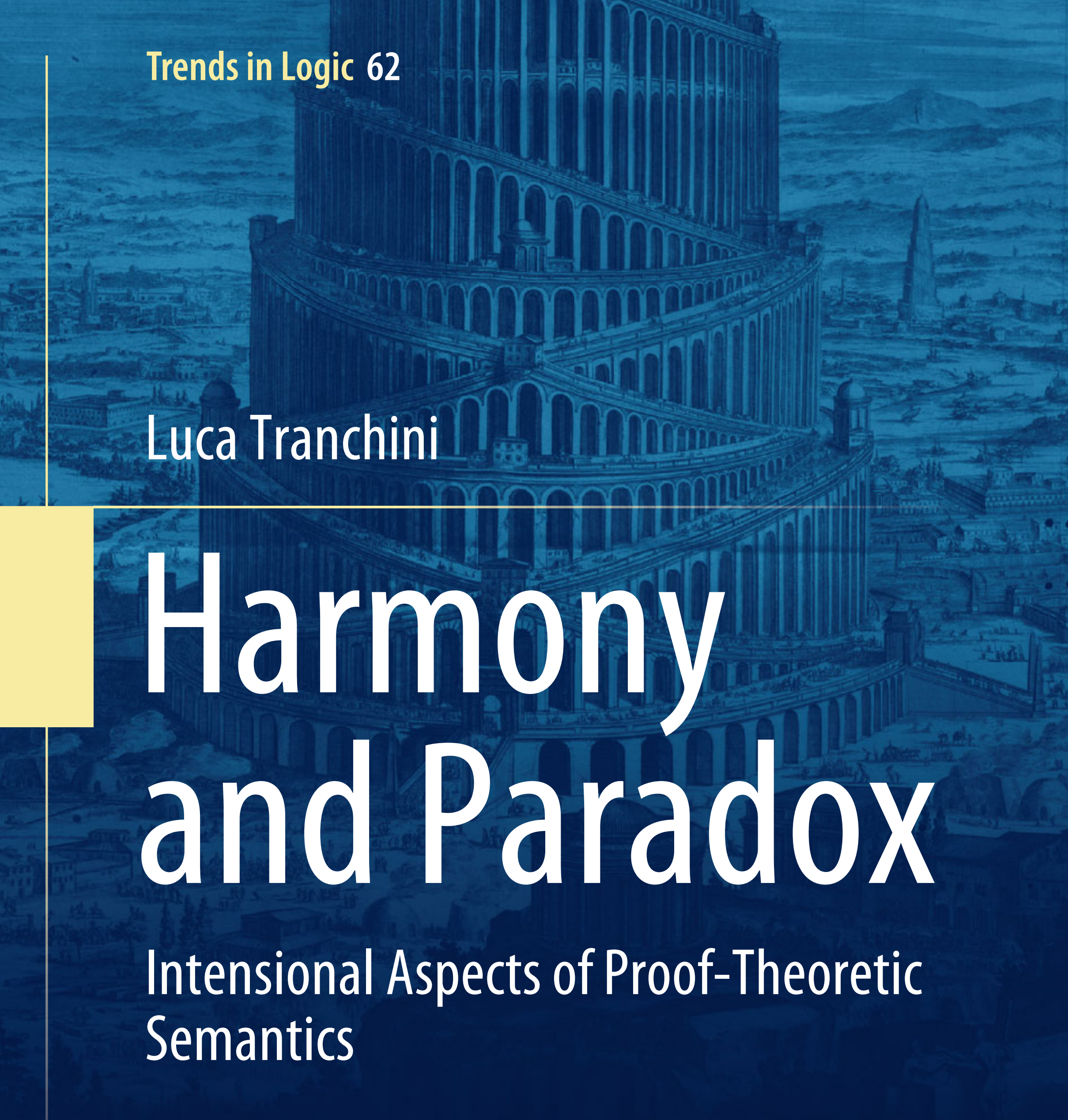

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Luca Tranchini

Harmony and Paradox

Intensional Aspects of Proof-Theoretic Semantics

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Harmony and Paradox

Intensional Aspects of Proof-Theoretic Semantics

With a contribution by Peter Schroeder-Heister

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*To Karin, Franka, Ada, Piero,
Raul, Evo and Irene*

Preface

The present monograph collects together the main results of two projects funded by the German Research Foundation (DFG). The first one, titled "Logical Consequence. Epistemological and Proof-Theoretic Perspectives" (TR1112/1-1) ran from 2012 to 2015. The second one, a continuation of the previous one, was titled "Logical Consequence and Paradoxical Reasoning" (TR112/3-1) and ran from 2015 to 2018.

The material here presented is largely based on a series of articles that constitute part of the output of the two projects.

In particular, Chaps. 1 and 3 and Appendix A are based on

1. Tranchini, 2018: "Stabilizing quantum disjunction" *Journal of Philosophical Logic*, 47 (6): 1029–1047.
2. Tranchini, 2021: "Proof-theoretic harmony: Towards an intensional account", *Synthese*, 198 (Suppl 5): 1145–1176.

Chapter 5 is a substantial reworking of the content of the articles:

3. Tranchini, 2015: "Harmonising harmony" *The Review of Symbolic Logic*, 8 (3): 411–423.
4. Tranchini, 2016: "Proof-theoretic semantics, paradoxes and the distinction between sense and denotation", *Journal of Logic and Computation*, 26 (2): 495–512.
5. Tranchini, 2019: "Proof, meaning and paradox. Some remarks" *Topoi*, 38 (3): 591–603.

Chapter 4 is based on material taken from items 3–5, and Chap. 2 is partly based on material coming from items 2, 4 and 5 and it partly consists of the original material.

Finally, Chap. 6 is based on

6. Schroeder-Heister and Tranchini, 2017: "Ekman's paradox" *Notre Dame Journal of Formal Logic*, 58 (4): 567–581.
7. Schroeder-Heister and Tranchini, 2018: "How to Ekman a Crabbé-Tennant" *Synthese*, Online First.

I thank the publishers for the permission to reuse this content. I am particularly grateful to Peter Schroeder-Heister for his agreeing to have the results of our joint and previously published work appear in revised form as a chapter of the present volume.

I wish to thank all the colleagues with whom I had the chance to interact over the years, for contributing one way or another to the development of the ideas presented in the book. Among all of them, I would like to mention Pablo Cobreros, Alberto Naibo and Mattia Petrolo for discussions, collaboration, exchanges, hospitality and friendship. A special thanks goes to Paolo Pistone not only for the intense and stimulating collaboration over the years, but also for the numerous discussions, insightful comments and suggestions which substantially contributed to shape and clarify my ideas on many of the topics covered in the present work. I am also grateful to two anonymous referees for detailed and constructive feedback on a first draft of the volume.

I would not have realized the significance of the notion of identity of proofs without several interactions with Göran Sundholm, whom I thank also for his warm hospitality during a short stay in Leiden, and with the late Kosta Došen from whom I wish I could have had the chance to learn more.

I would like to thank again Peter Schroeder-Heister not only for his agreeing to figure as a contributor to the present volume, but also for his unconditional support throughout my years in Tübingen and for his guidance and advice from which I could profit on countless occasions.

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At different stages of the preparation of this monograph, I received funding by the DFG as part of the projects “Logical Consequence. Epistemological and Proof-Theoretic Perspectives” (DFG Tr1112/1-1), “Logical Consequence and Paradoxical Reasoning” (DFG Tr1112/3-1) and “Falsity and Refutations: Understanding the Negative Side of Logic” (DFG Tra1112/4-1); by the DFG and ANR as part of the project “Hypothetical Reasoning. Its Proof-Theoretic Analysis” (Schr275/16-2) and “Beyond Logic: Hypothetical Reasoning in Philosophy of Science, Informatics, and Law” (Schr 275/17-1); and by the Government of Spain as part of the projects “Non-Transitive Logics” (FFI2013-46451-P) and “Logic and Substructurality” (FFI2017-84805-P).

Tübingen, Germany Luca Tranchini
September 2023

Introduction

The original contribution of the present monograph is that of articulating in an explicit way the role played by the notion of *identity of proofs* in proof-theoretic semantics.

Although identity of proofs is a topic of active research in more mathematically oriented strands of proof theory, it has been—with very few exceptions—either ignored or only implicitly hinted at in the more philosophically oriented literature so far. On the understanding of proof-theoretic semantics that underlies the present work, this is mostly unfortunate. Identity of proofs is here taken to be indispensable to properly formulate the scopes and goals of proof-theoretic semantics.

In particular, it will be shown that identity of proofs plays a key role both to clarify some core notions of proof-theoretic semantics, such as that of harmony, and to broaden the range of the phenomena which can be analyzed using the tools of this semantic paradigm, so as to include for instance paradoxes.

In the first chapter, the concept of harmony is shown to arise by considerations about the role of assertion in the theory of meaning. When applied to rules in the format of natural deduction, harmony can be explained by making reference to certain transformations on derivations, called reductions and expansions. These are the central ingredient of the proofs of normalization for natural deduction calculi and of related results. These provide the formal basis for the considerations to be developed in the following chapters. The chapter ends with a brief discussion of other accounts of harmony.

In the second chapter, proof-theoretic semantics is presented as primarily concerned with the relationship between proofs (understood as abstract entities) and derivations (the linguistic representations of proofs). This relationship is developed in analogy with that between names and (abstract) objects in Frege's semantic picture. On this Fregean conception of proof-theoretic semantics, reductions and expansions should be viewed as identity-preserving operations on derivations. These operations thus induce an equivalence relation on derivations (where equivalent derivations denote the same proof), which in turn can be used to define an equivalence relation on formulas stricter than interderivability, called isomorphism. Identity of proofs and formula isomorphism show the intensional nature of this conception of proof-theoretic semantics. The chapter ends with a comparison between this Fregean conception of proof-theoretic semantics and the one advocated by Dummett and Prawitz, which is based on a notion of validity of derivations.

The third chapter, which concludes the first part of the work, discusses how the intensional account of harmony sketched in the first chapter can be developed in a systematic way for a class of connectives whose rules are obtained in a uniform way from a principle of *inversion*. We discuss and compare different ways of formulating inversion principles and finally we investigate the prospects of developing an account of harmony for connectives whose rules do not obey inversion, pointing at the weakness of the approaches proposed in the literature so far.

The fourth chapter introduces the topic of the second part of the work, namely the Prawitz-Tennant analysis of paradoxes. According to it, paradoxes are derivations of a contradiction which cannot be brought into normal form, due to "loops" (or more generally, other patterns resulting in non-termination) arising in the process of reduction. After presenting Prawitz's original formulation of Russell's paradox, a simplified presentation of it is introduced. Finally, the chapter discusses the relevance for the Prawitz-Tennant analysis of the difference between intuitionistic and classical logic, and of structural properties of derivability.

The fifth chapter deals with the following question: Which modifications does the account of proof-theoretic semantics developed in the second chapter need to undergo for it to be applicable to languages containing paradoxical expressions? The intensional account of proof-theoretic semantics is enriched by introducing a notion of sense alongside the one of denotation. Paradoxical derivations in proof-theoretic semantics are then shown to play a role analogous to that of singular terms endowed with sense but lacking a denotation in Frege's semantic picture. The question of which class of derivations should be regarded as having a denotation is reconsidered. The choice of different criteria of identity of proofs is shown to have far-reaching consequences for the analysis of languages containing paradoxical expressions. In order to maintain that a derivation is valid iff it denotes a proof, Dummett and Prawitz's definition must be substantially modified. Two alternative accounts of validity, tied to two distinct conceptions of identity of proofs, are discussed and compared.

The last chapter of the second part of the work discusses two distinct kinds of phenomena, first observed by Crabbé and Ekman, showing that the Tennant-Prawitz criterion for paradoxicality overgenerates, that is, there are derivations which are intuitively non-paradoxical but which fail to normalize. We argue that a solution to "Ekman's paradox" consists in restricting the set of admissible reduction procedures to those that do not yield a trivial notion of identity of proofs. We then discuss a different kind of solution, due to von Plato, and recently advocated by Tennant, consisting in reformulating natural deduction elimination rules in general (or parallelized) form. Developing intuitions of Ekman, we show that the adoption of general rules has the consequence of hiding redundancies within derivations. Once reductions to get rid of the hidden redundancies are devised, it is clear that the adoption of general elimination rules offers no remedy to the overgeneration of the Prawitz-Tennant analysis. In this way, we indirectly provide further support for our own solution to Ekman's paradox.

In the concluding chapter, we summarize the main results of the investigation. We stress the composite nature of the relationship between proofs and meaning. In particular, the question of which criteria of identity of proofs should be accepted can be answered in different ways, and different answers mirror different ways of conceiving certain general features of a theory of meaning on proof-theoretic basis.

An appendix spells out the formal details of the calculus of higher level rules first introduced by Schroeder-Heister, together with the definitions of the technical notions used throughout the work.

Contents

Part I
Harmony

Chapter 1
Harmony via Reductions and Expansions

Abstract The notion of harmony arises by considerations about the notion of assertion in the theory of meaning. When applied to rules in the format of natural deduction, harmony can be explained by making reference to certain transformations on derivations, called reductions and expansions. Reductions are a key ingredient of the proof of normalization for the calculus of natural deduction for intuitionistic logic. In this calculus, normal derivations that are closed (i.e. such that their conclusions depend on no assumption) end with an introduction rule, a fact which is referred to as the *canonicity* of closed normal derivations. The condition for the canonicity of normal derivations in a more general setting are discussed. The chapter ends with a brief discussion of other accounts of harmony.

1.1 Meaning Theory and Harmony

A theory of meaning for a given language is a description of what a subject needs to *know* to qualify as a competent speaker of that language. In spite of widespread agreement on this general characterization, the question of what does 'to know' mean in this context has received very different answers.

One of the most influential answers to this question is that of Dummett [11, 12]. His starting point is the observation that the competent speakers of a language are able to interact with each other using a wide range of speech acts such as questions, commands, and—most importantly—assertions (see, e.g. p. 417, [11]).[1] The key aspects of the practice of assertion are the abilities of speakers to make assertions under appropriate conditions and to react appropriately to assertions made by other speakers. Thus, an essential task of a theory of meaning is that of accounting, on the one hand, of the knowledge of the conditions under which a proposition is correctly asserted, i.e. the *assertibility conditions* of the proposition; and, on the other hand, of the knowledge of the *consequences* that can be drawn from the assertion of a proposition.

Since the practice of assertion is rational, there must be a close connection between these two aspects of assertion, a connection to which Dummett refers as the principle of *harmony*:

L. Tranchini, *Harmony and Paradox*, Trends in Logic 62,
https://doi.org/10.1007/978-3-031-46921-3_1

Harmony: Informal statement 1 *The consequences that can be drawn from the assertion of a proposition can be neither more nor less than those that are guaranteed by the satisfaction of its assertibility conditions.*

To clarify this idea, Dummett (see [11], p. 454) discusses an example aimed at showing that the violation of harmony induces irrational elements in linguistic practices. The example considered is 'boche', a derogatory term with which the Anglo-Americans referred to German people during the First World War. The conditions for applying the predicate to a person (and thus for asserting his being a boche) are that the person is of German nationality, while the consequences that can be drawn from the assertion of 'He is a boche.' are the barbarism and cruelty of the subject of the proposition. The disharmony between the two aspects of statements of this type shows the non-rationality of the use of the term 'boche' and the need to modify the linguistic practices that involve it.

Harmony therefore has not only a descriptive value, but also a normative component, i.e. if the assertibility conditions of a proposition and the consequences that can be drawn from its assertion do not coincide, then the linguistic community ought to change its practices.[2]

As Dummett himself admits, the universal validity of the principle of harmony is a very strong demand, and it is doubtful that the two aspects of any possible assertion in a given language are in perfect harmony. Nonetheless, given that we are willing to concede that the linguistic practices in which we are involved as speakers are rational (at least for the most part), it seems natural to expect a theory of meaning to satisfy certain general conditions that guarantee that the practices it describes are (at least in principle) harmonious.

1.2 Harmony and Natural Deduction

Some of these conditions depend on another essential aspect of language, namely that a competent speaker of a language is able to produce and understand a potentially infinite set of distinct utterances, starting from his understanding of a finite set of minimal linguistic units (words) endowed with meaning. This aspect of language, perhaps the only one distinguishing human language from other forms of animal language, is made possible by the existence of expressions which can be used to build complex expressions starting from simpler expressions. One class of these expressions is that of logical constants, which in particular allow the formation of complex propositions starting from one or more simpler propositions.

As on the syntactic level complex propositions are obtained by composing simpler propositions, on the semantic level the meaning of complex propositions will depend on the meaning of their components and on the way in which they are composed. This principle, called *compositionality*, is embodied in a theory of meaning of the type outlined by Dummett by rules that specify the assertibility conditions

Table 1.1 The natural deduction calculus NI

$$\frac{A \qquad B}{A \wedge B}\wedge\mathrm{I} \qquad\qquad \frac{A \wedge B}{A}\wedge\mathrm{E}_1 \quad \frac{A \wedge B}{B}\wedge\mathrm{E}_2$$

$$\frac{\begin{array}{c}[A]\\ B\end{array}}{A \supset B}\supset\mathrm{I} \qquad\qquad \frac{A \supset B \qquad A}{B}\supset\mathrm{E}$$

$$\frac{A}{A \vee B}\vee\mathrm{I}_1 \quad \frac{B}{A \vee B}\vee\mathrm{I}_2 \qquad \frac{A \vee B \qquad \begin{array}{c}[A]\\ C\end{array} \qquad \begin{array}{c}[B]\\ C\end{array}}{C}\vee\mathrm{E}$$

$$\frac{}{\top}\top\mathrm{I} \qquad\qquad \text{no elimination rule for } \top$$

$$\text{no introduction rule for } \bot \qquad\qquad \frac{\bot}{C}\bot\mathrm{E}$$

(respectively the consequences of the assertions) of complex propositions in terms of the asseribility conditions (resp. the consequences of the assertions) of their components.

The paradigm for these rules are those of the calculus of natural deduction for intuitionistic logic NI [20, 65]. In this calculus (whose rules are depicted in Table 1.1), two types of rules are associated with each logical constant: *introduction rules* and *elimination rules*. The introduction rules for a logical constant † are those which allow one to infer a complex proposition having † as main operator, and thus they specify the assertibility conditions of such propositions; in the elimination rules for †, a complex proposition having † as main operator acts as the main premise[3] of the rules, and these rules thus specify which consequences can be drawn from its assertion.

In the case of conjunction, the introduction rule $\wedge$I allows one to infer from two propositions their conjunction,[4] and thus expresses the fact that the assertibility conditions of a conjunction are satisfied when those of both conjuncts are. The two elimination rules $\wedge\mathrm{E}_1$ and $\wedge\mathrm{E}_2$ allow one to infer from a conjunction each of the two conjuncts (respectively), and thus express the fact that the consequences that can be drawn from the assertion of a conjunction are all those that can be obtained from the two conjuncts.

In order to guarantee the harmony between the two aspects of assertion, the rules of introduction and elimination that govern the logical constants cannot be chosen arbitrarily. In particular, an inappropriate choice of introduction and elimination rules for a connective may result in a situation analogous to that of ‘boche’. Exemplary in this sense is the binary connective `tonk` introduced by Prior [76], governed by the following pair of introduction and elimination rules:

$$\frac{A}{A\ \texttt{tonk}\ B}\ \texttt{tonkI} \qquad \frac{A\ \texttt{tonk}\ B}{B}\ \texttt{tonkE}$$

As in the case of 'boche', the consequences that can be drawn from the assertion of a complex proposition governed by `tonk` do not coincide with what is warranted by the fulfillment of its assertibility conditions, and the disharmony between the two aspects of the practice of assertion deprives this of rationality: given the rules of `tonk` every proposition can be inferred from any other.

As (and even more than) in the case of boche, the strong intuition that `tonk` is "semantically defective" shows that not every collection of introduction and elimination rules for a connective is apt to determine the meaning of complex propositions in which the connective is the main operator.

Hence, in natural deduction, the requirement of harmony as applying to logically complex propositions becomes a condition that should be satisfied by the collections of introduction and elimination rules. We can informally state it as follows[5]:

Harmony: Informal statement 2 *What can be inferred from a logically complex proposition by means of the elimination rules for its main connective is* no more and no less *than what has to be established in order to infer that very logically complex proposition using the introduction rules for its main connective.*

The rules of `tonk` display no match between what can be inferred using the elimination rule and what is needed to establish the premise of the elimination rule using the introduction rule.[6] Thus, even if A and B are meaningful statements, their "contonktion" A `tonk` B is nonsense since the rule governing `tonk` are ill-formed.[7] In contrast to the rules of `tonk`, the rules for conjunction of `NI` display a perfect match.

1.3 Harmony, Reductions and Expansions

Both informal characterizations of harmony given above make clear that harmony is two-fold condition, and we will refer to its two components as the 'no more' and 'no less' aspect of harmony.[8] The two aspects of harmony are closely connected with two different kinds of deductive patterns.

Patterns of the first kind are those giving rise to *maximal formulas occurrences*, sometimes referred to as 'local peaks' [12] or 'hillocks' (used by von Plato to translate Gentzen's original '*Hügel*' [64]). These are formula occurrences which are the major premise of an application of an elimination rule and that are the consequence of an application of one of the introduction rules.

When the rules for a connective are in harmony, configurations of this kind are clearly redundant. In particular, the possibility of "leveling" these local peaks shows that harmonious elimination rules allow one to infer *no more* than what has to be established to infer their major premise by introduction.

Prawitz [65] defined certain operations on derivations called *reductions* that, when applied to a derivation, transform it into another one by getting rid of a single maximal formula occurrence.[9]

In the case of conjunction, there are two patterns of this kind, of which one can get rid as follows:

$$\dfrac{\dfrac{\begin{matrix}\mathscr{D}_1\\ A\end{matrix}\qquad\begin{matrix}\mathscr{D}_2\\ B\end{matrix}}{A\wedge B}\wedge\mathrm{I}}{A}\wedge\mathrm{E}_1\quad\text{reduces to}\quad\begin{matrix}\mathscr{D}_1\\ A\end{matrix}\qquad\qquad\dfrac{\dfrac{\begin{matrix}\mathscr{D}_1\\ A\end{matrix}\qquad\begin{matrix}\mathscr{D}_2\\ B\end{matrix}}{A\wedge B}\wedge\mathrm{I}}{B}\wedge\mathrm{E}_2\quad\text{reduces to}\quad\begin{matrix}\mathscr{D}_2\\ B\end{matrix}$$

Patterns of the other kind are those in which the premises of applications of introduction rules have been obtained applying the corresponding elimination rules.[10] These patterns could be described as *local valleys* since they result when one infers a complex proposition from itself by first eliminating and then reintroducing its main connective. Prawitz [66] defined operations that are, in a sense, the dual of reductions, called (immediate) *expansions* to introduce such valleys within a derivation. The possibility of "expanding" a derivation via a local valley amounts to the fact that harmonious elimination rules allow one to infer *no less* than what is needed to infer their major premise by introduction.[11] In the case of conjunction, the expansion is the following:

$$\begin{matrix}\mathscr{D}\\ A\wedge B\end{matrix}\quad\text{expands to}\quad\dfrac{\dfrac{\begin{matrix}\mathscr{D}\\ A\wedge B\end{matrix}}{A}\wedge\mathrm{E}_1\qquad\dfrac{\begin{matrix}\mathscr{D}\\ A\wedge B\end{matrix}}{B}\wedge\mathrm{E}_2}{A\wedge B}\wedge\mathrm{I}$$

In the case of implication, we have the following reduction and expansion:[12,13]

$$\dfrac{\langle u\rangle\,\dfrac{\begin{matrix}\overset{u}{[A]}\\ \mathscr{D}\\ B\end{matrix}}{A\supset B}\supset\mathrm{I}\qquad\begin{matrix}\mathscr{D}'\\ A\end{matrix}}{B}\quad\text{reduces to}\quad\begin{matrix}\mathscr{D}'\\ [A]\\ \mathscr{D}\\ B\end{matrix}$$

$$\begin{matrix}\mathscr{D}\\ A\supset B\end{matrix}\quad\text{expands to}\quad\langle u\rangle\,\dfrac{\dfrac{\begin{matrix}\mathscr{D}\\ A\supset B\end{matrix}\qquad\overset{u}{A}}{B}\supset\mathrm{E}}{A\supset B}\supset\mathrm{I}$$

(with u *fresh* for $\mathscr{D}$)

Most of the literature on harmony has focused on logical constants. However, there seems to be no principled reason to restrict the account of harmony just sketched to these expressions only. For any inductively definable n-ary predicate P, it is possible to formulate introduction and elimination rules for atomic propositions $P(t_1,\ldots,t_n)$, so that an introduction rule for a primitive n-ary predicate P yields a derivation having $P(t_1,\ldots,t_n)$ as conclusion (where $t_1,\ldots,t_n$ are singular terms), while an elimination rule for a primitive n-ary predicate P is one that, applied to a derivation having $P(t_1, \ldots, t_n)$ as conclusion and, possibly, other derivations, yields a derivation of some other proposition. For example, the following are the introduction and elimination rules for the unary predicate $\mathtt{Nat}$ expressing the property of being a natural number (in the rules, we use S as a unary function symbol for the successor function and with $A(t/x)$ we indicate capture-avoiding substitution of t for x in A):[14]

$$\frac{}{\mathtt{Nat}\,0}\,\mathtt{Nat}\,\mathrm{I}_1 \qquad \frac{\mathtt{Nat}\,x}{\mathtt{Nat}\,Sx}\,\mathtt{Nat}\,\mathrm{I}_2 \qquad \frac{\mathtt{Nat}\,t \qquad A(0/x) \qquad \begin{matrix}[A][\mathtt{Nat}\,x]\\ A(Sx/x)\end{matrix}}{A(t/x)}\,\mathtt{Nat}\,\mathrm{E}$$

(where x is an *eigenvariable*)

Whenever the major premise $\mathtt{Nat}\,t$ is the consequence of an application of one of the two introduction rules for $\mathtt{Nat}$, a reduction is readily defined, and an expansion can be defined as well, by applying the elimination rule taking A to be $\mathtt{Nat}\,x$.[15]

In the following we will (with a few exceptions) restrict ourselves to rules governing propositional connectives, thereby disregarding the exact nature of the non-logical vocabulary. The application of the ideas presented in the next chapters to specific languages, such as the one of arithmetic, represents an interesting challenge, but goes beyond the scope of the present work.

1.4 Some Formal Definitions

We will henceforth write $\mathscr{D}_1 \overset{1\supset\beta}{\triangleright} \mathscr{D}_2$ (respectively $\mathscr{D}_1 \overset{1\supset\eta}{\triangleleft} \mathscr{D}_2$) when $\mathscr{D}_2$ is obtained by one application of the reduction (respectively expansion) for implication from $\mathscr{D}_1$.[16] This is to be understood to mean that

- either $\mathscr{D}_1$ and $\mathscr{D}_2$ are of the form depicted to the left-hand and right-hand side of the reduction (resp. expansion) above;
- or that $\mathscr{D}_2$ is obtained by replacing in $\mathscr{D}_1$ one of its *subderivations* having the form depicted on the left-hand side of the reduction (resp. expansion) with a derivation having the form depicted on the right-hand side of the reduction (resp. expansion). (This latter case will be referred to as the *congruence condition* for $\supset$-reduction (resp. expansion).)

We will refer to these relations as *one-step* $\supset\beta$*-reduction* and *one-step* $\supset\eta$*-expansion*.

We will use a similar notation for one-step reductions and expansions of conjunction and of the other connectives introduced below. For connectives with more than one introduction (resp. elimination) rule we use subscripts (e.g. $\mathscr{D}_1 \overset{1\wedge\beta_1}{\triangleright} \mathscr{D}_2$ and $\mathscr{D}_1 \overset{1\wedge\beta_2}{\triangleright} \mathscr{D}_2$ in the case of conjunction) to distinguish between the relations induced by the reductions getting rid of maximal formula occurrences which are the consequences (resp. major premises) of different introduction (resp. elimination) rules.

Sometimes, we will omit the subscripts and/or the indication of the connective (thus writing e.g. $\mathscr{D} \overset{1\beta}{\triangleright} \mathscr{D}'$ and $\mathscr{D} \overset{1\eta}{\triangleleft} \mathscr{D}'$) where the omission of the connective (and possibly of the subscript) indicates that $\mathscr{D}_2$ can be obtained from $\mathscr{D}_1$ using some reduction (resp. expansion) for some connective.

If a β-reduction is available to get rid of a maximal formula occurrence whose main connective is †, the latter will be referred to as a †β-redex (contraction of *reducible expression*).

We indicate with $\overset{\beta}{\triangleright}$ the relation of β*-reduction* which is the reflexive and transitive closure of the relation of one-step β-reduction. That is, we write $\mathscr{D} \overset{\beta}{\triangleright} \mathscr{D}'$ when for some $n \geq 1$ there is a sequence of n-derivation $\mathscr{D}_1, \ldots, \mathscr{D}_n$ (to which we will refer to as a β*-reduction sequence* for $\mathscr{D}$) such that $\mathscr{D}_1 = \mathscr{D}$, $\mathscr{D}_n = \mathscr{D}'$ and $\mathscr{D}_{i-1} \overset{1\beta}{\triangleright} \mathscr{D}_i$ for each $1 < i \leq n$. Similar notation and terminology will be adopted for η-expansion as well.

Sometimes it will be useful to refer to the inverses of these relations as well, that we will indicate with $\overset{(1)\supset\beta}{\triangleleft}$ (respectively $\overset{(1)\supset\eta}{\triangleright}$). Clearly, also these relations may be dubbed (one-step) expansions and reductions respectively, and hence '$\supset$-reduction' is actually ambiguous between $\overset{\supset\beta}{\triangleright}$ and $\overset{\supset\eta}{\triangleright}$ (and '$\supset$-expansion' between $\overset{\supset\eta}{\triangleleft}$ and $\overset{\supset\beta}{\triangleleft}$). When precision is required we will speak of $\supset\beta$-reduction and $\supset\eta$-reduction (and $\supset\eta$-expansion and $\supset\beta$-expansion), but as we already did until now, we will however use '$\supset$-reduction' (resp. '$\supset$-expansion') to indicate the relation $\overset{\supset\beta}{\triangleright}$ (resp. $\overset{\supset\eta}{\triangleleft}$).

We observe that we will often speak of "applications" of $\supset\beta$-reduction (and similarly for other reductions/expansion), thereby treating $\supset\beta$ as a function that given a derivation $\mathscr{D}$ and a particular maximal formula occurrence of the form $A \supset B$ in $\mathscr{D}$ yields the result of reducing that maximal formula occurrence. We take this notion of reduction as function as clear enough and we omit its precise definition.

We will use the term *conversion* to refer to β-reductions, β-expansions, η-reductions, η-expansions, as well as to further transformations on derivations to be introduced in the following chapters.

1.5 Some Formal Results

A derivation $\mathscr{D}$ is called β*-normal* iff it is not possible to β-reduce it any further (i.e. iff $\mathscr{D} \overset{\beta}{\triangleright} \mathscr{D}'$ implies $\mathscr{D}' = \mathscr{D}$). In the $\{\supset, \wedge\}$-fragment of $\mathtt{NI}$ (we will indicate this fragment as $\mathtt{NI}^{\wedge\supset}$) Prawitz [65] showed how any given derivation $\mathscr{D}$ can be transformed into a β-normal one by successive applications of the $\supset\beta$- and $\wedge\beta$-reductions.

The proof is non-trivial, since an application of $\supset\beta$-reduction to a given derivation may yield a derivation containing the same number of maximal formula occurrences, or even more. (For an example consider the result of β-reducing the encircled occurrence of $A \supset B$ in the derivation of Table 1.2 below.) However, it is always possible

Table 1.2 A derivation illustrating the non-triviality of normalization

$$\dfrac{\dfrac{\dfrac{\dfrac{A \supset (A \supset B) \qquad \overset{1}{A}}{A \supset B}\,{\supset}\mathrm{E} \qquad \overset{1}{A}}{B}\,{\supset}\mathrm{E}}{A \supset B}\,\langle 1\rangle\,{\supset}\mathrm{I} \qquad \dfrac{\dfrac{\overset{2}{A}}{B \supset A}\,\langle 2\rangle\,{\supset}\mathrm{I} \qquad B}{A}\,{\supset}\mathrm{E}}{B}\,{\supset}\mathrm{E}$$

to find a maximal formula occurrence such that, by β-reducing it, the number of maximal formula occurrences of maximum degree in the resulting derivation is lower than in the original derivation, where the *degree* of a maximal formula occurrence is the number of logical constants it contains.[17] Therefore, for every derivation $\mathscr{D}$ in $\mathtt{NI}^{\wedge\supset}$ there is a β-reduction sequence starting with $\mathscr{D}$ and ending with a β-normal derivation.

This result, known as the *weak normalization* theorem for β-reduction in $\mathtt{NI}^{\wedge\supset}$, has been strengthened using a method introduced by Tait [107] to what is nowadays called the *strong normalization* theorem for β-reduction in $\mathtt{NI}^{\wedge\supset}$, namely that in $\mathtt{NI}^{\wedge\supset}$ there are no infinite β-reduction sequences (that is, in the process of β-reducing a derivation, no matter which maximal formula occurrence is chosen at any step, if one keeps on β-reducing one will always reach a β-normal derivation).

Given their significance for the following, we discuss some properties of β-normal derivations in $\mathtt{NI}^{\wedge\supset}$.

First, each derivation in $\mathtt{NI}^{\wedge\supset}$ has a *unique* β-normal form, i.e. no matter how a derivation is reduced, one will always end up with the same β-normal derivation. This property is an immediate consequence of the *confluence* property (sometimes Church-Rosser property) of the relation of β-reduction, that is the fact that if $\mathscr{D} \overset{\beta}{\triangleright} \mathscr{D}_1$ and $\mathscr{D} \overset{\beta}{\triangleright} \mathscr{D}_2$, then there is $\mathscr{D}'$ such that both $\mathscr{D}_1 \overset{\beta}{\triangleright} \mathscr{D}'$ and $\mathscr{D}_2 \overset{\beta}{\triangleright} \mathscr{D}'$. (See Table 1.3 for an example of two β-reduction sequences for the same derivation ending with the same β-normal derivation. In the table, the target of each arrow is a derivation obtained by β-reducing one of the maximal formula occurrences in the derivation which is the source of that arrow.)

Second, β-normal derivations in $\mathtt{NI}^{\wedge\supset}$ enjoy the *subformula property*, that is every formula occurring in a β-normal derivation is either a subformula of the conclusion or of one of the undischarged assumptions of the derivation.[18] The subformula property is an immediate consequence of the peculiar form of β-normal derivations in $\mathtt{NI}^{\wedge\supset}$. This can be described using the notion of *track*, where a track is a sequence of formula occurrences in a derivation such that (i) the first is an assumption of the derivation; (ii) all other members of the sequence are the consequence of an application of an inference rule of which the previous member is one of the premises; (iii) none of them is the minor premise of an application of ${\supset}$E.

Table 1.3 An example of confluence

$$\dfrac{\langle 2\rangle\,\dfrac{\dfrac{\langle 1\rangle\,\dfrac{\dfrac{\overset{2}{A\supset B}\qquad \overset{1}{A}}{B}\,{\supset}\mathrm{E}}{A\supset B}\,{\supset}\mathrm{I}\qquad A}{B}\,{\supset}\mathrm{E}}{(A\supset B)\supset B}\,{\supset}\mathrm{I}\qquad \dfrac{\langle 3\rangle\,\dfrac{\overset{3}{A\supset B}}{(A\supset B)\supset(A\supset B)}\,{\supset}\mathrm{I}\qquad A\supset B}{A\supset B}\,{\supset}\mathrm{E}}{B}\,{\supset}\mathrm{E}$$

$$\dfrac{\langle 1\rangle\,\dfrac{\dfrac{\dfrac{\langle 3\rangle\,\dfrac{\overset{3}{A\supset B}}{(A\supset B)\supset(A\supset B)}\,{\supset}\mathrm{I}\qquad A\supset B}{A\supset B}\,{\supset}\mathrm{E}\qquad \overset{1}{A}}{B}\,{\supset}\mathrm{E}}{A\supset B}\,{\supset}\mathrm{I}\qquad A}{B}\,{\supset}\mathrm{E}$$

$$\dfrac{\langle 2\rangle\,\dfrac{\dfrac{\langle 1\rangle\,\dfrac{\dfrac{\overset{2}{A\supset B}\qquad \overset{1}{A}}{B}\,{\supset}\mathrm{E}}{A\supset B}\,{\supset}\mathrm{I}\qquad A}{B}\,{\supset}\mathrm{E}}{(A\supset B)\supset B}\,{\supset}\mathrm{I}\qquad A\supset B}{B}\,{\supset}\mathrm{E}$$

$$\dfrac{\dfrac{\langle 3\rangle\,\dfrac{\overset{3}{A\supset B}}{(A\supset B)\supset(A\supset B)}\,{\supset}\mathrm{I}\qquad A\supset B}{A\supset B}\,{\supset}\mathrm{E}\qquad A}{B}\,{\supset}\mathrm{E}$$

$$\dfrac{\langle 2\rangle\,\dfrac{\dfrac{\overset{2}{A\supset B}\qquad A}{B}\,{\supset}\mathrm{E}}{(A\supset B)\supset B}\,{\supset}\mathrm{I}\qquad A\supset B}{B}\,{\supset}\mathrm{E}$$

$$\dfrac{A\supset B\qquad A}{B}\,{\supset}\mathrm{E}$$

In each track of a β-normal derivation in $\mathtt{NI}^{\wedge\supset}$, all eliminations precede the introductions (otherwise the track, and hence the derivation would contain a maximal formula occurrence). The two parts (either of which is possibly empty) of a track are separated by a minimal part. This is a formula occurrence which is both the consequence of an elimination and the premise of an introduction. Furthermore, each formula occurrence in the elimination part is a subformula of the preceding formula occurrence in the track, and each formula occurrence in the introduction part is a subformula of the next formula occurrence in the track (since the premises of introduction rules are of lower complexity than the consequences, and the consequences of elimination rules are of lower complexity than the (major) premise).

A bit more formally, we have the following:

Fact 1 (*The form of tracks*) Each track $A_1 \ldots A_{i-1}, A_i, A_{i+1}, \ldots A_n$ in a β-normal derivation in $\mathtt{NI}^{\wedge\supset}$ contains a minimal formula A_i such that

- If $i > 1$ then A_j (for all $1 \leq j < i$) is the major premise of an application of an elimination rule of which A_{j+1} is the consequence and thereby A_{j+1} is a subformula of A_j.
- If $n > i$ then A_j (for all $i \leq j < n$) is the premise of an application of an introduction rule of which A_{j+1} is the consequence and thereby A_j is a subformula of A_{j+1}.

Proof For a derivation to be β-normal, all applications of elimination rules must precede all applications of introduction rules in all of its tracks: This warrants the existence of a minimal formula in each track. Since a track ends whenever it "encounters" the minor premise of an application of $\supset$E, the subformula relationships between the members of a track hold (as it can be easily verified by checking the shape of the rules of $\mathtt{NI}^{\wedge\supset}$). $\square$

From this it follows (almost) immediately that β-normal derivations in $\mathtt{NI}^{\wedge\supset}$ enjoy the subformula property: each formula in a β-normal derivation is the subformula either of the conclusion or of one of the undischarged assumptions of the derivation.

Fact 2 (*Subformula property*) All formulas in a β-normal derivation in $\mathtt{NI}^{\wedge\supset}$ are subformulas either of the conclusion or of some undischarged assumption.

Proof The proof of the theorem is by induction on the order of tracks, where the order of a track is defined as follows: The tracks to which the conclusion belong are of order 0. A track is of order n if its last formula is the minor premise of an application of $\supset$E whose major premise belong to a track of order $n-1$ (see for details [65], Chap. III, Sect. 2). $\square$

In the calculus $\mathtt{NI}^{\wedge\supset}$, strong normalization and confluence hold for η-reduction as well.[19] These results hold moreover for the relation of $\beta\eta$-reduction (notation $\overset{\beta\eta}{\triangleright}$), that is defined as the reflexive and transitive closure of one-step $\beta\eta$-reduction (notation $\overset{1\beta\eta}{\triangleright}$), which is the union of $\overset{1\beta}{\triangleright}$ and $\overset{1\eta}{\triangleright}$.

For the relation resulting by putting together β-reduction and η-expansion, Prawitz [66] established a weak normalization theorem by showing that by successively applying expansions it is possible to transform any given a β-normal derivation in $\mathtt{NI}^{\wedge\supset}$ into one in which all minimal formula occurrences of all tracks are atomic. This relation is also confluent, but not strongly normalizing (due to the possibility of constructing looping infinite chains of η-expansions followed by β-reductions). However, in the case of the purely implicational fragment of $\mathtt{NI}$ (we refer to it as $\mathtt{NI}^{\supset}$) Mints [49] has shown that strong normalization can be recovered by disallowing η-expansions of formulas of the form $A \supset B$ which are the consequence of $\supset$I or the major premise of $\supset$E (see also [35] for a discussion).

1.6 Canonicity

It is not the case that β-normal derivations have the subformula property in every calculus of natural deduction consisting of harmonious rules.[20] Consider for example the calculus $\mathtt{NI}^{2\wedge\supset}$, the extension of $\mathtt{NI}^{\wedge\supset}$ with quantification over propositions governed by the following rules (we indicate with $A(B/X)$ the result of substituting the free occurrences of X in A with B):[21]

$$\frac{A}{\forall X.A}\ \forall^2\mathrm{I} \qquad\qquad \frac{\forall X.A}{A(B/X)}\ \forall^2\mathrm{E}$$

(where X is an *eigenvariable*)

The rules are in harmony as testified by the possibility of defining the following reduction and expansion (in the reduction, $\mathscr{D}(B/X)$ indicates the derivation that results by uniformly substituting all free occurrences of X in $\mathscr{D}$ with B):

$$\dfrac{\dfrac{\begin{matrix}\mathscr{D}\\ A\end{matrix}}{\forall X.A}\ \forall^2\mathrm{I}}{A(B/X)}\ \forall^2\mathrm{E} \quad \overset{\forall^2\beta}{\rhd} \quad \begin{matrix}\mathscr{D}(B/X)\\ A(B/X)\end{matrix} \qquad\qquad \begin{matrix}\mathscr{D}\\ \forall X.A\end{matrix} \quad \overset{\forall^2\eta}{\lhd} \quad \dfrac{\dfrac{\begin{matrix}\mathscr{D}\\ \forall X.A\end{matrix}}{A}\ \forall^2\mathrm{E}}{\forall X.A}\ \forall^2\mathrm{I}$$

In contrast to the other elimination rules so far encountered, the consequence of an application of $\forall^2$E might be of higher complexity than its premise, due to the fact that the complexity of the formula B (called the witness of the rule application) can be arbitrary. This features complicates significantly the proof of normalization of β-reduction in $\mathtt{NI}^{2\wedge\supset}$ (strong normalization of β-reduction in a calculus akin to $\mathtt{NI}^{2\wedge\supset}$ was first established by Girard [22] using a generalization of the method of Tait mentioned above). Moreover, β-normal derivations in $\mathtt{NI}^{2\wedge\supset}$ do not enjoy the subformula property, as shown by the following β-normal derivation (in the example, the witness of the application of $\forall^2$E is $\forall X.X \supset C$ which is also the premise of the rule application):

$$\dfrac{\dfrac{\forall X.X \supset C}{(\forall X.X \supset C) \supset C}\ \forall^2\mathrm{E} \qquad \forall X.X \supset C}{C}\ \supset\mathrm{E}$$

There is however another, though much weaker, property which is warranted by the harmonious setup of the rules of the calculus. This property, to which we will refer to as *canonicity*, is not enjoyed by any β-normal derivation, but only by those β-normal derivations which are also *closed* (i.e. such that all their assumptions are discharged): In harmonious calculi, any such derivation ends with an introduction rule.

