alpha\dot{\alpha}}$), and as a multiplicative operator ($p_{\alpha\dot{\alpha}}$) in an n-particle helicity spinor space. This representation is natural, as amplitudes are functions in this space. All these generators annihilate the scattering amplitudes. We have verified this explicitly for the MHV amplitudes. The representation obeys the commutation relations of the conformal algebra $\mathfrak{so}(2,4)$,

$$\begin{aligned}
&[d, p^{\alpha\dot{\alpha}}] = p^{\alpha\dot{\alpha}}\,, \qquad [d, k_{\alpha\dot{\alpha}}] = -k_{\alpha\dot{\alpha}}\,, \qquad [d, m_{\alpha\beta}] = 0 = [d, \overline{m}_{\dot{\alpha}\dot{\beta}}]\,, \\
&\qquad [k_{\alpha\dot{\alpha}}, p^{\beta\dot{\beta}}] = \delta_{\alpha}{}^{\beta}\,\delta_{\dot{\alpha}}{}^{\dot{\beta}}\,d + m_{\alpha}{}^{\beta}\,\delta_{\dot{\alpha}}{}^{\dot{\beta}} + \overline{m}_{\dot{\alpha}}{}^{\dot{\beta}}\,\delta_{\alpha}{}^{\beta}\,,
\end{aligned} \tag{2.107}$$

plus the Poincaré commutators discussed in Sect. 1.1.

The origin of this helicity spinor space representation becomes clear if one looks at the more familiar representation of the conformal group in configuration space x^μ. For scalar fields, it reads

$$\begin{aligned}
M_{\mu\nu} &= \mathrm{i}\left(x_\mu\,\partial_\nu - x_\nu\,\partial_\mu\right), & P_\mu &= -\mathrm{i}\,\partial_\mu\,, \\
\mathcal{D} &= -\mathrm{i}\,(x^\mu\,\partial_\mu + \Delta)\,, & K_\mu &= \mathrm{i}\left[x^2\,\partial_\mu - 2\,x_\mu\,(x^\nu\partial_\nu + \Delta)\right],
\end{aligned} \tag{2.108}$$

where Δ is the scaling dimension and $\partial_\mu := \partial/\partial x^\mu$. In quantum field theory the conformal symmetry is distinguished by the Haag-Lopuszanski-Sohnius theorem [26] as the maximal bosonic extension of the space-time symmetry of the S-matrix—generalising the Poincaré algebra.

A Fourier transform $\int d^4x\, e^{ip\cdot x} O(x, \partial_x)$ brings this representation into momentum space. From this point of view, it is clear why $p^{\alpha\dot{\alpha}}$ becomes a multiplication operator and $k_{\alpha\dot{\alpha}}$ a second-order derivative operator in momentum space as seen above. The momentum-space representation of the conformal symmetry as it applies to scattering amplitudes then takes the form

$$\begin{aligned} m^{\mu\nu} &= p^\mu\,\partial_p^\nu - p^\nu\,\partial_p^\mu\,, & p^\mu &= p^\mu\,, \\ d &= p_\mu\,\partial_p^\mu + \bar{\Delta}\,, & k^\mu &= p^\mu\partial_p^2 - 2\big(p_\nu\partial_p^\nu + \bar{\Delta}\big)\partial_p^\mu\,, \end{aligned} \tag{2.109}$$

where $\bar{\Delta} = 4 - \Delta$. This representation may be mapped to the helicity-spinor one discussed above.

▶ **The Conformal Generators in Spinor Helicity Space** Here we collect the generators of the conformal algebra in their single-particle action (with $\bar{\Delta} = 1$, which is relevant for gauge bosons and scalar fields):