Canonicity holds for any arbitrary natural deduction calculus provided it satisfies the following two conditions:

Definition 1.1 (*Harmonious calculus*) A natural deduction calculus[22] is said to be *harmonious* iff:

1. All rules of the calculus are either introduction or elimination rules.
   We stress that
   - No restriction is imposed on introduction rules (in particular no complexity condition has to be satisfied for the result to hold);
   - only a *mild requirement* is imposed on elimination rules, namely that no elimination rule is such that its applications can discharge assumptions in the derivation of the major premise.

2. For every maximal formula there is a β-reduction to get rid of it (i.e. every maximal formula is a β-redex).

Fact 3 (*Canonicity*) In an harmonious natural deduction calculus, every closed and β-normal derivation ends with an application of an introduction rule.

Proof We check by induction on the number of inference rules applied in a derivation that either the antecedent of the theorem is false or the consequent is true. If the derivation consists only of an assumption it is not closed. The inductive case of a derivation consisting of $n + 1$ rule applications falls into two sub-cases. The derivation ends either with an introduction rule or with an elimination rule. In the former case, we are done. In the latter case, we have to show that the derivation is either open or it is not β-normal. We apply the induction hypothesis to the subderivation of the major premise and we distinguish two sub-cases: either the subderivation is open or it is not β-normal and then so is the whole derivation (because of the mild requirement on elimination rules in condition 1 above); or the subderivation ends with an introduction rule, but then the whole derivation is not β-normal as the major premise of the application of the elimination rule yielding the conclusion of the derivation is obtained by introduction. □

Henceforth, we will refer to derivations ending with an introduction rule as *canonical derivations*.[23]

1.7 Normalization, Subformula Property, Canonicity and Harmony

In every harmonious calculus, β-normal derivations can be equivalently characterized as those containing no maximal formula occurrence.

In calculi which are not harmonious, however, this is not in general the case. Consider the extension of $\mathtt{NI}^{\wedge\supset}$ with the rules for $\mathtt{tonk}$, which we will refer to as $\mathtt{NI}^{\wedge\supset\mathtt{tonk}}$. The calculus is not harmonious in that it fails to satisfy the condition 2 of Definition 1.1: In such a calculus, it is not possible to devise a reduction to get rid of any maximal formula occurrence whose main connective is $\mathtt{tonk}$, as the following example shows (in the example A is an arbitrary proposition and p an atomic one):

$$\dfrac{\dfrac{\dfrac{\overset{u}{A}}{A \supset A}\supset\mathrm{I}\ \langle u\rangle}{(A \supset A)\ \mathtt{tonk}\ p}\mathtt{tonkI}}{p}\mathtt{tonkE} \qquad (\mathbf{T})$$

The occurrence of $(A \supset A)$ $\mathtt{tonk}$ p in **T** is maximal. Nonetheless, there is no way to further β-reduce the derivation, which therefore qualifies as β-normal.

To consider the above derivation as β-normal may appear counterintuitive at first. The reason is that, on the one hand, the notion of normal derivation is usually presented as meant to capture the intuitive idea of a derivation containing no redundancy; and, on the other hand, maximal formula occurrences are usually taken to constitute a kind of conceptual redundancy within derivations. Sticking to these intuitions, one may expect a necessary condition for a derivation of any calculus (and not just of an harmonious one) to qualify as normal to be that the derivation does not contain any maximal formula occurrences.

It is however doubtful that maximal formula occurrences should always count as constituting a redundancy. This is certainly the case in $\mathtt{NI}^{\wedge\supset}$, where consecutive applications of $\supset$I and $\supset$E, or of $\wedge$I and either $\wedge\mathrm{E}_1$ or $\wedge\mathrm{E}_2$ do constitute a conceptual detour. But what about a calculus containing the rules for $\mathtt{tonk}$? The rules for $\mathtt{tonk}$ are clearly not in harmony. This is tantamount to denying that we had already a derivation of the consequence of an application of the elimination rule, provided that the premise had been established by introduction. In other words, when we establish something passing through a complex formula governed by $\mathtt{tonk}$, we are not making an unnecessary detour. The fact that the rules for $\mathtt{tonk}$ are not in harmony means exactly that in some (actually most) cases it is only by appealing to its rules that we can establish a deductive connection between two propositions not involving $\mathtt{tonk}$. This is the opposite of the claim that maximal formula occurrences having $\mathtt{tonk}$ as main connective constitute a redundancy. Rather, they are the most essential ingredient for establishing a wide range of derivability claims. For example, in the derivation **T**, the maximal formula occurrence $(A \supset A)\mathtt{tonk}\,p$ is in no way redundant: without passing through it, it would have been impossible to establish the conclusion p.[24]

Hence, we do not take the fact that β-normal derivations in a calculus like $\mathtt{NI}^{\wedge\supset\mathtt{tonk}}$ might contain maximal formula occurrence as showing that there is something amiss with the definition of normality.

Of course this is not to deny that there is something amiss with $\mathtt{NI}^{\wedge\supset\mathtt{tonk}}$ and in fact the notion of β-normal derivation can be used to make clear what is amiss with this calculus. Although β-reduction is normalizing in $\mathtt{NI}^{\wedge\supset\mathtt{tonk}}$ (this can be easily proved in the same way as it was done for $\mathtt{NI}^{\wedge\supset}$), neither do β-normal derivations have the sub-formula property, nor are all closed β-normal derivations canonical.

In the next chapters, we will show that the canonicity of closed β-normal derivations of harmonious calculi plays a crucial role for their (proof-theoretic) semantic interpretation. The fact that not every β-normal derivation in a calculus like $\mathtt{NI}^{\wedge\supset\mathtt{tonk}}$ is canonical thus shows its semantic defectiveness.

It is worth stressing that among harmonious calculi we find not only calculi such as $\mathtt{NI}^{2\wedge\supset}$—in which β-normal derivations do not possess the subformula property—but also calculi in which β-reduction is not (weakly) normalizing. Like the failure of the subformula property for β-normal derivation in $\mathtt{NI}^{2\wedge\supset}$, the failure of normalization of β-reduction does not invalidate Fact 3, i.e. in every harmonious calculus, even those in which not every derivation can be reduced to a β-normal derivation, those derivations which are both closed and β-normal are canonical as well.

We conclude by observing that all (standard) natural deduction calculi for classical logic are obtained by the addition of one or more rules to $\mathtt{NI}$ which are neither introduction nor elimination rules. Hence these calculi do not comply with the definition of 'harmonious calculus' given above. It is by now commonplace that classical logic can be given an harmonious presentation by either abandoning natural deduction (typically in favor of sequent calculus) or by enriching natural deduction in different ways (typically, by allowing derivations to have multiple conclusions, or by considering "refutation" rules along side standard "proof" rules). Although we do not exclude the possibility of systematically applying the ideas to be developed in the next chapters (possibly in modified form) to these other formal settings, in the present work we will restrict our attention to standard natural deduction calculi, and therefore leave classical logic out of the picture.

1.8 A Quick Comparison with Other Approaches

The account of harmony sketched in Sect. 1.3 differs from the account of harmony stemming from Belnap [2], who cashed out the no more and no less aspects of the informal definition of harmony in terms of conservativity and uniqueness respectively.[25]

Following Dummett (who refers to conservativity as 'global' harmony and to the availability of reductions as 'intrinsic' harmony), for some authors (see e.g. [94], pp. 1204–5) the distinctive feature of Belnap's conditions is their being "global", in contrast with other "local" ways of rendering the informal definition of harmony, such as the one in terms of reductions and expansions.[26]

In our opinion, however, what crucially distinguishes the account of harmony sketched in the previous section from the one of Belnap—as well as from those of other authors, such as Tennant's (see, e.g. [108])—is something else: Both conservativity and uniqueness are defined in terms of derivability (i.e. of *what* can be derived by means of the rules for a connective) and not in terms of properties involving the internal structure of derivations (i.e. of *how* something can be derived). We propose to refer to accounts of harmony based on derivability as *extensional*, while those making explicit reference to the internal structure of derivations will be referred to as *intensional*.

The choice of this terminology will become clear in the course of the next chapter, in which it shown how starting from reductions and expansions one naturally arrives at a notion of identity of proofs and of formula isomorphism.

Notes to This Chapter

1. For a recent critical discussion of the thesis that assertion plays a distinguished role among speech acts, see [81].
2. The exact meaning of harmony is however open to different interpretations. In particular, in the subsequent literature, there is no agreement on whether harmony should be considered as a descriptive or a normative criterion; nor whether it should be considered as a criterion of "significance" or of "logicality" (that is, if expressions governed by rules that are not in harmony should be considered as meaningless; or as meaningful but not belonging to the logical vocabulary) or of something else. In the present chapter, we will stick to the reading of harmony as meaningfulness. A more specific characterization of the significance of harmony will be given in Sect. 3.6. See also Note 7 below.
3. More precisely, we call the *major premise* of an application of an elimination rule the one which corresponds, in the rule schema, to the premise in which the connective to be eliminated occurs.
4. I am therefore taking for granted that what is established by a proof, and what one can draw inferences from, is a proposition, rather than, say, a judgment that a proposition is true. The distinction between judgment and proposition, on which some authors particularly insist (see, e.g. [106]), will not play any significant role in the present work. The reason for the choice here made is of mere convenience.
5. Terminologically, [12] uses 'harmony' sometimes to refer only to the no more aspect of this condition (see, e.g. pp. 247–248, [12]) and sometimes to refer to both (see, e.g. p. 217, [12]). Later on (see, e.g. p. 287, [12]), Dummett introduces the term 'stability' to cover both aspects. Here, we will follow Jacinto and Read [34] and use 'stability' to refer to the no less aspect of harmony, where 'harmony' is understood as covering both aspects. We also remark that sometimes one refers to *what has to established in order to infer a proposition A* as *the (direct, or canonical) grounds for A* and harmony is informally stated as the requirement that "Whatever follows from the direct grounds for deriving a proposition must

follow from that proposition" (this formulation is due to Negri and von Plato [53], p. 6) to which one may add, '(and nothing else).' in order to stress the two-fold nature of the requirement. Negri and von Plato [53] refer to their informal formulation of harmony as 'inversion principle'. We will however reserve the term 'inversion principle' for functions yielding collections of elimination rules as outputs when applied to collections of introduction rules as inputs. Three distinct inversion principles in our sense will be discussed in Sects. 3.4, 3.7 and 3.9 (on related terminological issues, see also Note 6 to Chap. 3). Finally, we observe that, although Dummett himself stressed that harmony is a two-fold condition, the proof-theoretic semantic literature has been mostly concerned with the no more aspect of it (but see e.g. [7, 8, 52] for notable exceptions), thus making our informal characterization of harmony, to some extent, non-standard.

6. Whereas `tonk`'s rules fail to meet both the no less and the no more aspect of the informal characterization of harmony, there are connectives which fail to satisfy only one of the two. A connective with the same introduction rule as conjunction and the same elimination rule as implication is an example of a connective failing to satisfy the no less aspect, but satisfying the no more aspect (see pp. 158–159, [52]). For another example, consider the connective whose rules are obtained from those of conjunction by dropping one of the two elimination rules. For a connective satisfying the no less aspect but failing to satisfy the no more aspect one may consider a variant of `tonk` with two introduction rules (corresponding to both introduction rules for disjunction). In this case using the elimination rule one would obtain no less than what is needed to introduce the connective again using the second introduction rule.
7. A referee objected that from our diagnosis of A `tonk` B as nonsense it looks "as if harmony was a criterion for meaningfulness, although perhaps it is best interpreted as a criterion for logicality (in line with Dummett's own admission that it cannot be reasonably asked for all the expressions of the language)." The objection is fair but a full evaluation of it, though of the utmost importance for the current debates on proof-theoretic semantics, goes beyond the scope of the present work. Here we only remark that: (i) In spite of Dummett's own admissions, it is undeniable that he is at least strongly sympathetic to the equation between harmony and meaningfulness. (One of) Dummett's [12] aim(s) is to recast Brouwer's criticisms of classical mathematics (namely that of being incomprehensible, viz. meaningless) by showing that the rules for the logical constants in classical logic are not harmonious. (We do not thereby want to commit ourselves either to the cogency of Dummett's arguments, nor to the tenability of Brouwer's views.) (ii) Even if harmony is not a criterion for meaningfulness, its applicability goes certainly beyond that of logical expressions, as shown by the rules for the predicate 'x is a natural number' which we briefly discussed at the end of Sect. 1.3. (We are here implicitly endorsing the view on which the natural number predicate is a non-logical expression. Though widespread, this view has been notoriously challanged by logicist and neo-logicist, see [114].) See also Note 2 above.

8. Schroeder-Heister [94] refers to the two aspects of harmony as the 'criterion of reduction' and the 'criterion of recovery' respectively. Steinberger [102] refers to collections of rules that fail to meet the no more and no less aspects of harmony as cases of 'E-strong' (or equivalently 'I-weak') disarmony and cases of 'E-weak' (or equivalently 'I-strong') disarmony respectively.
9. However, new ones may be generated in the process, see below.
10. For rules discharging assumptions, we distinguish between their premises and their immediate premises (see Definition A.1 in the appendix). In the case of an instance of $\supset$I with consequence $A \supset B$, the immediate premise is B, while the premise is the (concrete) rule $A \Rightarrow B$. The distinction is relevant for showing that in the expansion for implication given below, the local valley accords with the general description just given. Through a single application of $\supset$E one obtains a derivation of B from A and $A \supset B$, that by Definition A.6 (see Sect. A.5 in the appendix) counts as a derivation of the rule $A \Rightarrow B$ (from the assumption $A \supset B$).
11. The idea that expansions express the no less aspect of harmony has been first explicitly formulated by Pfenning and Davies [58].
12. In actual and schematic derivations, discharge is indicated with natural numbers placed above the discharged assumptions and in angle brackets to the left of the inference line at which the assumptions are discharged. Sometimes, u, v possibly with subscripts are used in place of numbers. As detailed in Appendix A, according to the "official" definition of derivations all assumptions (and not only those that are discharged) actually carry a numerical label. Which assumptions count as discharged thus depends only on the numerical labels in angle brackets to the left of inference rules. With very few exceptions (see e.g. Footnote 5 to Chap. 2), the labels above undischarged assumptions are irrelevant for the issues described in the present work, and hence they will be mostly omitted. In schematic derivations, a formula in square brackets indicates an arbitrary number (≥ 0) of occurrences of that formula, if the formula is in assumption position, or of the whole subderivation having the formula in brackets as conclusion. Square brackets are also used in rule schemata to indicate the form of the assumptions that can be discharged by rule applications.
13. That u is fresh for $\mathscr{D}$ means that the application of $\supset$I in the expanded derivation discharges no assumptions of the form A in $\mathscr{D}$.
14. That x is an eigenvariable means that x does not occur free in any assumption of the derivation of the minor premise $A(Sx/x)$ other than those discharged by the rule.
15. For a general pattern to produce rules for inductively defined predicates covering the identity relation, the predicate 'being a natural number' and other more complex notions as special cases, see [42].
16. The choice of the notation is motivated by the Curry-Howard correspondence (between derivations in the implicational fragment of `NI` and terms of the simply typed λ-calculus), under which the reduction and expansion for implication correspond (respectively) to steps of β-reduction and η-expansions on λ-terms:

$$(\lambda x.t)s \overset{\beta}{\rightsquigarrow} t[s/x] \qquad t \overset{\eta}{\rightsquigarrow} \lambda x.tx$$

17. An application of $\supset\beta$-reduction introduces new maximal formula occurrences whose degree is not lower than the one cut away only when: (i) the derivation of the minor premise of the relevant application of $\supset$E contains at least one maximal formula occurrence whose degree is not lower than the one of the maximal formula occurrence cut away by the application of $\supset\beta$; and (ii) the relevant application of $\supset$I discharges more than one assumption. Choose among the maximal formula occurrences in a derivation in $\mathtt{NI}$ one of maximal degree which does not fulfill condition (i) above (such a formula occurrence can always be found). Let n be the degree of the chosen formula. By cutting away such a maximal formula occurrence with $\supset\beta$, the number of maximal formula occurrences of degree n necessarily decreases by one.
18. For a precise definition of the notion of undischarged assumption of a derivation, see Definition A.4 in Appendix A.
19. The proof of strong normalization for $\overset{\eta}{\triangleright}$ in $\mathtt{NI}^{\wedge\supset}$ can be given by induction on the number of "local valleys", since there are no complications analogous to those connected with $\supset\beta$-reduction. The proof of confluence for $\overset{\eta}{\triangleright}$ is also immediate since it follows from the confluence of $\overset{1\eta}{\triangleright}$ (which is immediate) by a simple induction on the length of η-reduction sequences. In contrast, the proof of confluence for $\overset{\beta}{\triangleright}$ is more involved and it requires the introduction of a relation "between" $\overset{1\beta}{\triangleright}$ and $\overset{\beta}{\triangleright}$ which can be (almost) immediately shown to be confluent.
20. A general definition of what is here understood by a 'calculus', although restricted to purely propositional languages (without quantification) is given in the Appendix, see Sect. A.4. At a few points, calculi equipped with rules for first-, or second-order quantification will be mentioned, as in the present section, but no general characterization of them will be provided.
21. That X is an eigenvariable here means that X does not occur free in A or in any undischarged assumption on which A depends.
22. A precise formulation of what is here understood by 'introduction rule' and 'elimination rule' is given in the Appendix, see Definition A.12 in Sect. A.9.
23. Prawitz refers to derivations ending with an introduction rule as 'canonical' starting from [68]. The term 'canonical' is used in the same way in Martin-Löf's constructive type theory, although it does not appear as late as [44] in Martin-Löf's writings. Dummett (see [12], pp. 260–261) defines canonical derivations in a more stringent way, by requiring canonical derivations to consist (roughly said) of introduction rules with the *exception of their open subderivations* on which no restriction is placed. In particular, the closed derivations in $\mathtt{NI}^{\wedge\supset}$ that qualify as canonical in Dummett's sense are the β_w-normal derivations to be discussed in Sect. 2.7 below. On canonical derivations in Dummett's sense, see also Notes 6 and 14 to Chap. 2.

24. In Chap. 6 we will actually provide arguments to reject the assumption that for a derivation to be normal it must be redundancy-free. This is however irrelevant for the present point.
25. The fact that uniqueness is a way of rendering the no less aspect of harmony may not be obvious at first, but see [94], pp. 1204–5. Observe moreover that Belnap's aim is that of providing conditions that a collection of rules has to satisfy in order to be able to qualify as implicit definitions of a connective, rather than that of defining harmony.
26. For a contrasting opinion on the globality of uniqueness, however, see [52], p. 151.

Chapter 2
Identity of Proofs

Abstract Proof-theoretic semantics is here presented as primarily concerned with the investigation of the relationship between proofs (understood as abstract entities) and derivations (the linguistic representations of proofs). This relationship is taken to be analogous to that between names and (abstract) objects in Frege. On this conception of proof-theoretic semantics, reductions and expansions should be viewed as identity-preserving operations on derivations and thus as inducing an equivalence relation on derivations such that equivalent derivations denote the same proof. Using this equivalence on derivations it is possible to define an equivalence relation on formulas that is stricter than interderivability, called isomorphism. We argue that identity of proofs and formula isomorphism show the intensional nature of this conception of proof-theoretic semantics. Finally, this conception is compared to the one advocated by Dummett and Prawitz, which is based on a notion of validity of derivations.

2.1 Proof-Theoretic Semantics

Semantic theories are alternatively presented either as a definition of a semantic predicate or as consisting in a mapping from linguistic entities onto semantic values. In traditional semantic theories, in which 'true' is the central semantic predicate, the semantics can be seen as mapping true sentences onto the truth-value 'Truth' (in Fregean terms) or onto facts (in a more Russellian or Wittgensteinian fashion).

According to Schroeder-Heister [87], the core of proof-theoretic semantics (henceforth PTS) is a definition of the predicate 'valid'. Unlike 'true', which applies to sentences, 'valid' applies to more complex linguistic structures: derivations.[1]

In analogy with traditional semantic theories, and following suggestions implicit in the work of Prawitz and Martin-Löf, it seem plausible that PTS could also be viewed as mapping syntactic expressions onto semantic values. In the case of PTS, the syntactic expressions and the semantic values in question are, respectively, derivations and proofs. That is, valid derivations are regarded as denoting, or representing, proofs.

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Thus, in PTS the syntactic category of expressions to which meaning is primarily assigned is that of derivations rather than propositions. Meaning is assigned to propositions only derivatively, by saying that the semantic value of a proposition is the set of its proofs.

As in traditional semantics 'true' (and 'false') are qualifying predicates (in contrast to the modifying usage of 'false', as in 'false friend'), one would expect 'valid' to play the role of distinguishing the class of derivations that denote proofs from a broader class of derivations, among which one could find also some invalid ones. In the second part of the present work (see in particular Chap. 5), the derivations of contradictions arising in languages containing paradoxical expressions will be taken to be the prototype of invalid derivations. However, this way of understanding invalid derivations differs substantially from the one arising from (several) definition(s) of validity advanced by Prawitz [66–68, 70, 73] and Dummett [12].

A discussion of the definition of validity raises several issues which complicate the discussion of the idea that in PTS proofs are the semantic values of (some, i.e. possibly not all) derivations. Moreover, in the first part of the present work, we will indeed focus on natural deduction calculi in which all derivations can be safely taken to denote proofs. For these reasons, in the present chapter, we will initially leave the notion of validity out of the picture, by assuming that all derivations of the natural deduction calculi to be considered *do* denote proofs, henceforth dropping the qualification 'valid'. The version of PTS endorsed by Dummett and Prawitz, in which the notion of validity plays a central role, will be presented in the final sections of the present chapter (see in particular Sects. 2.9–2.11).

2.2 Proofs as Constructions

The idea that proofs are the semantic values of formal derivations is particularly fitting for a specific conception of proofs, namely the one developed in the context of the intuitionistic philosophy of mathematics. According to intuitionism (see [32], for a survey) mathematics is not an activity of discovery, but of creation. Thus, mathematical objects are not entities populating some platonic realm existing independently of us. They are rather conceived as the result of an activity of *construction*, which intuitionists assume to be performed by an opportunely idealized knowing subject.

Proofs themselves are regarded as forming a particular variety of mathematical objects and a mathematical object qualifies as a proof of a certain proposition only if it satisfies certain conditions. Some of these conditions depend on the logical form of the proposition in question and they constitute what is nowadays called the Brouwer-Heyting-Kolmogorov (henceforth BHK) explanation.

Each clause of the BHK explanation (see, e.g. [123], Sect. 3.1) specifies a condition that a mathematical entity has to satisfy in order to qualify as a proof of a logically complex sentence of a given form. The clauses are formulated using certain basic operations which are assumed to be available to the creative subject in their activity of construction. For instance, the clause for conjunction says that a proof of a conjunction $A \wedge B$ is obtained by pairing together a proof of A and a proof of B. Traditionally the explanation is silent as to what counts as a proof of atomic propositions, with the exception of identity statements, whose proofs are taken to be computations of some sort.[2]

The BHK clause for implication requires some explanation: in traditional formulations, a proof of an implication $A \supset B$ is said to be a general method of construction transforming any proof of A into a proof of B, where by 'general method' one understands essentially a function from proofs of A to proofs of B. The use of the word 'function' is, however, somewhat misleading, in that a proof of an implication is not a function in Frege's sense of an unsaturated entity, but rather (to stick to Frege's terminology) a function as 'course of values'. For Frege, courses of values are objects that are associated with unsaturated functions by means of a specific operation $\dot{\varepsilon}\,\Phi(\varepsilon)$that takes an unsaturated function $f(\xi)$ as argument and that yields its course of values $\dot{\varepsilon} f(\varepsilon)$as value. The operation $\dot{\varepsilon}\,\Phi(\varepsilon)$is understood by Frege as an unsaturated function as well, though of higher level (thus, essentially like a quantifier, but applicable not just to first-level concepts, but to arbitrary first-level functions). Two distinct notions of application are associated with the two notions of function: application of a genuine (i.e. unsaturated) function $f(\xi)$ to an argument a simply consists in filling the argument-place of the function (henceforth referred to as its *slot*) with the argument a, thereby yielding $f(a)$; the application of a function as course-of-values, on the other hand, is itself a two-place (unsaturated) function $\mathtt{app}(\xi, \zeta)$, which applied to a function as course of values $\dot{\varepsilon} f(\varepsilon)$and to its argument a yields as value the same object that one would obtain by placing the second argument of the application-function in the slot of the unsaturated function from which the course of values is obtained, i.e. $\mathtt{app}(\dot{\varepsilon} f(\varepsilon), a) = f(a)$(for a thorough discussion of the distinction between the two notions of function, see [11, Chap. 8]).

In a fully analogous way, a proof of an implication $A \supset B$ is to be understood as the (abstract) object that results by applying an operation of higher-level to an "unsaturated proof entity", where an unsaturated proof entity is a function that filled with (applied to) a proof of A yields a proof of B [see 106]. Such a function, which may be described as "a proof of B whose construction depends on a proof of A", is what Prawitz [75] calls an 'unsaturated ground' for B, and Sundholm [105, 106] a 'dependent proof-object'[3]; and the operation that constructs out of it a proof of $A \supset B$ is (λ-)abstraction.

2.3 Derivations and Proofs

The idea that derivations in formal systems represent proofs is most fitting for natural deduction calculi in the style of Gentzen and Prawitz. In particular the idea of viewing derivations in these calculi as linguistic representations of proofs can be articulated in close analogy with Frege's traditional picture of the relationship between language and reality.

A key ingredient of Frege's conception is the distinction between singular terms, which denote objects, and predicates or, more generally, functional expressions, which denote concepts or, more generally, functions (understood as unsaturated entities as detailed in the previous section). In PTS this distinction is mirrored by the one between closed and open derivations. Only a closed derivation can be said to denote a proof(-object).[4] The semantic value of an open derivation is not a proof, but rather an (unsaturated) function that yields proofs of its conclusion when it is saturated using proofs of its assumptions. To clarify this point, consider an open derivation $\mathscr{D}$ of B having A as its only undischarged assumption[5]:

$$\begin{array}{c} [A] \\ \mathscr{D} \\ B \end{array}$$

Let $\mathscr{D}'$ be a closed derivation of A (thus denoting some proof of A). If we replace each undischarged occurrence of A in $\mathscr{D}$ with a copy of $\mathscr{D}'$, we obtain a closed derivation of B (denoting some proof of B). The derivation $\mathscr{D}$ can thus be seen as a means of mapping each proof of A (denoted by some derivation $\mathscr{D}'$) onto the particular proof B denoted by the composition of $\mathscr{D}$ with $\mathscr{D}'$, that we indicate with:

$$\begin{array}{c} \mathscr{D}' \\ [A] \\ \mathscr{D} \\ B \end{array}$$

In other words, we can say that $\mathscr{D}$ encodes (or denotes) a function from the set of proofs of A to the set of proofs of B.

A second ingredient of Frege's picture is that the same object can be denoted by distinct singular terms. For Frege, the sense of a singular term is "the way of giving" its denotation. The possibility of there being distinct singular terms denoting the same object reflects the fact that the same object can be given in different ways. As extensively argued by Dummett [11], a "way of giving an object" can be understood as an epistemic attitude towards the object, a way of epistemically accessing it.

The philosophical literature mostly focused on Frege's famous example of 'The morning star' and 'The evening star'—the two traditional singular terms used to refer to the same astronomical object, the planet Venus. It seems however that the idea that the same object can be denoted in different ways fully unwinds its potential when it is applied to abstract objects.

For example numbers, like proofs, are abstract objects, and in the language of arithmetic we have that the same number, e.g. fourteen, can be denoted by distinct numerical expressions such as '$(3 \times 4) + 2$', '$12 + 2$' and '14'. In contrast to 'The morning star'/'The evening star' example, examples like these make clear the possibility of distinguishing between "more direct" and "less direct" ways of denoting a given object. Looking at the three numerical expressions just considered we can say that, although they all denote the same number, the first does it in a less direct way than the second, and the second in a less direct way than the third. In general whereas numerals denote numbers in a direct way, complex numerical expressions, such as '$(3 \times 4) + 2$', denote numbers in a less direct way. When a particular numerical notation is adopted, such as the one of Heyting or Peano arithmetic, this comes out even more clearly. In these formal systems, the syntactic structure of a numeral '$S \ldots S0$' can be thought of as directly reflecting the process by means of which the number denoted by the numeral is obtained, i.e. by repeatedly applying the successor operation starting from zero. Numerals can thus be said to denote numbers in the most direct way possible.

In the case of derivations and proofs, a strikingly close correspondence can be observed between the BHK clauses and the introduction rules. The correspondence can be expressed by saying that the introduction rules encode the operations on proofs underlying the BHK clauses. For example. given two closed derivations $\mathscr{D}_1$ and $\mathscr{D}_2$, which we assume to denote a proof of A and a proof of B respectively, the following closed derivation:

$$\dfrac{\begin{array}{cc}\mathscr{D}_1 & \mathscr{D}_2 \\ A & B\end{array}}{A \wedge B}\,\wedge\mathrm{I}$$

can be thought of as denoting the pair of the two proofs. That is, the introduction rule for conjunction can be viewed as encoding the operation of pairing, and the derivation can be viewed as denoting the proof that results by applying this operation to the proofs denoted by the derivations $\mathscr{D}_1$ and $\mathscr{D}_2$.

When a closed derivation ends with an introduction rule, we will say that it denotes a proof in a *canonical* manner, where this means that the structure of the derivation corresponds, in its last step, to the last step of the process of construction of the denoted proof.

These considerations offer a way of understanding Gentzen's famous *dictum* "The introductions represent, as it were, the 'definitions' of the symbols concerned" [20, p. 80], namely as saying that what is a proof of a logically complex proposition governed by the connective † is *by definition* what is obtained by applying one of the operations encoded by the introduction rules of † to appropriate arguments.

It is important to stress that, since canonical derivations are simply derivations ending with an introduction rule, the structure of a closed canonical derivation reflects that of the denoted proof only in its last step, but no more than that.[6] This contrasts with the case of numerals, whose structure reflects the process of construction of the number throughout (and not only in the last step). In the analogy between proofs and numbers, canonical derivations thus correspond to what one may call *canonical numerical expressions*, that is expressions of the form 'St', where 't' is not necessarily a numeral.

One may therefore ask whether the analogy can be pushed further by identifying a class of derivations which stand to proofs as numerals to numbers. For a derivation to belong to this class it should denote a proof in the most direct way possible. That is, the structure of the whole derivation (and not just its last step) should correspond to the structure of the denoted proof (or, better, to the structure of the process of construction of which the proof is the result).

It seems tempting to say that in harmonious calculi (i.e. those for which Fact 3 holds) closed β-normal derivations are those that denote proofs in the *most direct* way possible. In the case of the derivation ending with an application of $\wedge$I considered above, Fact 3 warrants that, if the whole derivation is β-normal, than $\mathscr{D}_1$ and $\mathscr{D}_2$ will also end with an introduction rule (since they are closed and β-normal as well), so that the structure of the derivation will reflect the process of construction of the proof not just in its last step, but also in those immediately preceding it (i.e. those represented by the last rules applied in $\mathscr{D}_1$ and $\mathscr{D}_2$). In an harmonious calculus closed β-normal derivation are thus not only canonical, but reflect the structure of the denoted proof in a more thorough way.

As we will see in Sects. 2.7 and 2.8, further elements have to be take into consideration in order to answer the question of whether closed β-normal derivations can rightly be regarded as those representing proofs in the most direct ways.

We conclude this section by observing that the conception of PTS developed so far is certainly heavily influenced by that of Dummett and Prawitz. However, it aims to clarify some confusion which is pervasive in Dummett and Prawitz writings. Although both authors ascribe a central role to the distinction between direct and indirect ways of verification, they seem to identify ‘direct’ with ‘canonical’. As we argued, the structure of a canonical derivation does represent that of the process which yields the proof denoted by the derivation, but only in its last step. Moreover, Dummett and Prawitz do not always clearly distinguish between proofs and their linguistic representations. The closest formulation of the view presented here can probably be found in Dummett [e.g. 10, p.32]. Dummett draws a distinction between proofs as mental constructions and derivations as linguistic entities (which Dummett refers to as ‘demonstrations’) and makes the distinction canonical/non-canonical overlap with the one proofs/demonstrations, in the sense that proofs are canonical and demonstrations are non-canonical.

> We thus appear to require a distinction between a proof proper—a canonical proof—and the sort of argument which will normally appear in a mathematical article or textbook, an argument which we may call a ‘demonstration’[.] [10, p. 32]

In the view of PTS developed here, however, no issue of ‘canonicity’ (or normality) applies to proofs in themselves, but only to their linguistic presentations, viz. derivations. Prawitz [73] locates the distinction canonical/non-canonical both at the level of derivations (to which Prawitz refers as ‘arguments’) *and* of proofs and he blames Heyting for not stressing the distinction in the case of proofs [e.g. 71, 139]. (In more recent work, Prawitz has changed his views and now he agrees that the distinction does not apply to proofs (now referred to as ‘grounds’) but only to their linguistic representations.) Again, the distinction canonical/non-canonical is here applied only to derivations, whereas the BHK clauses are here viewed as characterizing, albeit informally, the notion of proof *tout court*.

2.4 From Reductions and Expansions to Equivalence

Some among the axioms of Peano or Heyting arithmetic, namely the following:

$$\begin{aligned} n + 0 &= n \\ m + Sn &= S(m + n) \\ n \times 0 &= 0 \\ m \times Sn &= m + (m \times n) \end{aligned}$$

can be "oriented", so to obtain "rewrite rules" thanks to which any expression denoting a number that is formed out of 0, S, $+$ and $\times$ can be transformed into a numeral (e.g. $(SS0 \times S0) + SS0$ into $SSSS0$). Obviously, when an expression is transformed into another using the rewrite rules, the identity of the denoted number is preserved, that is the original expression and the one obtained by rewriting denote the same number.

The analogy between numbers and proofs suggests the idea of regarding β-reductions as "rewriting rules" on derivations that preserve the identity of the denoted proof. But how plausible is this suggestion?

In the previous section, we observed that the introduction rules can be seen as the linguistic representations of these operations on proofs underlying the BHK explanation. It is not implausible to think that certain operations on proofs are associated with the elimination rules as well. For example, the rule $\wedge \mathrm{E}_1$ can be seen as encoding the operation that applied to a pair (of proofs) yields the first member of the pair as value. Given this reading of the rules, the left-hand side of the reduction associated with $\wedge \mathrm{E}_1$:

$$\dfrac{\dfrac{\begin{matrix}\mathscr{D}_1 \\ A\end{matrix} \qquad \begin{matrix}\mathscr{D}_2 \\ B\end{matrix}}{A \wedge B}\,\wedge\mathrm{I}}{A}\,\wedge\mathrm{E}_1$$

is a derivation that denotes the first member of the pair consisting of the proofs denoted by $\mathscr{D}_1$ and $\mathscr{D}_2$. But this is nothing but a cumbersome description of the very proof denoted by $\mathscr{D}_1$, i.e. by the right-hand side of the reduction. Thus, when a derivation $\mathscr{D}$ is transformed into another derivation $\mathscr{D}'$ using the reduction associated with $\wedge \mathrm{E}_1$, the identity of the proof denoted by $\mathscr{D}$ is preserved, i.e. $\mathscr{D}'$ denotes the same proof as $\mathscr{D}$.

Analogous considerations can be worked out not only for the β-reductions for the other connectives, but also for the η-expansions (although some authors have stressed some important differences, as detailed below in Sect. 2.7). For example, in the case of the expansion for conjunction, the expansion of a given derivation $\mathscr{D}$ of $A \wedge B$ denotes the proof which is obtained by pairing together the first and second projection of the proof denoted by $\mathscr{D}$. This again is nothing but a cumbersome description of the very same proof denoted by $\mathscr{D}$.[7]

Given the considerations developed so far, besides the relation of β-reduction we consider a relation of β-equivalence (notation $\stackrel{\beta}{\equiv}$), which we introduce as the reflexive and transitive closure of the relation of one-step β-equivalence (notation $\stackrel{1\beta}{\equiv}$). The latter is defined by saying that $\mathscr{D} \stackrel{1\beta}{\equiv} \mathscr{D}'$ iff either $\mathscr{D} \stackrel{1\beta}{\triangleright} \mathscr{D}'$ or $\mathscr{D}' \stackrel{1\beta}{\triangleright} \mathscr{D}$, and the former by saying that $\mathscr{D} \stackrel{\beta}{\equiv} \mathscr{D}'$ when for some $n \geq 1$ there is a sequence of n-derivations $\mathscr{D}_1, \ldots, \mathscr{D}_n$ such that $\mathscr{D}_1 = \mathscr{D}$, $\mathscr{D}_n = \mathscr{D}'$ and $\mathscr{D}_{i-1} \stackrel{1\beta}{\equiv} \mathscr{D}_i$ for each $1 < i \leq n$.

Analogous notions of equivalence can be introduced for each reduction relation considered in the previous chapter.

A crucial property of the relations of β- and $\beta\eta$-equivalence in the $\{\supset, \wedge, \top\}$-fragment of $\texttt{NI}$ (we indicate it as $\texttt{NI}^{\wedge\supset\top}$) is their non-triviality. That an equivalence relation on derivations is non-trivial means that there are at least one formula A and two derivations of A belonging to distinct equivalence classes. A typical example of two derivations of the same formula which are not $\beta\eta$-equivalent (and hence, a fortiori, that are not β-equivalent either) is this:

$$
\dfrac{\langle u \rangle\ \dfrac{\stackrel{u}{A}}{A \supset A}\supset\mathrm{I}}{A \supset (A \supset A)}\supset\mathrm{I}
\qquad\qquad
\langle u \rangle\ \dfrac{\dfrac{\stackrel{u}{A}}{A \supset A}\supset\mathrm{I}}{A \supset (A \supset A)}\supset\mathrm{I}
$$

These two derivations are $\beta\eta$-normal and the fact that they do belong to two different $\beta\eta$-equivalence classes is a consequence of the confluence property of $\beta\eta$-reduction. (We remind the reader that confluence means that if one derivation $\mathscr{D}$ reduces in a finite number of steps to two distinct derivations $\mathscr{D}_1$ and $\mathscr{D}_2$, then there should be a third one to which the both $\mathscr{D}_1$ and $\mathscr{D}_2$ reduce in a finite numbers of steps.) As for any two distinct $\beta\eta$-normal derivations there is no further derivation to which both reduce, it is also not the case that they could be obtained by reducing the same derivation. Hence, it is not possible to transform one derivation into the other by a chain of $\beta\eta$-reductions and $\beta\eta$-expansions. That is they belong to different equivalence classes. The above example thus shows that the notion of identity of proofs induced by $\beta\eta$-conversions is not trivial.

It is worth stressing that some formulas have infinitely many distinct proofs. Examples of such formulas are those of the form $(A \supset A) \supset (A \supset A)$. Three dinstinct proofs of formulas of this form are represented by the following $\beta\eta$-normal derivations[8]:

$$
\dfrac{\langle v \rangle\ \dfrac{\stackrel{v}{A}}{A \supset A}\supset\mathrm{I}}{(A \supset A) \supset (A \supset A)}\supset\mathrm{I}
\qquad
\langle u \rangle\ \dfrac{\stackrel{u}{A \supset A}}{(A \supset A) \supset (A \supset A)}\supset\mathrm{I}
\qquad
\langle u \rangle\ \dfrac{\langle v \rangle\ \dfrac{\dfrac{\stackrel{u}{A \supset A} \qquad \dfrac{\stackrel{u}{A \supset A} \qquad \stackrel{v}{A}}{A}\supset\mathrm{E}}{A}\supset\mathrm{E}}{A \supset A}\supset\mathrm{I}}{(A \supset A) \supset (A \supset A)}\supset\mathrm{I}
$$

It is easy to see how an infinite list of $\beta\eta$-normal derivations for formulas of this form can be obtained by repeating the addition of an assumption of the form $A \supset A$ and of an application of $\supset$E.[9]

In $\mathtt{NI}^{\wedge\supset\top}$, $\beta\eta$-equivalence plays a distinguished role, as this is the maximum among the non-trivial equivalence relations definable on $\mathtt{NI}^{\wedge\supset\top}$-derivations.[10] As Došen [5] and Widebäck [124] argued, the maximality of an equivalence relation on the derivations of a calculus $\mathtt{K}$ can be taken as supporting the claim that it is the *correct* way of analyzing the notion of identity of proofs underlying $\mathtt{K}$.

The notion of $\beta\eta$-equivalence in $\mathtt{NI}^{\wedge\supset\top}$ is well-understood: the decidability of $\beta\eta$-equivalence is an immediate consequence of normalization and confluence of $\beta\eta$-reduction in $\mathtt{NI}^{\wedge\supset}$, and its maximality was established by Statman [101], see also Došen and Petrić [6] and Widebäck [124].

2.5 Formula Isomorphism

Given an equivalence relation on derivations, it is possible to use it to define an equivalence relation on propositions that is, in general, stricter than interderivability and that is commonly referred to as *isomorphism*. Let E be an equivalence relation on derivations of a natural deduction calculus $\mathtt{K}$. The notion of E-isomorphism in $\mathtt{K}$ is defined as follows:

Definition 2.1 (*Isomorphism*) Two propositions A and B are E-isomorphic in $\mathtt{K}$ (notation $A \overset{E}{\simeq} B$) if and only if there is a pair of $\mathtt{K}$-derivations $\mathscr{D}_1$ and $\mathscr{D}_2$, called the *witness* of the isomorphism, such that:

- $\mathscr{D}_1$ is a derivation of A from B and $\mathscr{D}_2$ is a derivation of B from A (i.e. A and B are interderivable in $\mathtt{K}$);
- and the two compositions of $\mathscr{D}_1$ and $\mathscr{D}_2$ are E-equivalent to the derivations consisting only of the assumptions of A and of B respectively:

$$B \overset{E}{\equiv} \begin{array}{c} [B] \\ \mathscr{D}_1 \\ [A] \\ \mathscr{D}_2 \\ B \end{array} \qquad \begin{array}{c} [A] \\ \mathscr{D}_2 \\ [B] \\ \mathscr{D}_1 \\ A \end{array} \overset{E}{\equiv} A$$

Why is this notion called isomorphism? The derivation consisting of the assumption of a formula A can be viewed as representing the identity function on the set of proofs of A. Hence, the second condition of the definition of isomorphism can be expressed by saying that the two derivations $\mathscr{D}_1$ and $\mathscr{D}_2$ represent two functions from proofs of A to proofs of B and vice versa which are the inverse of each other. This in turn means that the set of proofs of A and of B are in bijection.[11]

Clearly, a necessary condition for some notion of E-isomorphism not to collapse onto that of interderivability is that the equivalence relation E used in the definition is non-trivial. In particular, if any two derivations $\mathscr{D}_1$ and $\mathscr{D}_2$ of any formula A from itself were E-equivalent, the second condition on the witness in the definition of

E-isomorphism would be vacuously satisfied (i.e. any pair of derivations $\mathscr{D}_1$ and $\mathscr{D}_2$ of A from B and vice versa would witness the E-isomorphism of A and B).

In order to establish that two propositions are E-isomorphic in $\mathtt{K}$ one can proceed syntactically (i.e. by constructing two derivations witnessing the isomorphism). Typical examples of $\beta\eta$-isomorphic formulas in $\mathtt{NI}^{\wedge\supset}$ are pairs of formulas of the form $(A \wedge B) \wedge C$ and $A \wedge (B \wedge C)$, or $(A \wedge B) \supset C$ and $A \supset (B \supset C)$, as can be checked by easily constructing appropriate witnesses.

To show that two propositions are not E-isomorphic in $\mathtt{K}$ one usually argues by constructing a counter-model, consisting in a categorial interpretation of $\mathtt{K}$ that validates E-equivalence and in which the two propositions in question are not isomorphic. For $\beta\eta$-isomorphism in $\mathtt{NI}^{\wedge\supset\top}$, a typical example is provided by interpretations in the category of finite sets obtained by mapping

- atomic propositions on arbitrary finite sets,
- $\top$ on some singleton set,
- the conjunction and implication of two propositions A and B on the cartesian product and the function space of the interpretations of A and B.

It easily checked that each inference rule of $\mathtt{NI}^{\wedge\supset\top}$ can be interpreted as a family of maps between sets so that the β- and η-equations are validated by the interpretation.

Using interpretations of this kind it easy to show that propositions of the form $A \wedge A$ and A are, in general, non-isomorphic. Whenever A is interpreted as a finite set of proofs of cardinality $\kappa > 1$, the interpretations of A and of $A \wedge A$ are sets of different cardinalities (as the cardinality of the interpretation of $A \wedge A$ is κ^2), and thus there cannot be an isomorphism between the two.

Another example of pairs of interderivable but not necessarily isomorphic propositions, which is of relevance for the results to be presented in Chap. 3, is constituted by pairs of propositions of the form $((A \supset B) \wedge (B \supset A)) \wedge A$ and $((A \supset B) \wedge (B \supset A)) \wedge B$. Whenever A and B are interpreted on sets of different cardinalities, the interpretations of the two propositions will also be sets of different cardinalities.

In fact, the category of finite sets plays a distinguished role for the notion of $\beta\eta$-isomorphism in $\mathtt{NI}^{\wedge\supset\top}$: as shown by Solov'ev [99], two formulas of $\mathtt{NI}^{\wedge\supset\top}$ are $\beta\eta$-isomorphic if and only if they are interpreted on sets of equal cardinality in every interpretation in the category of finite sets. From this, it follows that $\beta\eta$-isomorphism in $\mathtt{NI}^{\wedge\supset\top}$ is decidable and finitely axiomatizable.

2.6 An Intensional Picture

When inference rules are equipped with reduction and expansions, and thus a notion of identity of proofs is available, the notion of logical consequence is no longer to be understood as a relation but rather as a graph: beside being able to tell whether a proposition is provable we can discriminate between essentially different ways in which a proposition is provable.

Moreover, using identity of proofs we could introduce the notion of isomorphism. This has been proposed (notably by Došen [5]) as a formal *explicans* of the informal notion of synonymy, i.e. identity of meaning. Intuitively, interderivability is only a necessary, but not sufficient condition for synonymy. Isomorphic formulas can be regarded as synonymous in the sense that:

> They behave exactly in the same manner in proofs: by composing, we can always extend proofs involving one of them, either as assumption or as conclusion, to proofs involving the other, so that nothing is lost, nor gained. There is always a way back. By composing further with the inverses, we return to the original proofs. (Došen [5], p. 498)

The fact that the relationship of isomorphism is stricter than that of mere interderivability makes isomorphism more apt than interderivability to characterize the intuitive notion of synonymy in an inferentialist setting. For instance, whereas on an account of synonymy as interderivability all provable propositions are synonymous, this can be safely denied on an account of synonymy as isomorphism (e.g. $A \supset A$ is not $\beta\eta$-isomorphic to $\top$, whenever A has more than one proof).

We may therefore say that when synonymy is explained via isomorphism, we attain a truly intensional account of meaning.

This picture contrasts with the one arising from other accounts of harmony, such as Belnap's (see above Sect. 1.8). Belnap accounts for harmony using the notions of conservativity and uniqueness, which are defined merely in terms of derivability, rather than by appealing to any property of the structure of derivations. Hence, Belnap's criteria for harmony do not yield (nor require) any notion of identity of proofs analogous to the one induced by reductions and expansions. Thus, no notion of isomorphism is in general available for propositions which are governed by harmonious rules in Belnap's sense. Thus on Belnap's approach to harmony, it is not obvious that an account of synonymy other than the one in terms of interderivability is available and this vindicates the claim that his approach to harmony delivers a merely extensional account of logical consequence and meaning.[12]

2.7 Weak Notions of Reduction and Equivalence

In Sect. 2.4 the view of reductions and expansions as identity-preserving operations was defended by appealing to the nature of the operations associated with the introduction and elimination rules.

In the case of conjunction, what is expressed by the β-reductions (viewed as equivalences) is that the operations associated with the elimination rules are those operations that applied to a pair of proofs yield (respectively) the first and second member of the pair.

On reflection, it seems that this is not something that "follows" from the properties of the operations involved, but it may rather be taken as a definition of what the projection operations *are*. In other words, in the case of reductions, it does not seems that they are "justified" by the nature of the operations associated with the conjunction rules (pairing and projections). Rather, the reductions for conjunction can be seen as *definitions* of the projection operations associated with the elimination rules.

The same does not seem to apply in the case of η-expansions. If we consider the expansion for conjunction, it seems correct to say that its reading in terms of identity of proof is only justified by the nature of the operations underlying the inference rules for conjunction (and does not play the role of defining them).

Moreover, whether expansions for other connectives can be seen as identity preserving, seems to hinge on further assumptions. This is typically the case for implication. To assume $\supset\eta$-expansion to be identity preserving is to assume a principle of extensionality for the proofs of propositions of the form $A \supset B$, namely that two such proofs (which are functions from proofs of A to proofs of B) are the same iff they yield the same values for each of their arguments.[13] Although such an assumption is certainly in line with Frege's conception of functions (extensionality is nothing but Frege's infamous *Basic Law V* of his *Grundgetzte*), some authors [e.g. 44] have argued against it.

When functions are identified by *what* they do (i.e. by which values they associate with their arguments), extensionality is certainly an uncontroversial assumption. Not so when functions are understood as procedures to obtain certain values given certain arguments. On such a conception of functions, it is very natural to allow for different functions (i.e. procedures) to deliver the same result.

On an understanding of functions as procedures it is thus dubious that every instance of η-expansion is identity preserving. In fact, it dubious that even every instance β-reduction is identity preserving. For example, let $\mathscr{D}_1$ and $\mathscr{D}_2$ be the following two derivations:

$$\mathscr{D}_1 \quad = \quad \cfrac{\langle 1\rangle\ \cfrac{\overset{1}{A}}{A \supset A}\ {\supset}\mathrm{I}}{B \supset (A \supset A)}\ {\supset}\mathrm{I} \qquad\qquad \langle 2\rangle\ \cfrac{\langle 1\rangle\ \cfrac{\cfrac{\cfrac{\overset{1}{A} \qquad \overset{2}{B}}{A \wedge B}}{A}}{A \supset A}\ {\supset}\mathrm{I}}{B \supset (A \supset A)}\ {\supset}\mathrm{I} \quad = \quad \mathscr{D}_2$$

The two derivations are obviously β-equivalent, since $\mathscr{D}_2$ β-reduces in one step to $\mathscr{D}_1$. But do they denote the same proof? The derivation $\mathscr{D}_1$ denotes (the course of values of) a constant function f that assigns to any proof of B the (course of values of the) identity function on the set of proofs of A. The derivation $\mathscr{D}_2$ denotes a function g that assigns to any proof of B the (course of values of the) function h that does the following: it takes a proof of A as input and outputs the first projection of the pair consisting of the proof of A taken as input and the proof of B that acts as argument of the overall proof. The identity function on the set of proofs of A and the function h—i.e. the values of the functions f and g (whose courses of values are) denoted by $\mathscr{D}_1$ and $\mathscr{D}_2$—are clearly extensionally equivalent functions. However, if one chooses as a criterion of identity of functions the set of instructions by means of which it is specified how their values are computed, one should conclude that h and the identity function on the proofs of A are two distinct (though extensionally equivalent) functions. Hence, so are the two functions f and g, as they yield distinct values when applied to the same arguments.

What this example shows is that, on an intensional understanding of functions, not only η-equivalence, but also β-equivalence is not always an identity preserving relation. In order to capture the notion of identity of proofs faithful to such an understanding of functions, β-equivalence has to be weakened, and one possibility consists, essentially, in restricting the applications of β-reduction to closed derivations (i.e. derivations in which all assumptions are discharged). As derivations depending on assumptions do not denote proofs, but rather functions from proofs of the (undischarged) assumptions to proofs of the conclusion, the removal of local peaks from such derivations is not identity preserving: the functions denoted by the reduced derivations are not the same as those denoted by the derivations containing the local peaks.

This weakening on the notion of β-reduction/equivalence is motivated by the fact that the notion of reduction is of any semantic significance only for closed derivations. Only these represent (proof-)objects, and only for objects it is natural to distinguish between more and less direct ways to refer to them. In the case of functions, it is doubtful whether such a distinction makes sense at all, at least when functions are understood intensionally rather than extensionally. In the latter case, i.e. when the criterion of identity for functions is their yielding the same values for the same arguments, it may be plausible to speak of more and less direct ways of computing certain values. However, if functions are understood as the procedures by means of which the values are calculated, a "more direct" procedure to determine the values from the arguments will simply count as a different function.[14]

Formally, we indicate the relation of one-step weak β-reduction as $\mathscr{D} \overset{1\beta_w}{\triangleright} \mathscr{D}'$. It is common to define one-step weak β-reduction by modifying only the congruence condition of the definition of one-step β-reduction (see above Sect. 1.4), by replacing 'subderivation' with 'closed subderivation'. In this way one *can* β-reduce open derivations, so that in the example given above the immediate subderivation of $\mathscr{D}_2$ does weakly β-reduce in one step to the immediate subderivation of $\mathscr{D}_1$. Nonetheless, the definition achieves its goal for closed derivations, in that $\mathscr{D}_2$ does not weakly β-reduce in one step to $\mathscr{D}_1$ (as its maximal formula occurrence belongs to the open subderivation of $\mathscr{D}_2$).

The relation of weak β-reduction $\overset{\beta_w}{\triangleright}$ is defined as the reflexive and transitive closure of $\overset{1\beta_w}{\triangleright}$ and the relations of weak one-step β-equivalence $\overset{1\beta_w}{\equiv}$ and weak β-equivalence $\overset{\beta_w}{\equiv}$ as the symmetric closure of $\overset{1\beta_w}{\triangleright}$ and $\overset{\beta_w}{\triangleright}$ respectively.

In the calculus $\mathtt{NI}^{\wedge\supset}$, most of the results holding for $\overset{\beta}{\triangleright}$ also hold for $\overset{\beta_w}{\triangleright}$.

A derivation $\mathscr{D}$ is called β_w-normal iff it is not possible to β_w-reduce it (i.e. iff $\mathscr{D} \overset{\beta_w}{\triangleright} \mathscr{D}'$ implies $\mathscr{D} = \mathscr{D}'$).[15] Weak β-reduction in $\mathtt{NI}^{\wedge\supset}$ is strongly normalizing (i.e. there are no infinite weak β-reduction sequences), and it is confluent. Moreover, not only closed β-normal but also closed β_w-normal derivations are canonical (i.e. Fact 3 holds if one replaces β with β_w). However, the subformula property fails for β_w-normal derivations in $\mathtt{NI}^{\wedge\supset}$ (as testified for instance, by the derivation $\mathscr{D}_2$ in the above example).

In the first part of the present work we will be mostly assuming $\beta\eta$-equivalence, rather than this weakening of β-equivalence as the proper analysis of identity of proofs. In the second part, we will argue that the adoption of one or the other makes a substantial difference for the analysis of paradoxical phenomena to be worked out there.[16]

2.8 Derivations and Proofs, Again

At the end of Sect. 2.3, we hinted at the possibility of taking β-normal derivations as the class of derivations representing proofs in the most direct way possible. In the light of the considerations developed in the previous section, however, it should now be clear that the choice of the class of derivation that should be considered as "the most direct way" of representing proofs is not an absolute one, but it depends on the choice of a particular notion of identity of proof. In particular, on an extensional understanding of functions, it may be more natural to take $\beta\eta$-normal derivations to be those that most directly represent proofs. Alternatively, on an intensional understanding of functions, weak-β-normal derivations could be the most natural choice: on such a conception of functions, although a β_w-normal derivation may contain some maximal formula occurrences (in one of its open subderivations), it would still represent the proof it denotes in the most direct way: by further β-reducing the derivation, one would obtain a derivation denoting a different proof.

We also do not exclude the possibility of finding arguments in favor of accepting a middle ground position (such as e.g. claiming that β-normal derivations, rather than β_w or $\beta\eta$-normal ones are those that denote proofs in the most direct way).

It is however important to stress that any such option makes sense only when one considers derivations in harmonious calculi. As discussed in Sect. 1.7, when the rules of a calculus are not in harmony, as for instance in the case of $\mathtt{NI}^{\wedge\supset\mathtt{tonk}}$, the notion of β-normal derivation (and, for similar reasons, that of $\beta\eta$- and β_w-normal derivation) is devoid of semantic significance. The starting point for viewing normal derivations as the most direct way of representing proofs is their canonicity. In a calculus in which normal derivations are not canonical, we lack any reason to regard normal derivations as playing a distinguished semantic role.

Although less dramatic, a further assumption underlying the identification is the confluence of the chosen reduction relation. In a calculus like $\mathtt{NI}^{\wedge\supset}$ all of β-, $\beta\eta$- and β_w-reduction are confluent. As we will detail in the next chapter, this is not so in general (in fact $\beta\eta$-reduction is not confluent already in $\mathtt{NI}^{\wedge\supset\top}$, see [122] Exercise 8.3.6C) and when this is not the case it does not seem to make much sense to speak of *the* most direct way of denoting a proof. A derivation may have several distinct normal forms, and in this case there does not seem to be a criterion to choose one among them as *the* derivation denoting the proof in *the* most direct way.

Although derivations and proofs are entities of different sorts, when derivations in normal form (for *some* notion of normal form) can be viewed as representing the denoted proof in the most direct way possible, it is tempting to identify this particular derivation with the proof it denotes.[17] Given such an identification, the semantic significance of reduction can be formulated in a distinctive way. Namely, the process of reducing a derivation to normal form can be viewed as the process of assigning to the derivation its denotation, i.e. of interpreting it.

In the contexts in which normal forms play this distinguished role we can say the following: As normal derivations are—or, more properly, represent in the most direct way—their own denotation, for them interpretation is just identity. In the case of arbitrary derivations, to interpret them is to reduce them to normal form.

2.9 Validity

As anticipated at the beginning of the chapter, in this and the next sections we will give a concise presentation of the approach to PTS developed by Prawitz and Dummett (see [12, 66–68, 70, 74]).

The core of Prawitz-Dummett PTS is a definition of validity for derivations. The essential idea of the definition is that closed derivations that result by applying an introduction rule to one or more valid derivations are valid "by definition" (see also Sect. 2.3 above): more precisely, a closed canonical derivation will be said to be valid iff its immediate subderivations are valid. On the other hand, the validity of an arbitrary closed derivation consists in the possibility of reducing it to a valid closed derivation in canonical form.

Due to the fact that some introduction rule, such as $\supset$I, can discharge assumptions, the immediate subderivation(s) of a closed canonical derivation need not be closed. Thus the validity of a closed canonical derivation may depend on that of an open derivation: thus the validity predicate should apply not only to closed derivations, but to open derivations as well. On Dummett and Prawitz's definition, an open derivation is said to be valid iff every derivation that results from replacing its open assumptions with closed valid derivations of the assumptions (we will call these derivations the *closed instances* of the open derivation) is valid. This characterization reflects the idea that an open derivation denotes a function that takes proofs of the assumptions as arguments, and yields proofs of the conclusion as values.

In Prawitz and Dummett's original approach, 'validity' is meant as a distinguishing feature selecting a subset of linguistic structures that denote proofs from a broader class (analogously to what happens in truth-conditional semantics, where truth is a distinguishing feature of some, but not all, sentences of a language). Accordingly, Prawitz and Dummett propose to generalize the notion of derivation in a specific formal calculus to what they refer to as 'arguments'. Given a language $\mathcal{L}$, arguments are not generated just by a collection of introduction and elimination rules specified beforehand, but by arbitrary inference rules such as, e.g.,

$$\frac{A \wedge (B \wedge C)}{B}\, R_1 \qquad \frac{A \supset (B \supset C)}{(A \wedge B) \supset C}\, R_2 \qquad \frac{A \supset B}{A}\, R_3$$

which may also encode intuitively unacceptable principles of reasoning, as in the case of R_3. We will however refer also to arguments as derivations, keeping in mind that they are not to be understood as generated only using a fixed set of inference rules, but any arbitrary rule over $\mathcal{L}$. (In the case of a propositional language $\mathcal{L}$, what counts as an arbitrary rule can be made precise as in Definition A.1 in Appendix A.)

Beside reductions to eliminate redundant configurations constituted by introduction and elimination rules, further reduction procedures transforming derivations into derivations can be considered and the validity of a derivation is judged relative to the choice of a set of reduction procedures.

Reduction procedures are taken to be rewriting operations on derivations generalizing those associated with the elimination rules for the standard connectives. Only very minimal conditions are imposed on reductions, namely, that the result of reducing a given derivation $\mathscr{D}$ must be a (distinct) derivation $\mathscr{D}'$ having the same conclusion of $\mathscr{D}$ and possibly fewer, but no more, undischarged assumptions than $\mathscr{D}$, and that satisfy a condition analogous to the weakening of the congruence condition used in the definition of β_w-reduction in Sect. 2.7 (this is essentially a condition of closure under substitution, see [87] for more details).

In Dummett and Prawitz's original approach, validity is also relative to an *atomic system*, i.e. to a set of rules involving atomic propositions only (Dummett [12] refers to these as 'boundary rules'), which specifies which deductive relationships hold among atomic propositions. Dummett and Prawitz seem to restrict these rules to what in computer science are called production rules, i.e. rules of the form

$$\frac{A_1 \qquad \dots \qquad A_n}{B}$$

for some $n \geq 0$ in which all A_is and B are atomic propositions.[18] Examples of rules that might figure in an atomic system are the introduction rules (but not the elimination rule) for `Nat` given at the end of Sect. 1.3. Note that since rules with no premises are allowed, in some atomic systems it might be possible to give closed derivation of atomic propositions.

We thus have the following definition:

Definition 2.2 (*Prawitz's validity of a derivation*) A derivation $\mathscr{D}$ is valid with respect to a set $\mathcal{J}$ of reduction procedures and to an atomic system $\mathcal{S}$ iff:

- It is closed and
  - either its conclusion is an atomic proposition and it $\mathcal{J}$-reduces to a derivation of $\mathcal{S}$;
  - or its conclusion is a complex proposition and it $\mathcal{J}$-reduces to a canonical derivation whose immediate subderivations are valid with respect to $\mathcal{J}$ and $\mathcal{S}$;

- or it is open and for all $\mathcal{S}' \supseteq \mathcal{S}$ and $\mathcal{J}' \supseteq \mathcal{J}$, all closed instances of $\mathscr{D}$ (i.e. all derivations obtained by replacing the undischarged assumptions of $\mathscr{D}$ with closed derivations that are valid with respect to $\mathcal{S}'$ and $\mathcal{J}'$) are valid with respect to $\mathcal{S}'$ and $\mathcal{J}'$.

The definition is understood to proceed by induction on the joint complexity of the conclusion and the open assumptions of derivations. Thus, in order for the definition to be well-founded, introduction rules are assumed to satisfy what Dummett [12, p. 258] proposes to call the *complexity condition*: namely, that in every application of an introduction rule the consequence must be of higher complexity than all immediate premises and all assumptions discharged by the rule application.

The process of checking the validity of a closed derivation can be described as follows: (i) If the derivation is not canonical, try to reduce it to a (closed) canonical one; (ii) if a canonical derivation is obtained check for the validity of its subderivations: for closed subderivations repeat (i); for open subderivations check the validity of all their closed instances.[19]

Note that in order to show the validity of a given closed derivation, only its closed subderivations are reduced. The kind of reduction that derivations undergo is thus a generalization of the notion of weak reduction discussed in Sect. 2.7.[20] Given a derivation, to show that it is valid one has first to compute a weak normal form of it (relative to the set of reduction procedures under considerations) and then one has to additionally show the validity of all open subderivations of the weak normal form obtained.[21]

2.10 Correctness of Rules

In standard (i.e. truth-conditional) semantics the notion of truth is used to define logical consequence. Analogously, in Prawitz-Dummett PTS the notion of validity can be used to define what we will call the *correctness of an inference rule* (we therefore follow the terminology of [90], rather than of Prawitz, who uses 'validity' also for this notion).

As for Tarski logical consequence is truth preservation, Prawitz proposes to define the correctness of an inference as preservation of validity:

> An *inference rule* may be said to be *valid* when each application of it preserves validity of arguments. (Prawitz [70], p. 165)

More precisely, Prawitz defines the correctness of an inference rule (schema) as follows:

Definition 2.3 (*Prawitz's correctness of an inference rule*) An inference rule of the form

$$\frac{A_1 \qquad \dots \qquad A_n}{B}\, R$$

is correct iff there is a set of reduction procedures $\mathcal{J}$ such that for every extension $\mathcal{J}'$ of $\mathcal{J}$, for every atomic system $\mathcal{S}$ and for any collection of closed derivations $\mathscr{D}_1, \dots, \mathscr{D}_n$, such that each $\mathscr{D}_i$ is a derivation of conclusion A_i $(1 \leq i \leq n)$ that is valid with respect to $\mathcal{J}'$ and $\mathcal{S}$,

$$\dfrac{\begin{matrix}\mathscr{D}_1 & & \mathscr{D}_n \\ A_1 & \dots & A_n\end{matrix}}{B}\,R$$

is a derivation of B valid relative to $\mathcal{J}'$ and $\mathcal{S}$.

An inference rule schema is valid iff all its instances are valid.

In the case of introduction rules, their correctness is "automatic"—in Dummett's [12] terminology, they are "self-justifying". For example, the rule of conjunction introduction $\wedge$I is correct iff it yields a closed valid derivation of its conclusion whenever applied to closed valid derivations of its premises. Suppose we are given two derivations $\mathscr{D}_1$ and $\mathscr{D}_2$ of A and B respectively that are valid relative to $\mathcal{J}$ and $\mathcal{S}$. As the derivation

$$\dfrac{\begin{matrix}\mathscr{D}_1 & \mathscr{D}_2 \\ A & B\end{matrix}}{A \wedge B}\,\wedge\mathrm{I}$$

is a closed canonical derivation with valid immediate subderivations, then it is valid. Thus, whenever we apply $\wedge$I to closed valid derivations we obtain a derivation which is also valid (and this holds for any $\mathcal{J}$ and any $\mathcal{S}$). Hence, $\wedge$I is correct.[22]

Whereas the correctness of introduction rules is "automatic", to show the correctness of an elimination rule we need to make explicit reference to reduction procedures. Consider the left conjunction elimination $\wedge\mathrm{E}_1$. Again, its correctness amounts to its yielding a closed valid derivation of its conclusion whenever applied to a closed valid derivation of its premise. Suppose we have a closed derivation of $A \wedge B$ that is valid relative to $\mathcal{J}$ and $\mathcal{S}$. By definition of validity, this derivation $\mathcal{J}$-reduces to a closed canonical derivation valid relative to $\mathcal{J}$ and $\mathcal{S}$, i.e. to a closed derivation ending with an introduction rule whose immediate subderivations $\mathscr{D}_1$ and $\mathscr{D}_2$ are valid relative to $\mathcal{J}$ and $\mathcal{S}$. By applying $\wedge\mathrm{E}_1$ to it, one gets a derivation—call it $\mathscr{D}$—which is not canonical. But the reduction for conjunction:

$$\mathscr{D} \;=\; \dfrac{\dfrac{\begin{matrix}\mathscr{D}_1 & \mathscr{D}_2 \\ A & B\end{matrix}}{A \wedge B}\,\wedge\mathrm{I}}{A}\,\wedge\mathrm{E}_1 \quad \overset{\wedge\beta_1}{\triangleright} \quad \begin{matrix}\mathscr{D}_1 \\ A\end{matrix}$$

allows us to reduce $\mathscr{D}$ to $\mathscr{D}_1$, which we know to be valid. Thus, the set of reduction procedures $\mathcal{J}$ consisting of $\wedge\beta_1$ is such that whenever we apply $\wedge\mathrm{E}_1$ to derivations which are valid relative to any extension $\mathcal{J}'$ of $\mathcal{J}$ and any $\mathcal{S}$, we obtain a derivation which is valid relative to $\mathcal{J}'$ and relative to $\mathcal{S}$. Hence, the rule $\wedge\mathrm{E}_1$ is correct.

The definitions of validity and correctness can be used to show the correctness of inference rules which are neither introduction or elimination rules. For instance, in the case of the rule R_1 mentioned in Sect. 2.9, one can show its validity by considering the set of reductions consisting of all instances of the following schema:

$$\dfrac{\dfrac{\begin{matrix}\mathscr{D}_1 \\ A\end{matrix} \quad \dfrac{\begin{matrix}\mathscr{D}_2 & \mathscr{D}_3 \\ B & C\end{matrix}}{B \wedge C}\,\wedge\mathrm{I}}{A \wedge (B \wedge C)}\,\wedge\mathrm{I}}{B}\,R_1 \qquad \triangleright \qquad \begin{matrix}\mathscr{D}_2 \\ B\end{matrix}$$

As observed by Schroeder-Heister [87], however, in contrast to the standard elimination rules and simple generalizations thereof, the reductions needed to justify a rule may need to make reference to other non "self-justifying" rules. For example, all instances of the following schema can be used to justify the rule R_2 mentioned in the previous section:

$$\dfrac{\langle n\rangle\ \dfrac{\begin{matrix}\overset{n}{[A]}\\ \mathscr{D}\\ B\supset C\end{matrix}}{A\supset(B\supset C)}\supset\mathrm{I}}{(A\wedge B)\supset C}R_2 \qquad \rhd \qquad \langle n\rangle\ \dfrac{\dfrac{\begin{matrix}\dfrac{\overset{n}{A\wedge B}}{[A]}\wedge\mathrm{E}_1\\ \mathscr{D}\\ B\supset C\end{matrix} \qquad \dfrac{\overset{n}{A\wedge B}}{B}\wedge\mathrm{E}_2}{C}\supset\mathrm{E}}{(A\wedge B)\supset C}\supset\mathrm{I}$$

However, as the reduction is formulated using the elimination rules for conjunction and implication, to show the correctness of R_2 one has to consider the set of reductions containing not only all instances of the schema just given, but $\wedge\beta_1$, $\wedge\beta_2$ and $\supset\beta$ as well.

In fact, as observed by Schroeder-Heister [87] and by Prawitz [75] himself, the generalization of the notion of reduction is to some extent trivial as one can simply define the reduction for a rule R which is neither an introduction nor an elimination rule by "inflating" a derivation ending with an application of the rule in question with a derivation of the instance of the rule using the introduction and elimination rules. In the case of R_1 and R_2 these "inflating" reductions would be the following:

$$\dfrac{\begin{matrix}\mathscr{D}\\ A\wedge(B\wedge C)\end{matrix}}{B}R_1 \qquad \rhd \qquad \dfrac{\dfrac{\begin{matrix}\mathscr{D}\\ A\wedge(B\wedge C)\end{matrix}}{B\wedge C}\wedge\mathrm{E}_2}{B}\wedge\mathrm{E}_1$$

$$\dfrac{\begin{matrix}\mathscr{D}\\ A\supset(B\supset C)\end{matrix}}{(A\wedge B)\supset C}R_1 \qquad \rhd \qquad \langle n\rangle\ \dfrac{\dfrac{\dfrac{\begin{matrix}\mathscr{D}\\ A\supset(B\supset C)\end{matrix} \qquad \dfrac{\overset{n}{A\wedge B}}{A}\wedge\mathrm{E}_1}{B\supset C}\supset\mathrm{E} \qquad \dfrac{\overset{n}{A\wedge B}}{B}\wedge\mathrm{E}_2}{C}\supset\mathrm{E}}{(A\wedge B)\supset C}\supset\mathrm{I}$$

The rule R_1 and R_2 can then be shown to be correct using the set of reduction procedures consisting of the instances of the relevant "inflating" reductions together with the standard β-reductions. Inflating reductions thus deprive the whole idea of using reductions to justify inference rules of its significance, apart from the basic case in which β-reductions are used to justify the elimination rules.

Moreover, inflating reductions induce the expectation that introduction and elimination rules are complete with respect to correctness i.e. that if a rule is correct then it is derivable in `NI` (given inflating reductions, the correctness of a rule boils down to its being derivable from the introduction and elimination rules). The conjecture can be equivalently understood as expressing the fact that the elimination rules are the strongest rules that can be justified given the introduction rules, in the sense that any correct rule is derivable from the introduction rules together with the elimination rules.

Although completeness holds for the conjunctive-implicational language fragment, as shown by Piecha and Schroeder-Heister [60], it does not hold in presence of disjunction. The result does not hinge on the notion of reduction but rather on particular features of atomic systems.

Whereas we will ignore the issue of validity in the first part of the present work, in the second part we will consider the possibility of applying the notion of validity to derivations of specific natural deduction calculi. As anticipated, in order to show that validity is a notion that applies to some, but not all derivations, it will not be necessary to consider derivations built up using arbitrary inference rules and evaluating their validity using arbitrary reductions as proposed by Prawitz and Dummett.

Rather we will consider a calculus extending `NI` with specific rules characterizing paradoxical expressions, for which a clear notion of reduction is available. For this calculus, we will show that (an appropriately modified) notion of validity plays the role of selecting a subset of derivations which can be said to denote proofs of their conclusions. The definition of validity will however differ from the one of Prawitz and Dummett in significant respects. In particular, it will reject the relative priority of the notion of validity and correctness that underlies the Prawitz-Dummett approach. This aspect has been recognized as problematic already by Prawitz, as we detail in the next section.

2.11 The Relative Priority of Correctness and Validity

The definition of correctness of an inference as transmission of validity yields a perfect analogy between PTS and truth-conditional semantics. However, the resulting relationship between validity of derivations and correctness of inferences appears odd. One would naturally define the validity of a derivation in terms of the correctness of the inferences out of which it is constituted. In Prawitz-Dummett PTS it is rather the other way around. The validity of a derivation is independently defined in terms of its being reducible to canonical form. And the correctness of an inference is defined, we could say, in terms of the validity of the derivations in which it is applied.

Prawitz himself acknowledges that this way of defining the validity of derivations and the correctness of inferences is not the most intuitive. Nonetheless, he stresses that although being constituted by correct inferences is rejected as a definition of validity, the following intuitive principle still holds:

(V) A derivation is valid if it is constituted by applications of correct inferences.

In Prawitz's words:

> If all the inferences of an argument are applications of valid inference rules [...], then it is easily seen that also the argument must be valid, namely with respect to the justifying operations [viz. the reduction procedures] in virtue of which the rules are valid. But this is not the way we have defined validity of arguments. On the contrary, the validity of an inference rule is explained in terms of validity of arguments (although once explained in this way, an argument may be shown to be valid by showing that all the inference rules applied in the argument are valid). (Prawitz, [70] p. 169)

Unfortunately, the availability of reduction procedures is sufficient to show the correctness of an elimination rule only in what we may call 'standard' cases. Without undertaking the task of making the notion of 'standard' fully precise, it will be clear that whenever we have to deal with paradoxical phenomena we are *not* in a standard case.

As a result, the availability of reduction procedures will no more be sufficient for a rule to be correct in Prawitz's sense. In the second part of the present work this will be taken as a reason to relax the definition of the correctness of an inference. The weaker notion of correctness of an inference to be introduced there will be shown to open the way to the application of PTS to languages containing paradoxical expressions.

Notes to This Chapter

1. One of the referees asks why validity applies to derivations, rather than to the sequents they establish (i.e. to derivability claims $\Gamma \Rightarrow A$, or more generally, using the terminology of Appendix A, to rules). The reason is both historical and conceptual. Historically, as detailed below in Sect. 2.9, Prawitz [66] introduced the notion of validity as applying primarily to derivations, and only derivatively rules are said to be valid, namely iff there is a valid derivation establishing them. On the intensional conception of PTS here defended, the priority ascribed to derivations is no accident. Intensional PTS is not simply concerned with *what* is derivable from a given set of inference rules, but rather with *how* what is derivable can in fact be derived. We do not thereby wish to deny either the viability or the significance of an extensional conception of PTS in which validity primarily applies to what is established by derivations, rather than to derivations themselves. However, such an approach will not be pursued here.
2. However, in line with Martin-Löf [43, 45], one may take proofs of the different kinds of atomic propositions to be generated (like those for logically complex propositions) by the availability of some basic constructions and operations.
3. Thus Frege's terminology, according to which functions and objects are unsaturated and saturated entities respectively, is reversed in the context of constructive type theory: dependent objects are unsaturated entities and the functions proving $A \supset B$ are saturated entities.

4. We here understand 'proof' broadly enough so to also allow (closed) atomic propositions to have proofs. For example, using the rules for `Nat` depicted at the end of Sect. 1.3, one can easily construct a closed derivation of `Nat` $SS0$, which we regard as denoting a proof of this atomic proposition. Admittedly, the deductive content of this proof is rather thin, and in this case, 'computation' rather than 'proof' could be more appropriate. See also Note 2 above.
5. As described in Note 12 to Chap. 1, in schematic derivations a formula in square brackets simply indicates an arbitrary number of occurrences (possibly zero) of that formula in assumption position. That is, the derivation $\mathscr{D}$ we are considering is a derivations of B in which an undischarged assumption of the form A may occur a finite (possibly zero) number of times. Given the official definition of derivation and composition of derivations in Appendix A, we are implicitly assuming all occurrences of A to be labelled by the same numerical index, which is here being omitted.
6. As remarked in Note 23 to Chap. 1, Dummett gives a more stringent definition of canonical derivations to which the present remark does not apply. (Thanks to a referee for stressing this point.) We wish however to point out that, although more stringent, when applied to closed derivations in $\mathtt{NI}^{\wedge\supset}$, canonicity in Dummett's sense is a less stringent requirement than β-normality. That is, all closed β-normal $\mathtt{NI}^{\wedge\supset}$-derivations qualify as canonical in Dummett's sense, but the converse does not hold. Some closed $\mathtt{NI}^{\wedge\supset}$-derivations that are canonical in Dummett's sense are not β-normal. This is due to the fact that Dummett allows local peaks to occur in his canonical derivations, namely in the subderivations depending on assumptions that are later on discharged, such as the subderivations of the premises of applications of $\supset$I (see again Note 23 to Chap. 1). On Dummett's canonical derivations cf. also Note 14 below.
7. A setting in which the considerations developed in the present section can be developed in a formally rigorous manner is Martin-Löf's [45] constructive type theory. In constructive type theory, the operations encoded by the inference rules are made explicit by decorating the consequence of each inference rule with a term constructed out of the terms decorating the premises by means of a specific operation associated with the rule, like this:

$$\frac{t : A \qquad s : B}{\langle t, s\rangle : A \wedge B} \qquad\qquad \frac{t : A \wedge B}{\pi_1(t) : A} \quad \frac{t : A \wedge B}{\pi_2(t) : B}$$

$$\frac{\begin{matrix}[x : A]\\ t : B\end{matrix}}{\lambda x.t : A \supset B}\,\supset\mathrm{I} \qquad \frac{t : A \supset B \qquad s : A}{\mathtt{app}(t, s) : B}\,\supset\mathrm{E}$$

That is, premises and consequences of inference are not propositions, but judgments of the form $t : A$, whose informal reading is 't is a proof of A'. The result is that the conclusion of a derivation is decorated by a term which encodes the whole sequence of operations associated with each of the inferences constituting the derivation. Whereas in standard natural deduction reductions and expansions, or the equations on derivations associated with them, are defined in the metalanguage, the richer syntax of constructive type theory permits to "internalize" them as a further kind of rule whose consequences are judgments of the form $t \equiv s : A$. The informal reading of judgments of this form is that t and s denote the same proof of A. The equality from which the reduction for implication is obtained by "orientation" becomes the following rule:

$$\dfrac{s : A \qquad \begin{matrix}[x : A]\\ t : B\end{matrix}}{\mathtt{app}(\lambda x.t, s) \equiv t(s/x) : B}\,\supset\!\beta$$

whose informal reading is as follows: if s is a proof of A and t is a proof of B depending on a proof of A (i.e. a function as unsaturated entity from proofs of A to proofs B) then the proof which one obtains by applying the course-of-values of t to s is the same proof that results by filling the slot of t with s. Although several parts of the material covered in the present work could be naturally reconstructed in the setting of intuitionistic type theory, we prerer to avoid this formalism in order to make the presentation accessible also to readers unfamiliar with it.

8. These derivations correspond to the Church encoding of the numerals 0, 1 and 2 in the λ-calculus. The two derivations of the previous example correspond to the Church encoding of the Booleans.
9. Both this example and the previous one rely in an essential way on the availability of the structural rules of weakening and contraction, i.e. on the availability of vacuous discharge and of simultaneous discharge of more copies of one assumption. The implicit availability of the structural rule of exchange also triggers the existence of distinct proofs of certain formulas. For example, formulas of the form $(A \wedge A) \supset (A \wedge A)$ can be proved either by means of the identity function or by means of the function that maps every ordered pair of proofs of A onto the pair in which the order of the members has been exchanged. The fact that for some formulas there are distinct proofs is however independent of the availability of structural rules. For example, whenever A is some provable formula and $\mathscr{D}$ is a closed $\beta\eta$-normal proof of A, the following derivations denote distinct proofs of $(A \supset (A \supset B)) \supset (A \supset B)$, no matter which formula is taken for B:

$$\langle u\rangle\;\dfrac{\langle v\rangle\;\dfrac{\dfrac{\dfrac{\overset{u}{A \supset (A \supset B)} \qquad \begin{matrix}\mathscr{D}\\ A\end{matrix}}{A \supset B}\,\supset\!\mathrm{E} \qquad \overset{v}{A}}{B}\,\supset\!\mathrm{E}}{A \supset B}\,\supset\!\mathrm{I}}{(A \supset (A \supset B)) \supset (A \supset B)}\,\supset\!\mathrm{I} \qquad\qquad \langle u\rangle\;\dfrac{\langle v\rangle\;\dfrac{\dfrac{\dfrac{\overset{u}{A \supset (A \supset B)} \qquad \overset{v}{A}}{A \supset B}\,\supset\!\mathrm{E} \qquad \begin{matrix}\mathscr{D}\\ A\end{matrix}}{B}\,\supset\!\mathrm{E}}{A \supset B}\,\supset\!\mathrm{I}}{(A \supset (A \supset B)) \supset (A \supset B)}\,\supset\!\mathrm{I}$$

Note that in these two derivations no implicit appeal to any structural rule is made. Rather than to structural rules, the availability of more than one proof of a proposition A in $\mathtt{NI}^{\wedge\supset\top}$ is closely connected to the number of occurrences of atoms of various polarity in A. As first proved by Mints [48], if a formula A in $\mathtt{NI}^{\wedge\supset\top}$ is *balanced* (which means that each atom in A occurs exatly twice, once in positive and once in negative position) then A has a unique proof [see, for details, 122, Sects. 8.4 and 8.5.2–3]. For further references and for a recent investigation on the number of proofs of formulas in presence of disjunction, see Scherer and Rémy [84]. Thanks a lot to Paolo Pistone for a thorough discussion of this point and for providing the example given in this footnote.

10. This result is what in the typed λ-calculus corresponds to a corollary of Böhm's theorem for the untyped λ-calculus.
11. It is worth stressing that the definition of the notion of isomorphism relies in an essential way on a generalization to open derivations of the ideas presented in the previous section. That is, it is not only assumed that distinct closed derivations in the same equivalence class denote the same proof, but also that distinct *open* derivations in the same equivalence class denote the same *function* from proofs to proofs. As detailed below in Sect. 2.7, this generalization is not uncontroversial.
12. In calculi in which the replacement theorem does not hold and/or in which transitivity fails, one can define an equivalence relation $A \approx B$ stricter than interderivability by requiring the interderivability of every pair of formulas C and D, such that D is obtained from C by replacing some (possibly all) occurrences of A with occurrences of B (Smiley [98] proposed this notion as the correct analysis of the intuitive notion of synonymy). Observe however that, like interderivability, this notion is also extensional in character, in that it is formulated without making reference to identity of proofs. In calculi in which a non-trivial notion of identity of proofs is available, it is therefore plausible that isomorphism is stricter than Smileyan synonymy. An example is the natural deduction system for Nelson's [54] logic $\mathtt{N4}$ (see [65], Appendix B, Sect. 2). This calculus is an extension of $\mathtt{NI}$ with introduction and elimination rules for the 'strong' negation of each kind of logically complex formulas. In $\mathtt{N4}$, the interderivability of A and B alone does not warrant that $A \approx B$: A and B are Smileyan synonymous only if both A and B *and* $\sim A$ and $\sim B$ are interderivable (where $\sim$ is the strong negation). The relation of $\beta\eta$-equivalence for derivations of this calculus can defined by extending in the obvious way the definition for $\mathtt{NI}$ (the generalized η-expansions to be introduced in Chap. 3 below have to be used both in the case of the "positive rules" of disjunction and of the "negative rules" for conjunction). In $\mathtt{N4}$, A and $A \wedge A$ are Smileyan synoynmous, but they are not $\beta\eta$-isomorphic, i.e. $A \approx A \wedge A$ but $A \overset{\beta\eta}{\not\simeq} A \wedge A$. (Note however that in $\mathtt{N4}$ an equivalence relation on formulas A and B even stricter than isomorphism could be defined by requiring *both* $A \overset{\beta\eta}{\simeq} B$ *and* $\sim A \overset{\beta\eta}{\simeq} \sim B$.) Another example of a calculus in which replacement fails is Tennant's [113] Core Logic. In contrast to what happens in $\mathtt{N4}$, in Core Logic no sufficient condition is known that warrants Smileyan synonymy. We conjecture however, that A and $A \wedge A$ are Smileyan synonymous in Core Logic as well. To show that $A \wedge A$ and A are not isomorphic in Core Logic would require the definition of an equivalence relation on derivations in Core Logic. (For some

difficulties that might be encountered, see the remarks at the end of Sect. 3.5 and in particular Note 16 to Chap. 3.) If the resulting notion of identity proofs is close enough to the one resulting from $\beta\eta$-equivalence in $\mathtt{NI}$, A and $A \wedge A$ would fail to qualify as isomorphic in Core Logic as well.