$$\begin{aligned} p^{\alpha\dot{\alpha}} &= \lambda^\alpha\tilde{\lambda}^{\dot{\alpha}}\,, & k_{\alpha\dot{\alpha}} &= \partial_\alpha\partial_{\dot{\alpha}}\,, \\ m_{\alpha\beta} &= \lambda_{(\alpha}\partial_{\beta)}\,, & \overline{m}_{\dot{\alpha}\dot{\beta}} &= \tilde{\lambda}_{(\dot{\alpha}}\partial_{\dot{\beta})}\,, \\ d &= \tfrac{1}{2}\lambda^\alpha\partial_\alpha + \tfrac{1}{2}\tilde{\lambda}^{\dot{\alpha}}\partial_{\dot{\alpha}} + 1\,, \end{aligned} \tag{2.110}$$

where $r_{(\alpha\beta)} := (r_{\alpha\beta} + r_{\beta\alpha})/2$ denotes symmetrisation of the indices. The helicity generator is given by $h = -\frac{1}{2}\lambda^\alpha\partial_\alpha + \frac{1}{2}\tilde{\lambda}^{\dot{\alpha}}\partial_{\dot{\alpha}}$. It commutes with all generators of the conformal algebra.

Scaleless quantum field theories—such as pure Yang-Mills or massless QCD—enjoy conformal symmetry at the tree-level. Yet, this symmetry is usually broken at the loop level, as the need to regularise divergencies introduces a scale into the quantum theory. It is manifested by a non-vanishing β-function of the coupling g. In fact, understanding the implications of broken conformal symmetry for loop amplitudes is an area of ongoing research. In the maximally supersymmetric generalisations of Yang-Mills theory, $\mathcal{N} = 4$ super Yang-Mills theory, this property is prominently absent. It is a quantum conformal field theory, with far-reaching consequences, including a hidden infinite dimensional symmetry known as the Yangian symmetry. In fact, tree-level amplitudes are invariant under this extension of the super-conformal group [27] and the hidden integrability of the leading colour limit of the theory allows for exact non-perturbative results, see [28] for a review.

Exercise 2.7 (Conformal Algebra) Show that the representation constructed in the above Eq. (2.110) indeed obeys the commutation relations of the conformal algebra given in Eq. (2.107). For the solution see Chap. 5.

Exercise 2.8 (Inversion and Special Conformal Transformations) The generator K_μ of Eq. (2.108) generates infinitesimal special conformal transformations. A finite special conformal transformation is given by

$$x^\mu \to x'^\mu = \frac{x^\mu - a^\mu\, x^2}{1 - 2\,a\cdot x + a^2\, x^2}\,, \tag{2.111}$$

where a^μ is the transformation parameter.

(a) An intuition on the character of these transformations may be found by noting that the action of K^μ may be also written as $K^\mu = I\,P^\mu\,I$, i.e. as the composition of an inversion $I\,x^\mu = x^\mu/x^2$, followed by a translation $P^\mu x = x^\mu - a^\mu$, followed by another inversion. Show that $K^\mu = I\,P^\mu\,I$ is equivalent to Eq. (2.111).

(b) A scalar field $\Phi(x)$ transforms under special conformal transformations $x \to x'$ as

$$\Phi(x) \longrightarrow \Phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/4} \Phi(x)\,, \tag{2.112}$$

where $|\partial x'/\partial x|$ is the Jacobian of the transformation, and Δ is the scaling dimension. Show that the generator K_μ of Eq. (2.108) indeed generates the transformation of Eq. (2.111) for a scalar field. In other words, prove that

$$\Phi'(x) = \Big[1 - \mathrm{i}\, a^\mu\, K_\mu + O\big(a^2\big)\Big]\, \Phi(x)\,. \tag{2.113}$$

Hint: in order to compute the Jacobian factor, treat the finite special conformal transformation as the composition of inversion, translation and inversion.

For the solution see Chap. 5.

2.7 Double-Copy Relations for Gluon and Graviton Amplitudes

So far we have discussed gauge theories and gravity rather in parallel. While their Lagrangians and Feynman rules look very different, there exist intriguing relationships between gluon amplitudes and graviton amplitudes that suggest a deeper relationship of these two theories—at least in their perturbative domain. In a nutshell, they express gravity as the square of Yang-Mills theory, a property we already saw at the level of the polarisations and of the three-point amplitudes.