13. This is fully analogous to the role played by η-conversion in the untyped λ-calculus—see Note 16 to Chap. 1.
14. This weaker notion of β-reduction is essentially the one advocated by Martin-Löf [44], although with a partly different motivation. As stressed in Note 23 to Chap. 1 and in Note 6 to the present chapter, Dummett [12] gives a different definition of canonical derivations. For closed derivations in $\mathtt{NI}^{\wedge\supset}$, being canonical in Dummett's sense essentially coincides with being normal with respect to the presenly discussed weakening of β-reduction. In the present context, Dummett's claim that canonical derivations (in his sense) are the most direct way of establishing their conclusions can thus be taken to be the claim that closed derivations which are normal with respect to this weakening of β-reduction are the most direct ways of representing the proofs they denote. As discussed in this section, this is tantamount to adopt an intensional conception of functions and to deny that all instances of β-reduction are identity preserving.
15. What we call β_w-normal derivations are what correspond to *fully evaluated* terms in type theory (see, e.g. [55]).
16. The fact that the η-equations express a principle of extensionality for functions is not in conflict with the claim of Sect. 2.6 that the account of harmony based on reduction and expansions is intensional. Reductions and expansions yield an intensional account of consequence (by allowing the possibility of establishing the same proposition by means of different proofs) and of meaning even though the criteria of identity for functions that one adopts are extensional.
17. This is at least implicitly done by Dummett, as discussed at the end of Sect. 2.3 above.
18. A more general notion of atomic systems allowing rules of higher-level is discussed in [59].
19. To check the validity of an open derivation with respect to $\mathcal{S}$ and $\mathcal{J}$ one has to check the validity of *all* its closed instances relative to *all* possible extensions of $\mathcal{S}$ and $\mathcal{J}$. As pointed out by one of the referees, it is therefore dubious that the notion of validity can be checked effectively. In some specific cases, it is however easy to show the validity of certain open derivations. A few examples are given in the next section.
20. We speak of a generalization since weak reduction was defined for β-reduction only, whereas here we consider an arbitrary set of reduction procedures.
21. The resulting notion of validity could be strengthened, by requiring that all open subderivations of every weak normal derivation obtained from a given one should be valid. Since Prawitz considers arbitrary sets of reduction procedures, there is no warrant that confluence holds, and hence a given derivation may have several different weak normal forms (see [87], for details).
22. It is worth stressing that, although Dummett refers to the introduction rules as 'self-justifying', strictly speaking they are not correct by fiat, i.e. their validity

is not simply stipulated. Both Dummett and Prawitz agree in reducing their correctness to the notion of validity of arguments. However, it is true that given the role played by closed canonical derivations in the definition of validity, introduction rules always turn out to be correct "automatically".

Chapter 3
Towards an Intensional Notion of Harmony

Abstract In this chapter we discuss how the intensional account of harmony sketched in the first chapter can be developed in a systematic way for a class of connectives whose rules are obtained in a uniform way using an inversion principle. To handle disjunction and disjunction-like connectives, the formulation of the expansions requires particular care. We discuss and compare two different ways of formulating the inversion principle and finally we investigate the prospects of developing an account of harmony for connectives whose rules do not obey inversion, pointing at the weakness of the approaches proposed in the literature so far.

3.1 Disjunction: A Problem for Stability

The account of harmony in terms of reductions and expansions sketched in Sect. 1.3 encounters a difficulty when one tries to apply it to the rules of disjunction:

$$\frac{A}{A \vee B}\,\vee\mathrm{I}_1 \qquad\qquad \frac{B}{A \vee B}\,\vee\mathrm{I}_2 \qquad\qquad \frac{A \vee B \quad \begin{matrix}[A]\\ C\end{matrix} \quad \begin{matrix}[B]\\ C\end{matrix}}{C}\,\vee\mathrm{E}$$

As in the case of conjunction, it is quite uncontroversial that the rules satisfy both aspects of the informal characterization of harmony, and in fact deductive patterns of the two kinds discussed, as well as reductions and expansions can be exhibited in this case as well:

$$\langle u_1, u_2\rangle\,\frac{\dfrac{\begin{matrix}\mathscr{D}\\ A_i\end{matrix}}{A_1 \vee A_2}\,\vee\mathrm{I}_i \quad \begin{matrix}\overset{u_1}{[A_1]}\\ \mathscr{D}_1\\ C\end{matrix} \quad \begin{matrix}\overset{u_2}{[A_2]}\\ \mathscr{D}_2\\ C\end{matrix}}{C}\,\vee\mathrm{E} \quad \overset{\vee\beta_i}{\rhd} \quad \begin{matrix}\mathscr{D}\\ [A_i]\\ \mathscr{D}_i\\ C\end{matrix} \qquad (\vee\beta_i)$$

(for $i = 1, 2$)

$$\begin{matrix}\mathscr{D}\\ A \vee B\end{matrix} \quad \overset{\vee\eta}{\lhd} \quad \langle u_1, u_2\rangle\,\frac{\begin{matrix}\mathscr{D}\\ A \vee B\end{matrix} \quad \dfrac{\overset{u_1}{A}}{A \vee B}\,\vee\mathrm{I}_1 \quad \dfrac{\overset{u_2}{B}}{A \vee B}\,\vee\mathrm{I}_2}{A \vee B}\,\vee\mathrm{E} \qquad (\vee\eta)$$

(with u_1 and u_2 *fresh* for $\mathscr{D}$)

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Besides these rules for disjunction (which in most formulations are common to both intuitionistic and classical logic), Dummett [12] discusses also those for quantum disjunction (more commonly referred to as lattice disjunction) $\overline{\vee}$. The rules for this connective differ from those of standard disjunction in that the elimination rule comes with a restriction, to the effect that the rule can be applied only when the minor premises C depend on no other assumptions apart from those that get discharged by the rule application (we indicate this using double square brackets in place of the usual ones):

$$\dfrac{A}{A \mathbin{\overline{\vee}} B}\,\overline{\vee}\mathrm{I}_1 \qquad\qquad \dfrac{B}{A \mathbin{\overline{\vee}} B}\,\overline{\vee}\mathrm{I}_2 \qquad\qquad \dfrac{A \mathbin{\overline{\vee}} B \qquad \begin{matrix}[\![A]\!]\\ C\end{matrix} \qquad \begin{matrix}[\![B]\!]\\ C\end{matrix}}{C}\,\overline{\vee}\mathrm{E}$$

Using the elimination rule for quantum disjunction one can derive from $A \mathbin{\overline{\vee}} B$ *less* than what one can derive from $A \vee B$ using $\vee\mathrm{E}$. Thus, on the assumption that the standard rules for disjunction are in perfect balance, we expect the rules for quantum disjunction *not* to be in perfect harmony.[1] In particular, we expect the *no less* aspect of harmony *not* to be met.[2] That is, we expect the rules for $\overline{\vee}$ to be *unstable*.[3]

However, and here is the problem, reductions and expansions are readily available in the case of quantum disjunction as well (again with double square brackets we indicate that no other assumption (apart from those indicated) occurs in $\mathscr{D}_1$ and $\mathscr{D}_2$):

$$\langle u_1, u_2\rangle\ \dfrac{\dfrac{\begin{matrix}\mathscr{D}\\ A_i\end{matrix}}{A_1 \mathbin{\overline{\vee}} A_2}\,\overline{\vee}\mathrm{I}_i \qquad \begin{matrix}\overset{u_1}{[\![A_1]\!]}\\ \mathscr{D}_1\\ C\end{matrix} \qquad \begin{matrix}\overset{u_2}{[\![A_2]\!]}\\ \mathscr{D}_2\\ C\end{matrix}}{C}\,\overline{\vee}\mathrm{E} \quad \overset{\beta_i}{\rhd} \quad \begin{matrix}\mathscr{D}\\ [\![A_i]\!]\\ \mathscr{D}_i\\ C\end{matrix} \qquad (\text{for } i = 1, 2)$$

$$\begin{matrix}\mathscr{D}\\ A \mathbin{\overline{\vee}} B\end{matrix} \quad \overset{\eta}{\lhd} \quad \langle u_1, u_2\rangle\ \dfrac{\begin{matrix}\mathscr{D}\\ A \mathbin{\overline{\vee}} B\end{matrix} \qquad \dfrac{\overset{u_1}{A}}{A \mathbin{\overline{\vee}} B}\,\overline{\vee}\mathrm{I}_1 \qquad \dfrac{\overset{u_2}{B}}{A \mathbin{\overline{\vee}} B}\,\overline{\vee}\mathrm{I}_2}{A \mathbin{\overline{\vee}} B}\,\overline{\vee}\mathrm{E}$$

(with u_1 and u_2 fresh for $\mathscr{D}$)

Observe in particular that the restriction on $\overline{\vee}\mathrm{E}$ is satisfied by the applications of the rule in the expanded derivations and thus that, though unstable, the restricted elimination rule allows one to derive from $A \mathbin{\overline{\vee}} B$ no less than what is needed in order introduce $A \mathbin{\overline{\vee}} B$ back again.

Most authors (see, e.g. [4, 103]) have taken the modest stance of regarding instabilities of this kind as too subtle to be ruled out by the existence of expansions. Here we will argue against this modest stance, by showing that, once properly formulated, expansions are perfectly capable of detecting instabilities of this kind.

3.2 A “Quantum-Like” Implication

Before reformulating the expansion pattern for disjunction, we would like to point at some independent evidence in favor of the view that the existence of expansions should rule out instabilities of this kind. In so doing, we hope to dispel the possibly mistaken impression that our solution to the problem of quantum disjunction is merely ad hoc.

Evidence in favor of our bolder stance towards stability and expansions arises when one considers a restriction on $\supset$I analogous to the one yielding quantum disjunction also briefly discussed by Dummett [12, p. 289] as a case of instability. Let $\overline{\supset}$ be the “quantum-like” implication connective governed by the following rules:

$$\dfrac{\begin{matrix}[\![A]\!]\\ B\end{matrix}}{A \mathbin{\overline{\supset}} B}\,\overline{\supset}\mathrm{I} \qquad\qquad \dfrac{A \mathbin{\overline{\supset}} B \qquad A}{B}\,\overline{\supset}\mathrm{E}$$

where the introduction rule is restricted to the effect that it can be applied only when the premise B depends on no other assumptions apart from those that get discharged by the rule application.

The restricted introduction rule sets higher standards for inferring a proposition of the form $A \mathbin{\overline{\supset}} B$ than those set by $\supset$I to derive a proposition of the form $A \supset B$. Thus, on the assumption that $\supset$E is in perfect harmony with the standard introduction rule, we expect $\overline{\supset}$E *not* to be in perfect harmony with the restricted introduction rule. In particular, as the restricted introduction rule sets higher standards to derive an implication, we expect that using $\overline{\supset}$E we cannot derive from $A \mathbin{\overline{\supset}} B$ all that is needed to introduce $A \mathbin{\overline{\supset}} B$ again using its introduction rule. In other words, we expect also in this case the *no less* aspect of harmony not to be met. That is we expect the rules for $\overline{\supset}$ to be *unstable*, the kind of instability at stake being the same as the one flawing the rules for $\overline{\vee}$.[4]

As in the case of $\overline{\vee}$, in the case of $\overline{\supset}$ we have a reduction readily available which shows that—as expected—the *no more* aspect of harmony is satisfied:

$$\dfrac{\langle u\rangle\,\dfrac{\begin{matrix}\overset{u}{[\![A]\!]}\\ \mathscr{D}\\ B\end{matrix}}{A \mathbin{\overline{\supset}} B}\,\overline{\supset}\mathrm{I} \qquad \begin{matrix}\mathscr{D}'\\ A\end{matrix}}{B}\,\overline{\supset}\mathrm{E} \quad \overset{\beta}{\triangleright} \quad \begin{matrix}\mathscr{D}'\\ [\![A]\!]\\ \mathscr{D}\\ B\end{matrix}$$

Moreover, it is easy to see that the expansion pattern for standard implication does *not* work for quantum-like implication, as the application of $\overline{\supset}$I would violate the restriction that the premise B depends on no other assumptions than the one that get discharged by the rule: In the expanded pattern B would depend not only on A but also on all assumptions on which $A \mathbin{\overline{\supset}} B$ depends:

$$\begin{matrix}\mathscr{D}\\ A \mathbin{\overline{\supset}} B\end{matrix} \quad \text{does } \textit{not} \text{ expand to} \quad \langle u\rangle\,\dfrac{\dfrac{\begin{matrix}\mathscr{D}\\ A \mathbin{\overline{\supset}} B\end{matrix} \qquad \overset{u}{A}}{B}\,\overline{\supset}\mathrm{E}}{A \mathbin{\overline{\supset}} B}\,\overline{\supset}\mathrm{I}$$

The requirement that it should be possible to equip the rules with both reductions and expansions is thus capable of detecting the instability of the rules of quantum-like implication. We take this as a reason to consider an alternative pattern for the expansion of disjunction, namely one capable of detecting the disharmony induced by the restriction on the quantum disjunction elimination rule.[5]

3.3 Generalizing the Expansions for Disjunction

The expansion pattern for disjunction ($\vee\eta$) we considered above—which was first proposed by Prawitz [66]—gives the instructions to expand a derivation in which the disjunctive proposition figures as conclusion of the whole derivation.

The idea behind the alternative pattern is that an expansion operates on a formula which is not, in general, the conclusion of a derivation, but on one that occurs at some point in the course of a derivation. Consider a derivation $\mathscr{D}$ in which the formula $A \vee B$ may occur at some point. Such a derivation may be depicted as follows:

$$\mathscr{D}\left\{\begin{array}{c}\mathscr{D}'\\ [A\vee B]\\ \mathscr{D}''\\ C\end{array}\right.$$

—that is, it may be viewed as the result of plugging a (certain number $k \geq 0$) of copies of a derivation $\mathscr{D}'$ of $A \vee B$ on top of a derivation $\mathscr{D}''$ of C depending on (k copies of) the assumption $A \vee B$, possibly alongside other assumptions Γ.

It is certainly true that Prawitz's expansion ($\vee\eta$) can also be used to expand a derivation $\mathscr{D}$ of this form: To expand $\mathscr{D}$, we can apply Prawitz's expansion to the upper chunk $\mathscr{D}'$ of $\mathscr{D}$ (in which $A \vee B$ figures as conclusion), and then we can plug the result of the expansion on top of the lower chunk $\mathscr{D}''$ of $\mathscr{D}$, thereby obtaining the following:

$$\begin{array}{c}\langle u_1,u_2\rangle\,\dfrac{\begin{array}{c}\mathscr{D}'\\ A\vee B\end{array}\qquad \dfrac{\overset{u_1}{A}}{A\vee B}\,\vee\mathrm{I}\qquad \dfrac{\overset{u_2}{B}}{A\vee B}\,\vee\mathrm{I}}{[A\vee B]}\,\vee\mathrm{E}\\ \mathscr{D}''\\ C\end{array}$$

It is however possible to define an alternative procedure to directly expand the whole of $\mathscr{D}$, namely the following:

$$\begin{array}{c}\mathscr{D}'\\ [A\vee B]\\ \mathscr{D}''\\ C\end{array}\quad \overset{\vee\eta_g}{\lhd}\quad \langle u_1,u_2\rangle\,\dfrac{\begin{array}{c}\mathscr{D}'\\ A\vee B\end{array}\qquad \begin{array}{c}\dfrac{\overset{u_1}{A}}{[A\vee B]}\,\vee\mathrm{I}\\ \mathscr{D}''\\ C\end{array}\qquad \begin{array}{c}\dfrac{\overset{u_2}{B}}{[A\vee B]}\,\vee\mathrm{I}\\ \mathscr{D}''\\ C\end{array}}{C}\,\vee\mathrm{E} \qquad (\vee\eta_g)$$

(with u_1 and u_2 fresh for $\mathscr{D}'$)

In the alternatively expanded derivation, the application of the elimination rule $\vee$E is postponed to the effect that its minor premises are not the two copies of $A \vee B$ obtained respectively by $\vee\mathrm{I}_1$ and $\vee\mathrm{I}_2$ (as in Prawitz's expansion), but rather two copies of C which are the conclusions of two copies of the lower chuck $\mathscr{D}''$ of $\mathscr{D}$ that now constitute the main part of the derivations of the minor premises of the application of $\vee$E.

This alternative pattern, first proposed by Seely [97], is a generalization of Prawitz's expansion pattern: Each instance of Prawitz's pattern ($\vee\eta$) corresponds to an instance of the alternative pattern ($\vee\eta_g$) in which the lower chunk $\mathscr{D}''$ of the derivation $\mathscr{D}$ is "empty" i.e. it just consists of the proposition $A \vee B$.

Moreover, it is easy to see that we cannot replace $\vee$ with $\overline{\vee}$ in the above pattern for generalized expansions, since the application of $\overline{\vee}$E in the expanded derivation would violate the quantum restriction: The minor premises C would not depend only on the assumptions of the form A and B that get discharged by the rules, but also on all other undischarged assumptions of $\mathscr{D}''$:

$$\begin{array}{c}\mathscr{D}' \\ [A \,\overline{\vee}\, B] \\ \mathscr{D}'' \\ C\end{array} \quad \text{does } \textit{not} \text{ expand to} \quad \langle n, m\rangle\ \dfrac{\begin{array}{c}\mathscr{D}' \\ A \,\overline{\vee}\, B\end{array} \qquad \begin{array}{c}\dfrac{\overset{n}{A}}{[A \,\overline{\vee}\, B]}\,\overline{\vee}\mathrm{I} \\ \mathscr{D}'' \\ C\end{array} \qquad \begin{array}{c}\dfrac{\overset{m}{B}}{[A \,\overline{\vee}\, B]}\,\overline{\vee}\mathrm{I} \\ \mathscr{D}'' \\ C\end{array}}{C}\,\overline{\vee}\mathrm{E}$$

(with n and m fresh for $\mathscr{D}$)

Thus, quantum disjunction *does* turn out to be unstable (in accordance with what we would expect), provided that stability is understood as the existence of generalized expansions.

What the alternative expansion expresses is a generalization of the *no less* aspect of harmony that could be roughly approximated as follows:

> The elimination rule allows one to derive no less than what is needed to derive all consequences from a logically complex proposition of a given form.

Starting from the alternative formulation of harmony given by Negri and von Plato [53]: "whatever follows from the direct grounds for a proposition must follow from that proposition." (see also Note 5 to Chap. 1 above) one may propose the following as the proper way of understanding harmony:

Harmony: Informal statement 1 *What can be inferred from the direct grounds for a proposition A together with further propositions Γ should be no more and no less that what follows from A together with Γ.*

3.4 Harmony: Arbitrary Connectives and Quantifiers

Prawitz [69] first proposed a general procedure to map any arbitrary collection of introduction rules onto a specific collection of elimination rules which is in harmony with the given collection of introduction rules. We will refer to such procedures as *inversion principles*.[6] Prawitz's procedure has been refined by Schroeder-Heister [85, 86, 94] in a deductive framework that generalizes the key ingredients of standard natural deduction calculi called the calculus of higher-level rules. We will therefore refer to the Prawitz–Schroeder-Heister procedure to generate elimination rules from a given collection of introduction rules as PSH-inversion.

The details of the calculus of higher-level rules and a more general presentation of PSH-inversion for rules of propositional connectives will be given in the Appendix A (see in particular Sect. A.9). In the present section, we informally give these results for the simplest case which does not require rules of higher level, and we suggest how PSH-inversion could be generalized to cover rules for arbitrary first-order quantifiers as well.

In this chapter we will assume $\dagger$ to be an n-ary connective, and $\dagger\mathbf{I}$ to be a collection of $r \geq 0$ distinct introduction rules for $\dagger$ of the following form:

$$\dfrac{B_{k1} \qquad \ldots \qquad B_{km_k}}{\dagger(A_1, \ldots, A_n)}\ \dagger\mathrm{I}_k$$

satisfying the following condition: for all $1 \leq k \leq r$, either $m_k = 0$ or for all $1 \leq j \leq m_k$ there is an $1 \leq i \leq n$ such that B_{kj} is syntactically identical to A_i. (Introduction rules of a more general kind are discussed in Appendix A, see in particular Sect. A.9.)

Definition 3.1 (PSH-*inversion*) Let $\dagger$ and $\dagger\mathbf{I}$ be as above. We indicate with PSH($\dagger\mathbf{I}$) the collection of elimination rules containing only the following rule:

$$\dfrac{\dagger(A_1, \ldots, A_n) \qquad \begin{matrix}[B_{11}] \ \ldots\ [B_{1m_1}] \\ C\end{matrix} \qquad \ldots \qquad \begin{matrix}[B_{r1}] \ \ldots\ [B_{rm_r}] \\ C\end{matrix}}{C}\ \dagger\mathrm{E}^{\mathtt{PSH}}$$

where C is distinct from each A_i.

We say that $\dagger$ is a PSH-connectives in a calculus K iff K includes $\dagger\mathbf{I}$ and PSH($\dagger\mathbf{I}$), and $\dagger$ does not occur in any other of the primitive rules of K.[7]

Let K be a calculus in which $\dagger$ is a PSH-connective. In every K-derivation, one can get rid of any maximal formula occurrence governed by $\dagger$ as follows (we abbreviate $\dagger(A_1, \ldots, A_n)$ with $\dagger$):

$$\langle u_{11}, \ldots, u_{rm_r}\rangle\ \dfrac{\dfrac{\begin{matrix}\mathscr{D}_{k1} \\ B_{k1}\end{matrix} \quad \ldots \quad \begin{matrix}\mathscr{D}_{km_k} \\ B_{km_k}\end{matrix}}{\dagger}\ \dagger\mathrm{I}_k \qquad \begin{matrix}\overset{u_{11}}{[B_{11}]} \ \ldots\ \overset{u_{1m_1}}{[B_{1m_1}]} \\ \mathscr{D}'_1 \\ C\end{matrix} \qquad \ldots \qquad \begin{matrix}\overset{u_{r1}}{[B_{r1}]} \ \ldots\ \overset{u_{rm_r}}{[B_{rm_r}]} \\ \mathscr{D}'_r \\ C\end{matrix}}{C}\ \dagger\mathrm{E}^{\mathtt{PSH}}$$

$$\overset{\dagger\beta_k}{\rhd}$$

$$\begin{array}{ccc} \mathscr{D}_{k1} & & \mathscr{D}_{km_k} \\ [B_{k1}] & \ldots & [B_{km_k}] \\ & \mathscr{D}'_k & \\ & C & \end{array}$$

Moreover, given a K-derivation $\mathscr{D}'$ of C depending on some assumptions of the form $\dagger(A_1, \ldots, A_n)$ and a K-derivation $\mathscr{D}$ of $\dagger(A_1, \ldots, A_n)$ we can define the following generalized η-expansion (again we abbreviate $\dagger(A_1, \ldots, A_n)$ with $\dagger$; freshness conditions on the discharge indeces will be left implicit henceforth):

$$\begin{array}{c} \mathscr{D} \\ [\dagger] \\ \mathscr{D}' \\ C \end{array} \quad \overset{\eta_g}{\lhd} \quad \langle u_{11}, \ldots, u_{rm_r} \rangle\ \dfrac{\begin{array}{c} \mathscr{D} \\ \dagger \end{array} \quad \begin{array}{c} \dfrac{\overset{u_{11}}{B_{11}} \ \ldots \ \overset{u_{1m_1}}{B_{1m_1}}}{[\dagger]}\,\dagger\mathrm{I}_1 \\ \mathscr{D}' \\ C \end{array} \quad \ldots \quad \begin{array}{c} \dfrac{\overset{u_{r1}}{B_{r1}} \ \ldots \ \overset{u_{rm_r}}{B_{rm_r}}}{[\dagger]}\,\dagger\mathrm{I}_r \\ \mathscr{D}' \\ C \end{array}}{C}\,\dagger\mathrm{E}^{\mathrm{PSH}}$$

The disjunction rules of NI are obtained by instantiating these schemata for $r = 2$ and $m_1 = m_2 = 1$. To give a further example, for $r = 1$ and $m_1 = 2$ we obtain the well-known variant of the rules for conjunction in which the two elimination rules of NI are replaced by the so-called general elimination rule for conjunction[8]:

$$\dfrac{A \qquad B}{A \wedge B}\,\wedge\mathrm{I} \qquad\qquad \dfrac{A \wedge B \qquad \begin{array}{c} [A][B] \\ C \end{array}}{C}\,\wedge\mathrm{E}^{\mathrm{PSH}}$$

for which we have the following reduction and (generalized) expansion:

$$\langle u_1, u_2 \rangle\ \dfrac{\dfrac{\begin{array}{c} \mathscr{D}_1 \\ A \end{array} \qquad \begin{array}{c} \mathscr{D}_2 \\ B \end{array}}{A \wedge B}\,\wedge\mathrm{I} \qquad \begin{array}{c} \overset{u_1}{[A]}\overset{u_2}{[B]} \\ \mathscr{D}' \\ C \end{array}}{C}\,\wedge\mathrm{E}^{\mathrm{PSH}} \quad \overset{\beta}{\rhd} \quad \begin{array}{c} \mathscr{D}_1 \quad \mathscr{D}_2 \\ [A] \quad [B] \\ \mathscr{D}' \\ C \end{array} \qquad (\wedge\beta^{\mathrm{PSH}})$$

$$\begin{array}{c} \mathscr{D} \\ [A \wedge B] \\ \mathscr{D}' \\ C \end{array} \quad \overset{\eta_g}{\lhd} \quad \langle u_1, u_2 \rangle\ \dfrac{\begin{array}{c} \mathscr{D} \\ A \wedge B \end{array} \qquad \begin{array}{c} \dfrac{\overset{u_1}{A} \qquad \overset{u_2}{B}}{[A \wedge B]}\,\wedge\mathrm{I} \\ \mathscr{D}' \\ C \end{array}}{C}\,\wedge\mathrm{E}^{\mathrm{PSH}} \qquad (\wedge\eta_g^{\mathrm{PSH}})$$

We intend the schema to cover also the limit case of $m = 0$ that gives us the standard intuitionistic rules for $\bot$:

$$\text{no introduction} \qquad\qquad \dfrac{\bot}{C}\,\bot\mathrm{E}$$

for which we have no reduction but the following (generalized) expansion:

$$\begin{array}{c} \mathscr{D} \\ [\bot] \\ \mathscr{D}' \\ C \end{array} \quad \overset{\eta_g}{\lhd} \quad \begin{array}{c} \mathscr{D} \\ \dfrac{\bot}{C}\,\bot\mathrm{E} \end{array}$$

As we detail in the Appendix, in the setting of Schroeder-Heister's calculus of higher-level rules the above schemata cover also the cases in which the B_{kj}s are not formulas (i.e. rules of lowest level), but rules of arbitrary (finite) level.[9] For example, take $\supset\mathbf{I}$ to be the collection of introduction rules consisting only of the standard introduction rules for implication $\supset$I:

$$\dfrac{\begin{array}{c}[A] \\ B\end{array}}{A \supset B}\supset\mathrm{I}$$

The collection of elimination rules $\mathtt{PSH}(\supset\mathbf{I})$ consists of the following rule:

$$\dfrac{A \supset B \qquad \begin{array}{c}[A \Rightarrow B] \\ C\end{array}}{C}\supset\mathrm{E}^{\mathtt{PSH}}$$

Using the notation that is introduced in the appendix, the reduction associated with these rules can be depicted as follows:

$$\begin{array}{l}\langle u_1\rangle \\ \langle u_2\rangle\end{array} \dfrac{\dfrac{\begin{array}{c}\overset{u_1}{[A]} \\ \mathscr{D} \\ B\end{array}}{A \supset B}\supset\mathrm{I} \qquad \begin{array}{c}\overset{u_2}{[A \Rightarrow B]} \\ \mathscr{D}' \\ C\end{array}}{C}\supset\mathrm{E}^{\mathtt{PSH}} \quad \overset{\beta}{\rhd} \quad \begin{array}{c}\mathscr{D} \\ [A \Rightarrow B] \\ C\end{array}$$

To give the reader at least an informal clarification of the notation involved in the reduction, we observe the following. A derivation of the rule $A \Rightarrow B$ is equated by definition with a derivation of B from A, thus the derivation $\mathscr{D}$ of B from A is *ipso facto* a derivation of $A \Rightarrow B$. The result of substituting the derivation $\mathscr{D}$ of $A \Rightarrow B$ for the rule assumption $A \Rightarrow B$ in $\mathscr{D}'$ can be informally described as the derivation which results by removing all applications of the assumption rule $A \Rightarrow B$ in $\mathscr{D}'$ and inserting $\mathscr{D}$ to fill the gap, i.e. by successively replacing all patterns of the form on the right with patterns of the form on the left.

$$\begin{array}{c}\vdots \\ \dfrac{A}{B}\,A \Rightarrow B \\ \vdots\end{array} \qquad \begin{array}{c}\vdots \\ [A] \\ \mathscr{D} \\ B \\ \vdots\end{array}$$

Observe that depending on the number of copies of the assumption A in $\mathscr{D}$ and the number of applications of $A \Rightarrow B$ in $\mathscr{D}'$, the operation requires a quite involved transformation already for a rule of almost lowest level such as $A \Rightarrow B$. For exact definitions (covering rules of arbitrary level) see Appendix A.

The (generalized) expansion is the following:

$$\begin{array}{c}\mathscr{D}\\ [A \supset B]\\ \mathscr{D}'\\ C\end{array} \quad \overset{\eta_g}{\lhd} \quad \langle u_2\rangle\;\dfrac{\begin{array}{c}\mathscr{D}\\ A \supset B\end{array} \qquad \begin{array}{c}\langle u_1\rangle\;\dfrac{\dfrac{\overset{u_1}{A}}{B}\; A \overset{u_2}{\Rightarrow} B}{[A \supset B]}\supset\mathrm{I}\\ \mathscr{D}'\\ C\end{array}}{C}\supset\mathrm{E}^{\mathtt{PSH}}$$

It should be clear that the schemata above can be generalized to apply to more general collections of introduction rules which may not only be of arbitrary high level but which may also contain propositional quantification as in Schroeder-Heister [94]. A generalization covering also first-order quantification is possible too and expectedly straightforward. We limit ourselves to discuss a (hopefully suggestive) example, the rules characterizing a binary quantifier encoding the I corner of the traditional square of oppositions ("Some A are B"). The collections of introduction and elimination rules $I\mathbf{I}$ and $\mathtt{PSH}(I\mathbf{I})$ consist of the following two rules respectively:

$$\dfrac{A(t/x) \qquad B(t/x)}{I_x(A, B)}\,I\mathrm{I} \qquad\qquad \dfrac{I_x(A, B) \qquad \begin{array}{c}[A(y/x)][B(y/x)]\\ C\end{array}}{C}\,I\mathrm{E}$$

(where y is an *eigenvariable*)

Also for these collections of rules we have a reduction and a (generalized) expansion following the pattern of those of the rules of conjunction with the elimination rule in general form:

$$\langle u_1, u_2\rangle\;\dfrac{\dfrac{\begin{array}{c}\mathscr{D}_1\\ A(t/x)\end{array} \qquad \begin{array}{c}\mathscr{D}_2\\ B(t/x)\end{array}}{I_x(A, B)}\,I\mathrm{I} \qquad \begin{array}{c}[\overset{u_1}{A(y/x)}][\overset{u_2}{B(y/x)}]\\ \mathscr{D}'(y/x)\\ C\end{array}}{C}\,I\mathrm{E} \quad \overset{\beta}{\rhd} \quad \begin{array}{c}\mathscr{D}_1 \qquad \mathscr{D}_2\\ [A(t/x)]\;\;[B(t/x)]\\ \mathscr{D}'(t/x)\\ C\end{array}$$

$$\begin{array}{c}\mathscr{D}\\ [I_x(A, B)]\\ \mathscr{D}'\\ C\end{array} \quad \overset{\eta_g}{\lhd} \quad \langle u_1, u_2\rangle\;\dfrac{\begin{array}{c}\mathscr{D}\\ I_x(A, B)\end{array} \qquad \begin{array}{c}\dfrac{\overset{u_1}{A(y/x)} \qquad \overset{u_2}{B(y/x)}}{[I_x(A, B)]}\,I\mathrm{I}\\ \mathscr{D}'\\ C\end{array}}{C}\,I\mathrm{E}$$

3.5 Stability and Permutations

When calculi containing connectives with rules obeying PSH-inversion are considered, such as the full calculus NI, one usually considers further conversions besides reductions and expansions. A typical example of further conversions are permutative conversions. The result of applying such conversions is a change in the order of application of certain rules within derivations. Permutative conversions were first introduced by Prawitz [65, Chap. IV] with the goal of extending the subformula property of β-normal derivations to the whole of NI. Although β-normal (and β_w-normal) derivations in NI are canonical (i.e. Fact 3 holds for NI), β-normal derivations might contain occurrences of formulas which are neither a subformula of the undischarged assumptions nor of the conclusion. The following derivation in NI displays this fact (the example is taken from [26]):

$$\dfrac{\langle 1,2\rangle\ \dfrac{A \vee A \qquad \dfrac{\overset{1}{A} \qquad \overset{1}{A}}{A \wedge A}\wedge\mathrm{I} \qquad \dfrac{\overset{2}{A} \qquad \overset{2}{A}}{A \wedge A}\wedge\mathrm{I}}{A \wedge A}\vee\mathrm{E}}{A}\wedge\mathrm{E}_1$$

The failure of the subformula property is triggered by the fact that, due to its peculiar form, the application of $\vee$E "hides" the fact that the formula $A \wedge A$ is introduced (in both derivations of the minor premises of the application of $\vee$E) and then eliminated.

Prawitz [65] characterized the "hidden" redundancies of this kind in NI as *maximal segments*. Segments are defined as follows:

Definition 3.2 A *segment* (of length n) in a derivation is a sequence of formula occurrences $A_1, \ldots, A_n$ of the same formula A such that

1. for $n > 1$, for all $i < n$ A_i is a minor premise of an application of $\vee$E with conclusion A_{i+1}.
2. A_n is not the minor premise of an application of $\vee$E;
3. A_1 is not the consequence of an application of $\vee$E.

Note that in the above derivation the only two segments of length 2 are those consisting (respectively) of one of the two minor premises followed by the consequence of the application of $\vee$E. All other formula occurrences in the derivations constitute segments of length 1.

Definition 3.3 A segment $A_1, \ldots A_n$ is *maximal* iff A_1 is the consequence of an application of an introduction rule or of $\bot$E and A_n is the major premise of an application of an elimination rule.[10]

In the above derivation both segments of lengths 2 are maximal. Note that maximal formula occurrences in $\mathrm{NI}^{\wedge\supset}$ are a limit case of maximal segments (i.e. maximal segments of length 1).

Whereas applications of β-reductions allow one to get rid of maximal formula occurrences (i.e. maximal segments of length 1), permutative conversions can be used to shorten the length of maximal segments. The permutative conversions for disjunction (which we will indicate as $\vee\gamma$) can be schematically depicted as follows (in the derivation schemata, $\langle\mathscr{D}\rangle$ indicates the possible presence of minor premises in $\dagger$E, where $\dagger$E stands for the elimination rule of some connective $\dagger$)[11]:

$$\dfrac{\dfrac{\begin{matrix}\mathscr{D}_1\\ A\vee B\end{matrix}\quad \begin{matrix}\mathscr{D}_2\\ C\end{matrix}\quad \begin{matrix}\mathscr{D}_3\\ C\end{matrix}}{C}\,\vee\mathrm{E}\qquad \langle\mathscr{D}\rangle}{D}\,\dagger\mathrm{E}\quad \overset{\vee\gamma}{\triangleright}\quad \dfrac{\begin{matrix}\mathscr{D}_1\\ A\vee B\end{matrix}\qquad \dfrac{\begin{matrix}\mathscr{D}_2\\ C\end{matrix}\quad \langle\mathscr{D}\rangle}{D}\,\dagger\mathrm{E}\qquad \dfrac{\begin{matrix}\mathscr{D}_3\\ C\end{matrix}\quad \langle\mathscr{D}\rangle}{D}\,\dagger\mathrm{E}}{D}\,\vee\mathrm{E}\qquad (\vee\gamma)$$

An application of $(\vee\gamma)$ to the above derivation yields the following one:

$$\langle 1,2\rangle\ \dfrac{A\vee A\qquad \dfrac{\dfrac{\overset{1}{A}\quad \overset{1}{A}}{A\wedge A}\,\wedge\mathrm{I}}{A}\,\wedge\mathrm{E}_1\qquad \dfrac{\dfrac{\overset{2}{A}\quad \overset{2}{A}}{A\wedge A}\,\wedge\mathrm{I}}{A}\,\wedge\mathrm{E}_1}{A}\,\vee\mathrm{E}$$

which by two applications of $\wedge\beta_1$ reduces to

$$\langle 1,2\rangle\ \dfrac{A\vee A\qquad \overset{1}{A}\qquad \overset{2}{A}}{A}\,\vee\mathrm{E}$$

Counterexamples to the subformula property in NI are triggered by applications of $\bot$E as well, as shown by the following derivation:

$$\dfrac{\dfrac{\bot}{A\supset B}\,\bot\mathrm{E}\qquad A}{B}\,\supset\mathrm{E}$$

which contains an occurrence of a formula—viz. $A\supset B$—which is not a subformula of either the undischarged assumptions or the conclusion. This formula occurrence constitutes a segment which qualifies as maximal according to Definition 3.3. Using the following permutative conversions for $\bot$ one can get rid of such segments[12]:

$$\dfrac{\dfrac{\begin{matrix}\mathscr{D}\\ \bot\end{matrix}}{C}\,\bot\mathrm{E}\qquad \langle\mathscr{D}\rangle}{D}\,\dagger\mathrm{E}\quad \overset{\bot\gamma}{\triangleright}\quad \dfrac{\begin{matrix}\mathscr{D}\\ \bot\end{matrix}}{D}\,\bot\mathrm{E}\qquad (\bot\gamma)$$

In NI, $\beta\gamma$-reduction is not only weakly normalizing, but also confluent and strongly normalizing. Moreover, by generalizing the notion of *track* used for $\mathtt{NI}^{\wedge\supset}$ (see Sect. 1.5 above), it is possible to establish a result analogous to Fact 1, from which a result analogous to Fact 2 follows immediately, i.e. the subformula property for $\beta\gamma$-normal derivations in the full calculus NI.[13]

For future reference we observe that if one restricts oneself to the calculus for the $\{\supset,\bot\}$-language fragment, to which we will refer as $\mathtt{NI}^{\supset\bot}$, there is no need to modify the notion of track used for $\mathtt{NI}^{\wedge\supset}$ and the subformula property for $\beta\gamma$-normal derivations follows from the following:

Fact 4 (*The form of tracks*) Each track $A_1\ldots A_{i-1}, A_i, A_{i+1},\ldots A_n$ in a $\beta\gamma$-normal derivation in $\mathtt{NI}^{\supset\bot}$ contains a minimal formula A_i such that

- If $i > 1$ then A_j (for all $1 < j < i$) is the premise of an application of an elimination rule of which A_{j+1} is the consequence and thereby A_{j+1} is a subformula of A_j.
- If $n > i$ then A_i is the premise of either an application of $\bot$E or of an introduction rule.
- If $n > i$ then A_j (for all $i < j < n$) is the premise of an application of an introduction rule of which A_{j+1} is the consequence and thereby A_j is a subformula of A_{j+1}.

Proof For a derivation to be $\beta\gamma$-normal, in each track all applications of elimination rules must precede all applications of introduction rules, and if $\bot$E is applied in the track, its consequence is either the last element of the track or the premise of an introduction rule. This warrants the existence of a minimal formula in each track. Since a track ends whenever it "encounters" the minor premise of an application of $\supset$E, the subformula relationships between the members of a track hold (as it can be easily verified by checking the shape of the rules of $\mathtt{NI}^{\supset\bot}$). $\square$

It is worth observing that permutative conversions can be "simulated" using generalized η-expansions and β-reductions. In particular, in the case of $(\vee\gamma)$ one can obtain the derivation on the right-hand side from the one on the left-hand side as follows. First apply a generalized expansion to the derivation on the left-hand side of $(\vee\gamma)$ by instantiating in the schema for the generalized expansion $(\vee\eta_g)$ $\mathscr{D}'$ with $\mathscr{D}_1$ and $\mathscr{D}''$ with

$$\dfrac{\dfrac{A \vee B \qquad \begin{matrix}\mathscr{D}_2\\ C\end{matrix} \qquad \begin{matrix}\mathscr{D}_3\\ C\end{matrix}}{C}\,\vee\mathrm{E} \qquad \langle\mathscr{D}\rangle}{D}\,\dagger\mathrm{E}$$

thereby obtaining the following derivation:

$$\langle u_1, u_2\rangle\,\dfrac{\begin{matrix}\mathscr{D}_1\\ A \vee B\end{matrix} \qquad \dfrac{\dfrac{\dfrac{\overset{u_1}{A}}{A \vee B} \quad \begin{matrix}\mathscr{D}_2\\ C\end{matrix} \quad \begin{matrix}\mathscr{D}_3\\ C\end{matrix}}{C}\,\vee\mathrm{E} \qquad \langle\mathscr{D}\rangle}{D}\,\dagger\mathrm{E} \qquad \dfrac{\dfrac{\dfrac{\overset{u_2}{B}}{A \vee B} \quad \begin{matrix}\mathscr{D}_2\\ C\end{matrix} \quad \begin{matrix}\mathscr{D}_3\\ C\end{matrix}}{C}\,\vee\mathrm{E} \qquad \langle\mathscr{D}\rangle}{D}\,\dagger\mathrm{E}}{D}\,\vee\mathrm{E}$$

in which the two rightmost occurrences of $A \vee B$ constitute two local peaks. Their reduction yields the derivation on the right-hand side of $(\gamma\vee)$. (In the case of $\bot$ the permutation is just an instance of the generalized η-expansion; for other connectives, see Sect. 3.7 below.)

The relevance of permutations to harmony and in particular to stability was already pointed out by Dummett [12] and has been recently stressed by Francez [17]. The fact that permutative conversions can be recovered from generalized expansions thus provides on the one hand further evidence for the significance of permutative conversions for an inferential account of meaning, and on the other hand it offers further reasons to view generalized expansions as the proper way to capture stability.

Actually, using the alternative expansion pattern one can recover a more general form of permutation, which we indicate as $\vee\gamma_g$-conversion, in which any chunk of derivation (and not only applications of elimination rule) can be permuted-up across an application of disjunction elimination [97]:

$$\dfrac{\begin{array}{c}\mathscr{D}_1\\ A\vee B\end{array}\quad\begin{array}{c}\mathscr{D}_2\\ C\end{array}\quad\begin{array}{c}\mathscr{D}_3\\ C\end{array}}{\begin{array}{c}[C]\\ \mathscr{D}\\ D\end{array}}\vee\mathrm{E}\quad\overset{\vee\gamma_g}{\rhd}\quad\dfrac{\begin{array}{c}\mathscr{D}_1\\ A\vee B\end{array}\quad\begin{array}{c}\mathscr{D}_2\\ [C]\\ \mathscr{D}\\ D\end{array}\quad\begin{array}{c}\mathscr{D}_3\\ [C]\\ \mathscr{D}\\ D\end{array}}{D}\vee\mathrm{E}\qquad(\vee\gamma_g)$$

Conversely, this general form of permutation coupled with Prawitz's simple form of expansion is as strong as the alternative form of expansion. More precisely, the equivalence relation induced by β-reductions, and generalized η_g-expansions is equivalent to the one induced by β-reductions, Prawitz's η-expansions and the generalized permutative γ_g-conversions ([97], for a proof see [40]).

As it has been recently established [83], $\beta\eta_g$-equivalence (or equivalently $\beta\eta\gamma_g$-equivalence) is the maximum non-trivial notion of equivalence in the full language of `NI`. The notion of $\beta\eta_g$-isomorphism (see Sect. 2.5) has been shown to decidable, but only in the absence of $\bot$, whereas it is still an open question whether $\beta\eta_g$-isomorphism is decidable in the full language of `NI`, though it is known that it is not finitely axiomatizable [33].

In contrast to $\beta\gamma$-reduction, $\beta\gamma_g$ reduction is neither strongly normalizing nor confluent [1, 21, 40]. Due to the lack of confluence, it is thus difficult to make sense of the idea that $\beta\gamma_g$-normal derivations in `NI` represent proofs in the most direct way possible. The same proof may be represented by more than one $\beta\gamma_g$-normal derivation. Hence, without further ado there is no criterion of selecting one among the normal derivations belonging to the same equivalence class as "the" most direct way of representing a given proof.[14] It therefore seems that in the case of the full language of `NI` it is hard to reconcile, on the one hand, the idea that maximality is the criterion to select the "correct" way of analyzing identity of proofs and, on the other hand, that conversions are means to transform a less direct representation of a proof into a more direct one. For these reasons, Girard famously referred to permutative conversions in `NI` as the 'defects of the system' (see [26], Sect. 10.1).