2.7.1 Lower-Point Examples

The squaring relation between gravity and gluon amplitudes is manifest at the level of three-point amplitudes:

$$\begin{aligned} M_3^{\text{tree}}(1^{--}, 2^{--}, 3^{++}) &= \frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 31\rangle^2} = \left[A_3^{\text{tree}}(1^-, 2^-, 3^+)\right]^2, \\ M_3^{\text{tree}}(1^{--}, 2^{++}, 3^{++}) &= \frac{[23]^6}{[12]^2[31]^2} = \left[A_3^{\text{tree}}(1^-, 2^+, 3^+)\right]^2. \end{aligned} \tag{2.114}$$

For simplicity we set all couplings to unity here and absorbed a factor of i in the amplitudes. Hence, for any choice of polarisations we find

$$M_3^{\text{tree}}(1, 2, 3) = A_3^{\text{tree}}(1, 2, 3)^2\,. \tag{2.115}$$

Turning to the four-point case, we need to compare the MHV_4 gluon amplitude and the MHV_4 four-graviton amplitude of Eq. (2.66). We first expose the s, t, u poles in the MHV colour-ordered gluon amplitude explicitly. For $A_4(1^-, 2^-, 3^+, 4^+)$ we have

$$\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle} = -\frac{1}{s_{12}}\frac{\langle 12\rangle^3[34]}{\langle 23\rangle\langle 41\rangle} = -\frac{1}{s_{12}s_{23}}\frac{\langle 12\rangle^2\,\overbrace{\langle 12\rangle[32]}^{\langle 14\rangle[43]}[34]}{\langle 41\rangle} = -\frac{\langle 12\rangle^2[34]^2}{s_{12}s_{23}}\,.$$

This gets close to the four-graviton amplitude (2.66) but is not just a simple square. Looking at $A_4(1^-, 2^-, 4^+, 3^+)$, that is obtained by swapping $3 \leftrightarrow 4$ in the above, we arrive at

$$M_4^{\text{tree}}(1^{--}, 2^{--}, 3^{++}, 4^{++}) = s_{12}\, A_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+)\, A_4^{\text{tree}}(1^-, 2^-, 4^+, 3^+)\,. \tag{2.116}$$

Again resulting in a squaring relation between the two. In fact, such squaring relations turn out to be generally true for all multiplicities.

2.7.2 Colour-Kinematics Duality: A Four-Point Example

The relations (2.115) and (2.116) suggest a general squaring relation of the structure $M_n \sim A_n^2$. This can be made precise in the context of the colour-kinematic duality, which we now want to discuss in a four-point example.

For this, let us look at the *coloured* tree-level four-gluon amplitude in D dimensions using the polarisation ϵ_i^μ and momentum p_i^μ vectors with $i = 1, \ldots, 4$ to describe the kinematics. It may be written as

$$\mathbf{A}_4^{\text{tree}} = -\mathrm{i}g^2 \left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_s}{u} \right) , \tag{2.117}$$

split into the s, t and u channels, with

$$c_s = -2 f^{a_1 a_2 e} f^{e a_3 a_4} , \tag{2.118}$$

$$\begin{aligned} n_s = -\frac{1}{2} \Big\{ & \big[(\epsilon_1 \cdot \epsilon_2)\, p_1^\mu + 2(\epsilon_1 \cdot p_2)\, \epsilon_2^\mu - (1 \leftrightarrow 2) \big] \\ & \times \big[(\epsilon_3 \cdot \epsilon_4)\, p_{3,\,\mu} + 2(\epsilon_3 \cdot p_4)\, \epsilon_{4,\,\mu} - (3 \leftrightarrow 4) \big] \\ & + s \big[(\epsilon_1 \cdot \epsilon_3)(\epsilon_2 \cdot \epsilon_4) - (\epsilon_1 \cdot \epsilon_4)(\epsilon_2 \cdot \epsilon_3) \big] \Big\} , \end{aligned} \tag{2.119}$$

and

$$c_t\, n_t = c_s\, n_s \Big|_{1 \to 2 \to 3 \to 1} , \qquad c_u\, n_u = c_s\, n_s \Big|_{1 \to 3 \to 2 \to 1} . \tag{2.120}$$