We conclude this section by observing that some authors (notably [63, 110]) argued in favour of calculi in which *all* elimination rules follow the pattern of disjunction elimination (Tennant refers to rules of this form as 'parallelized elimination rules', while von Plato as 'general elimination rules'). However, to avoid the complications of rules of higher level, the following equivalent[15] rule is adopted instead of the `PSH`-elimination rule for $\supset$:

$$\dfrac{A\supset B\qquad A\qquad \begin{array}{c}[B]\\ C\end{array}}{C}\supset\mathrm{E}_g$$

The resulting natural deduction calculus, to which we will refer as $\mathtt{NI}_g$, bears a close correspondence to sequent calculus (in fact, a much closer correspondence than the original NI) and as such is particularly suited for automated proof search (see, e.g., [113], Sects. 2.3.4 and 2.3.7).

For reasons analogous to those discussed in connection with the rule of disjunction elimination in NI, in $\mathtt{NI}_g$ permutative conversions are associated to all elimination rules. As in the case of disjunction, one can distinguish between permutative conversions involving only elimination rules (analogous to $\vee\gamma$ above) and permutative conversions of a more general form (analogous to $\vee\gamma_g$).

As in the case of NI, $\beta\gamma$-reduction in $\mathtt{NI}_g$ is strongly normalizing and confuent (see [37] for a proof of strong normalization for the disjunction-implication fragment and [36] for a proof of confluence for the implication fragment alone; Matthes [46] claims that the proofs carry over to the full calculus), and the $\beta\gamma$-normal derivations can be concisely described as those derivations in which all major premises of elimination rules are in assumption position (or, in Tennant's terminology 'stand proud'). Neither strong normalization nor confluence however hold for the more general kind of permutative conversions.

Observe finally, that the adoption of elimination rules in general form jeopardizes the idea that normal derivations are the most direct way of denoting proofs already in the purely conjunctive fragment. For example, the following $\mathtt{NI}^{\wedge\supset}$ β-normal derivation:

$$\dfrac{\dfrac{\dfrac{A\wedge B}{A}\wedge\mathrm{E}_1 \qquad \dfrac{C\wedge D}{C}\wedge\mathrm{E}_1}{A\wedge C}\wedge\mathrm{I} \qquad \dfrac{\dfrac{A\wedge B}{B}\wedge\mathrm{E}_2 \qquad \dfrac{C\wedge D}{D}\wedge\mathrm{E}_2}{B\wedge D}\wedge\mathrm{I}}{(A\wedge C)\wedge(B\wedge D)}\wedge\mathrm{I}$$

corresponds to the following two distinct $\beta\gamma$-normal derivations in the calculus in which $\wedge\mathrm{E}_{\mathrm{PSH}}$ replaces $\wedge\mathrm{E}_1$ and $\wedge\mathrm{E}_2$:

$$\langle n_1,n_2\rangle\,\dfrac{C\wedge D \qquad \langle m_1,m_2\rangle\,\dfrac{A\wedge B \qquad \dfrac{\dfrac{\overset{m_1}{A} \qquad \overset{n_1}{C}}{A\wedge C}\wedge\mathrm{I} \qquad \dfrac{\overset{m_2}{B} \qquad \overset{n_2}{D}}{B\wedge D}\wedge\mathrm{I}}{(A\wedge C)\wedge(B\wedge D)}\wedge\mathrm{I}}{(A\wedge C)\wedge(B\wedge D)}\wedge\mathrm{E}_{\mathrm{PSH}}}{(A\wedge C)\wedge(B\wedge D)}\wedge\mathrm{E}_{\mathrm{PSH}}$$

$$\langle m_1,m_2\rangle\,\dfrac{A\wedge B \qquad \langle n_1,n_2\rangle\,\dfrac{C\wedge D \qquad \dfrac{\dfrac{\overset{m_1}{A} \qquad \overset{n_1}{C}}{A\wedge C}\wedge\mathrm{I} \qquad \dfrac{\overset{m_2}{B} \qquad \overset{n_2}{D}}{B\wedge D}\wedge\mathrm{I}}{(A\wedge C)\wedge(B\wedge D)}\wedge\mathrm{I}}{(A\wedge C)\wedge(B\wedge D)}\wedge\mathrm{E}_{\mathrm{PSH}}}{(A\wedge C)\wedge(B\wedge D)}\wedge\mathrm{E}_{\mathrm{PSH}}$$

Although they are both $\beta\gamma$-normal (and hence they are not $\beta\gamma$-equivalent), these two derivations are γ_g-equivalent: each can be obtained from the other by exchanging the order of the last two applications of $\wedge\mathrm{E}_{\mathrm{PSH}}$.

The two derivations are two different representations of the same (function from) proof(s of the undischarged assumptions to proofs of the conclusion). However, neither of the two derivations can be said to represent their common denotation more directly than the other one.

In this sense (although maybe not in other respects), the natural deduction calculus $\mathtt{NI}^{\wedge\supset}$ can be deemed superiour to the conjunction-implication fragment of $\mathtt{NI}_g$: the syntax of $\mathtt{NI}^{\wedge\supset}$ "filters out" inessential differences such as the order in which inference rules are applied within derivations by enabling a more canonical representation of proofs.[16]

3.6 The Meaning of Harmony

As argued in Sect. 2.3, collections of introduction rules play the role of definitions. This can be understood as meaning not only that each introduction rule for † expresses a sufficient condition to prove a proposition having † as main operator, but also that these conditions are jointly necessary. The joint necessity of these conditions is not captured by any of the introduction rules, but it is the content of the PSH-elimination rule. That is, the content of the PSH-elimination rules is that the introduction rules for a given kind of propositions encode all possible means of constructing proofs of propositions of that kind. Equivalently, the content of PSH-elimination rules is that there are no other means of constructing proofs of their major premises other than those encoded by the corresponding introduction rules.

The PSH-elimination rules thus play the same role of the final clauses of inductive definitions. This is best understood by resorting again to the analogy between numbers and proofs that was developed in the previous chapter. In the case of the natural numbers, their inductive definition consists of the following three clauses:

(i) 0 is a natural number;
(ii) if n is a natural number then Sn is a natural number as well;
(iii) nothing else is a natural number.

If we look again at the rules for the predicate 'x is a natural number' given at the end of Sect. 1.3, the first two clauses are captured by the two introduction rules, whereas the third is captured by the elimination rule. The content of the latter is namely that in order to infer that a certain property $C(x)$ holds of a natural number t, it is enough to show that it holds of 0 and that if it holds of a number, it holds of its successor as well.

In the case of disjunction, the rule $\vee$E tells us that what follows from both A and B follows from $A \vee B$ as well. What warrants that a proof of C can be obtained given a proof of $A \vee B$ and given means of obtaining a proof C from either a proof of A or a proof of B? The answer seems to be that the only way of obtaining a proof of $A \vee B$ is by applying one of the two operations (usually called injections) associated with the introduction rules for $\vee$ to a proof of A or a proof of B.

We wish to stress that neither the final clause of the inductive definition of natural numbers nor the elimination rule for $\mathtt{Nat}$ warrant that every natural number can be reached by 0 using the successor function in a *finite* number of steps. In fact, the existence of non-standard elements of the set of natural number is compatible with the introduction and elimination rules, since there may be non-standard natural numbers which are the successor of some other (again non-standard) natural number.[17]

Moreover, the formulation of the inductive principle encoded by the elimination rules for a given kind of propositions does not require that the inductive process specifying how to construct the set of proofs for that kind of propositions satisfies any well-foundedness condition.

The understanding of `PSH`-elimination rules as final clauses of inductive definitions was first proposed by Martin-Löf [42] and it constitutes one of the cornernstones of constructive type theory. Hallnäs [29] and Hallnäs and Schroeder-Heister [30, 31] explored the possibility of extending Martin-Löf's ideas to cover the case of non-wellfounded inductive definitions in the context of logic programming. In the second part of the present work the conception of PTS developed in the previous chapter will be used to connect the understanding of harmony described in the present section with the analysis of paradoxical expresssions in the setting of natural deduction proposed by Prawitz [65] and Tennant [109].

3.7 Comparison with Jacinto and Read's GE-Stability

Jacinto and Read [34] have also recently pointed out the need of generalizing the usual formulation of the *no less* aspect of harmony in order to properly capture stability. In particular, they refer to the original formulation of the no less aspect of harmony as 'local completeness' (thereby following [58]) and propose to replace it in favor of what they call 'generalized local completeness'.

Rather than cashing out generalized local completeness by formulating generalized expansions as we did, Read and Jacinto formalize this notion as a complex requirement on derivability. Remarkably, to establish the generalized local completeness of the collections of introduction and elimination rules that they consider, they show how to construct derivations that closely resemble the (generalized) "expanded" derivations obtained by our generalized expansions. Thus, although the work of Jacinto and Read is not based on the idea of transformations on derivations or identity of proofs, the two approaches are quite close to each other.

In this section we explore the possibility of recasting Jacinto and Read's account of generalized local completeness using generalized η-expansions. Some difficulties will be encountered, whose source is that Jacinto and Read (following [77, 78]) consider elimination rules whose form differs from the one discussed in the previous sections of this chapter.

We wish to stress however that the considerations to be developed below do not undermine the results of Jacinto and Read. Rather, these considerations show how their results can be given an intensional reformulation by taking the notion of identity of proofs into account.

To spell out the issue in a more precise way, we begin by presenting the elimination rules considered by Jacinto and Read:

Definition 3.4 (JR-*inversion*) Let † and †**I** be as in Definition 3.1.

We indicate with JR(†**I**) the collection of elimination rules consisting of $\prod_{k=1}^{r} m_k$ rules, each of which has the following form:

$$\dfrac{\dagger(A_1,\ldots,A_n) \quad \begin{matrix}[f_h(1)]\\ C\end{matrix} \quad \ldots \quad \begin{matrix}[f_h(r)]\\ C\end{matrix}}{C}\,\dagger\mathrm{E}_h^{\mathrm{JR}} \qquad (\mathrm{JR})$$

where f_h is the hth choice function that selects one of the premises of each of the r introduction rules of †, that is for each $1 \leq k \leq r$, $f_h(k) = B_{kj}$ for some $1 \leq j \leq m_k$[18] and where C is distinct from each A_i.

We say that † is a JR-connectives in a calculus K iff K includes †**I** and JR(†**I**), and † does not occur in any other of the primitive rules of K.[19]

When each rule in a collection of introduction rules †**I** has at most one premise (as in the case of disjunction), then PSH-inversion and JR-inversion yield the same collection of elimination rules, i.e. PSH(†**I**) = JR(†**I**). Not so if at least one of the introduction rules has more than one premise. For example, in the case of conjunction we obtain yet another variant of the collection of elimination rules consisting of the following two rules:

$$\dfrac{A\wedge B \quad \begin{matrix}[A]\\ C\end{matrix}}{C}\,\wedge\mathrm{E}_1^{\mathrm{JR}} \qquad \dfrac{A\wedge B \quad \begin{matrix}[B]\\ C\end{matrix}}{C}\,\wedge\mathrm{E}_2^{\mathrm{JR}}$$

The definition of reductions to get rid of local peaks is straightforward in the case of JR-elimination rules (although one has to specify one reduction for each elimination rule):

$$\langle u\rangle\,\dfrac{\dfrac{\begin{matrix}\mathscr{D}_1\\ A_1\end{matrix} \quad \begin{matrix}\mathscr{D}_2\\ A_2\end{matrix}}{A_1\wedge A_2}\,\wedge\mathrm{I} \quad \begin{matrix}\overset{u}{[A_i]}\\ \mathscr{D}'\\ C\end{matrix}}{C}\,\wedge\mathrm{E}_i^{\mathrm{JR}} \quad \overset{\beta}{\rhd} \quad \begin{matrix}\mathscr{D}_i\\ [A_i]\\ \mathscr{D}'\\ C\end{matrix} \qquad (\text{for } i = 1,2) \qquad (\wedge\beta_i^{\mathrm{JR}})$$

In establishing that the introduction rule for conjunction ∧I and the JR-elimination rules $\wedge\mathrm{E}_1^{\mathrm{JR}}$ and $\wedge\mathrm{E}_2^{\mathrm{JR}}$ satisfy the condition for generalized local completeness, Jacinto and Read show how to construct the following derivation:

$$\langle u_2\rangle\,\dfrac{\begin{matrix}\mathscr{D}\\ A\wedge B\end{matrix} \quad \langle u_1\rangle\,\dfrac{\begin{matrix}\mathscr{D}\\ A\wedge B\end{matrix} \quad \begin{matrix}\dfrac{\overset{u_1}{A} \quad \overset{u_2}{B}}{[A\wedge B]}\,\wedge\mathrm{I}\\ \mathscr{D}'\\ C\end{matrix}}{C}\,\wedge\mathrm{E}_1^{\mathrm{JR}}}{C}\,\wedge\mathrm{E}_2^{\mathrm{JR}}$$

Although they do not define an operation of expansion, one could argue that they *could* have defined it, by positing the derivation just given as what one obtains by expanding the following:

$$\begin{array}{c} \mathscr{D} \\ [A \wedge B] \\ \mathscr{D}' \\ C \end{array}$$

However, there is no principled reason why the expansion of this derivation should not rather be the following:

$$\dfrac{\begin{array}{c}\mathscr{D}\\ A \wedge B\end{array} \quad \dfrac{\begin{array}{c}\mathscr{D}\\ A \wedge B\end{array} \quad \begin{array}{c}\dfrac{\overset{u_1}{A} \quad \overset{u_2}{B}}{[A \wedge B]}\wedge\mathrm{I}\\ \mathscr{D}'\\ C\end{array}}{C}\wedge\mathrm{E}_2^{\mathrm{JR}}\langle u_2\rangle}{C}\wedge\mathrm{E}_1^{\mathrm{JR}}\langle u_1\rangle$$

in which the elimination rules are applied in a different order.

By looking at more complex collections of introduction rules, one immediately realizes that the problem is not just the order in which the different elimination rules are applied. Consider the collection of introduction rules $\odot\mathbf{I}$ for the quaternary connective $\odot$ consisting of the following two rules:

$$\dfrac{A \quad B}{\odot(A, B, C, D)}\odot\mathrm{I}_1 \qquad \dfrac{C \quad D}{\odot(A, B, C, D)}\odot\mathrm{I}_2$$

The collection of elimination rules $\mathtt{JR}(\odot\mathbf{I})$ associated with $\odot\mathbf{I}$ by $\mathtt{JR}$-inversion consists of the following four rules:

$$\dfrac{\odot(A, B, C, D) \quad \begin{array}{c}[A]\\ E\end{array} \quad \begin{array}{c}[C]\\ E\end{array}}{E}\odot\mathrm{E}_1^{\mathrm{JR}} \qquad \dfrac{\odot(A, B, C, D) \quad \begin{array}{c}[A]\\ E\end{array} \quad \begin{array}{c}[D]\\ E\end{array}}{E}\odot\mathrm{E}_2^{\mathrm{JR}}$$

$$\dfrac{\odot(A, B, C, D) \quad \begin{array}{c}[B]\\ E\end{array} \quad \begin{array}{c}[C]\\ E\end{array}}{E}\odot\mathrm{E}_3^{\mathrm{JR}} \qquad \dfrac{\odot(A, B, C, D) \quad \begin{array}{c}[B]\\ E\end{array} \quad \begin{array}{c}[D]\\ E\end{array}}{E}\odot\mathrm{E}_4^{\mathrm{JR}}$$

Using Jacinto and Read's recipe, one can cook up the first derivation displayed in Table 3.1 (in which $\odot(A, B, C, D)$ is abbreviated with $\odot$). However, in this case also there is no principled reason to claim that a derivation of the form:

$$\begin{array}{c} \mathscr{D} \\ [\odot] \\ \mathscr{D}' \\ E \end{array}$$

should expand that way rather than, say, as in the second derivation of the table, where not only the order, but also the number of applications of the different elimination rules changes. By considering connectives with more complex introduction rules, the situation quickly becomes unwieldy as to the number of possible ways of expanding a given derivation.

In light of this, it is not obvious how to cash out Jacinto and Read's generalization of the *no less* aspect of harmony in terms of (generalized) expansions. For the

Table 3.1 Two possible expansion patterns using $\mathtt{JR}(\odot\mathbf{I})$

$$
\langle u_9, u_{10}\rangle\;
\dfrac{
\begin{array}{c}\mathscr{D}\\ \odot\end{array}
\quad
\langle u_3,u_4\rangle\;
\dfrac{
\begin{array}{c}\mathscr{D}\\ \odot\end{array}
\quad
\begin{array}{c}\dfrac{\overset{u_3}{A}\quad\overset{u_9}{B}}{[\odot]}\,\odot\mathrm{I}_1\\ \mathscr{D}'\\ E\end{array}
\quad
\langle u_1,u_2\rangle\;
\dfrac{
\begin{array}{c}\mathscr{D}\\ \odot\end{array}
\quad
\begin{array}{c}\dfrac{\overset{u_1}{A}\quad\overset{u_9}{B}}{[\odot]}\,\odot\mathrm{I}_1\\ \mathscr{D}'\\ E\end{array}
\quad
\begin{array}{c}\dfrac{\overset{u_2}{C}\quad\overset{u_4}{D}}{[\odot]}\,\odot\mathrm{I}_2\\ \mathscr{D}'\\ E\end{array}
}{E}\,\odot\mathrm{E}_1^{\mathtt{JR}}
}{E}\,\odot\mathrm{E}_2^{\mathtt{JR}}
\quad
\langle u_7,u_8\rangle\;
\dfrac{
\begin{array}{c}\mathscr{D}\\ \odot\end{array}
\quad
\langle u_5,u_6\rangle\;
\dfrac{
\begin{array}{c}\mathscr{D}\\ \odot\end{array}
\quad
\begin{array}{c}\dfrac{\overset{u_5}{A}\quad\overset{u_7}{B}}{[\odot]}\,\odot\mathrm{I}_1\\ \mathscr{D}'\\ E\end{array}
\quad
\begin{array}{c}\dfrac{\overset{u_6}{C}\quad\overset{u_{10}}{D}}{[\odot]}\,\odot\mathrm{I}_2\\ \mathscr{D}'\\ E\end{array}
}{E}\,\odot\mathrm{E}_1^{\mathtt{JR}}
\quad
\begin{array}{c}\dfrac{\overset{u_8}{C}\quad\overset{u_{10}}{D}}{[\odot]}\,\odot\mathrm{I}_2\\ \mathscr{D}'\\ E\end{array}
}{E}\,\odot\mathrm{E}_3^{\mathtt{JR}}
}{E}\,\odot\mathrm{E}_4^{\mathtt{JR}}
$$

$$
\langle u_9, u_{10}\rangle\;
\dfrac{
\begin{array}{c}\mathscr{D}\\ \odot\end{array}
\quad
\langle u_3,u_4\rangle\;
\dfrac{
\begin{array}{c}\mathscr{D}\\ \odot\end{array}
\quad
\begin{array}{c}\dfrac{\overset{u_9}{A}\quad\overset{u_3}{B}}{[\odot]}\,\odot\mathrm{I}_1\\ \mathscr{D}'\\ E\end{array}
\quad
\langle u_1,u_2\rangle\;
\dfrac{
\begin{array}{c}\mathscr{D}\\ \odot\end{array}
\quad
\begin{array}{c}\dfrac{\overset{u_9}{A}\quad\overset{u_1}{B}}{[\odot]}\,\odot\mathrm{I}_1\\ \mathscr{D}'\\ E\end{array}
\quad
\begin{array}{c}\dfrac{\overset{u_4}{C}\quad\overset{u_2}{D}}{[\odot]}\,\odot\mathrm{I}_2\\ \mathscr{D}'\\ E\end{array}
}{E}\,\odot\mathrm{E}_4^{\mathtt{JR}}
}{E}\,\odot\mathrm{E}_3^{\mathtt{JR}}
\quad
\langle u_7,u_8\rangle\;
\dfrac{
\begin{array}{c}\mathscr{D}\\ \odot\end{array}
\quad
\langle u_5,u_6\rangle\;
\dfrac{
\begin{array}{c}\mathscr{D}\\ \odot\end{array}
\quad
\begin{array}{c}\dfrac{\overset{u_5}{A}\quad\overset{u_7}{B}}{[\odot]}\,\odot\mathrm{I}_1\\ \mathscr{D}'\\ E\end{array}
\quad
\begin{array}{c}\dfrac{\overset{u_6}{C}\quad\overset{u_{10}}{D}}{[\odot]}\,\odot\mathrm{I}_2\\ \mathscr{D}'\\ E\end{array}
}{E}\,\odot\mathrm{E}_1^{\mathtt{JR}}
\quad
\begin{array}{c}\dfrac{\overset{u_8}{C}\quad\overset{u_{10}}{D}}{[\odot]}\,\odot\mathrm{I}_2\\ \mathscr{D}'\\ E\end{array}
}{E}\,\odot\mathrm{E}_3^{\mathtt{JR}}
}{E}\,\odot\mathrm{E}_2^{\mathtt{JR}}
$$

family of connectives they consider, there is no operation (i.e. function) to expand a given derivation, since in these cases the process of expansion would be highly non-deterministic.

It is instructive to compare the collection of elimination rules $\mathtt{JR}(\odot\mathbf{I})$ obtained by $\mathtt{JR}$-inversion, with the collection $\mathtt{PSH}(\odot\mathbf{I})$ obtained by $\mathtt{PSH}$-inversion and consisting of the following rule:

$$\dfrac{\odot(A,B,C,D) \qquad \begin{matrix}[A][B]\\ E\end{matrix} \qquad \begin{matrix}[C][D]\\ E\end{matrix}}{E}\;\odot\mathrm{E}^{\mathtt{PSH}}$$

To show that $\mathtt{PSH}(\odot\mathbf{I})$ is in harmony with $\odot\mathbf{I}$ one only need two reductions (instead of the eight reductions needed for $\mathtt{JR}(\odot\mathbf{I})$), and the way in which a derivation should be expanded is unequivocally determined: in the expanded derivation each of the introduction rules and the elimination rule is applied exactly once, and each rule application discharges exactly one copy of each dischargeable assumption:

$$\begin{matrix}\mathscr{D}\\ [\odot]\\ \mathscr{D}'\\ E\end{matrix} \quad \underset{\lhd}{\odot\eta_g^{\mathtt{PSH}}} \quad \langle u_1,u_2,u_3,u_4\rangle\;\dfrac{\odot \qquad \begin{matrix}\dfrac{\overset{u_1}{A} \qquad \overset{u_2}{B}}{[\odot]}\,\odot\mathrm{I}_1\\ \mathscr{D}'\\ E\end{matrix} \qquad \begin{matrix}\dfrac{\overset{u_3}{C} \qquad \overset{u_4}{D}}{[\odot]}\,\odot\mathrm{I}_2\\ \mathscr{D}'\\ E\end{matrix}}{E}\;\odot\mathrm{E}^{\mathtt{PSH}}$$

Despite the more bureaucratic character of $\mathtt{JR}$-inversion, one can nonetheless argue that, from the intensional standpoint we advocate, $\mathtt{JR}$-elimination rules are as harmonious as $\mathtt{PSH}$-elimination rules.

The reason is that it is possible to recover permutative conversions for the $\mathtt{JR}$-elimination rules using a combination of expansions and reductions in (almost) the same way as for disjunction (see above Sect. 3.5).

Observe first that the strategy used to simulate the permutation for the disjunction elimination rule generalizes straightforwardly to all $\mathtt{PSH}$-elimination rules. For instance, the permutative conversion for $\wedge\mathrm{E}^{\mathtt{PSH}}$[20]:

$$\begin{matrix}\langle u_1,u_2\rangle\;\dfrac{\begin{matrix}\mathscr{D}\\ A\wedge B\end{matrix} \qquad \begin{matrix}\overset{u_1}{[A]}\;\overset{u_2}{[B]}\\ \mathscr{D}'\\ C\end{matrix}}{[C]}\;\wedge\mathrm{E}^{\mathtt{PSH}}\\ \mathscr{D}''\\ D\end{matrix} \quad \text{permutes to} \quad \langle u_1,u_2\rangle\;\dfrac{\begin{matrix}\mathscr{D}\\ A\wedge B\end{matrix} \qquad \begin{matrix}\overset{u_1}{[A]}\;\overset{u_2}{[B]}\\ \mathscr{D}'\\ [C]\\ \mathscr{D}''\\ D\end{matrix}}{D}\;\wedge\mathrm{E}_1$$

can be simulated using the reduction $(\wedge\beta^{\mathtt{PSH}})$ and the generalized expansion $(\wedge\eta_g^{\mathtt{PSH}})$, since the derivation on the left-hand side of the permutation expands using $(\wedge\eta_g^{\mathtt{PSH}})$ to the following derivation:

$$
\langle u_1,u_2\rangle\,\dfrac{\begin{array}{c}\mathscr{D}\\ A\wedge B\end{array}\qquad \begin{array}{c}\langle u_1,u_2\rangle\,\dfrac{\dfrac{\overset{u_1}{A}\qquad \overset{u_2}{B}}{A\wedge B}\wedge\mathrm{I}\qquad \begin{array}{c}\overset{u_1}{[A]}\overset{u_2}{[B]}\\ \mathscr{D}'\\ C\end{array}}{[C]}\wedge\mathrm{E}^{\mathtt{PSH}}\\ \mathscr{D}''\\ D\end{array}}{D}\wedge\mathrm{E}^{\mathtt{PSH}}
$$

which in turn reduces using $(\wedge\beta^{\mathtt{PSH}})$ to the derivation on the right-hand side of the permutation.

In the case of $\mathtt{JR}$-connectives, in order to simulate permutations one needs not only β-reductions and (generalized) η-expansions but their inverse operations as well. Suppose one stipulates that the "official" way of performing generalized expansions involving the $\mathtt{JR}$-elimination rules for $\wedge$ is the following:

$$
\begin{array}{c}\mathscr{D}\\ [A\wedge B]\\ \mathscr{D}'\\ C\end{array}\quad \overset{\wedge\eta_g^{\mathtt{JR}}}{\lhd}\quad \langle u_2\rangle\,\dfrac{\begin{array}{c}\mathscr{D}\\ A\wedge B\end{array}\qquad \langle u_1\rangle\,\dfrac{\begin{array}{c}\mathscr{D}\\ A\wedge B\end{array}\qquad \begin{array}{c}\dfrac{\overset{u_1}{A}\qquad \overset{u_2}{B}}{[A\wedge B]}\wedge\mathrm{I}\\ \mathscr{D}'\\ C\end{array}}{C}\wedge\mathrm{E}_1^{\mathtt{JR}}}{C}\wedge\mathrm{E}_2^{\mathtt{JR}} \qquad (\wedge\eta_g^{\mathtt{JR}})
$$

The (general) permutative conversion for $\wedge\mathrm{E}_1^{\mathtt{JR}}$:

$$
\begin{array}{c}\langle u\rangle\,\dfrac{\begin{array}{c}\mathscr{D}\\ A\wedge B\end{array}\qquad \begin{array}{c}\overset{u}{[A]}\\ \mathscr{D}'\\ C\end{array}}{[C]}\wedge\mathrm{E}_1^{\mathtt{JR}}\\ \mathscr{D}''\\ D\end{array}\quad \text{permutes to}\quad \langle u\rangle\,\dfrac{\begin{array}{c}\mathscr{D}\\ A\wedge B\end{array}\qquad \begin{array}{c}\overset{u}{[A]}\\ \mathscr{D}'\\ [C]\\ \mathscr{D}''\\ D\end{array}}{D}\wedge\mathrm{E}_1
$$

can be simulated as depicted in Table 3.2.[21] A permutation for the other elimination rule $\wedge\mathrm{E}_2^{\mathtt{JR}}$ can be obtained in a similar manner.

We observe that we could have simulated the permutations in a similar way also if we had stipulated the other possible expansion pattern to be the "official" one:

$$
\begin{array}{c}\mathscr{D}\\ [A\wedge B]\\ \mathscr{D}'\\ C\end{array}\quad \overset{\wedge\eta_g^{\mathtt{JR}'}}{\lhd}\quad \langle m\rangle\,\dfrac{\begin{array}{c}\mathscr{D}\\ A\wedge B\end{array}\qquad \langle n\rangle\,\dfrac{\begin{array}{c}\mathscr{D}\\ A\wedge B\end{array}\qquad \begin{array}{c}\dfrac{\overset{n}{A}\qquad \overset{m}{B}}{[A\wedge B]}\wedge\mathrm{I}\\ \mathscr{D}'\\ C\end{array}}{C}\wedge\mathrm{E}_1^{\mathtt{JR}}}{C}\wedge\mathrm{E}_2^{\mathtt{JR}} \qquad (\wedge\eta_g^{\mathtt{JR}'})
$$

In fact, the left-hand sides of the expansions $(\wedge\eta_g^{\mathtt{JR}})$ and $(\wedge\eta_g^{\mathtt{JR}'})$ differ only up to the order of application of the two elimination rules, and hence they would belong to the same equivalence class of derivations induced by the β-reductions together with either of the two expansions. Thus the two equational theories induced by coupling the β-reductions with either of the two generalized η-expansions induce the same equivalence classes of derivations.[22]

Table 3.2 Simulating the permutation for $\wedge\mathrm{E}_1^{\mathrm{JR}}$

$$\langle u\rangle\,\dfrac{\begin{array}{c}\mathscr{D}\\ A\wedge B\end{array}\qquad\begin{array}{c}\overset{u}{[A]}\\ \mathscr{D}'\\ C\end{array}}{\begin{array}{c}[C]\\ \mathscr{D}''\\ D\end{array}}\,\wedge\mathrm{E}_1^{\mathrm{JR}}\quad\overset{\wedge\eta_g^{\mathrm{JR}}}{\triangleleft}\quad\langle u_2\rangle\,\dfrac{\begin{array}{c}\mathscr{D}\\ A\wedge B\end{array}\qquad\langle u_1\rangle\,\dfrac{\begin{array}{c}\mathscr{D}\\ A\wedge B\end{array}\qquad\begin{array}{c}\langle u\rangle\,\dfrac{\dfrac{\overset{u_1}{A}\quad\overset{u_2}{B}}{A\wedge B}\qquad\begin{array}{c}\overset{u}{[A]}\\ \mathscr{D}'\\ C\end{array}}{[C]}\,\wedge\mathrm{E}_1^{\mathrm{JR}}\\ \mathscr{D}''\\ D\end{array}}{D}\,\wedge\mathrm{E}_1^{\mathrm{JR}}}{D}\,\wedge\mathrm{E}_2^{\mathrm{JR}}\quad\overset{\wedge\beta_1^{\mathrm{JR}}}{\triangleright}$$

$$\overset{\wedge\beta_1^{\mathrm{JR}}}{\triangleright}\quad\dfrac{\begin{array}{c}\mathscr{D}\\ A\wedge B\end{array}\qquad\langle u_1\rangle\,\dfrac{\begin{array}{c}\mathscr{D}\\ A\wedge B\end{array}\qquad\begin{array}{c}\overset{u_1}{[A]}\\ \mathscr{D}'\\ [C]\\ \mathscr{D}''\\ D\end{array}}{D}\,\wedge\mathrm{E}_1^{\mathrm{JR}}}{D}\,\wedge\mathrm{E}_2^{\mathrm{JR}}\quad\overset{\wedge\beta_1^{\mathrm{JR}}}{\triangleleft}\quad\langle u_2\rangle\,\dfrac{\begin{array}{c}\mathscr{D}\\ A\wedge B\end{array}\qquad\langle u_1\rangle\,\dfrac{\begin{array}{c}\mathscr{D}\\ A\wedge B\end{array}\qquad\langle u\rangle\,\dfrac{\dfrac{\overset{u_1}{A}\quad\overset{u_2}{B}}{A\wedge B}\qquad\begin{array}{c}\overset{u}{[A]}\\ \mathscr{D}'\\ [C]\\ \mathscr{D}''\\ D\end{array}}{D}\,\wedge\mathrm{E}_1^{\mathrm{JR}}}{D}\,\wedge\mathrm{E}_1^{\mathrm{JR}}}{D}\,\wedge\mathrm{E}_2^{\mathrm{JR}}\quad\overset{\wedge\eta_g^{\mathrm{JR}}}{\triangleright}\quad\langle u\rangle\,\dfrac{\begin{array}{c}\mathscr{D}\\ A\wedge B\end{array}\qquad\begin{array}{c}\overset{u}{[A]}\\ \mathscr{D}'\\ [C]\\ \mathscr{D}''\\ D\end{array}}{D}\,\wedge\mathrm{E}_1$$

Summing up, it is true that there is no unique way of defining expansion patterns, and it is also true that one cannot simulate permutations for the JR-elimination rules using β-reduction and generalized η-expansions only (as one needs their inverses as well). Nonetheless, the different expansion patterns (coupled with the $\beta^{\mathtt{JR}}$-reductions) yield the same equivalence relation on derivations. Hence the choice among different expansions patterns is irrelevant for the resulting notion of identity of proofs.

We conclude this section by observing that JR-inversion shows a true deficiency if one tries to apply it to first-order quantifiers. Apart from the case of collections of introduction rules with at most one premise (in which case JR-inversion yields the same elimination rule as PSH-inversion) JR-inversion delivers elimination rules which are not stable. Let's reconsider the quantifier expressing the I corner of the square of opposition. With the collection $I\mathbf{I}$ consisting of the introduction rule II, JR-inversion may be expected to associate the collection of elimination rules $\mathtt{JR}(I\mathbf{I})$ consisting of the following two rules:

$$\dfrac{I_x(A,B) \qquad \begin{matrix}[A(y/x)]\\ C\end{matrix}}{C}\,I\mathrm{E}_1^{\mathtt{JR}} \qquad\qquad \dfrac{I_x(A,B) \qquad \begin{matrix}[B(y/x)]\\ C\end{matrix}}{C}\,I\mathrm{E}_2^{\mathtt{JR}}$$

with an *eigenvariable* condition on y in the two elimination rules. Contrary to the case of $\mathtt{PSH}(I\mathbf{I})$, when $I\mathbf{I}$ is coupled with $\mathtt{JR}(I\mathbf{I})$ it is not even possible to formulate an expansion following the simpler pattern of Prawitz, since the application of the first elimination rule would violate the *eigenvariable* condition:

$$\begin{matrix}\mathscr{D}\\ I_x(A,B)\end{matrix} \quad \text{does } \textit{not} \text{ expand to} \quad \dfrac{\begin{matrix}\mathscr{D}\\ I_x(A,B)\end{matrix} \qquad \dfrac{\begin{matrix}\mathscr{D}\\ I_x(A,B)\end{matrix} \qquad \dfrac{\overset{u_1}{A(y/x)} \quad \overset{u_2}{B(y/x)}}{I_x(A,B)}\,I\mathrm{I}}{I_x(A,B)}\,\langle u_1\rangle\, I\mathrm{E}_1^{\mathtt{JR}}}{I_x(A,B)}\,\langle u_2\rangle\, I\mathrm{E}_2^{\mathtt{JR}}$$

Thus in general, it does not seem that JR-inversion can deliver stable rules for quantifiers apart from those instances in which it coincides with PSH-inversion.

3.8 Harmony by Interderivability

Although implicitly acknowledged by most authors, it was only recently observed in an explicit manner [56, 93, 94] that the specification of an inversion principle cannot constitute an exhaustive characterization of harmony.

To see why, it is sufficient to reconsider the rules for conjunction discussed so far. The collection of rules consisting of $\wedge\mathrm{E}_1$ and $\wedge\mathrm{E}_2$ is not the one that one would obtain from the Prawitz–Schroeder-Heister inversion principle, namely the one consisting of the unique elimination rule in general form:

$$\frac{A \wedge B \qquad \begin{matrix}[A][B]\\ C\end{matrix}}{C}\wedge\mathrm{E}^{\mathtt{PSH}}$$

However, neither Prawitz nor Schroeder-Heister seem to be willing to deny that the elimination rules $\wedge\mathrm{E}_1$ and $\wedge\mathrm{E}_2$ are in harmony with $\wedge$I as much as the rule $\wedge\mathrm{E}^{\mathtt{PSH}}$.

Similarly, when Read [77] argues that the rules:

$$\frac{A \wedge B \qquad \begin{matrix}[A]\\ C\end{matrix}}{C}\wedge\mathrm{E}_1^{\mathtt{JR}} \qquad\qquad \frac{A \wedge B \qquad \begin{matrix}[B]\\ C\end{matrix}}{C}\wedge\mathrm{E}_2^{\mathtt{JR}}$$

are in harmony with $\wedge$I he does not seem to be willing to deny that the other collections of elimination are in harmony with $\wedge$I as well.

Thus, the collection of elimination rules generated by some inversion principle from a given collection of introduction rules is not, in general, the only one which is in harmony with it.[23]

But what do all these alternative—but, intuitively at least, equally harmonious—collections of elimination rules have in common? What all mentioned authors explicitly observe is that the alternative collections of elimination rules are interderivable with each other (we indicate interderivability between (collections of) rules using $\dashv\vdash$).[24] For instance, both $\wedge\mathrm{E}_1$ and $\wedge\mathrm{E}_2$ (resp. $\wedge\mathrm{E}_1^{\mathtt{JR}}$ and $\wedge\mathrm{E}_2^{\mathtt{JR}}$) are derivable from $\wedge\mathrm{E}^{\mathtt{PSH}}$, and conversely the latter rule is derivable from the former ones. Similarly, both $\wedge\mathrm{E}_1^{\mathtt{JR}}$ and $\wedge\mathrm{E}_2^{\mathtt{JR}}$ are derivable from $\wedge\mathrm{E}_1$ and $\wedge\mathrm{E}_2$ and vice versa.

Although not stated in an explicit manner, this seems to be the reason why the rules of $\wedge$ of `NI` are considered as much in harmony as those obtained by `PSH`- and `JR`-inversion.

That is, it seems plausible to claim that Prawitz and Schroeder-Heister, and Jacinto and Read (at least implicitly) endorse the following notions of harmony:

Definition 3.5 (`PSH`-*harmony by interderivability*) Given two collections $\dagger\mathbf{I}$ and $\dagger\mathbf{E}$ of introduction and elimination rules for a connective $\dagger$, we say that $\dagger\mathbf{I}$ and $\dagger\mathbf{E}$ are in *PSH-harmony via interderivability* if and only if

$$\dagger\mathbf{E} \dashv\vdash \mathtt{PSH}(\dagger\mathbf{I})$$

Definition 3.6 (`JR`-*harmony by interderivability*) Given two collections $\dagger\mathbf{I}$ and $\dagger\mathbf{E}$ of introduction and elimination rules for a connective $\dagger$, we say that $\dagger\mathbf{I}$ and $\dagger\mathbf{E}$ are in *JR-harmony via interderivability* if and only if

$$\dagger\mathbf{E} \dashv\vdash \mathtt{JR}(\dagger\mathbf{I})$$

In fact, for any collection of introduction rules $\dagger\mathbf{I}$, $\mathtt{PSH}(\dagger\mathbf{I}) \dashv\vdash \mathtt{JR}(\dagger\mathbf{I})$.[25] Thus the same collections of rules qualify as harmonious according to the two definitions.

The invariance of harmony with respect to the choice of the inversion principle has been taken by Schroeder-Heister as a reason for defining harmony without making reference to any inversion principle at all. In fact Schroeder-Heister [93, 94] proposed two different accounts of harmony, on the basis of which he then *demonstrated* that the rules obeying PSH-inversion satisfy the proposed condition for harmony. Both notions of harmony are equivalent with each other, and moreover they are equivalent to those resulting from Definitions 3.5 and 3.6.

We will say that the rules satisfying these notions of harmony are in *harmony by interderivability*.

We fully agree with Schroeder-Heister on the need for a notion of harmony going beyond the specification of an inversion principle. However, it is doubtful whether rules which are in harmony by interderivability can, in general, be equipped with plausible reductions and expansions. In other words, it is doubtful whether the account of harmony obtained by coupling inversion with interderivability can still qualify as intensional.

After introducing in the next Section a further example of inversion principles, in Sect. 3.10 we will present an example justifying this claim.

3.9 Yet Another Inversion Principle

In this section we consider a further inversion principle. Its range of applicability is limited to the restricted case of a collection of introduction rules consisting of a single introduction rule, which is however allowed to discharge assumptions, in contrast to the inversion principles discussed in Sects. 3.4 and 3.7. Nonetheless, whereas the two inversion principles of the previous chapter can be generalized to cover also introduction rules that can discharge assumptions (as shown in the Appendix, see Sect. A.9), the inversion principle introduced in this section cannot be generalized to cover more than one introduction rule. For this reason, we will referred to it as '*toy* inversion', henceforth T-inversion. In spite of its limited range of applicability, it will be useful to establish the negative result in the final part of the present chapter, namely that harmony by interderivability is not intensional. As in the case of PSH- and JR-inversion, the collection of rules obtained by T-inversion from a given collection of introduction rules is interderivable with those obtained by the other two inversion principles.

Definition 3.7 (T-*inversion*) Let $\dagger$ be an n-ary connective and let $\dagger\mathbf{I}$ consist of a single introduction rules for $\dagger$ of the following form:

$$\dfrac{\begin{matrix}[B_{11}]\ldots[B_{1p_1}] \\ C_1\end{matrix} \qquad \ldots \qquad \begin{matrix}[B_{m1}]\ldots[B_{mp_m}] \\ C_m\end{matrix}}{\dagger(A_1,\ldots,A_n)}\,\dagger\mathrm{I}$$

where either $m = 0$ or $m \geq 1$ and both of the following two conditions hold: (i) for all $1 \leq j \leq m$ and either $p_j = 0$ or $p_j \geq 1$ and for all $1 \leq k \leq p_j$ there is an $1 \leq i \leq n$ such that B_{jk} is syntactically identical to A_i; (ii) for all $1 \leq j \leq m$ there is an $1 \leq i \leq n$ such that C_j is syntactically identical to A_i.

We indicate with $\mathtt{T}(\dagger\mathbf{I})$ the collection of elimination rules consisting of m elimination rules, each of which has the following form (we take $\mathtt{T}(\dagger\mathbf{I})$ to be empty if $m = 0$):

$$\dfrac{\dagger \qquad B_{j1} \qquad \dots \qquad B_{jp_j}}{C_j}\ \dagger\mathrm{E}_j$$

We say that $\dagger$ is a $\mathtt{T}$-connectives in a calculus $\mathtt{K}$ iff $\mathtt{K}$ includes $\dagger\mathbf{I}$ and $\mathtt{T}(\dagger\mathbf{I})$, and $\dagger$ does not occur in any other of the primitive rules of $\mathtt{K}$.[26]

Clearly, $\supset$, $\wedge$ and $\top$ are $\mathtt{T}$-connectives in $\mathtt{NI}$.