In writing the amplitude in this fashion we have split up contact terms emerging from the four-gluon vertex (1.66) into the s, t and u channels by multiplying the corresponding colour factors n_s by $\frac{s}{s}$, and so on. These are the terms proportional to s in the last line of Eq. (2.119). It is instructive to study the gauge invariance of leg 4. Replacing $\epsilon_4 \to p_4$ yields

$$n_s \Big|_{\epsilon_4 \to p_4} = \frac{s}{2} \Big[(\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot p_{12}) + \text{cyclic}(1, 2, 3) \Big] =: s\, \alpha(\{\epsilon_i, p_i\}) , \tag{2.121}$$

where $p_{12} = p_1 - p_2$. Crucially, the function α is cyclically invariant. Therefore the gauge transformations of the other kinematical numerators read

$$n_t \Big|_{\epsilon_4 \to p_4} = t\, \alpha(\{\epsilon_i, p_i\}) , \qquad n_u \Big|_{\epsilon_4 \to p_4} = u\, \alpha(\{\epsilon_i, p_i\}) . \tag{2.122}$$

Hence, the numerators are not gauge invariant individually. This is to be expected, as only the full amplitude and not individual graphs (or parts thereof) are gauge invariant. How does $\mathbf{A}_4^{\text{tree}}$ then become gauge invariant? We have

$$\mathbf{A}_4^{\text{tree}}\Big|_{\epsilon_4 \to p_4} = (c_s + c_t + c_u)\,\alpha(\{\epsilon_i, p_i\})\,, \tag{2.123}$$

which is zero by virtue of Jacobi's identity (1.135),

$$c_s + c_t + c_u = -2\left(f^{a_1a_2e}\,f^{ea_3a_4} + f^{a_2a_3e}\,f^{ea_1a_4} + f^{a_3a_1e}\,f^{ea_2a_4}\right) = 0\,. \tag{2.124}$$

Remarkably, one also sees that the kinematical numerators n_i obey a Jacobi-like identity,

$$n_s + n_t + n_u = 0\,, \tag{2.125}$$

upon using the on-shell identities. This is known as the *kinematical Jacobi identity*. This property allows us to construct a gauge invariant object that has all the required properties to be the four-graviton amplitude: we simply replace the colour numerators c_i by the kinematical ones n_i in Eq. (2.117), obtaining

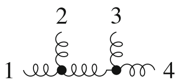
$$M_4^{\text{tree}} = \mathbf{A}_4^{\text{tree}}\Big|_{\substack{c_i \,\to\, n_i \\ g \,\to\, \kappa/2}} = -\mathrm{i}\left(\frac{\kappa}{2}\right)^2\left(\frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}\right). \tag{2.126}$$

Clearly, it is bi-linear in the polarisation vectors ϵ_i^μ and displays a consistent pole structure, which are necessary ingredients for it to be a graviton amplitude. Also the gauge invariance may be tested straightforwardly using Eqs. (2.121) and (2.122),

$$M_4^{\text{tree}}\Big|_{\epsilon_4 \to p_4} = 2\,(n_s + n_t + n_u)\,\alpha(\{\epsilon_i, p_i\})\,, \tag{2.127}$$

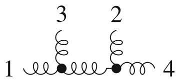
which now vanishes by virtue of the kinematical Jacobi identity (2.125). We shall show momentarily that the result (2.126) is equivalent to Eq. (2.116). In order to do so, let us express the gluon amplitude (2.117) in a minimal colour and kinematical basis. Going to the DDM basis of Sect. 1.10 amounts to eliminating c_t via $c_t = -c_s - c_u$ as

$$c_s = f^{a_1a_2e}f^{ea_3a_4} =$$



,

$$-c_u = f^{a_1a_3e}f^{ea_2a_4} =$$



. (2.128)