Connectives whose rules obey $\mathtt{T}$-inversion satisfy the informal statement of harmony, as it is shown by the possibility of formulating the following β- and η-equations for $\mathtt{K}$-derivations in a calculus $\mathtt{K}$ in which $\dagger$ is a $\mathtt{T}$-connective (in the schemata we abbreviate $\dagger(A_1, \dots, A_n)$ with $\dagger$):

$$\dfrac{\langle u_{11},\dots,u_{mp_m}\rangle\ \dfrac{\begin{matrix}\overset{u_{11}}{[B_{11}]}\dots\overset{u_{1p_1}}{[B_{1p_1}]}\\ \mathscr{D}_1\\ C_1\end{matrix} \qquad \dots \qquad \begin{matrix}\overset{m1}{[B_{m1}]}\dots\overset{u_{mp_m}}{[B_{mp_m}]}\\ \mathscr{D}_m\\ C_m\end{matrix}}{\dagger}\,\dagger\mathrm{I} \qquad \begin{matrix}\mathscr{D}'_1\\ B_{j1}\end{matrix}\ \dots\ \begin{matrix}\mathscr{D}'_{p_j}\\ B_{jp_j}\end{matrix}}{C_j}\,\dagger\mathrm{E}_j \qquad \overset{\dagger\beta_j^{\mathtt{T}}}{\rhd} \qquad \begin{matrix}\mathscr{D}'_1 \qquad\quad \mathscr{D}'_{p_j}\\ [B_{j1}]\ \dots\ [B_{jp_j}]\\ \mathscr{D}_j\\ C_j\end{matrix}$$

for $1 \leq j \leq m$

$$\dfrac{\langle u_{11},\dots,u_{1p_1},\dots,u_{m1},\dots,u_{mp_m}\rangle\ \dfrac{\begin{matrix}\mathscr{D}\\ \dagger\end{matrix} \quad \overset{u_{11}}{B_{11}} \quad \dots \quad \overset{u_{1p_1}}{B_{1p_1}}}{C_1}\,\dagger\mathrm{E}_1 \qquad \dots \qquad \dfrac{\begin{matrix}\mathscr{D}\\ \dagger\end{matrix} \quad \overset{u_{m1}}{B_{m1}} \quad \dots \quad \overset{u_{mp_m}}{B_{mp_m}}}{C_m}\,\dagger\mathrm{E}_m}{\dagger}\,\dagger\mathrm{I} \qquad \overset{\dagger\eta^{\mathtt{T}}}{\lhd} \qquad \begin{matrix}\mathscr{D}\\ \dagger\end{matrix}$$

with $u_{11} \dots u_{mp_m}$ fresh for $\mathscr{D}$

We conclude this section by presenting two collections of rules for two connectives $\sharp$ and $\flat$ such that in a calculus $\mathtt{K}$ consisting of both collections of rules as primitive both $\sharp$ and $\flat$ are $\mathtt{T}$-connectives. Moreover for any pair of propositions A and B, the propositions $A\sharp B$ and $A\flat B$ are interderivable but not $\beta\eta$-isomorphic in $\mathtt{K}$. In the next section we will then consider the collection of rules for a third connective $\natural$ having the same collection of introduction of rules of $\sharp$ and the same collection of elimination rules of $\flat$. We will show that the rules of $\natural$ are in harmony as interderivability. However, although it is possible to define reductions and expansions for $\natural$, the most obvious candidates for these equations trivialize the notion of isomorphism.

Let's first consider the following collections of rules $\sharp\mathbf{I}$ and $\sharp\mathbf{E}$ for $\sharp$:

$$\sharp\mathbf{I}: \quad \frac{\begin{array}{c}[A]\\ B\end{array}\qquad \begin{array}{c}[B]\\ A\end{array}\qquad A}{A\sharp B}\,\sharp\mathrm{I}$$

$$\sharp\mathbf{E}: \quad \frac{A\sharp B \qquad A}{B}\,\sharp\mathrm{E}_1 \qquad \frac{A\sharp B \qquad B}{A}\,\sharp\mathrm{E}_2 \qquad \frac{A\sharp B}{A}\,\sharp\mathrm{E}_3$$

Clearly, $\sharp\mathbf{E} = \mathtt{T}(\sharp\mathbf{I})$, and the harmonious nature of the rules is displayed by the β- and η-equations of Table 3.3.

Using these equations, it easy to show that $A\sharp B$ is $\beta\eta$-isomorphic to $((A \supset B) \wedge (B \supset A)) \wedge A$ in $\mathtt{NI}^{\wedge\supset\sharp}$, the extension of $\mathtt{NI}^{\wedge\supset}$ with $\sharp\mathbf{I}$ and $\sharp\mathbf{E}$.

Consider now the collections of rules $\flat\mathbf{I}$ and $\flat\mathbf{E}$ for the connective $\flat$. These two collection of rules differ from those of $\sharp$ in having B instead of A as third premise of the only introduction rule, and, correspondingly, in having B instead of A as consequence of the third elimination rule:

$$\flat\mathbf{I}: \quad \frac{\begin{array}{c}[A]\\ B\end{array}\qquad \begin{array}{c}[B]\\ A\end{array}\qquad B}{A\flat B}\,\flat\mathrm{I}$$

$$\flat\mathbf{E}: \quad \frac{A\flat B \qquad A}{B}\,\flat\mathrm{E}_1 \qquad \frac{A\flat B \qquad B}{A}\,\flat\mathrm{E}_2 \qquad \frac{A\flat B}{B}\,\flat\mathrm{E}_3$$

These two collections of rules also obey $\mathtt{T}$-inversion and thus β- and η-equations that follow the same pattern of those of $\sharp$ are available. Using them, it easy to show that $A\sharp B$ is $\beta\eta$-isomorphic to $((A \supset B) \wedge (B \supset A)) \wedge B$ in $\mathtt{NI}^{\wedge\supset\flat}$, the extension of $\mathtt{NI}^{\wedge\supset}$ with $\flat\mathbf{I}$, $\flat\mathbf{E}$.

It is moreover easy to see that in the calculus consisting of $\sharp\mathbf{I}$, $\sharp\mathbf{E}$, $\flat\mathbf{I}$, $\flat\mathbf{E}$ we have that $A\sharp B \dashv\vdash A\flat B$ and $A\sharp B \not\simeq A\flat B$. To establish the latter fact it suffices to adapt to this system the interpretation of $\mathtt{NI}^{\wedge\supset}$ in the category of finite sets (see above Sect. 2.6), by interpreting $A\sharp B$ and $A\flat B$ as the sets of triples whose first two members are functions from the set interpreting A to that interpreting B and vice versa, and whose third members are elements of the interpretation of A and of B respectively. Whenever A and B are interpreted on finite sets of different cardinalities so are $A\sharp B$ and $A\flat B$.

Table 3.3 β and η-equations for $\sharp$

$$\dfrac{{\scriptstyle\langle u,v\rangle}\,\dfrac{\begin{matrix}\overset{u}{[A]} & \overset{v}{[B]} & \\ \mathscr{D}_1 & \mathscr{D}_2 & \mathscr{D}_3 \\ B & A & A\end{matrix}}{A\sharp B}\,{\scriptstyle\sharp\mathrm{I}} \qquad \begin{matrix}\mathscr{D}_4\\ A\end{matrix}}{B}\,{\scriptstyle\sharp\mathrm{E}_1} \quad \overset{\sharp\beta_1^{\mathrm{T}}}{\rhd} \quad \begin{matrix}\mathscr{D}_4\\ [A]\\ \mathscr{D}_1\\ B\end{matrix}$$

$$\dfrac{{\scriptstyle\langle u,v\rangle}\,\dfrac{\begin{matrix}\overset{u}{[A]} & \overset{v}{[B]} & \\ \mathscr{D}_1 & \mathscr{D}_2 & \mathscr{D}_3 \\ B & A & A\end{matrix}}{A\sharp B}\,{\scriptstyle\sharp\mathrm{I}} \qquad \begin{matrix}\mathscr{D}_4\\ B\end{matrix}}{A}\,{\scriptstyle\sharp\mathrm{E}_2} \quad \overset{\sharp\beta_2^{\mathrm{T}}}{\rhd} \quad \begin{matrix}\mathscr{D}_4\\ [B]\\ \mathscr{D}_2\\ A\end{matrix}$$

$$\dfrac{{\scriptstyle\langle u,v\rangle}\,\dfrac{\begin{matrix}\overset{u}{[A]} & \overset{v}{[B]} & \\ \mathscr{D}_1 & \mathscr{D}_2 & \mathscr{D}_3 \\ B & A & A\end{matrix}}{A\sharp B}\,{\scriptstyle\sharp\mathrm{I}}}{A}\,{\scriptstyle\sharp\mathrm{E}_3} \quad \overset{\sharp\beta_3^{\mathrm{T}}}{\rhd} \quad \begin{matrix}\mathscr{D}_3\\ A\end{matrix}$$

$$\dfrac{{\scriptstyle\langle u,v\rangle}\,\dfrac{\begin{matrix}\mathscr{D}\\ A\sharp B\end{matrix} \quad \overset{u}{A}}{B}\,{\scriptstyle\sharp\mathrm{E}_1} \qquad \dfrac{\begin{matrix}\mathscr{D}\\ A\sharp B\end{matrix} \quad \overset{v}{B}}{A}\,{\scriptstyle\sharp\mathrm{E}_2} \qquad \dfrac{\begin{matrix}\mathscr{D}\\ A\sharp B\end{matrix}}{A}\,{\scriptstyle\sharp\mathrm{E}_3}}{A\sharp B}\,{\scriptstyle\sharp\mathrm{I}} \quad \overset{\sharp\eta^{\mathrm{T}}}{\lhd} \quad \begin{matrix}\mathscr{D}\\ A\sharp B\end{matrix}$$

(with u and v fresh for $\mathscr{D}$)

3.10 Harmony by Interderivability is Not Intensional

To show the limit of harmony by interderivability we now consider a collection of rules for a third connective, we call it $\natural$, which is obtained by "crossing over" the collections of rules of $\sharp$ and $\flat$: The collection $\natural\mathbf{I}$ consists of the introduction rule obtained by replacing $\sharp$ with $\natural$ in $\sharp$I; and the collection $\natural\mathbf{E}$ consists of the elimination rules obtained by replacing $\flat$ with $\natural$ in $\flat E_1$, $\flat E_2$ and $\flat E_3$.

Clearly, $\natural\mathbf{I}$ and $\natural\mathbf{E}$ do not obey $\mathtt{T}$-inversion, due to the mismatch between the third premise A of the introduction rule and the consequence B of the third elimination rule. We list all of $\natural\mathbf{I}$, $\natural\mathbf{E}$, and $\mathtt{T}(\natural\mathbf{I})$:

$\natural\mathbf{I}$	$\natural\mathbf{E}$	$\mathtt{T}(\natural\mathbf{I})$
$\dfrac{\begin{matrix}[A] \\ B\end{matrix} \quad \begin{matrix}[B] \\ A\end{matrix} \quad A}{A\natural B}\,\natural\mathrm{I}$	$\dfrac{A\natural B \quad A}{B}\,\natural E_1$	$\dfrac{A\natural B \quad A}{B}\,\natural E_1$
	$\dfrac{A\natural B \quad B}{A}\,\natural E_2$	$\dfrac{A\natural B \quad B}{A}\,\natural E_2$
	$\dfrac{A\natural B}{B}\,\natural E_3$	$\dfrac{A\natural B}{A}\,\natural E_3^*$

Although $\natural\mathbf{E} \neq \mathtt{T}(\natural\mathbf{I})$, the two collections of rules are clearly interderivable. Hence, in spite of the fact that they do not obey $\mathtt{T}$-inversion, the two collections of rules $\natural\mathbf{I}$ and $\natural\mathbf{E}$ do qualify as in harmony by interderivability.

The question that we want to address now is the following: Can we define appropriate β- and η-equations for the derivations of a calculus $\mathtt{K}$ in which the rules governing $\natural$ are $\natural\mathbf{I}$ and $\natural\mathbf{E}$?

Whereas the β-reductions for local peaks generated by $\natural$I and $\natural E_1$ and $\natural E_2$ follow the pattern of those of $\sharp$ and $\flat$, one may doubt that a reduction for the peak generated by $\natural$I and $\natural E_3$ can be found. A moment of reflection however dispels this doubt, since one can come up with the following reduction:

$$\dfrac{\langle u, v\rangle\ \dfrac{\begin{matrix}\overset{u}{[A]} \\ \mathscr{D}_1 \\ B\end{matrix} \quad \begin{matrix}\overset{v}{[B]} \\ \mathscr{D}_2 \\ A\end{matrix} \quad \begin{matrix}\mathscr{D}_3 \\ A\end{matrix}}{A\natural B}\,\natural\mathrm{I}}{B}\,\natural E_3 \quad \overset{\natural\beta_3}{\rhd} \quad \begin{matrix}\mathscr{D}_3 \\ [A] \\ \mathscr{D}_1 \\ B\end{matrix}$$

This reduction shows that, in spite of the mismatch between the third premise of $\natural$I and the consequence of $\natural E_3$, this elimination rule allows one to derive *no more* than what is needed in order to infer its premise by the introduction rule.

Similarly, although the expansion pattern cannot simply be constituted by applications of the three elimination rules followed by an application of the introduction rule, the following η-expansion shows that what one gets from $A\natural B$ using the elimination rules is *no less* than what is needed to reintroduce $A\natural B$ by means of its introduction rule:

$$\langle u,v\rangle\ \dfrac{\dfrac{\begin{matrix}\mathscr{D}\\ A\natural B\end{matrix}\quad \overset{u}{A}}{B}\,\natural\mathrm{E}_1 \qquad \dfrac{\begin{matrix}\mathscr{D}\\ A\natural B\end{matrix}\quad \overset{v}{B}}{A}\,\natural\mathrm{E}_2 \qquad \dfrac{\begin{matrix}\mathscr{D}\\ A\natural B\end{matrix}\quad \dfrac{\begin{matrix}\mathscr{D}\\ A\natural B\end{matrix}}{B}\,\natural\mathrm{E}_3}{A}\,\natural\mathrm{E}_2}{A\natural B}\,\natural\mathrm{I} \qquad \overset{\natural\eta}{\lhd} \qquad \begin{matrix}\mathscr{D}\\ A\natural B\end{matrix}$$

In spite of the fact that these conversions show that the rules for $\natural$ satisfy the informal statement of harmony, they are inadmissible from the viewpoint of the intensional approach to inferentialism that we advocated.

To see why, consider the derivation obtained by expanding a given $\mathtt{K}$-derivation $\mathscr{D}$ of $A\natural B$ ending with an introduction rule. The form of $\mathscr{D}$ is the following:

$$\langle u_1,u_2\rangle\ \dfrac{\begin{matrix}\overset{u_1}{[A]}\\ \mathscr{D}_1\\ B\end{matrix}\qquad \begin{matrix}\overset{u_2}{[B]}\\ \mathscr{D}_2\\ A\end{matrix}\qquad \begin{matrix}\mathscr{D}_3\\ A\end{matrix}}{A\natural B}\,\natural\mathrm{I}$$

and that of the derivation $\mathscr{D}'$ obtained by $\natural\eta$-expanding $\mathscr{D}$ is depicted in Table 3.4. In such a derivation all occurrences of $A\natural B$ (apart from the conclusion) constitute local peaks. By β-reducing them we do not obtain the derivation $\mathscr{D}$ of which the derivation considered is an expansion, but instead the following:

$$\langle v_1,v_2\rangle\ \dfrac{\begin{matrix}\overset{v_1}{[A]}\\ \mathscr{D}_1\\ B\end{matrix}\qquad \begin{matrix}\overset{v_2}{[B]}\\ \mathscr{D}_2\\ A\end{matrix}\qquad \begin{matrix}\mathscr{D}_3\\ [A]\\ \mathscr{D}_1\\ [B]\\ \mathscr{D}_2\\ A\end{matrix}}{A\natural B}\,\natural\mathrm{I}$$

By symmetry and transitivity of the equivalence relation induced by the β- and η-conversions for $\natural$ we thus have the following equivalence:

$$\langle u_1,u_2\rangle\ \dfrac{\begin{matrix}\overset{u_1}{[A]}\\ \mathscr{D}_1\\ B\end{matrix}\qquad \begin{matrix}\overset{u_2}{[B]}\\ \mathscr{D}_2\\ A\end{matrix}\qquad \begin{matrix}\mathscr{D}_3\\ A\end{matrix}}{A\natural B}\,\natural\mathrm{I} \qquad \overset{\beta\eta}{\equiv} \qquad \langle v_1,v_2\rangle\ \dfrac{\begin{matrix}\overset{v_1}{[A]}\\ \mathscr{D}_1\\ B\end{matrix}\qquad \begin{matrix}\overset{v_2}{[B]}\\ \mathscr{D}_2\\ A\end{matrix}\qquad \begin{matrix}\mathscr{D}_3\\ [A]\\ \mathscr{D}_1\\ [B]\\ \mathscr{D}_2\\ A\end{matrix}}{A\natural B}\,\natural\mathrm{I}$$

This means that all instances of these two derivation schemata (obtained by replacing $\mathscr{D}_1$, $\mathscr{D}_2$ and $\mathscr{D}_3$ with actual derivations) pairwise belong to the same equivalence classes induced by β- and η-equations. This is problematic and the reason is that the derivations obtained by appending an application of $\natural\mathrm{E}_3$ to the conclusions of the previous ones will belong to the same equivalence classes as well:

Table 3.4 The $\natural\eta$-expansion of a derivation ending with $\natural$I

$$
\dfrac{
\dfrac{
\langle u_1, u_2\rangle\,\dfrac{\begin{matrix}\overset{u_1}{[A]} & \overset{u_2}{[B]} & \\ \mathscr{D}_1 & \mathscr{D}_2 & \mathscr{D}_3 \\ B & A & A\end{matrix}}{A\natural B}\,\natural\mathrm{I} \qquad \overset{v_1}{A}
}{B}\,\natural\mathrm{E}_1
\qquad
\dfrac{
\langle u_1, u_2\rangle\,\dfrac{\begin{matrix}\overset{u_1}{[A]} & \overset{u_2}{[B]} & \\ \mathscr{D}_1 & \mathscr{D}_2 & \mathscr{D}_3 \\ B & A & A\end{matrix}}{A\natural B}\,\natural\mathrm{I} \qquad \overset{v_2}{B}
}{A}\,\natural\mathrm{E}_2
\qquad
\dfrac{
\langle u_1, u_2\rangle\,\dfrac{\begin{matrix}\overset{u_1}{[A]} & \overset{u_2}{[B]} & \\ \mathscr{D}_1 & \mathscr{D}_2 & \mathscr{D}_3 \\ B & A & A\end{matrix}}{A\natural B}\,\natural\mathrm{I} \qquad \dfrac{\langle u_1, u_2\rangle\,\dfrac{\begin{matrix}\overset{u_1}{[A]} & \overset{u_2}{[B]} & \\ \mathscr{D}_1 & \mathscr{D}_2 & \mathscr{D}_3 \\ B & A & A\end{matrix}}{A\natural B}\,\natural\mathrm{I}}{B}\,\natural\mathrm{E}_3
}{A}\,\natural\mathrm{E}_2
}{A\natural B}\,\natural\mathrm{I}\ \langle v_1, v_2\rangle
$$

$$\dfrac{\langle u_1, u_2\rangle\ \dfrac{\begin{array}{c} \overset{u_1}{[A]} \\ \mathscr{D}_1 \\ B \end{array} \quad \begin{array}{c} \overset{u_2}{[B]} \\ \mathscr{D}_2 \\ A \end{array} \quad \begin{array}{c} \mathscr{D}_3 \\ A \end{array}}{A \natural B}\,\natural\mathrm{I}}{B}\,\natural\mathrm{E}_3 \quad \overset{\beta\eta}{\equiv} \quad \dfrac{\langle v_1, v_2\rangle\ \dfrac{\begin{array}{c} \overset{v_1}{[A]} \\ \mathscr{D}_1 \\ B \end{array} \quad \begin{array}{c} \overset{v_2}{[B]} \\ \mathscr{D}_2 \\ A \end{array} \quad \begin{array}{c} \mathscr{D}_3 \\ [A] \\ \mathscr{D}_1 \\ [B] \\ \mathscr{D}_2 \\ A \end{array}}{A \natural B}\,\natural\mathrm{I}}{B}\,\natural\mathrm{E}_3$$

By reducing using $\natural\beta_3$ on both side of this equivalence we obtain the following:

$$\begin{array}{c} \mathscr{D}_3 \\ [A] \\ \mathscr{D}_1 \\ B \end{array} \quad \overset{\beta\eta}{\equiv} \quad \begin{array}{c} \mathscr{D}_3 \\ [A] \\ \mathscr{D}_1 \\ [B] \\ \mathscr{D}_2 \\ [A] \\ \mathscr{D}_1 \\ B \end{array}$$

and in the limit case in which $\mathscr{D}_1$ and $\mathscr{D}_3$ simply consist of the assumption of some formula (in which case, $A = B = C$ for some C) these schemata boil down to the following:

$$C \quad \overset{\beta\eta}{\equiv} \quad \begin{array}{c} [C] \\ \mathscr{D}_2 \\ C \end{array}$$

This means that in presence of the reductions and expansion for $\natural$, for any formula C, any derivation from C to C is equated with the derivation consisting only of the assumption of C (i.e. the identity function on the set of proofs of C). But this means that in presence of $\natural$ any two interderivable formulas are also isomorphic, since the compositions of the proofs establishing that each is derivable from the other are equated to the identity functions (cf. Sect. 2.5 above).

In other words, the addition of $\natural$ to any calculus $\mathtt{K}$, even one containing interderivable but not isomorphic formulas, has the result of making formula isomorphism collapse onto interderivability.

We take this fact to show that the rules for $\natural$ equipped with the reductions and expansion discussed in this section should *not* qualify as harmonious, at least not on the intensional account of harmony we want to advocate. As we argued in Sect. 2.6, what is characteristic of this conception of harmony is that it is not formulated merely in terms of derivability conditions, but rather on the availability of certain proof-transformation that induce a notion of identity of proofs, which in turn makes room for the definition of the notion of isomorphism, that is of an equivalence relation on formulas which is finer grained than intederivability. Given the dramatic consequences for identity of proofs and isomorphism of the rules and equations associated with $\natural$, it seems that a genuine intensional account of harmony should disqualify these rules as genuinely harmonious.

This shows that, on an intensional account of harmony, in order for a collection of elimination rules to qualify as in harmony with a certain collection of introduction rules, one should require more than just its interderivability with the collection of elimination rules generated by inversion from the given collection of introduction rules. The notion of harmony as intederivability cannot be taken as a satisfactory definition of an intensional account of harmony.

Notes to This Chapter

1. It is worth observing already now that we are *not* assuming that given a collection of introduction rules, at most one collection of elimination rules can be in perfect harmony with it. We only assume that if a collection of elimination rules is in harmony with a given collection of introduction rules, a *necessary* condition for another collection of elimination rules to be also in harmony with that collection of introduction rules is that the two collections of elimination rules are interderivable. (For an exact definition of the notion of derivability of rules, see Appendix A below.) That this condition is not *sufficient* is suggested by the operator governed by the following (doubtly harmonious) rules:

$$\frac{A \qquad B}{A \otimes B}\,\otimes\mathrm{I} \qquad\qquad \frac{A \otimes B}{A}\,\otimes\mathrm{E}_1 \quad \frac{A \otimes B \qquad A}{B}\,\otimes\mathrm{E}_2$$

   A thorough discussion of this point will be the content of Sect. 3.10.
2. In Steinberger [102] terminology, the rules for quantum disjunction are a case of E-weak (or equivalently I-strong) disharmony.
3. Another difficulty with quantum disjunction consists in the fact that in a calculus containing both standard disjunction and quantum disjunction the two connectives are interderivable, i.e. they "collapse" into one another. It is true that the collapse of quantum disjunction into standard disjunction is clearly related to the problem of stability. However, at least according to Dummett (see [12], p. 290), this is "another interesting phenomenon illustrated by the restricted 'or'-elimination rule", rather than an illustration of the failure of stability itself. In fact, stability (in Dummett's sense at least) seems to be orthogonal to this kind of "collapses": An analogous collapse takes place if one considers a natural deduction calculus with both intuitionistic and classical negation, but this is usually (and certainly by Dummett) not taken as a reason for deeming intuitionistic negation as unstable. Conversely, as is shown by the example discussed in the next section, we may have instability (in Dummett's sense) without an analogous collapse taking place (see below Note 5). Although the point is certainly debatable, the given reconstruction is, at least, sound to Dummett's intent.
4. One of the referees objects that one might expect in the case of $\overline{\supset}$ that it is the no more aspect of harmony which is not met, since "$\overline{\supset}$E permits to draw *more*

conclusions from $A \mathbin{\overline{\supset}} B$ than we should be able to: applying $\overline{\supset}$E immediately after $\overline{\supset}$I results in a deduction of B from all kinds of assumptions, whereas $\overline{\supset}$I requires a deduction from A alone." [emphasis added]. We agree with the referee in pointing out that using $\overline{\supset}$E one can infer B from A only given further assumptions, namely $A \mathbin{\overline{\supset}} B$ (or, possibly, the assumptions on which $A \mathbin{\overline{\supset}} B$ depends). This is however "less" than what is needed to infer $A \mathbin{\overline{\supset}} B$ using the introduction rule, which is why we claim that it is the no less aspect of harmony the one which is not met. For now, that C is "less" than D can be understood in terms of deductive strength, i.e. in the sense in which for instance, $A \supset B$ is less than B, or $\Gamma, A \Rightarrow B$ is less than $A \Rightarrow B$: whenever B is derivable so is $A \supset B$ but not necessarily the other way around, and similarly whenever $A \Rightarrow B$ is derivable so is $\Gamma, A \Rightarrow B$ but not necessarily the other way around. The considerations to be developed in the final section of the chapter suggest however that in order to attain a genuinely intensional account of harmony a finer grained understanding of 'more' and 'less' might be required.

5. It is worth remarking that, contrary to what happens in the case of the two disjunctions, 'quantum like' implication does not collapse on standard implication in a calculus containing both connectives (although $A \supset B$ is implied by $A \mathbin{\overline{\supset}} B$, the converse is not true). As for Dummett the rules of 'quantum-like' implication give rise to a situation of instability analogous to the one of quantum disjunction, for him the collapse issue cannot be the heart of the problem (cf. Note 3 above).
6. Although Prawitz (see [65], Chap. 2) actually uses this term to refer to the "no more" aspect of the informal characterization of harmony given at the beginning of Sect. 1.3 above, our way of using the term is certainly in the spirit of Lorenzen [41], who coined it to refer to a particular principle of reasoning (whose role corresponds essentially to that of an elimination rule) which he obtained by "inverting" a certain collection of defining conditions for an expression (whose role corresponds essentially to that of introduction rules). For more details, see Moriconi and Tesconi [50].
7. The rules governing `PSH`-connectives are referred to by Kürbis (see [38], Sect. 2.8) 'rules of type 2'.
8. For a discussion on the origin of the terminology, see Schroeder-Heister [91].
9. For a definition of how the schemata

$$\begin{array}{ccc} \mathscr{D}_{k1} & & \mathscr{D}_{km_k} \\ [B_{k1}] & \dots & [B_{km_k}] \\ & \mathscr{D}'_k & \\ & C & \end{array} \qquad \text{and} \qquad \dfrac{B_{k1} \quad \dots \quad B_{km_k}}{\dagger}\,\dagger\mathrm{I}_k$$

are to be understood in the context of the calculus of higher-level rules see the Appendix.

10. Prawitz [65] restricts the rule $\bot$E to atomic conclusions and shows that in the variant of `NI` with restricted $\bot$E, all instances of the unrestricted rule are derivable. Normalization is then established for the modified system. As in the modified system no maximal segments beginning with $\bot$E can arise, Prawitz drops the

wording 'or of $\bot$E' from his definition of maximal segment. Details of normalization for the calculus with the unrestricted rule can be found in Troelstra and Schwichtemberg [122]. Gentzen's unpublished normalization result is also worked out using the unrestricted rule, see von Plato [64]. We stick to the unrestricted rule to stress the fact that the rules for $\bot$ in NI follow the pattern of Definition 3.1, i.e. that $\bot$ is a PSH-connective in NI.

11. The notions of *(1-step)* $\vee\gamma$*-conversion*, to be indicated with $\overset{(1)\vee\gamma}{\triangleright}$ are to be understood in analogy with those of (one-step) β-reduction as introduced in Sect. 1.4, and similarly for the other connectives.
12. Due to the fact that $\bot$E has no minor premises, the permutative conversions in this case do not exchange the order of application of rules, but simply erase some rule applications.
13. The notions of $\beta\gamma$-reduction and $\beta\gamma$-normal derivation are to be understood in analogy with those of $\beta\eta$-reduction and $\beta\eta$-normal derivation introduced in Sect. 1.5.
14. The development of formalisms for the full language of NI in which it is possible to single out a unique representative for each equivalence class of derivations is an active field of research. These formalisms rely on advanced proof-theoretic techniques, such as (multi-)focusing [82] or normalization by evaluation [1].
15. The rule $\supset$E$_g$ and $\supset$E$_{\texttt{PSH}}$ are equivalent in the sense defined in Appendix A. In other contexts, notably in constructive type theory, the higher-level rule is however essentially stronger than both $\supset$E$_g$ and $\supset$E (see [19]).
16. The natural deduction calculus for Tennant's [113] Core Logic is obtained from NI$_g$ by allowing only derivations in $\beta\gamma$-normal form, and additionally, by dropping the rule of $\bot$E altogether and imposing on the other rules some restrictions which enforce the relevance of the assumptions for the conclusion of the derivations. Observe that the two derivations of $(A \wedge C) \wedge (B \wedge D)$ from $A \wedge B$ and $B \wedge C$ discussed in the text are perfectly accetable core proofs for Tennant's standards. Thus the criticism raised against NI$_g$ applies to Tennant's system as well. We thereby do not call into question Tennant's claim that core proofs capture the core of derivability, in the sense that whenever A follows from Γ in NI (or NI$_g$) there is a subset of Γ from which either A or $\bot$ can be shown to follow by a derivation in Core Logic. What the remarks in this section however call in to question is whether the derivations in the natural deduction calculus for Core Logic are really capable of representing the proofs they denote in the most direct way possible.
17. Non-standard elements are of course ruled out by the second-order definition ("The set of natural numbers is the *smallest* set X such that…") which is however strictly stronger than the first-order formulations given by clauses (i)–(iii) or encoded by the introduction and elimination rules.
18. Clearly, the number of these functions is the product of the numbers m_k $(1 \leq k \leq r)$ of premises of each of the r introduction rules
19. As in the case of PSH-inversion, also JR-inversion generalizes straightforwardly to collections of introduction rules of higher-level, by taking the B_{kj} to be rules

rather than formulas (see Appendix A for details). Read [77] and Francez and Dyckhoff [18] initially suggested an inversion principle that they claimed could generate harmonious elimination rules of level ≤ 2 (i.e. possibly discharging formulas but not rules) for collection of introduction of level ≤ 2. As shown by Dyckhoff [13] and Schroeder-Heister [91], the need of elimination rules of higher-level is however unavoidable and Jacinto and Read [34] rectify these initial attempts by using higher-level rules unrestrictedly.

20. Here and below we consider the more general form of permutations in which arbitrary chunks of derivations (and not just applications of elimination rules) are permuted upwards, corresponding to the more general form of permutations for disjunction discussed at the end of Sect. 3.5.
21. I thank Paolo Pistone for his help in getting this right.
22. The same is true in the case of more complex collection of rules as well, such as $\odot\mathbf{I}$ and $\mathtt{R}(\odot\mathbf{I})$, although showing that the alternative expansion patterns are equivalent is, in general, a lot more laborious.
23. An exception is Kürbis, who defines the rules of a connective as harmonious if they follow either the pattern of $\mathtt{PSH}$-inversion or that of $\mathtt{T}$-inversion to be defined below in Sect. 3.9 (see [38], Sect. 2.8, and Notes 7 and 26 to the present chapter). Thus, on Kürbis definition, $\mathtt{JR}$-connectives are not harmonious, unless one ascribes him the implicit adoption of a notion of harmony by interderivability along the lines introduced in the present section.
24. For an exact definition of structural derivability and $\mathtt{K}$-derivability of rules, and of collections of rules, see Appendix, in particular Sect. A.5. Whenever not otherwise stated, by (inter-)derivability of (collections of) rules we will always understand *structural* (inter-) derivability.
25. Although a proof of this fact is still missing in the literature, its precise presentation would require the introduction of further notation (needed to keep track of the indexes occurring in the $\mathtt{R}$-elimination rules) and for this reason will be omitted.
26. The rules governing $\mathtt{T}$-connectives are referred to by Kürbis (see [38], Sect. 2.8) as 'rules of type 1'.

Part II
Paradox

Chapter 4
Paradoxes: A Natural Deduction Approach

Abstract The chapter introduces the Prawitz-Tennant analysis of paradoxes, according to which paradoxes are derivations of a contradiction which cannot be brought into normal form, due to "loops" arising in the process of reduction. After presenting Prawitz's original formulation of Russell's paradox, we introduce a simplified presentation of it, and then discuss the relevance of the difference between intuitionistic and classical logic and of structural properties of derivability for the Prawitz-Tennant analysis.

4.1 The Prawitz-Tennant Analysis of Paradoxes

In Appendix A to his monograph on natural deduction, Prawitz [65] considered a calculus for naive set theory, we will refer to it as $\mathtt{NI}^{\supset\bot\in}$, obtained by extending $\mathtt{NI}^{\supset\bot}$ (the $\{\supset, \bot\}$-fragment of $\mathtt{NI}$) with an introduction and an elimination rule for formulas of the form $t \in \{x : A\}$ to express set-theoretical comprehension:

$$\dfrac{A(t/x)}{t \in \{x : A\}}\in\mathrm{I} \qquad \dfrac{t \in \{x : A\}}{A(t/x)}\in\mathrm{E}$$

where $A(t/x)$ is the result of substituting t for x in A.

The rules certainly satisfy the informal statements of harmony (given on Sects. 1.2, 1.3 and 3.4) and thus it seems quite uncontroversial to regard these as being in harmony[1]: a reduction and an expansion can straightforwardly be defined as follows:

$$\dfrac{\dfrac{\begin{array}{c}\mathscr{D}\\ A(t/x)\end{array}}{t \in \{x : A\}}\in\mathrm{I}}{A(t/x)}\in\mathrm{E} \quad \overset{\in\beta}{\rhd} \quad \begin{array}{c}\mathscr{D}\\ A(t/x)\end{array} \qquad\qquad \begin{array}{c}\mathscr{D}\\ t \in \{x : A\}\end{array} \quad \overset{\in\eta}{\lhd} \quad \dfrac{\dfrac{\begin{array}{c}\mathscr{D}\\ t \in \{x : A\}\end{array}}{A(t/x)}\in\mathrm{E}}{t \in \{x : A\}}\in\mathrm{I}$$

Using $\neg A$ to abbreviate $A \supset \bot$, choosing $\neg(x \in x)$ for A and $\{x : \neg(x \in x)\}$ for t we have that

$$\begin{array}{rcl} A(t/x) & = \neg(\{x : \neg(x \in x)\} \in \{x : \neg(x \in x)\}) & =_{\mathrm{def}} \neg\rho \\ t \in \{x : A\} = & \{x : \neg(x \in x)\} \in \{x : \neg(x \in x)\} & =_{\mathrm{def}} \rho \end{array}$$

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and hence we obtain the following instances of $\in$I, $\in$E and $\in\beta$ (we indicate these instances as ρI, ρE and $\rho\beta$ respectively):

$$\dfrac{\neg\rho}{\rho}\,\rho\mathrm{I} \qquad\qquad \dfrac{\rho}{\neg\rho}\,\rho\mathrm{E} \qquad\qquad \dfrac{\dfrac{\begin{matrix}\mathscr{D}\\ \neg\rho\end{matrix}}{\rho}\,\rho\mathrm{I}}{\neg\rho}\,\rho\mathrm{E} \ \overset{\rho\beta}{\rhd}\ \begin{matrix}\mathscr{D}\\ \neg\rho\end{matrix}$$

Using them it is very easy to construct a closed derivation, we call it $\neg\mathscr{R}$, of $\neg\rho$:

$$\langle 1\rangle\ \dfrac{\dfrac{\dfrac{\overset{1}{\rho}}{\neg\rho}\,\rho\mathrm{E} \qquad \overset{1}{\rho}}{\bot}\,\supset\mathrm{E}}{\neg\rho}\,\supset\mathrm{I} \tag{$\neg\mathscr{R}$}$$

By a further application of the instance ρI of $\in$I, one obtains a closed derivation $\mathscr{R}$ of ρ as well:

$$\dfrac{\langle 1\rangle\ \dfrac{\dfrac{\dfrac{\overset{1}{\rho}}{\neg\rho}\,\rho\mathrm{E} \qquad \overset{1}{\rho}}{\bot}\,\supset\mathrm{E}}{\neg\rho}\,\supset\mathrm{I}}{\rho}\,\rho\mathrm{I} \tag{$\mathscr{R}$}$$

and combining these two derivations with an application of $\supset$E yields Russell's paradox in the form of the following derivation:

$$\dfrac{\langle 1\rangle\ \dfrac{\dfrac{\dfrac{\overset{1}{\rho}}{\neg\rho}\,\rho\mathrm{E} \qquad \overset{1}{\rho}}{\bot}\,\supset\mathrm{E}}{\neg\rho}\,\supset\mathrm{I} \qquad \dfrac{\langle 1\rangle\ \dfrac{\dfrac{\dfrac{\overset{1}{\rho}}{\neg\rho}\,\rho\mathrm{E} \qquad \overset{1}{\rho}}{\bot}\,\supset\mathrm{E}}{\neg\rho}\,\supset\mathrm{I}}{\rho}\,\rho\mathrm{I}}{\bot}\,\supset\mathrm{E} \tag{$\mathbf{R}$}$$

This situation confirms the paradoxical nature of ρ: We can produce two closed derivations of ρ and its negation respectively. This obviously leads to contradiction, i.e. to a closed derivation of $\bot$.[2]

Observe however that, since the encircled occurrence of $\neg\rho$ is a maximal formula occurrence, this derivation is not β-normal. By applying the implication reduction $\supset\beta$ we obtain the following derivation:

$$\dfrac{\dfrac{\dfrac{\langle 1\rangle\ \dfrac{\dfrac{\dfrac{\overset{1}{\rho}}{\neg\rho}\,\rho\mathrm{E} \qquad \overset{1}{\rho}}{\bot}\,\supset\mathrm{E}}{\neg\rho}\,\supset\mathrm{I}}{\rho}\,\rho\mathrm{I}}{\neg\rho}\,\rho\mathrm{E} \qquad \dfrac{\langle 1\rangle\ \dfrac{\dfrac{\dfrac{\overset{1}{\rho}}{\neg\rho}\,\rho\mathrm{E} \qquad \overset{1}{\rho}}{\bot}\,\supset\mathrm{E}}{\neg\rho}\,\supset\mathrm{I}}{\rho}\,\rho\mathrm{I}}{\bot}\,\supset\mathrm{E} \tag{$\mathbf{R}'$}$$

Here the encircled occurrence of ρ is a maximal formula occurrence. By an application of $\rho\beta$ we obtain the derivation **R** we started with.

At each of the two steps there was only a single possibility to reduce the derivation, hence no β-reduction sequence starting from **R** ends with a β-normal derivation. That is, weak normalization does not hold for β-reduction in $\mathtt{NI}^{\supset\perp\in}$, since the process of reduction of **R** gets stuck in what Tennant [109] called an "oscillating loop". Prawitz proposed this to be the distinctive feature of Russell's paradox.[3]

Tennant [109] considers a rather rich variety of both semantic and set-theoretic paradoxes besides Russell's—the Liar and some of its relatives, Grelling's, and Tarski's quotational paradox—and shows that once the assumptions required for their formulation are spelled out in terms of natural deduction rules, they all generate derivations of $\perp$ (or, as in the case of Curry's paradox, of an arbitrary atomic proposition) which do not normalize.[4]

The steps playing the role of $\in$I and $\in$E are called *id est inferences*, as they result from extra-logical principles: In the case of Russell's paradox, from set-theoretic comprehension. In the case of the liar paradox, to take another example, analogous *id est* inferences would be based on the observation that a certain sentence says of itself that it is not true. Here, "observation" is not necessarily empirical inspection, but may result from some arithmetical referencing mechanism.

Schroeder-Heister and Tranchini [95] dubbed the 'Prawitz-Tennant analysis of paradox' the thesis that a paradoxical derivation is a derivation of an "unwanted" sentence (such as 'Santa Claus exists' in the case of Curry's paradox, or $\perp$) that fails to normalize.[5]

4.2 A Simplified Presentation

In the sequel we will mainly focus on a simplified formulation of the paradoxes obtained by assuming ρ to be a nullary connective governed by the rules ρI and ρE. We refer to the extension of $\mathtt{NI}^{\supset\perp}$ with the rules for ρ as $\mathtt{NI}^{\supset\perp\rho}$. The derivations of the previous section will therefore be viewed both as derivations of $\mathtt{NI}^{\supset\perp\in}$ *and* as derivation of $\mathtt{NI}^{\supset\perp\rho}$, the context always making clear what is meant.[6]

When ρ is treated as primitive, the labels introduction and elimination in the case of ρI and ρE may appear odd at first, since ρ figures both in the premise and in the conclusion of the rule. The labels are however justified, as ρ is the main operator of the conclusion of the introduction rule and the main operator of the premise of the elimination rule. (In particular they are instances of the general form of introduction and elimination rules given in the Appendix).

Being a nullary connective, ρ behaves like a proposition which is interderivable with its own negation. As is well known, a proposition like ρ is definable not only in the language of naive set theory but in other settings as well. Examples are languages which allow to refer to their own expressions—as arithmetic does by means of Gödel numbering, or natural language does by means of quotes—and that contain a transparent truth predicate—that is a predicate T governed by the following inference rules:

$$\frac{A}{T\ulcorner A\urcorner}\ T\mathrm{I} \qquad \frac{T\ulcorner A\urcorner}{A}\ T\mathrm{E}$$

where $\ulcorner A\urcorner$ is a name of the sentence A.

The brute stipulation that ρ is governed by the two rules and by the reduction above permits one to disregard the exact conditions under which a sentence like it can be defined, and to focus on what is essential for the analysis of paradoxes to be developed in this part of the present work.

Different simplified versions of paradoxes such as the one we will consider have been discussed against the background of certain extensions of both natural deduction and sequent calculus. Schroeder-Heister [89] considers extensions of both sequent calculi and natural deduction by means of clausal definitions. In this context, paradoxes are typical examples of partial inductive definitions [29] such as the following one, in which an atom R (only a notational variant of ρ) is defined through its own negation:

$$\mathcal{D} = \{R \Leftarrow \neg R$$

Definitions are "put into action" by inference rules, which are justified by a principle of definitional closure (yielding introduction rules in natural deduction and right rules in sequent calculus) and a principle of definitional reflection (yielding elimination rules and left rules respectively). The natural deduction rules putting definition $\mathcal{D}$ into action are just the rules for ρ.

A different but related approach is deduction modulo [9], where a given set of rewriting rules (essentially corresponding to a definition in the sense of Schroeder-Heister) is viewed as inducing a congruence relations on propositions, modulo which deductive reasoning takes place.[7]

The analysis of paradoxes to be developed here is by no means in contrast to the one arising from these settings and it is meant as being, at least in principle, applicable to them as well, as well as to those in which paradoxes are defined by the usual subtler means. This is, however, a task which will be left for further work.

4.3 Which Background Logic?

All paradoxes investigated by Tennant yield a non-normalizing derivation of $\perp$ using only intuitionistically acceptable inference rules (beyond those specific for the paradox). Therefore he concludes his analysis stressing that

> it appears to me an open question whether every paradoxical set of sentences [...] can be shown to be paradoxical by means of an intuitionistic proof with a looping reduction sequence. (Tennant [109], p. 285)

The focus on intuitionistic logic of the Prawitz-Tennant analysis of paradoxes has been criticized by Rogerson [80], who considers a formulation of Curry's paradox in classical logic and observes that the derivation fails to display the loopy feature called for by the Prawitz-Tennant analysis. We consider a slight variation of Rogerson's proof based on Russell's rather than Curry's paradox.