The amplitude then takes the form

$$\begin{aligned}\frac{\mathrm{i}}{g^2}\mathbf{A}_4^{\text{tree}} &= c_s\left(\frac{n_s}{s}-\frac{n_t}{t}\right)-c_u\left(\frac{n_t}{t}-\frac{n_u}{u}\right)\\ &= c_s\,A_4^{\text{tree}}(1,2,3,4)-c_u\,A_4^{\text{tree}}(1,3,2,4)\,,\end{aligned}\tag{2.129}$$

hence the two colour-ordered amplitudes are to be identified as

$$A_4^{\text{tree}}(1,2,3,4)=\frac{n_s}{s}-\frac{n_t}{t}\,,\qquad A_4^{\text{tree}}(1,3,2,4)=\frac{n_t}{t}-\frac{n_u}{u}\,.\tag{2.130}$$

Their gauge invariance follows from Eqs. (2.121) and (2.122), e.g.

$$A_4^{\text{tree}}(1,2,3,4)\Big|_{\epsilon_4\to p_4}=\left(\frac{s}{s}-\frac{t}{t}\right)\alpha=0\,.$$

This must be the case, as we argued before for the gauge invariance of the colour-ordered amplitudes. Due to the kinematical Jacobi identity, the above representation of the amplitude is not yet minimal. By eliminating (in analogy to c_t) now n_t as well via $n_t=-n_s-n_u$ in Eq. (2.129) we find the relation

$$\begin{pmatrix}A_4^{\text{tree}}(1,2,3,4)\\ A_4^{\text{tree}}(1,3,2,4)\end{pmatrix}=\begin{pmatrix}s^{-1}+t^{-1} & t^{-1}\\ -t^{-1} & -u^{-1}-t^{-1}\end{pmatrix}\begin{pmatrix}n_s\\ n_u\end{pmatrix}\,.\tag{2.131}$$

From this expression we learn that the two colour-ordered amplitudes $A_4^{\text{tree}}(1,2,3,4)$ and $A_4^{\text{tree}}(1,3,2,4)$ *cannot* be independent of each other: while they are gauge invariant, the kinematical numerators n_s and n_u are not, and hence the 2×2 matrix relating them is not invertible. The linear dependance reads

$$s\,A_4^{\text{tree}}(1,2,3,4)=u\,A_4^{\text{tree}}(1,3,2,4)\,.\tag{2.132}$$

This is the Bern-Carrasco-Johannson (BCJ) relation for a four-point amplitude. The general relation reads, cf. (1.158),

$$\sum_{i=2}^{n-1}p_1\cdot(p_2+\ldots+p_i)\,A_n^{\text{tree}}(2,\ldots,i,1,i+1,\ldots,n)=0\,,\tag{2.133}$$

which for $n=4$ reduces to Eq. (2.132). As a matter of fact, using this kinematical 2×2 matrix the coloured amplitude may be written as

$$\mathbf{A}_4^{\text{tree}}=-\mathrm{i}g^2\left(c_s\;-c_u\right)\begin{pmatrix}s^{-1}+t^{-1} & t^{-1}\\ -t^{-1} & -u^{-1}-t^{-1}\end{pmatrix}\begin{pmatrix}n_s\\ n_u\end{pmatrix}\,.\tag{2.134}$$

Returning to the conjectured form of the four-graviton amplitude (2.126), we can use there the relations $n_t = -n_u - n_s$ and $n_u = t A_4^{\rm tree}(1,2,3,4) + n_s u/s$. One finds

$$M_4^{\rm tree}(1,2,3,4) = -\mathrm{i}\frac{st}{u}\Big[A_4^{\rm tree}(1,2,3,4)\Big]^2 . \tag{2.135}$$

This very compact result may be transformed into the one we derived in Eq. (2.116) upon using the BCJ relation (2.132). Cyclically permute $1 \to 2 \to 3 \to 4 \to 1$ in (2.132) (under which $s \leftrightarrow t$ but u is inert) to reach

$$t\, A_4^{\rm tree}(2,3,4,1) = u\, A_4^{\rm tree}(2,4,3,1) . \tag{2.136}$$

With the cyclicity of the colour-ordered amplitudes, one has $A_4^{\rm tree}(1,2,4,3) = \frac{t}{u} A_4^{\rm tree}(1,2,3,4)$, which inserted in Eq. (2.116) yields Eq. (2.135).