In the presence of the classical rule of *reductio ad absurdum*

$$\dfrac{\begin{array}{c}[\neg A]\\ \bot\end{array}}{A}\,\mathrm{RAA}$$

the derivation of Russell's paradox **R** can be recast in a more symmetric fashion[8]:

$$\dfrac{\langle 1\rangle\,\dfrac{\dfrac{\dfrac{\overset{1}{\rho}}{\neg\rho}\,\rho\mathrm{E} \qquad \overset{1}{\rho}}{\bot}\,{\supset}\mathrm{E}}{\neg\rho}\,{\supset}\mathrm{I} \qquad \langle 1\rangle\,\dfrac{\dfrac{\overset{1}{\neg\rho} \qquad \dfrac{\overset{1}{\neg\rho}}{\rho}\,\rho\mathrm{I}}{\bot}\,{\supset}\mathrm{E}}{\rho}\,\mathrm{RAA}}{\bot}\,{\supset}\mathrm{E} \qquad (\mathbf{R}_{cl})$$

This derivation can be reduced by an application of $\supset\beta$-reduction to the following:

$$\dfrac{\dfrac{\langle 1\rangle\,\dfrac{\dfrac{\overset{1}{\neg\rho} \qquad \dfrac{\overset{1}{\neg\rho}}{\rho}\,\rho\mathrm{I}}{\bot}\,{\supset}\mathrm{E}}{\rho}\,\mathrm{RAA}}{\neg\rho}\,\rho\mathrm{E} \qquad \langle 1\rangle\,\dfrac{\dfrac{\overset{1}{\neg\rho} \qquad \dfrac{\overset{1}{\neg\rho}}{\rho}\,\rho\mathrm{I}}{\bot}\,{\supset}\mathrm{E}}{\rho}\,\mathrm{RAA}}{\bot}\,{\supset}\mathrm{E}$$

According to Rogerson, this derivation cannot be further reduced[9]:

> No standard reduction steps given by Prawitz [65] straightforwardly apply in this case as the use of the [set-forming] operator insulates the formulas from the normalization process. It seems plausible to conclude that this proof does not reduce to a normal form and does not generate a non-terminating reduction sequence in the sense of Tennant [109, 111]. Thus, Tennant's criterion for paradoxicality does not apply here. (This is not to say that it is inconceivable that someone might be able to define a reduction step applicable in this case that would induce a non-terminating reduction sequence.) (Rogerson [80], p. 174)

Although it is true that no standard reduction step given by Prawitz [65] applies to this derivation, it is also well known that the normalization strategy for classical logic devised there applies only to language fragments for which the application of RAA can be restricted to atomic conclusions. In richer languages, for example in languages containing disjunction and existential quantification, the conclusion of RAA cannot be restricted without loss of generality to atomic formulas, and in order for normal derivations to enjoy the subformula property a further (family of) reduction(s) has to be considered. This new reduction is based on the idea that the conclusion of an application of RAA which is also the major premise of an elimination rule counts as a redundancy to be eliminated. The reduction, proposed by Stålmark [100], can be depicted schematically as follows:

$$\dfrac{\langle n\rangle\,\dfrac{\begin{matrix}[\overset{n}{\neg A}]\\ \mathscr{D}\\ \bot\end{matrix}}{A}\,\text{RAA} \qquad \langle\mathscr{D}\rangle}{B}\,\dagger\text{E} \qquad \overset{\text{RAA}}{\rhd} \qquad \dfrac{\begin{matrix}\langle m\rangle\,\dfrac{\dfrac{\overset{n}{\neg B} \qquad \dfrac{\overset{m}{A} \qquad \langle\mathscr{D}\rangle}{B}\,\dagger\text{E}}{\bot}}{[\neg A]}\,\supset\text{I}\\ \mathscr{D}\\ \bot\end{matrix}}{B}\,\text{RAA}\;\langle n\rangle$$

where †E stands for an application of an elimination rule for some connective † belonging to the language fragment under consideration and $\langle\mathscr{D}\rangle$ stands for the (possibly empty) list of derivations of the minor premises of the application of †E.

In the language of naive set theory, the presence of the operators for the formation of set terms jeopardizes the notion of atomic sentence. Thus, a redundant conclusion of RAA is not always a non-atomic formula, but more generally any formula which can act as the major premise of an elimination rule. This makes it plausible to let, in the scheme for reduction proposed by Stålmarck, †E range over ρE as well. Once this is done, Rogerson's derivation can be further reduced, and the derivation after some steps reduces back to itself.[10]

Tennant formulated his criterion for paradoxicality with an emphasis on intuitionistic logic, by claiming that a paradoxical sentence is one governed by *id est* inferences such that, in the extension of intuitionistic logic obtained by this addition, there are derivations of $\bot$ that fail to normalize. As observed by Rogerson, the choice of intuitionistic logic is certainly motivated by the will of showing that non-constructive principles of reasoning do not play any significant role in the phenomenon of paradoxes. However, part of the reason for this choice is also the fact that the criterion for paradoxicality is formulated in terms of normalization, and intuitionistic logic (in its usual formulation at least) is well-behaved with respect to normalization. Given the crucial role played by normalization (not only from the formal, but also from the conceptual point of view), the 'base' calculus relative to which the non-normalizability effects of *id est* inference is to be tested must enjoy normalization.

Tennant may be wrong in restricting one's attention to intuitionistic logic, but we do not believe that extending the criterion beyond this logic is as problematic as Rogerson claims. For the case of classical logic, the above observations are sufficient to show that on a proper account of normalization for classical logic, Russell's (and Curry's paradox as well) display the looping effect called for by the Prawitz-Tennant analysis. Rogerson hints at other possible counterexamples, but, provided the logical frameworks in the background can be given a proper proof-theoretic presentation, Tennant's criterion should always be applicable.

In fact, the notion of *harmonious calculus* used to establish Fact 3 (see Sect. 1.6 above) provides some sufficient conditions for what should be meant by "a proper proof-theoretic presentation". Whenever the *id est* inferences display the kind of harmony needed in order for the resulting calculus to qualify as *harmonious*, Fact 3 warrants that every closed β-normal derivation ends with an introduction rule. Hence, if in a certain calculus a proposition has no introduction rules (as it is the case for $\bot$ in $\mathtt{NI}^{\supset\bot\rho}$), there cannot be closed β-normal derivations of it. Hence, by contraposition, if the calculus allows one to construct a closed derivation of a proposition without introduction rules, this derivation cannot be brought into β-normal form.

4.4 A Substructural Analysis

Current philosophical discussions on paradoxes have highlighted the key role played by "structural" properties of logical consequence, such a reflexivity, transitivity, monotonocity and contraction[11] in the derivations of contradictions within paradoxical languages, and proposed solutions of the paradoxes consisting in the rejection or restriction of one of these properties (see e.g. Zardini [126] for an overview of the different directions of investigations on the topic). The term 'structural' is used for these properties since the rules expressing them in sequent calculus (the identity initial sequents for reflexivity, the cut rule for transitivity, weakening for monotonicity) are commonly referred to as 'structural rules'.[12] Structural rules (like the properties they express) do not make reference to any particular piece of logical vocabulary, in contrast to the "operational" rules each of which governs a particular logical connective.

Given the correspondence between normalizability in natural deduction and the admissibility of the cut rule in sequent calculus, the Prawitz-Tennant analysis of paradoxes is closely connected to the analysis that focuses on the role played by the cut rule in sequent calculus. This has been advanced in the last decade by several authors (notably by Ripley [79], who adapted ideas going back to Girard [23, 24] to a language equipped with a transparent truth predicate).

Whereas the connection between cut and transitivity of logical consequence is obvious, the one between transitivity and normalization might require a short explanation. The transitivity of derivability is hard-wired in the natural deduction setting: In (almost)[13] any calculus of natural deduction, given a derivation $\mathscr{D}$ of B from A and another one $\mathscr{D}'$ of C from B, there is obviously one of C from A, namely the one resulting by plugging $\mathscr{D}$ in place of the open assumptions of $\mathscr{D}'$. However, it is the normalization theorem which warrants transitivity at the level of *normal* derivability in a given calculus: Given a β-*normal* derivation $\mathscr{D}$ of B from A and another one $\mathscr{D}'$ of C from B, the derivation resulting by plugging $\mathscr{D}$ in place of the open assumptions of $\mathscr{D}'$ is not necessarily normal (in the resulting derivation one or more occurrences of B could be maximal). It is only if normalization holds that we have the warrant that there is also a β-normal derivation of C from A. When normalization fails, as in the calculus $\mathtt{NI}^{\supset\bot\rho}$ there is no such warrant. In fact, a counterexample to the transitivity of β-normal derivability is easily obtained by considering the derivation $\neg\mathscr{R}$ and the one that resulting by "removing" from **R** the two occurrences of the subderivation $\neg\mathscr{R}$:

$$\dfrac{\neg\rho \qquad \dfrac{\neg\rho}{\rho}\,\rho\mathrm{I}}{\bot}\,\supset\mathrm{E}$$

This derivation of $\bot$ from $\neg\rho$ is β-normal, and so is the derivation $\neg\mathscr{R}$. However, by combining together the two derivations one obtains a derivation of $\bot$ from no undischarged assumption (viz. **R**) which is not β-normal and that does not reduce to a β-normal derivation. In fact, as observed above, in $\mathtt{NI}^{\supset\bot\rho}$ there cannot be any closed β-normal derivation of $\bot$ as a consequence of Fact 3.

Another structural rule of sequent calculus which plays an essential role in connection to paradoxes is contraction.[14] In the natural deduction setting, contraction corresponds to the possibility of discharging more than one copy of an assumption at once. Dropping contraction in sequent calculus corresponds to restricting $\supset$I by allowing at most one copy of an assumption to be discharged at a time in natural deduction. In the modified calculus, the non-normalizing derivation **R** is blocked, since it is impossible to derive $\bot$ using the rules of ρ and the 'restricted' implication rules. Actually it would not even be possible to derive either of ρ or $\neg\rho$. Moreover, all derivations would normalize (without contraction, both $\supset\beta$ and $\rho\beta$ make the size of the derivations (i.e. the number of applications of inference rules in a derivation) decrease, therefore one can show normalization to terminate by induction on the size rather than on the number of maximal formulas of maximal degree), i.e. transitivity would be fully restored.

That contraction is an essential ingredient for triggering Russell's paradox has already been observed by Fitch [16] who initiated the investigations of contraction-free logical settings. Formal attempts at developing naive set theory on a contraction-free base are due to Grišin [27], Girard [25] and Petersen [57]. More recently Zardini [125] investigated the possibility of developing a theory of truth in a contraction-free setting (therefore addressing also semantic paradoxes).

Contraction-free approaches to paradoxes are of great interest, but they will not be pursued in the present work. Given the semantic role of reduction in PTS, it seems that approaches to paradoxes focusing on normalization failure have the best prospects of being integrated into the PTS picture developed in Chap. 2.

Notes to This Chapter

1. For some authors, in order for a collection of rules to qualify as harmonious, it is essential that the introduction rules satisfy Dummett's complexity condition (see the remarks following Definition 2.2 in Sect. 2.9). Hence they would deny the harmony of these rules on these grounds. Note however that not only "problematic" rules such as $\in$E and $\in$E, but also the rules for e.g. the second-order existential quantifier (see Troelstra and Schwichtemberg [122], Chap. 11) would fail to qualify as harmonious on these grounds.
2. It is worth stressing that the nullary connective $\bot$ counts as a contradiction only if some sort of explosion principle is associated with it. This is the case in $\mathtt{NI}^{\supset\bot\in}$, where $\bot$E ensures that $\bot$ is indeed something "bad".
3. Observe that the above analysis as well as the considerations to be developed below apply if we replace the notions of β-reduction and β-normal derivation

with the notions of weak β-reduction and β_w-normal derivation (discussed in Sect. 2.7).

4. Instead of looping reduction sequences one can, more generally, consider non-terminating reduction sequences, which covers paradoxes such as Yablo's (see Tennant, [111]). In the following, we will throughout speak of the *looping feature* of paradoxical derivations, keeping in mind that "non-termination" of reduction sequences is the appropriate more general term.
5. In more recent work Tennant (see [112, 113, 115]) has proposed a different proof-theoretic analysis of Russell's paradox. On the revised analysis (which will be presented in some detail in Sect. 6.5 of Chap. 6), instead of a non-normalising derivation of $\bot$ from no assumption, one obtains a normal derivation of $\bot$ from the assumption that Russell's term $\{x : \neg x \in x\}$ possesses a denotation, i.e. a disproof of the existence of the set of all sets that do not belong to themselves. On these grounds, Tennant argues that the proof-theoretic anaysis of paradoxes aligns with Ramsey famous distinction between semantic and set-theoretic paradoxes. On Tennant revised view only semantic paradoxes give rise to non-normalizing derivations of $\bot$, whereas set-theoretic "paradoxes" merely yield disproofs of inconsistent assumptions, and hence do not qualify as paradoxes at all.
6. In the context of constructive type theory, we may introduce ρ directly by stipulating the following formation, introduction, elimination and equality rules (where ! stands for the operation associated with the introduction rule, and ¡ for the inverse operation to be associated with the elimination rule, whose meaning will be informally explained in Sect. 5.8):

Formation rule:
$$\dfrac{}{\rho\ \boldsymbol{set}}$$

Introduction rule: $\qquad$ Elimination rule:
$$\dfrac{t : \neg\rho}{!t : \rho} \qquad\qquad \dfrac{t : \rho}{\text{¡}t : \neg\rho}$$

Equality rule:
$$\dfrac{t : \neg\rho}{\text{¡}!t \equiv t : \neg\rho}$$

The derivations $\neg\mathscr{R}$ and $\mathscr{R}$ of $\neg\rho$ and of ρ are decorated (respectively) as follows:

$$\langle 1\rangle\ \dfrac{\dfrac{\dfrac{\overset{1}{x : \rho}}{\text{¡}x : \neg\rho}\,{\in}\mathrm{E} \qquad \overset{1}{x : \rho}}{\mathtt{app}(\text{¡}x, x) : \bot}\,{\supset}\mathrm{E}}{\lambda x.\mathtt{app}(\text{¡}x, x) : \neg\rho}\,{\supset}\mathrm{I} \qquad\qquad \langle 1\rangle\ \dfrac{\dfrac{\dfrac{\dfrac{\overset{1}{x : \rho}}{\text{¡}x : \neg\rho}\,{\in}\mathrm{E} \qquad \overset{1}{x : \rho}}{\mathtt{app}(\text{¡}x, x) : \bot}\,{\supset}\mathrm{E}}{\lambda x.\mathtt{app}(\text{¡}x, x) : \neg\rho}\,{\supset}\mathrm{I}}{!\lambda x.\mathtt{app}(\text{¡}x, x) : \rho}\,{\in}\mathrm{I}$$

That is, the two derivations $\neg\mathscr{R}$ and $\mathscr{R}$ correspond to the two closed β-normal terms decorating their respective conclusions:

$$\lambda x.\mathtt{app}(¡x, x) : \rho \supset \bot \qquad\qquad !\lambda x.\mathtt{app}(¡x, x) : \rho$$

and the derivation **R** corresponds to the following term:

$$\mathtt{app}(\ \lambda x.\mathtt{app}(¡x, x)\ ,\ !\lambda x.\mathtt{app}(¡x, x)\ )\ \ :\ \ \bot$$

whose loopy reduction can be written as:

$$\mathtt{app}(\ \lambda x.\mathtt{app}(¡x, x)\ ,\ !\lambda x.\mathtt{app}(¡x, x)\ )\ \ \lhd \rhd\ \ \mathtt{app}(\ ¡!\lambda x.\mathtt{app}(¡x, x)\ ,\ !\lambda x.\mathtt{app}(¡x, x)\ )$$

The reader familiar with untyped λ-calculus will easily recognize that **R** is just a typed version of the well-known loopy combinator $(\lambda x.xx)(\lambda x.xx)$ (cf. also Schroeder-Heister [89, Sect. 4]) and Note 7 below.

7. The two approaches are closely related, and their main difference is manifested when one considers the type theories that can be associated with them: In definitional reflection a type-constructor—yielding a term typed by the *definiens* (the head of the clause) when applied to terms typed by the *definiendum* (the body of the clause)—is associated with each clause of the definition together with an inverse operation of type annihilation for the given defined atom. On the other hand, in deduction modulo the types of the definiens and of the definiendum are just identified modulo the congruence induced by the set of rewriting rules. In this case there is thus no explicit operation on terms corresponding to the definitional steps. In both settings the proof-terms which are associated with the above derivation **R** of $\bot$ do not normalize. In deduction modulo, this term is just the same as the non-normalizing term $(\lambda x.xx)(\lambda x.xx)$ well-known from untyped λ-calculus. In the definitional setting the term has the somewhat more richer structure described in Note 6, due to the extra term-constructors associated with the definitional steps.
8. By 'symmetric' we mean that the two immediate subderivations of $\mathbf{R}_{cl}$ can be obtained from each other by replacing occurrences of ρ with occurrences of $\neg\rho$ (and vice versa) and by switching the order of the premises of ($\supset$E).
9. Although Rogerson speaks of a derivation based on Curry's paradox, the derivation we discuss can be viewed as obtained from the last derivation on p. 174 of [80] by replacing $a \in a$ with ρ and p with $\bot$, and moreover by (i) removing in both main subderivations redundant applications of RAA, i.e. applications allowing one to pass from $\bot$ to $\bot$ with no discharge; (ii) η-reducing in both subderivations $\dfrac{\langle 1 \rangle\ \dfrac{\neg\rho \qquad \overset{1}{\rho}}{\bot}}{\neg\rho}$ to $\neg\rho$. The considerations in this section apply exactly to Rogerson's original derivation as well.

10. Provided that, as usual, one also reduces redundant applications of RAA (see Note 9 above). Otherwise, one ends up with the more general kind of non-termination mentioned in Note 4.
11. When logical consequence is assumed (as it is usually done) to hold between sets the rule of contraction does not correspond to any distinguished property of logical consequence. Not so when consequence is understood as holding between multi-sets of formulas, in which case (if desired) it has to be explicitly formulated.
12. The term 'structural' is sometimes used only for the rules of contraction and weakening but we will follow here the nowadays more common usage (in the philosophical community) of referring also to cut and identity initial sequents as structural rules.
13. A notable exception is Tennant's calculus for Core Logic, in which only derivations in normal form are admitted (see Note 16 to Chap. 3). As a result of this restriction (and of other peculiarities of Core Logic), transitivity does not hold in general: given a derivation of B from Γ and one of C from B, Δ there might not be a derivation of C from Γ, Δ (although there is one of either C or $\perp$ from a subset of Γ, Δ.
14. This is true when sequents are defined as pairs of *multi-sets* or of *sequences* of formulas (whereby, in a single-conclusion setting, the second element of sequents is just a single formula). When sequents are taken to be pairs of *sets* of formulas, contraction becomes an implicit feature of sequent calculus, and thus it plays no explicit role in paradoxical derivations.

Chapter 5
Validity, Sense and Denotation in the Face of Paradoxes

Abstract Which modifications does the account of PTS developed in the second chapter need to undergo for it to be applicable to languages containing paradoxical expressions? We argue that one of the basic tenets of Prawitz-Dummett PTS—namely, the definition of the correctness of an inference as validity preservation—must be given up. As a result, the proposed account of PTS is enriched by introducing a notion of sense alongside the one of denotation. Paradoxical derivations are shown to act as the proof-theoretic analog of singular terms endowed with sense but lacking a denotation. The question of which class of derivations should be regarded as having a denotation is reconsidered, and we show the consequences of the choice of different criteria of identity of proofs for the analysis of languages containing paradoxical expressions.

5.1 Paradoxes as Non-denoting Derivations

The Prawitz-Tennant analysis of paradoxes is a way to characterize paradoxes by their proof-theoretic behavior, looking at the derivation of absurdity generated. Although this is not *per se* a solution to the paradoxes and Tennant stresses is should not be meant as such (see e.g. [109], p. 268) it can be naturally turned into a solution. This is implicitly suggested by both Prawitz and Tennant who write:

> In other words, the set-theoretical paradoxes are ruled out by the requirement that derivations shall be normal. (Prawitz [65], p. 95)

and

> The general loss of normalisability, confined as it is according to our conjecture above to just the paradoxical part of the semantically closed language, is a small price to pay for the protection it gives against paradox itself. Logic plays its role as an instrument of knowledge only insofar as it keeps proofs in sharp focus, through the lens of normality. Normalisability, in the context of semantically closed languages, is not to be pressed as a general pre-condition for the very possibility of talking sense; rather, normality of proof is to be pressed as a general pre-condition for the very possibility of telling the truth. (Tennant [109], p. 284)

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Neither Prawitz nor Tennant did develop these remarks any further.[1] However, the idea that requiring derivations to be normalisable can rule out paradoxes seems to fit, and induce a refinement of, the conception of derivations as linguistic representations of proofs that we developed in Chap. 2.

The relationship of derivations to proofs has been there developed by closely following an analogy with the relationship between numerical terms and numbers. The analogy can be further extended by taking into account an additional element. In the case of numerical expressions, nothing prohibits the possibility of considering languages which allow the formation of non-denoting expressions, such as '5 : 0', or expressions obtained using a definite description operator, such as 'the greatest even number'. Clearly, there is no way of rewriting expressions such as these onto a numeral. Given that numerals represent numbers in the most direct way, in a language for arithmetic allowing for the formation of non-denoting expressions, that a numerical expression denotes a number means that it can be rewritten into a numeral.

As numeral are the most direct expressions denoting numbers, in Sect. 2.8 we argued that in harmonious calculi closed normal derivations can be regarded as the most direct way of denoting proofs (although we stressed that there are different options as to what exactly 'normal' should be taken to mean). By analogy with the arithmetical case, that a derivation denotes a proof can be taken to mean that it can be rewritten into a normal derivation.

As we stressed in Sect. 2.8, the claim that normal derivations in harmonious calculi can be regarded as the most direct way of denoting proofs is backed by Fact 3 (see Sect. 1.7), i.e. by the fact that in harmonious calculi every closed normal derivation ends with an introduction rule. (As we stressed in Sect. 2.7, Fact 3 holds not only for β-normal—and hence also for $\beta\eta$-normal—derivations, but for β_w-normal derivations as well).

This suggests to take as a necessary (though not necessarily sufficient) condition for a closed derivation $\mathscr{D}$ of an *arbitrary* calculus (i.e. one that need not be harmonious) to "really" denote a proof, to be that $\mathscr{D}$ reduces to a canonical derivation.

Paradoxical derivations fail to normalize, thus they act as the proof-theoretic analogue of non-denoting singular terms: a paradoxical derivation such as the derivation **R** of Russell's paradox (see Sect. 4.1 above) fails to denote a proof, that is it lacks a denotation. This squares very well with the above remarks by Prawitz and Tennant. In Prawitz's calculus for naive set theory $\mathtt{NI}^{\supset\bot\in}$, or in its simplified relative $\mathtt{NI}^{\supset\bot\rho}$, although there are derivations of $\bot$, there is no closed *canonical* derivation of it. The reason is simply that there are no introduction rules for $\bot$ and thus all derivations of $\bot$ in Prawitz's calculus for naive set theory cannot be canonical nor reducible to a canonical derivation.

Hence, any such derivation will fail to denote a proof. This squares with the BHK interpretation as well, which tells us that $\bot$ is the proposition of which there is no proof!

5.2 Non-denoting Derivations and (In)validity

In Chap. 2, we presented PTS in a two-fold manner, either as being based on the idea that derivations denote proofs, or as based on the specification of a validity predicate. The intuitive connection between the two ways of presenting PTS is that a derivation is called ‘valid’ iff it denotes a proof.

In truth-conditional semantics, truth has to be a distinguishing feature of some but not all propositions if the semantics is to be of any interest at all. Analogously, validity-based presentations of PTS stress that validity should be a distinguishing feature of some but not all derivations, if PTS is to be of any interest at all.

To show that validity applies only to some, but not all derivations, Prawitz and Dummett consider derivations built up not just from the inference rules of a specific calculus, but rather from “arbitrary” inference rules (see Sect. 2.9 for details). Among these rules, one also finds rules which are intuitively not correct, such as the following:

$$\frac{A \supset B}{A}$$

By admitting non-correct inference rules, they can then distinguish between valid and invalid derivations, the latter being those in which non-correct inference rules have been applied. Note however, that Prawitz and Dummett do not define the validity of a derivation as its being constituted by applications of correct inference rules. Rather, it is the correctness of an inference that is defined in terms of the validity of the derivations in which it is applied (see Sects. 2.10 and 2.11).

In the present chapter we will argue that paradoxical languages offer another setting in which validity can be shown to apply only to some, but not all derivations available. In the previous section, building on remarks by Prawitz and Tennant we argued that paradoxical derivations do not denote proofs. As the validity predicate is meant to apply to a derivation iff this denotes a proof, paradoxical derivations in systems such as those discussed in the previous chapter are therefore the natural candidate for non-valid derivations.

As we will show in the next section, the notion of validity as defined by Prawitz cannot however be meaningfully applied to derivations of a calculus such as $\mathtt{NI}^{\supset\bot\rho}$. The main task of the present chapter will be that of proposing an alternative definition of validity suitable to be applied to the derivations of such a calculus. Rather than using validity of derivations to define correcntess of rules (as done by Dummett and Prawitz), the notion of validity to be proposed will rely on the notion of correctness of rules, which will therefore need to be defined first.

Since the analysis of paradoxes presented in the previous chapter goes back to Prawitz [65] himself, it might be somewhat surprising that his definition of validity cannot be applied to derivations in $\mathtt{NI}^{\supset\bot\rho}$. It should however be remarked that Prawitz never came back to discuss his treatment of paradoxes in any of his later writings. Similarly, considerations on the set-theoretic and semantic paradoxes play only a marginal role in Dummett’s writings. Thus, when framing the notion of validity, they were not in the least interested in having a notion that could be applied to paradoxical derivations.

5.3 The Need of Revising Prawitz's Validity

In this section, as well as later on in the chapter, we will repeatedly refer to the derivations $\neg\mathscr{R}$ and $\mathscr{R}$ of $\neg\rho$ and ρ as well as to the two derivations $\mathbf{R}$ and $\mathbf{R}'$ of $\bot$ from the previous chapter (the reader can find them in Sect. 4.1).

On Prawitz's definition of validity (see Definition 2.2 in Sect. 2.9), we immediately see that the closed derivation $\mathbf{R}$ of $\bot$ in $\mathtt{NI}^{\supset\bot\in}$ or $\mathtt{NI}^{\supset\bot\rho}$ fails to qualify as valid, since it cannot be reduced to a canonical derivation, not just relative to the set of reduction procedures $\mathcal{J}$ consisting of $\supset\beta$ and $\in\beta$, or $\rho\beta$, but on any extension of $\mathcal{J}$, as there is no canonical derivation of $\bot$ at all.

Troubles with Prawtiz's definition arise however when we consider the derivations $\neg\mathscr{R}$ and $\mathscr{R}$ of $\neg\rho$ and ρ. Although $\neg\mathscr{R}$ and $\mathscr{R}$ are both closed and canonical (and actually β-normal as well) if we try to evaluate their validity using the clauses of Prawitz's definition, we obtain contradictory results.

Consider the immediate subderivation of $\neg\mathscr{R}$:

$$\dfrac{\dfrac{\rho}{\neg\rho}\,\rho\mathrm{E} \qquad \rho}{\bot}\,\supset\mathrm{E} \tag{$*$}$$

Using Prawitz's definition, we can easily show that the derivation $(*)$ is valid relative to $\mathcal{J}$ and any arbitrary atomic system $\mathcal{S}$. Being an open derivation, it is valid relative to $\mathcal{J}$ and $\mathcal{S}$ iff for every closed derivation $\mathscr{D}$ of ρ that is valid relative to $\mathcal{J}' \supseteq \mathcal{J}$ and $\mathcal{S}$, the corresponding closed instance of $(*)$:

$$\dfrac{\dfrac{\begin{matrix}\mathscr{D}\\ \rho\end{matrix}}{\neg\rho}\,\rho\mathrm{E} \qquad \begin{matrix}\mathscr{D}\\ \rho\end{matrix}}{\bot}\,\supset\mathrm{E} \tag{$**$}$$

is valid relative to $\mathcal{J}'$ and $\mathcal{S}$ as well. The validity of $\mathscr{D}$ relative to $\mathcal{J}'$ and $\mathcal{S}$ means that $\mathscr{D}$ $\mathcal{J}'$-reduces to a valid canonical derivation, and hence that $(**)$ $\mathcal{J}'$-reduces to a derivation of the following form:

$$\dfrac{\dfrac{\dfrac{\langle n\rangle\,\dfrac{\begin{matrix}\overset{n}{[\rho]}\\ \mathscr{D}'\\ \bot\end{matrix}}{\neg\rho}\,\supset\mathrm{I}}{\rho}\,\rho\mathrm{I}}{\neg\rho}\,\rho\mathrm{E} \qquad \begin{matrix}\mathscr{D}\\ \rho\end{matrix}}{\bot}\,\supset\mathrm{E} \tag{$***$}$$

(where $\mathscr{D}'$ is valid relative to $\mathcal{J}'$ and $\mathcal{S}$). Now observe that $(***)$ β-reduces to:

$$\begin{matrix}\mathscr{D}\\ [\rho]\\ \mathscr{D}'\\ \bot\end{matrix}$$

which is warranted to be valid by the validity of $\mathscr{D}'$.

We have therefore shown that (∗)—i.e. the immediate subderivation of $\neg\mathscr{R}$—is valid relative to $\mathcal{J}$ and $\mathcal{S}$. Hence so are $\neg\mathscr{R}$ and, in turn, $\mathscr{R}$ as well (since these are closed canonical derivations with valid immediate subderivations).

Unfortunately, we can at this point easily establish the contradictory of these claims, namely that these derivations are also *not* valid. Consider again (∗∗) and look at what happens when we take the derivation $\mathscr{D}$ to be $\mathscr{R}$. We thereby obtain a particular closed instance of (∗), namely the derivation $\mathbf{R}'$ (see Sect. 4.2). As we know, $\mathbf{R}'$ does not reduce to canonical form, thus (∗) (and hence $\neg\mathscr{R}$ and $\mathscr{R}$ as well) are not valid.

The contradiction we arrived at has a clearly identifiable cause, namely the application of Prawitz's definition of validity to derivations of $\mathtt{NI}^{\supset\bot\in}$ and $\mathtt{NI}^{\supset\bot\rho}$. Prawitz's definition is by induction on the joint complexity of the conclusion and the undischarged assumptions of derivations. Thus, the induction underlying the definition is well-founded only when the introduction rules satisfy the following *complexity condition*: the consequence of any application of an introduction rule must be of higher logical complexity than its immediate premises and assumptions discharged by the rule application. This condition is not satisfied by either $\in$I or ρI. Hence, when applied to derivations of $\mathtt{NI}^{\supset\bot\in}$ or $\mathtt{NI}^{\supset\bot\rho}$ the definition has to be understood as a partial inductive definition in the sense of Hallnäs [29].[2]

This situation prompts to revise the definition of validity of Prawitz in order to be able to consistently apply it to the derivations of calculi such as $\mathtt{NI}^{\supset\bot\in}$ and $\mathtt{NI}^{\supset\bot\rho}$. We will in particular explore the prospects of defining a notion of validity on which

- as in Prawitz's definition, a necessary condition for a closed derivation to be valid is its reducing to a canonical derivation (and this is enough to rule out **R** as invalid);
- both $\neg\mathscr{R}$ and $\mathscr{R}$ qualify as valid.

This is not the only possible choice. One could for instance try to figure out a notion of validity on which not only **R**, but also both $\neg\mathscr{R}$ and $\mathscr{R}$ qualify as invalid. Such a notion of validity may be more congenial to the advocates of solutions to the problem of paradoxes based on the rejection of the structural rule of contraction (see Sect. 4.4). Yet a further option would be of course that of denying the acceptability of the introduction rule for ρ due to its violating the complexity condition, and thereby rejecting the idea of revising Prawitz's definition altogether. We take our choice to be the closest in spirit to the remarks of Tennant and Prawitz quoted at the beginning of this chapter. The derivations $\neg\mathscr{R}$ and $\mathscr{R}$ are normal and the remarks of Tennant and Prawitz seem to suggest that this is a sufficient condition to qualify as valid in a calculus like $\mathtt{NI}^{\supset\bot\rho}$.

The goal of this investigation is not that of arguing in favor of a formal system, such as that of naive set theory, in which normalization fails, but rather of bringing to light the assumptions that are needed to make sense of the core idea of proof-theoretic semantics (i.e. that a derivation is valid iff it denotes a proof) in the context of paradoxical languages. Whether such assumptions should be accepted or rejected, will be mainly left to the reader, but we will try to make clear what does the acceptance or rejection of these assumptions commit one to.

To regard both $\neg\mathscr{R}$ and $\mathscr{R}$ as valid is tantamount to commit oneself to accepting that both $\neg\rho$ and ρ have a proof, and thus to endorse a form of paraconsistency. As we will show, this gives rise to two distinct issues: the first concerns the way in which the BHK explanation should be extended, so as to provide an explanation of what a proof of ρ is; the second concerns the way in which the BHK clause for implication has to be understood in a paradoxical language.

Before giving an exact formulation of the alternative definition of validity and coping with these two issues, we will discuss a further problem arising by the adoption of a notion of validity on which both $\neg\mathscr{R}$ and $\mathscr{R}$ but not **R** qualify as valid: namely, that of a proper understanding of the correctness of rules.

5.4 The Local Correctness of an Inference

If we intend to revise the notion of validity along the lines envisaged, we immediately realize that we also need to revise the notion of correctness of an inference rule. Given the prospected revision of the notion of validity, the correctness of an inference rule cannot be any more defined as the fact that, given proofs of its premises, the rule yields a proof of its consequence: that would be too strong a requirement in the presence of paradoxes.

If we look again at **R**, we can observe that it is obtained by applying $\supset$E to two closed derivations that we want to regard as valid. As we want to deny the validity of **R**, we have that, on the revised notion of validity, by applying $\supset$E to two valid derivations one would obtain an invalid one. Thus this would be a case in which $\supset$E would not preserve validity.[3]

Hence, would we keep Prawitz's definition of correctness (see Definition 2.3 in Sect. 2.10)—according to which an inference is correct iff it yields closed valid derivations when applied to closed valid derivations—we would be forced to say that $\supset$E is *not* correct.

This we take as a reason to revise the notion of correctness as well. We do not want the adoption of the (currently only envisaged) notion of validity to force us to deny the correctness of $\supset$E: the proper diagnose for the fact that $\supset$E fails to preserve (the revised notion of) validity is not that $\supset$E is not correct but rather the presence of ρ. How can this intuition be spelled out?

As observed in Sect. 2.11, the availability of reduction procedures usually suffices to warrant the correctness of the elimination rules with which they are associated. It should now be clear that, when the language contains paradoxical expressions such as ρ, this is no more the case. To repeat, while in standard cases the existence of reduction procedures associated with the rule is enough to show that the rule preserves validity, this not so in general.

The problem with Prawitz and Dummett's definitions of validity and correctness is that they are too much tied to the standard cases. A way out of the problem is just to deny that preservation of validity is the right way of characterizing the correctness of rules. Although in standard cases correct rules do transmit validity, it is way too demanding to expect a correct rule to preserve validity in all cases.

The fact that the availability of reduction procedures for an inference rule suffices to warrant validity preservation tells us something important about the standard cases. However, from the fact that in general this does not happen, it should not follow that the availability of reduction procedures for an inference is not sufficient for the rule to be correct. Rather, it should only signal that we are not in a standard case.

The application of a reduction procedure "cuts away" two consecutive applications of an introduction rule and of an elimination rule. Plausibly, a necessary condition for "standardness" (i.e. for the availability of reduction procedures to be sufficient for the correctness of the rule in Prawitz's sense) is that all reduction procedures under consideration have the following property: The formulas which are conclusion of the application of the introduction and the major premise of the application of the elimination rule cut away by the application of the reduction procedures must be of a higher logical complexity than the formulas surrounding it. Clearly, $\rho\beta$ violates such a condition.[4]

As detailed in Sect. 2.10, given Prawitz and Dummett's definitions of validity and correctness, the correctness of different kinds of rules is shown in substantially different ways. The correctness of introduction rules is almost "automatic", and this reflects their "self-justifying" nature, i.e. the fact that they "define" the meaning of the logical constant involved (see in particular Note 22 to Chap. 2). Elimination rules, as well as other non-introductory inference rules, are shown to be correct by appealing to reduction procedures. There we stressed however that in the case of rules which are neither introduction nor elimination rules, reduction procedures boil down to derivations of the rules from the introduction and elimination rules. It is therefore natural to distinguish between the correctness of introduction rules (by definition), of elimination rules (which means that they are in harmony with the introduction rules) and of other rules (which means that they are derivable from introduction and elimination rules.

Rather than obtaining this articulation of correctness as a byproduct of the definition of correctness in terms of validity (as in Prawitz-Dummett PTS), we propose to use it as a direct definition of a notion of correctness (to distinguish it from Prawitz's definition, we refer to this notion as 'correctness*'):

Definition 5.1 (*Correctness* of an inference rule*) An inference rule schema is correct* iff

- It is an introduction rule;
- it is an elimination rule for † that belongs to the collection of elimination rules obtained by inversion from the collection of introduction rules for †.
- it is derivable from the introduction and elimination rules governing the expressions that occur in the rule.[5]

The adoption of Definition 5.1 instead of Prawitz's Definition 2.3 results in a better analysis of the situation. Even adopting a notion of validity on which both

$\neg\mathscr{R}$ and $\mathscr{R}$ count as valid and **R** as invalid, $\supset$E still qualifies as correct* since it is in harmony with $\supset$I.

It is true that $\supset$E does not preserve validity in all cases. But this is due to the presence of the paradoxical ρ (indeed, in standard cases $\supset$E *does* preserve validity).

5.5 Local Correctness Versus Global Validity

It may be retorted that the revision the definition of correctness according to Definition 5.1 has the drawback of forcing the rejection of the principle (V), according to which an argument is valid if it is constituted by correct rules (see Sect. 2.11). All rules in **R** are correct* but we want to deny that the argument as a whole is valid.

As we saw, although Prawitz rejected (V) as a definition of validity, he stressed that the principle still holds under his definitions of validity and correctness. On the contrary, by replacing Prawitz's Definition 2.3 with Definition 5.1 we are forced to give up principle (V).

Is this really an unwanted consequence? I do not think so.

The revised definition of the correctness of an inference makes the two notions of 'being valid' and 'being constituted by applications of correct* inference rules' diverge. As we saw, the semantic content of an argument being valid is its having a denotation. In the following, I will argue that, the notion of 'being constituted by applications of correct* inference rules' has also a genuine semantic content which, furthermore, should be kept distinct from that of having a denotation: Namely, having sense.

According to Dummett [11], what Frege called the sense of an expression is best understood as a procedure, i.e. a set of instructions, to determine its denotation. Without entering the details of the idea, it should be clear enough that in general, it may be the case that although one is in possession of a set of instructions to identify something, any attempt to carry out the instructions fails. We may refer to such a situation as one in which the set of instructions is inapplicable. The inapplicability may depend on there not being a something satisfying the conditions codified in the set of instructions. Or on factual contingencies (such as time and space limitations). But one may also conceive cases of, say, structural inapplicability of the instructions. That is, cases in which the instructions are shaped in such a way that one cannot successfully bring to the end the procedure they codify.

The core intuition underlying the notion of validity is that proofs are denoted by valid arguments. As we argued in Sect. 2.8, in harmonious calculi normal derivation can be seen as the most direct way of representing proofs. Although it is an abuse of language (since derivations and proofs belong to two distinct realms) it is tempting to "identify" proofs with normal derivations and thus, to view reduction to normal form as the process of interpreting derivations, i.e. as the process of ascribing them their denotation.

Dummett's model of sense perfectly applies to this picture: the set of instructions telling how a derivation is to be reduced to normal form is the set of instructions telling how to identify the denotation of the derivation, i.e. it is the sense of the derivation. In the case of **R**, we have a derivation that does not reduce to canonical form, it does not denote a proof, i.e. it lacks a denotation. However, as there is a reduction procedure associated with each of its elimination rules, we can say that also in this case there is a procedure to determine its denotation. Thus, there is a sense associated with the derivation. However, although each step of the procedure in which the sense of **R** consists can be carried out, it is not possible to bring the procedure to the end, due to its entering the oscillating loop.

The result of replacing Definition 2.3 with Definition 5.1 is thus that of making room, alongside the idea that derivations have a denotation, for the idea that they also have a sense, where having sense means that they are constituted by applications of inference rules which are correct*.

Taking 'having sense' as the semantic content of 'being constituted by applications of correct* inferences rules' and 'having a denotation' as the semantic content of 'being valid', we can express the failure of principle (V) as a feature of PTS: Namely that of allowing the existence of derivations endowed with sense which lack a denotation.

The alternative picture resulting from the adoption of Definition 5.1 returns an enlightening picture of paradoxes. Paradoxical derivations are not nonsense. On the contrary, what is paradoxical in them is exactly that they make perfectly sense. But putting this sense in action reveals their awkward features. For a derivation to be paradoxical, it must have sense. Its being paradoxical means that it does not denote a proof of its conclusion.

5.6 ρ Versus **tonk**

To really appreciate that the proposed distinction between sense and denotation is not an *ad hoc* solution to the problem of paradoxes, I believe it is worth comparing **R** with another kind of arguments for $\bot$, namely those that can be constructed in $\mathtt{NI}^{\supset\bot\mathtt{tonk}}$, the extension of $\mathtt{NI}^{\supset\bot}$ with tonk's rules. The following closed derivation of $\bot$ is obtained by replacing the atomic proposition p with $\bot$ in the derivation **T** discussed in Sect. 1.7:

$$\dfrac{\dfrac{\langle u\rangle \quad \dfrac{\overset{u}{A}}{A \supset A}\supset\mathrm{I}}{(A \supset A)\ \mathtt{tonk}\ \bot}\mathtt{tonkI}}{\bot}\mathtt{tonkE} \qquad (\mathbf{T}')$$

The common features of **R** and **T**′ are the following:

- they have $\bot$ as conclusion;
- they are both non-canonical, since they end with an elimination rule;
- they are both irreducible to canonical form.

Being both not reducible to canonical form, they both qualify as invalid, that is, they both fail to denote a proof.

Nonetheless, there is a crucial difference between **R** and **T**′. On the one hand, **R** cannot be reduced to canonical form because it is not normal and if one tries to apply the reduction procedures associated with its elimination rules one enters an oscillating loop. On the other hand, **T**′ does not reduce to canonical form because no reduction is (nor can be) associated with `tonk`'s rules, that is, because it is already a normal argument.

As the derivation **T** discussed in Sect. 1.7, **T**′ is normal, since there is no reduction procedure that can be applied to it. This of course does not depend on the set of reduction procedures one is considering because, under any plausible notion of reduction procedures, there cannot be a reduction procedure associated with patterns constituted by an application of `tonk` I followed immediately by an application of `tonk` E.

As observed, two are the features of `NI` which suggested the development of a semantics on proof-theoretic basis. The first one is that every derivation β-normalizes; the second is the canonicity of closed β-normal derivations. The additions of ρ and `tonk` to a well-behaving calculus (such as $\mathtt{NI}^{\supset\bot}$) pose two different kinds of problems. On the one hand, the addition of ρ blocks normalization, but does not threat the canonicity of closed β-normal derivations: Although not every argument normalizes (as exemplified by **R**), closed normal derivations always end with an introduction rule. On the other hand, in spite of the addition of `tonk` normalization still holds (**T**′ *is* normal since there is no reduction which can be applied to it); but the normality of **T**′ ruins the semantic significance of closed normal derivation, since it is no more the case that every closed normal derivation ends with an introduction.

Observe that although **R** is invalid, it is composed of applications of correct* inference rules. This is not the case for **T**′: as there is no reduction procedure associated with `tonk`'s elimination rule, the rule is neither correct nor correct*. In semantic terms, not only **T**′ (like **R**) lacks a denotation, but also a procedure to determine it, since we do not have any reduction procedure telling us how to transform it into canonical form. In other words, **T**′ not only lacks a denotation but also sense. An argument such as **T**′ thus fails to count as paradoxical. It is just nonsensical.