Exercise 2.9 (Kinematical Jacobi Identity) Prove the kinematical Jacobi identity (2.125) for the coloured tree-level four-gluon amplitude in D dimensions given in Eq. (2.117). For the solution see Chap. 5.

2.7.3 The Double-Copy Relation

The general statement of the duality between colour and kinematics is as follows. A general coloured n-gluon amplitude may be written as

$$\mathbf{A}_n^{\rm tree} = -\mathrm{i} g^{n-2} \sum_i \frac{c_i\, n_i}{\prod_{\alpha_i} D_{\alpha_i}} , \tag{2.137}$$

where the sum is over all n-point diagrams with a trivalent vertex structure. Here c_i denote the colour factors (made up of the structure constants and possibly of generators), n_i the numerators (made up of the momenta and polarisation vectors), and the $D_{\alpha_i} = p_{\alpha_i}^2 - m_{\alpha_i}^2$ are the inverse propagators. We note that any graph may be made tri-valent upon inserting the identity $1 = D_{\alpha_i}/D_{\alpha_i}$ in order to lift the four-gluon vertices to a sum of three s-t-u channel diagrams as dictated by the colour structure. Due to the Jacobi identity, the colour factors obey algebraic relations of the form $c_i - c_j = c_k$. Colour-kinematic duality now asserts that it is always possible to find a representation of the amplitude (2.137) in which also the kinematical numerators obey an analogous identity $n_i - n_j = n_k$. This may be reached by possibly adding overall zeros to the amplitude.

The n-graviton scattering amplitude is then obtained upon replacing colour by kinematics $c_i \to n_i$,

$$M_n^{\text{tree}} = -\mathrm{i}\left(\frac{k}{2}\right)^{n-2} \sum_i \frac{n_i^2}{\prod_{\alpha_i} D_{\alpha_i}} . \tag{2.138}$$

In general it is non-trivial to find numerators which satisfy the colour-kinematics duality. A possible route is to start out from an ansatz, which one then constrains to match the amplitude and to obey the duality. At tree level the duality has been proven [29–31], while at the Lagrangian level a full understanding is still lacking [30, 32, 33]. A comprehensive review of the double-copy relation is given in [34].

The n-point generalisation of the squaring relation (2.135) is known as the Kawai-Lewellen-Tye (KLT) relation and takes the form [35]

$$M_n^{\text{tree}} = \sum_{\sigma,\rho \in S_{n-3}} A_n^{\text{tree}}(1, \sigma, n-1, n)\, S[\sigma|\rho]\, A_n^{\text{tree}}(1, \rho, n, n-1) . \tag{2.139}$$

Here σ and ρ range over the $(n-3)!$ permutations of the elements $\{2, \ldots, n-2\}$. The KLT kernel $S[\sigma|\rho]$ are the entries of an $(n-3)! \times (n-3)!$ matrix of kinematic polynomials. A closed form expression reads [29, 36]

$$S[\sigma|\rho] = \prod_{i=2}^{n-2} \left[2\, p_1 \cdot p_{\sigma_i} + \sum_{j=2}^{i} 2\, p_{\sigma_i} \cdot p_{\sigma_j}\, \theta(\sigma_j, \sigma_i)_\rho \right] , \tag{2.140}$$

where $\theta(\sigma_j, \sigma_i)_\rho = 1$ when σ_j is before σ_i in the permutation ρ, and zero otherwise. For $n = 4$ the matrix degenerates to a scalar and reproduces Eq. (2.116).

Exercise 2.10 (Five-Point KLT Relation) Prove the five-point KLT relation

$$M_5^{\text{tree}}(1, 2, 3, 4, 5) = s_{12}\, s_{34}\, A_5^{\text{tree}}(1, 2, 3, 4, 5)\, A_5^{\text{tree}}(1, 2, 5, 3, 4) + (2 \leftrightarrow 3) . \tag{2.141}$$

For the solution see Chap. 5.

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