5.7 Paradox and Partial Functions

Considerations that are fully analogous to those that prompted the revision of the definition of correctness apply to the validity of open derivations.

On the revised (but so far only envisaged) notion of validity, on which both $\neg\mathscr{R}$ and $\mathscr{R}$ are to count as valid, we have that there are at least one proof of $\neg\rho$ and one of ρ. According to BHK a proof of $A \supset B$ is a function from proofs of A to proofs of B. As $\neg\rho$ is shorthand for $\rho \supset \bot$, a proof of $\neg\rho$ is a function from proofs of ρ to proofs of $\bot$.

The provability of both $\neg\rho$ and ρ together with the unprovability of $\bot$ forces the view that, in presence of paradoxical phenomena, the functions proving an implication must be understood as being sometimes partial. In particular, the proof of $\neg\rho$ denoted by $\neg\mathscr{R}$ is a function from proofs of ρ to proofs of $\bot$, but when we apply this function to the proof of ρ denoted by $\mathscr{R}$, we do not obtain a proof of $\bot$.

The derivation $\neg\mathscr{R}$ denotes a proof of $\neg\rho$ that is a function as course-of-value obtained by abstraction from the function as unsaturated entity denoted by the derivation $(*)$. Both the function as course-of-value and the function as unsaturated entity are however partial, in that they should yield proofs of $\bot$ when applied to proofs of ρ.

This situation brings to the fore the fact that, in the context of a paradoxical language, we cannot expect the validity of an open derivation to consist in the validity of its closed instances (i.e. that when its undischarged assumptions are replaced by closed valid derivations one obtains a closed valid derivation for the conclusion). In particular, the result of replacing the derivation $\mathscr{R}$ for the undischarged assumptions of $(*)$ yields the invalid derivation $\mathbf{R}'$. In these contexts, a weaker notion of validity for open arguments should be adopted, a notion of validity which is not based on the idea of validity-transmission.

A radical way of weakening the notion of validity for open derivations consists in simply requiring for an open derivation to qualify as valid that it is constituted by inference rules that are correct*. We therefore obtain the following (to distinguish it from Prawitz's definition, we refer to this notion as 'validity*').

Definition 5.2 (*Validity**) A derivation $\mathscr{D}$ is valid* with respect to a set of reduction procedures $\mathcal{J}$ iff:

- It is closed and it $\mathcal{J}$-reduces to a canonical derivation whose immediate subderivations are valid* with respect to $\mathcal{J}$;
- or it is open and it is constituted by correct* inference rules.

The revised definition achieves its goals, in that $\mathbf{R}$ qualifies as non-valid* (since it is closed but is not reducible to canonical form) whereas both $\neg\mathscr{R}$ and $\mathscr{R}$ qualify as valid* (they are closed canonical derivations, and their immediate subderivation are valid, being constituted by correct* inference rules. The derivation $\mathbf{T}'$ fails to be valid (since, like $\mathbf{R}$ it is closed but not reducible to canonical form), and so is the following derivation:

$$\dfrac{\dfrac{\dfrac{\overset{u}{A}}{A\ \texttt{tonk}\ B}\ \texttt{tonk}\,\mathrm{I}}{B}\ \texttt{tonk}\,\mathrm{E}}{A \supset B}\ \supset\mathrm{I}\ \langle u\rangle$$

This derivation is normal (i.e. cannot be reduced any further) and canonical, but its immediate subderivation is not valid, as it is constituted by an application of `tonk` E, an inference rule which is not correct*.

It is easy to check that, in calculi whose rules are in harmony, for a closed derivation to be valid* means to be reducible to a β_w-normal derivation, and conversely, derivations that are reducible to β_w-normal form qualify as valid*. This is the aspect in which the definition of validity* remains faithful to Prawitz's original definition (see Sect. 2.9).

The differences are however substantial. As detailed in Sect. 2.10, in the case of the Prawitz-Dummett approach it is the notion of correctness of an inference that depends on that of validity. On the contrary, it is our notion of validity* to depend on that of correctness*.

As we observed, Prawitz's definition of validity proceeds by induction on the joint complexity of the conclusion and the undischarged assumptions. Thus, for the induction to be well-founded, introduction rules should satisfy the complexity condition: in the introduction rules the consequence of the rule must be of higher logical complexity than all immediate premises and all dischargeable assumptions. The different clause for open derivations makes the definition of validity* well-founded even in cases in which Prawitz's definition is not. The reason is that in order to check whether a closed canonical derivation is valid* one has to check the validity* of its immediate subderivations (like in the case of Prawitz's validity). The difference is however that the validity* of open subderivation consists in their being constituted by correct* inference rules (and not, as in Prawitz's definition in transmitting closed validity from the assumptions to the conclusion). Hence, in order for the definition to be well-founded, it is enough that the introduction rules satisfy a weaker complexity condition: the consequence of any application of an introduction rule must be of higher logical complexity of those immediate premises which are the conclusion of closed subderivations.

The introduction rule for ρ does not satisfy even this weaker condition, in that it does not discharge any assumption and the complexity of the premise is higher than that of the consequence, thus inducing the need of checking the validity* of a closed derivation of $\neg\rho$ in evaluating the validity* of a closed canonical derivation of ρ, hence giving no warrant that the process of checking the validity of a given derivation in $\mathtt{NI}^{\supset\bot\rho}$ terminates. However, the well-foundedness of validity* can be warranted by considering the following revised version of the paradoxical ρ, which we call ρ^* (cf. also [77]):

$$\dfrac{\begin{array}{c}[\rho^*]\\ \bot\end{array}}{\rho^*}\,\rho^*\mathrm{I} \qquad\qquad \dfrac{\rho^* \qquad \rho^*}{\bot}\,\rho^*\mathrm{E}$$

The reduction for ρ^* is the following:

$$\dfrac{\langle n\rangle\,\dfrac{\begin{array}{c}\overset{n}{[\rho^*]}\\ \mathscr{D}_1\\ \bot\end{array}}{\rho^*}\,\rho^*\mathrm{I} \qquad \begin{array}{c}\mathscr{D}_2\\ \rho^*\end{array}}{\bot}\,\rho^*\mathrm{E} \qquad \overset{\rho^*\beta}{\rhd} \qquad \begin{array}{c}\mathscr{D}_2\\ [\rho^*]\\ \mathscr{D}_1\\ \bot\end{array}$$

Using ρ^* we can construct a valid closed normal derivation of ρ^* as follows:

$$\dfrac{\dfrac{\overset{1}{\rho^*} \qquad \overset{1}{\rho^*}}{\langle 1 \rangle \; \dfrac{\bot}{\rho^*} \; \rho^*\mathrm{I}} \; \rho^*\mathrm{E}}{} \tag{$\mathscr{R}^*$}$$

and by combining two copies of it using the elimination rule for ρ^* one obtains the following closed non-normalizing derivation of $\bot$:

$$\dfrac{\dfrac{\dfrac{\overset{1}{\rho^*} \qquad \overset{1}{\rho^*}}{\bot}\rho^*\mathrm{E}}{\rho^*}\langle 1 \rangle\,\rho^*\mathrm{I} \qquad \dfrac{\dfrac{\overset{1}{\rho^*} \qquad \overset{1}{\rho^*}}{\bot}\rho^*\mathrm{E}}{\rho^*}\langle 1 \rangle\,\rho^*\mathrm{I}}{\bot} \tag{$\mathbf{R}^*$}$$

As desirable, this derivation is not valid* because it is closed and it does not reduce to canonical form (as the reduction of its only maximal formula occurrence using $\rho^*\beta$ gives back the derivation itself).[6]

As it will be shown in the final sections of the chapter, the proposed modification of Prawitz's definition may still be found demanding in some respects. Before discussing these issues, however, we wish to address a further issue, namely the sense in which the rules for ρ can be said to endow it with meaning.

5.8 Meaning Explanations for Paradoxes

How can one state the proof-conditions for ρ or ρ^*? Following the pattern of explanation common to the standard logical constants, to answer this question we stipulate an operation on proofs such that the result of applying it to a proof of $\neg\rho$ is a proof of ρ, this operation reflecting the kind of negative self-reference encoded in the instance of $\in$I which entitles one to pass over from $\neg\rho$ to ρ. The introduction rule for ρ stipulates that a proof of ρ is what one obtains by applying this operation to a proof of $\neg\rho$. The elimination rule for ρ does no more than stating that this is the only means of constructing proofs of ρ (cf. Sect. 3.6).

Though weak, these principles are strong enough to establish interesting facts, such as for instance the existence of both a proof of ρ and of its negation. At the same time, these principles are not arbitrary, as shown by the fact that they do not allow to establish the existence of proofs of $\bot$.

Similarly, a BHK-clause-like explanation of the proof-conditions of ρ^* would be the following: a proof of ρ^* is the result of applying a self-referential abstraction-like operation to a function (as unsaturated entity) from proofs of ρ^* to proofs of $\bot$. The result of this operation are objects whose nature is similar to that of the functions as courses-of-value that constitute proofs of propositions of the form $A \supset B$, with the crucial difference that proofs of ρ^* take proofs of ρ^* as arguments and yields proofs of $\bot$ as values.

As in the case of the proof of $\neg\rho$ denoted by $\neg\mathscr{R}$, these functions must be sometimes be understood as partial. Moreover, it is easy to see that, in a calculus equipped with the rules for ρ^*, the proofs of implications must be understood as functions (as course-of-value) which sometimes fail to be total as well. For example, this is the case for the proof denoted by the following valid* derivation:

$$\dfrac{\langle 1\rangle\ \dfrac{\overset{1}{\rho^*} \qquad \overset{1}{\rho^*}}{\bot}\,\rho^*\mathrm{E}}{\rho^* \supset \bot}\,\supset\mathrm{I} \qquad (\neg\mathscr{R}^*)$$

which does not yield a proof of $\bot$ when applied to the proof of ρ^* denoted by $\mathscr{R}^*$ above, but only a derivation which fails to denote (being closed and not being reducible to canonical form).

In both cases, however, it is clear that compositionality is violated, since their introduction rules fail to explain the meaning of ρ and ρ^* in terms of that of propositions of lower complexity. This could be taken as constituting too big a departure from standard meaning explanation to qualify as acceptable.

It is however worth stressing, that Dummett himself conceives the compositionality of meaning as compatible with local forms of circular meaning-dependencies, presenting the names of colors as a typical example of words whose meaning is interdependent and cannot but be learned together. The case of an expression such as ρ^* is the limit case of a circular meaning-dependency in which to understand an expression one needs a previous understanding of that very expression. If the idea of self-dependence is too disturbing, it is easy to see that the formulation of paradoxes does not rely on it in an essential way. If one considers instead of ρ^* the pair of expressions σ and τ governed by the following rules:

$$\dfrac{\begin{matrix}[\tau]\\ \bot\end{matrix}}{\sigma}\,\sigma\mathrm{I} \qquad \dfrac{\sigma \qquad \tau}{\bot}\,\sigma\mathrm{E}$$

$$\dfrac{\sigma}{\tau}\,\tau\mathrm{I} \qquad \dfrac{\tau}{\sigma}\,\tau\mathrm{E}$$

one can easily reconstruct Jourdain's paradox and the considerations developed in the present chapter equally apply to a calculus equipped with both σ and τ (see [109], p. 281).

It is true that there is a fundamental difference between colors and paradoxes, namely that in the case of paradoxes we have chains of dependencies which make an expression depend "negatively" on itself. It is however far from obvious why compositionality should be compatible with positive but not with negative forms of dependency. It therefore seems that the burden of proof lays on the side of who wants to deny the viability of a PTS account of paradoxical expressions, rather than on the side of who wants to defend it.

5.9 Conservativity

As a matter of fact, the adoption of validity* for derivations in $\mathtt{NI}^{\supset\bot\rho}$ implies the existence of at least one proof of $A \supset B$ for any A and B, denoted by a derivation of the following form:

$$\langle n\rangle\ \dfrac{\dfrac{\dfrac{\overset{\mathbf{R}}{\bot}}{A \supset B}\,{\bot}\mathrm{E} \qquad \overset{n}{A}}{B}\,{\supset}\mathrm{E}}{A \supset B}\,{\supset}\mathrm{I} \tag{5.1}$$

Being closed and canonical, any derivation of this form is valid* iff its immediate subderivation is valid*, which in fact is the case since we defined an open derivation to be valid* iff it is constituted by correct rules, and $\supset$E, $\bot$E as well as all rules applied in **R** are correct since they are either introduction rules or harmonious elimination rules.

It is true that one may argue that this is unproblematic, by claiming that all such function are undefined for every argument . However, the need of introducing partial function in the explanation of the meaning of implication seems to be essentially tied to paradoxical expressions. Thus if A and B do not contain ρ as a subformula, one may expect that if there is a proof of $A \supset B$, the proof should be a function defined for all its arguments.

In other words, one may wish to adopt a notion of validity such that if there is a valid derivation of a ρ-free proposition A in $\mathtt{NI}^{\supset\bot\rho}$, there must be a valid derivation of A already in $\mathtt{NI}^{\supset\bot}$. This requirement is strongly reminiscent of Belnap's conservativity with a crucial different. While Belnap's requirement of conservativity is formulated using derivability (i.e. existence of derivations), we formulated the requirement for "valid derivability" (i.e. existence of *valid* derivations).

As we have seen in the previous chapter, although the rules for ρ are in harmony, they extend in a non-conservative way derivability in $\mathtt{NI}^{\supset\bot}$. In the extension $\mathtt{NI}^{\supset\bot\rho}$ of $\mathtt{NI}^{\supset\bot}$, the proposition $\bot$ (which belongs to the language of the restricted calculus) is derivable, although it is not derivable in the original calculus.

A conservativity result of the kind envisaged can however be established for $\beta\gamma$-normal derivations of $\mathtt{NI}^{\supset\bot}$: as we will now show, if neither A nor Γ contain ρ as subformula, then there is a $\beta\gamma$-normal derivation of A from Γ in $\mathtt{NI}^{\supset\bot\rho}$ if and only if there one in $\mathtt{NI}^{\supset\bot}$. This will be taken as a reason to consider a further notion of validity by strengthening the notion of valid* derivation to the effect that a derivation is valid iff it $\beta\gamma$-reduces to a $\beta\gamma$-normal derivation.

As we have seen, β-reduction—and a fortiori $\beta\gamma$-reduction—is not normalizing in $\mathtt{NI}^{\supset\bot\rho}$ with the derivation **R** providing a typical counterexample. In spite of this, $\beta\gamma$-normal derivations in $\mathtt{NI}^{\supset\bot\rho}$ also have the peculiar structure of $\beta\gamma$-normal derivations in $\mathtt{NI}^{\supset\bot}$ (see Sect. 3.5). Prawitz [65] already observed that in $\mathtt{NI}^{\supset\in}$, the tracks in β-normal derivations are still divided into an introduction and elimination part. This holds for $\beta\gamma$-normal derivation in $\mathtt{NI}^{\supset\bot\rho}$ as well. The reason is essentially the same as in $\mathtt{NI}^{\supset\bot}$: In order for the consequence of an application of an introduction (or of

$\bot$E) to act as the major premise of an application of an elimination, the derivation must be non-normal.

However, given the standard definition of subformula.

Definition 5.3 (*subformula*)

- For all A, A is a subformula of A;
- all subformulas of A and B are subformulas of $A \supset B$,

the neat subformula relationships between the formula occurrences constituting a track of a $\beta\gamma$-normal derivation are lost in $\mathtt{NI}^{\supset\bot\rho}$. To wit, both in $\neg\mathscr{R}$ and $\mathscr{R}$ we need to pass through $\neg\rho$ in order to establish $\bot$ from ρ. Thus, $\beta\gamma$-normal derivations in $\mathtt{NI}^{\supset\bot\rho}$ do not enjoy the subformula property.

The reason for this is that the premise of ρI is the formula $\neg\rho$ which is more complex than its consequence ρ. If we take the rules of a connective to codify semantic information, this situation is unsurprising. The rule $\supset$I gives the meaning of an implication in terms of its subformulas, and thus the semantic complexity of an implicational formula corresponds to its syntactic complexity. In the case of ρ, the rule ρI gives the meaning of ρ in terms of the more complex formula $\neg\rho$. Whereas the syntactic complexity of formulas in the $\{\supset, \bot, \rho\}$-language fragment is well-founded, one could say that their semantic complexity is not.

This informal remark can be spelled out by defining the following notion, which in lack of a better name we call 'pre-formula'. Intuitively, it reflects the semantic complexity of a formula, in the sense that the pre-formulas of a formula A are those formula one has to understand in order to understand A.

Definition 5.4 (*Pre-formula*)

- For all A, A is a pre-formula of A;
- all pre-formulas of A and B are pre-formulas of $A \supset B$;
- all pre-formulas of $\neg\rho$ are pre-formulas of ρ.

The seemingly inductive process by which pre-formulas are defined is clearly non-well-founded. However, this is not a reason to reject it as a definition.[7] Indeed, the notion of pre-formula turns out to be very useful in describing the structure of tracks in $\beta\gamma$-normal derivations in $\mathtt{NI}^{\supset\bot\rho}$: The neat subformula relationship holding between the members of a track in $\beta\gamma$-normal derivations in $\mathtt{NI}^{\supset\bot}$ are replaced by pre-formula relationships between members of a track in $\beta\gamma$-normal derivations in $\mathtt{NI}^{\supset\bot\rho}$.

By replacing 'subformula' with 'pre-formula', one can establish a fact analogous to to Fact 4 (see Sect. 3.5) for $\mathtt{NI}^{\supset\bot\rho}$, from which one obtains the following:

Fact 5 (*Pre-formula property*) All formulas in a $\beta\gamma$-normal derivation in $\mathtt{NI}^{\supset\bot\rho}$ are either pre-formulas of the conclusion or of some undischarged assumption.

Proof By induction on the order of tracks (see proof of Fact 2 in Sect. 1.5). ∎

We thus have that.

Fact 6 If Γ and A are ρ-free, then there is a $\beta\gamma$-normal derivation of A from Γ in $\mathtt{NI}^{\supset\perp\rho}$ iff there is one in $\mathtt{NI}^{\supset\perp}$.

Proof This follows immediately from Fact 5 together with the fact that if ρ does not occur in a formula than it is not a pre-formula of it (which can be established by induction on the degree of formulas). ■

That is, $\beta\gamma$-normal derivability in $\mathtt{NI}^{\supset\perp\rho}$ is a conservative extension of $\beta\gamma$-normal derivability in $\mathtt{NI}^{\supset\perp}$. More briefly, we will refer to this fact by saying that the rules for ρ are conservative over $\beta\gamma$-normal derivability in $\mathtt{NI}^{\supset\perp}$.[8]

Fact 6 suggests to strengthen the condition for the validity of open derivations by requiring not only that they are constituted by correct* inference rules, but also that they reduce to $\beta\gamma$-normal form. Accordingly, for closed derivations in a calculus consisting of correct* inference rules, in order to qualify as valid it would not be enough for them to reduce to β_w-normal derivations (which is what being valid* boils down to) but to reduce to a $\beta\gamma$-normal derivation. This would yield a further alternative to Prawitz's validity, which we may call validity**:

Definition 5.5 (*Validity***) A derivation is valid** iff it is constituted by correct inference rules and it reduces to a $\beta\gamma$-normal derivation.

Though valid*, the derivations of the form (5.1) are not valid** (as they fail to reduce to $\beta\gamma$-normal derivations). The adoption of validity** as the notion to characterize the closed derivations having a denotation would allow one to avoid to accept that any proposition of the form $A \supset B$ has a proof.

A general concern undermines however the notion of validity**, namely it requires to apply reductions not only to closed, but also to open derivations.

The validity* of open derivations is not defined in term of the validity* of its closed instances, and in this sense it represents a major deviation from the traditional notion of Prawitz. However, validity* remains faithful to the idea that reduction is of semantic significance only for closed derivations, and not for open derivations. This very fact, on the other hand, would be denied by the adoption of validity**, since validity** is defined in terms of reduction for open derivations as well.

Compared to validity*, the advantage of adopting validity** is the fact that the addition of the rules for a paradoxical expression like ρ does not enrich the set of proofs in the ρ-free language. On the other hand the addition of the rules for a paradoxical expression result in novel valid* closed derivations of propositions belonging to the ρ-free language. Whether this is a drawback for validity* is debatable, since in general it is not the case that the addition of a new expression (governed by harmonious rules) yields a conservative extension of the language to which it is added (a counterexample, mentioned by Prawitz [72], is provided by the rules for second-order quantifiers, which extend in a non-conservative way first-order arithmetic).

To conclude, we ended up with two possible accounts of validity which can be applied to a paradoxical language. One is validity*, that is more faithful to the basic philosophical tenet that reduction is of semantic significance only for closed derivations, but has the drawback of making room for unwanted proofs of large classes of propositions (in particular for each pair of propositions A and B there is at least one function from proofs of A to proofs of B).

The other is validity** which is more restrictive in that, although its adoption leads in some cases to the existence of both a proof of a proposition and of its negation (e.g. ρ^* and $\neg\rho^*$), it tames the proliferation of undesired proofs of propositions belonging to the paradox-free fragment. However, this requires a major departures from the Prawitz-Dummett conception of validity: namely it requires to ascribe semantic significance to the notion of reduction for open derivations, which requires the adoption of an extensional conception of functions from proof to proof.

Notes to This Chapter

1. In recent years Tennant has further developed his analysis of paradoxes by proof-theoretic means (see Note 4 to Chap. 4). However, he did not explored the connection between the proof-theoretic analysis of paradoxes and the idea that proofs are the denotations of formal derivations.
2. In other words, Prawitz's definition of validity as applied to $\mathtt{NI}^{\supset\bot\in}$ is a metalinguistic analog of the paradoxical definition of R discussed in Sect. 4.2.
3. In the type-theoretic setting (see Note 6 to Chap. 4), this means that $\mathtt{app}(x, y)$ does not denote a total function from proofs of $A \supset B$ and A to proofs of B, since when we replace for x and y the terms corresponding to the derivations $\neg\mathscr{R}$ and $\mathscr{R}$ we obtain the non-denoting term corresponding to $\mathbf{R}$. In the untyped λ-calculus, this corresponds to the fact that application does not preserve normalizability in the untyped setting.
4. In sequent calculus, this condition is essentially that reductions for a principal cut on a formula A (or, in one-sided calculi, on two formulas A and $A^\perp$ having the same complexity) yield one or more cuts on formulas of strictly lower complexity than A (and $A^\perp$). We also remark that, in the light of the considerations made in Sect. 4.4, it is clear that this condition is plausible only in presence of contraction. In its absence, it is unnecessary restrictive.
5. The idea that the correctness of an inference rule can be shown by deriving it from other, previously accepted, rules, is referred to by Dummett [12] as 'the proof-theoretic justification of first-grade'. A justification of this kind requires obviously that some other rules have been previously justified as correct. The notion of correctness* embodies this idea, in that introduction rules are correct* by fiat; elimination rules are correct* iff they are in harmony with the introduction rules; and all other rules are justified using introduction and elimination rules. Dummett [12] follows a different path: he rather argues that there must be more powerful means of justification which he calls justifications of second and of third degree respectively. A justification of third degree amounts to showing that whenever a rule is applied to valid closed canonical derivations for the premises it yields a valid closed canonical derivation of the conclusion. This essentially coincide with showing that the rule is correct in Prawitz's sense, as discussed in

Sect. 2.10 (with some caveats due to the different way in which Dummett and Prawitz define canonical derivations, see Note 23 to Chap. 1 and Notes 6 and 14 to Chap. 2).

6. In a type-theoretic setting, the introduction and elimination rules could be presented as follows:

$$\frac{\begin{array}{c}[x : \rho^*] \\ t : \bot\end{array}}{\rotatebox[origin=c]{180}{λ} x.t : \rho^*} \qquad \frac{s : \rho^* \qquad t : \rho^*}{\texttt{ddɐ}(s, t) : \bot}$$

and the reduction for ρ^* could be internalized with the following equality rule:

$$\frac{\begin{array}{c}[x : \rho^*] \\ t : \bot\end{array} \qquad s : \rho^*}{\texttt{ddɐ}(\rotatebox[origin=c]{180}{λ} x.t, s) = t(s/x) : \bot}$$

The derivation $\mathscr{R}^*$ would be encoded by the term $\rotatebox[origin=c]{180}{λ} x.\texttt{ddɐ}(x, x)$ and the derivation $\mathbf{R}^*$ by the term $\texttt{ddɐ}(\ \rotatebox[origin=c]{180}{λ} x.\texttt{ddɐ}(x, x)\ ,\ \rotatebox[origin=c]{180}{λ} x.\texttt{ddɐ}(x, x)\ )$ which resembles even more closely than the term associated with $\mathscr{R}$ the untyped loopy combinator $(\lambda x.xx)(\lambda x.xx)$ (see Note 6 to Chap. 4 above).

7. To see that there is nothing wrong with the notion of pre-formula one could first define the notion of *immediate pre-formula* as follows: (i) the immediate pre-formulas of $A \supset B$ are A and B ; (ii) the immediate pre-formula of ρ is $\neg\rho$. The notion of pre-formula could then be introduced as the reflexive and transitive closure of the one of immediate pre-formula.

8. As briefly recalled in Sect. 4.4, the notion of cut-free derivation roughly corresponds to the notion of normal derivation. The conservativity result for natural deduction parallels an analogous conservativity result in sequent calculus [79]. In that setting, one can take ρ to be governed by the following left and right rules:

$$\frac{\Gamma \neg\rho \Rightarrow \Delta}{\Gamma, \rho \Rightarrow \Delta}\,\mathrm{L}\rho \qquad \frac{\Gamma \Rightarrow \neg\rho, \Delta}{\Gamma \Rightarrow \rho, \Delta}\,\mathrm{R}\rho$$

Let $\mathtt{LK}_\rho$ be the extension of the (cut-free) implicative fragment of the sequent calculus for classical logic $\mathtt{LK}$, whose rules are:

$$\frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma', B \Rightarrow \Delta'}{A \supset B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\mathrm{L}\supset \qquad \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}\,\mathrm{R}\supset$$

together with identity, exchange, weakening and contraction (for the present scopes, one could equivalently consider an intuitionistic or minimal variant of the calculus). The following hold:

Fact 7 For Γ and Δ ρ-free: $\Gamma \Rightarrow \Delta$ is deducible in $\mathtt{LK}$ iff it is deducible in $\mathtt{LK}_\rho$.

Proof Given the rules for $\mathtt{LK}_\rho$ if there is no occurrence of ρ in the consequence of a rule-application then there is none in the premises of the rule-application. Thus if the end-sequent of a derivation is ρ-free, the whole derivation is. ■

There is however a remarkable difference between the natural deduction and sequent calculus approach. In sequent calculus, the same reasoning allows to establish a result analogous to Fact 7 for $\mathtt{LK}_{\mathtt{tonk}}$, the extension of $\mathtt{LK}$ with the following rules for $\mathtt{tonk}$:

$$\frac{\Gamma, B \Rightarrow \Delta}{\Gamma, A \;\mathtt{tonk}\; B \Rightarrow \Delta}\,\mathtt{Ltonk} \qquad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow A \;\mathtt{tonk}\; B, \Delta}\,\mathtt{Rtonk}$$

On the other hand, the rules for $\mathtt{tonk}$ are not conservative over $\beta\gamma$-normal derivations in $\mathtt{NI}^{\supset\bot}$: the derivation $\mathbf{T}'$ established $\bot$ in $\mathtt{NI}^{\supset\bot\mathtt{tonk}}$ by means of a $\beta\gamma$-normal derivation. The difference is due to the fact that in natural deduction the addition of $\mathtt{tonk}$ and ρ have different effects: the addition of $\mathtt{tonk}$ does not invalidate normalization, but invalidates the canonicity of closed normal derivations; on the other hand, the addition of ρ invalidates normalization but not the canonicity of closed normal derivations. To recover the full analogy with the natural deduction setting one can consider $\mathtt{LK}^*$, $\mathtt{LK}^*_\rho$ and $\mathtt{LK}^*_{\mathtt{tonk}}$, the calculi extending (respectively) $\mathtt{LK}$, $\mathtt{LK}_\rho$ and $\mathtt{LK}_{\mathtt{tonk}}$ with the cut rule. Whereas for the rules for $\supset$ and ρ opportune reductions can be defined to push applications of the cut rule towards the axioms, this cannot be done in the case of the rules for $\mathtt{tonk}$. Consequently, although cut is neither eliminable in $\mathtt{LK}^*_{\mathtt{tonk}}$ nor in $\mathtt{LK}^*_\rho$, this would be for different reasons: in $\mathtt{LK}^*_{\mathtt{tonk}}$ one would have derivations containing applications of the cut rule which cannot be further reduced; in $\mathtt{LK}^*_\rho$ one would have derivations containing applications of the cut rule to which reductions can be applied, but that cannot be brought into cut-free form due to a loop arising in the process of reduction. By introducing the notion of normal derivation as one to which no reduction can be further applied, it would be possible to show that whereas the rules for ρ are conservative over normal derivations in $\mathtt{LK}^*$, the rules for $\mathtt{tonk}$ are not.

Chapter 6
Two Kinds of Difficulties

(by Peter Schroeder-Heister and Luca Tranchini)

Abstract Two distinct kinds of cases, going back to Crabbé and Ekman, show that the Tennant-Prawitz criterion for paradoxicality overgenerates, that is, there are derivations which are intuitively non-paradoxical but which fail to normalize. We argue that a solution to "Ekman's paradox" consists in restricting the set of admissible reduction procedures to those that do not yield a trivial notion of identity of proofs. We then discuss a different kind of solution, due to von Plato, and recently advocated by Tennant, consisting in reformulating natural deduction elimination rules in general (or parallelized) form. Developing intuitions of Ekman we show that the adoption of general rules has the consequence of hiding redundancies within derivations. Once reductions to get rid of the hidden redundancies are devised, it is clear that the adoption of general elimination rules offers no remedy to the overgeneration of the Prawitz-Tennant analysis. In this way, we indirectly provide further support for our own solution to Ekman's paradox.

6.1 From Naive Comprehension to Separation

In this chapter we discuss two distinct cases in which the Tennant-Prawitz analysis overgenerates, i.e. in which it ascribes paradoxicality to derivations of $\perp$ that fail to normalize, although they belong to deductive settings of which we know that are too weak to allow for the formulation of paradoxes.

The first case of overgeneration arises in a consistent set theory in which Zermelo's separation axiom is formulated in rule form:

$$\frac{t \in s \qquad A(t/x)}{t \in \{x \in s : A\}} \in^z\mathrm{I} \qquad \frac{t \in \{x \in s : A\}}{A(t/x)} \in^z\mathrm{E} \qquad \frac{t \in \{x \in s : A\}}{t \in s} \in^z\mathrm{E}_2$$

(the second elimination rule will actually play no role in what follows). We call $\mathtt{NI}^{\supset\in^z}$ the system which results by adding these rules to $\mathtt{NI}^{\supset}$.[1]

An application of $\in^z$I followed by $\in^z$E constitutes a redundancy that can be eliminated according to the following reduction (we call the formula eliminated a *Zermelo-maximal formula occurrence*):

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$$\dfrac{\dfrac{\overset{\mathscr{D}_1}{t \in s} \qquad \overset{\mathscr{D}_2}{A(t/x)}}{t \in \{x \in s : A\}}}{A(t/x)} \quad \overset{\in^z\beta}{\rhd} \quad \overset{\mathscr{D}_2}{A(t/x)}$$

What if we now try to reconstruct Russell's reasoning in this setting? For any set y, we can construct a term denoting the Russell subset of y, i.e. the set of all elements of y which do not belong to themselves, $r_y =_{\text{def}} \{x \in y : x \notin x\}$. Taking now ρ_y to be $r_y \in r_y$, an application of $\in^z$E allows one to pass over from ρ_y to $\neg\rho_y$, but in order to pass over from $\neg\rho_y$ to ρ_y using an application of $\in^z$I one needs a further premise, namely $r_y \in y$:

$$\dfrac{r_y \in y \qquad \neg\rho_y}{\rho_y}\,\in^z\text{I} \qquad \dfrac{\rho_y}{\neg\rho_y}\,\in^z\text{E}$$

Thus, by following Russell's reasoning in $\mathtt{NI}^{\supset\in^z}$ one obtains a derivation of absurdity $\perp$ in $\mathtt{NI}^{\supset\in^z}$ that, contrary to **R**, depends on an assumption, namely the assumption $r_y \in y$ that is needed for the application of $\in^z$I (for visibility the assumption is boxed):

$$\dfrac{\langle 1\rangle\,\dfrac{\dfrac{\dfrac{\overset{1}{\rho_y}}{\neg\rho_y}\,\in^z\text{E} \qquad \overset{1}{\rho_y}}{\perp}\,\supset\text{E}}{\neg\rho_y}\,\supset\text{I} \qquad \dfrac{\boxed{r_y \in y} \qquad \langle 1\rangle\,\dfrac{\dfrac{\dfrac{\overset{1}{\rho_y}}{\neg\rho_y}\,\in^z\text{E} \qquad \overset{1}{\rho_y}}{\perp}\,\supset\text{E}}{\neg\rho_y}\,\supset\text{I}}{\rho_y}\,\in^z\text{I}}{\perp}\,\supset\text{E} \qquad (\mathbf{R^z})$$

Now assume that existential quantification with its standard rules is available. As y does not occur free in the conclusion nor in any undischarged assumption other than $r_y \in y$, by assuming $\exists y(r_y \in y)$ we can obtain by $\exists E$ a derivation of $\perp$ from $\exists y(r_y \in y)$ and by $\supset$I we can thereby establish $\neg\exists y(r_y \in y)$, that is that no set contains its own Russell subset:

$$\langle 3\rangle\,\dfrac{\langle 2\rangle\,\dfrac{\overset{3}{\exists y(r_y \in y)} \qquad \dfrac{\langle 1\rangle\,\dfrac{\dfrac{\dfrac{\overset{1}{\rho_y}}{\neg\rho_y}\,\in^z\text{E} \qquad \overset{1}{\rho_y}}{\perp}\,\supset\text{E}}{\neg\rho_y}\,\supset\text{I} \qquad \dfrac{\overset{2}{r_y \in y} \qquad \langle 1\rangle\,\dfrac{\dfrac{\dfrac{\overset{1}{\rho_y}}{\neg\rho_y}\,\in^z\text{E} \qquad \overset{1}{\rho_y}}{\perp}\,\supset\text{E}}{\neg\rho_y}\,\supset\text{I}}{\rho_y}\,\in^z\text{I}}{\perp}\,\supset\text{E}}{\perp}\,\exists\text{E}}{\neg\exists y(r_y \in y)}\,\supset\text{I}$$

That no set contains its own Russell subset is a perfectly acceptable conclusion in a consistent set theory like Zermelo's. It shows in particular that there is no set of all sets, which is something that any set theory based on the separation axiom should be able to prove. However, and here is the problem, the derivation $\mathbf{R}^{\mathbf{z}}$ of $\bot$ from $r_y \in y$ (and likewise the one of $\bot$ from $\exists y(r_y \in y)$) fails to normalize, for the same reason as Russell's original $\mathbf{R}$. By removing the encircled maximal formula occurrence of $\neg\rho_y$, a Zermelo-redundant formula is introduced, and by removing it, one gets back to $\mathbf{R}^{\mathbf{z}}$. So, on the Prawitz-Tennant analysis the derivation does not represent a real proof, and (as in the case of the derivation of $\bot$ in naive set theory) no other derivation fares better. That is, on the Prawitz-Tennant analysis, though we have derivations showing that there is no set of all sets in Zermelo set theory based on separation, these derivations are unacceptable as they qualify as paradoxical.[2]

These facts were first observed by Marcel Crabbé [3] in 1974 at the Logic Colloquium in Kiel (see [51]) and have been largely neglected in the philosophical literature (in particular by Tennant), except for a short reference to them by Sundholm [104]. However, they represent the starting point of modern proof-theoretic investigations of set theory see [14, 28].

6.2 Ekman's Paradox

The other kind of overgeneration arises in an even weaker setting: pure propositional logic. Suppose we have derived A by means of a derivation $\mathscr{D}$. By assuming $A \supset B$, $\mathscr{D}$ can be extended by $\supset$E to a derivation of conclusion B. By further assuming $B \supset A$ one can conclude A, but this had already been established by $\mathscr{D}$. The two applications of $\supset E$ just make one jump back and forth between A and B:

$$\dfrac{B \supset A \qquad \dfrac{A \supset B \qquad \begin{matrix}\mathscr{D}\\ A\end{matrix}}{B}\supset\mathrm{E}}{A}\supset\mathrm{E}$$

Ekman [15] observed that although the official reductions of $\mathtt{NI}^{\supset}$ do not allow one to get rid of patterns of this kind, such patterns constitute redundancies which can be easily removed by identifying the top and bottom occurrences of A and removing the two applications of $\supset$E between them. We refer to this conversion as `Ekman` and we will call an *`Ekman`-maximal formula occurrence* the occurrence of B acting as conclusion of the first application of $\supset$E and as minor premise of the second application of $\supset$E in the schema below[3]:

$$\dfrac{B \supset A \qquad \dfrac{A \supset B \qquad \begin{matrix}\mathscr{D}\\ A\end{matrix}}{B}\supset\mathrm{E}}{A}\supset\mathrm{E} \quad \overset{\mathtt{Ekman}}{\rhd} \quad \begin{matrix}\mathscr{D}\\ A\end{matrix} \qquad (\mathtt{Ekman})$$

Observe now that $\neg A$ follows from $A \supset \neg A$:

$$\dfrac{\dfrac{\dfrac{A \supset \neg A \qquad \overset{1}{A}}{\neg A}\supset\mathrm{E} \qquad \overset{1}{A}}{\bot}\supset\mathrm{E}}{\neg A}\supset\mathrm{I}\,\langle 1\rangle$$

By further assuming $\neg A \supset A$, the previous derivation can be extended by $\supset$E to a derivation of A from $A \supset \neg A$ and $\neg A \supset A$:

$$\dfrac{\neg A \supset A \qquad \langle 1\rangle\,\dfrac{\dfrac{\dfrac{A \supset \neg A \quad \overset{1}{A}}{\neg A}\supset\mathrm{E} \quad \overset{1}{A}}{\bot}\supset\mathrm{E}}{\neg A}\supset\mathrm{I}}{A}\supset\mathrm{E}$$

The two derivations can be joined together by an application of $\supset$E and the result is the following derivation of $\bot$ from $A \supset \neg A$ and $\neg A \supset A$ (here and below some of the rules labels will be omitted for readability):

$$\dfrac{\langle 1\rangle\,\dfrac{\dfrac{\dfrac{A \supset \neg A \quad \overset{1}{A}}{\neg A}\supset\mathrm{E} \quad \overset{1}{A}}{\bot}\supset\mathrm{E}}{\neg A}\supset\mathrm{I} \qquad \dfrac{\neg A \supset A \qquad \langle 1\rangle\,\dfrac{\dfrac{\dfrac{A \supset \neg A \quad \overset{1}{A}}{\neg A} \quad \overset{1}{A}}{\bot}}{\neg A}}{A}\supset\mathrm{E}}{\bot}\supset\mathrm{E} \tag{E}$$

The derivation **E** is not normal, since the encircled occurrence of $\neg A$ is a maximal formula occurrence. By applying $\supset$-`Red` one introduces a redundancy of the kind observed by Ekman (we encircle in the derivation the `Ekman`-maximal formula occurrence):

$$\dfrac{\dfrac{A \supset \neg A \qquad \dfrac{\neg A \supset A \qquad \langle 1\rangle\,\dfrac{\dfrac{\dfrac{A \supset \neg A \quad \overset{1}{A}}{\neg A} \quad \overset{1}{A}}{\bot}}{\neg A}\supset\mathrm{I}}{A}\supset\mathrm{E}}{\neg A}\supset\mathrm{E} \qquad \dfrac{\neg A \supset A \qquad \langle 1\rangle\,\dfrac{\dfrac{\dfrac{A \supset \neg A \quad \overset{1}{A}}{\neg A} \quad \overset{1}{A}}{\bot}}{\neg A}}{A}\supset\mathrm{E}}{\bot}\supset\mathrm{E} \tag{E$'$}$$

By applying the relevant instance of `Ekman`:

$$\dfrac{A \supset \neg A \qquad \dfrac{\neg A \supset A \qquad \overset{\mathscr{D}}{\neg A}}{A}\supset\mathrm{E}}{\neg A}\supset\mathrm{E} \quad \triangleright \quad \overset{\mathscr{D}}{\neg A} \tag{6.2.1}$$

we get back the derivation **E**.

Thus, on the natural extension of the set of conversions suggested by Ekman, we have a counterexample to (weak and hence strong) normalization already in $\mathtt{NI}^{\supset}$: **E** is not normal and does not normalize, since its reduction process enters a loop. Given the Prawitz-Tennant's analysis of paradoxes in term of non-normalizability, the phenomenon observed by Ekman should show that paradoxes already appear at the level of propositional logic.[4]

In fact, Ekman's paradox can be taken to show that the logical component of Russell's paradox can be fully described using propositional logic. The derivations of Russell's paradox **R** and **R**$'$ can be obtained from Ekman's derivations **E** and **E**$'$ by suppressing all occurrences of $A \supset \neg A$ and $\neg A \supset A$ and by replacing all occurrences of the schematic letter A with ρ: In this way, the applications of $\supset$E with major premise $\neg A \supset A$ and $A \supset \neg A$ become applications of $\in$I and $\in$E respectively. In other words, the *id est* inferences involved in the derivation of Russell's paradox are simulated by applications of $\supset$E in **E** and **E**$'$, and $\rho\beta$—the instance of $\in\beta$ used to transform **R**$'$ into **R**—is simulated by the instance (6.2.1) of `Ekman` used to transform **E**$'$ into **E**.

To repeat, the difference between $\rho\beta$ and the instance of `Ekman` triggering Ekmans's paradox consists only in the fact that the redundancy is in one case generated by *id est* inferences, whereas in Ekman's case it is mimicked in propositional logic by applications of modus ponens. The major premises of modus ponens represent the rule applied in *id est* inferences. As Ekman puts it:

> Whatever motivation we have for [$\rho\beta$] this motivation also applies to [the instance (6.2.1) of `Ekman`] since the two reductions, from an informal point of view, are one and the same, but expressed using two different formal systems.
>
> (Ekman [14], p. 148; Ekman [15], p. 78)

Given this observation, paradoxical derivations can be analyzed as consisting of an extra-logical construction which is plugged into a portion of purely propositional reasoning. The extra-logical part is constituted by *id est* inferences which allow one to pass over, for some specific ρ, from $\neg\rho$ to ρ and back. The logical part consists of the derivation E of absurdity $\bot$ from $\neg A \supset A$ and $A \supset \neg A$, for an unspecific (i.e. for all) A. Ekman's paradox would thus show that loops are not a feature of the extra-logical part, but of the logical part of paradoxical derivations. The looping feature would not depend on the possibility to move, for a certain ρ, from ρ to $\neg\rho$ and vice versa, but that we can move, for any formula A, from $\neg A \supset A$ and $A \supset \neg A$ to absurdity.

We do not take this to be the right conclusion to be drawn from the phenomenon observed by Ekman. Rather, we take Ekman's paradox to push the question of when a certain reduction counts as acceptable: Whether a derivation is normal depends on the collection of reductions adopted, and hence Tennant's criterion requires that one carefully considers what should be taken to be a good reduction. In particular, Ekman's phenomenon shows that on too loose a notion of reduction, one obtains a too coarse criterion of paradoxicality.

Thus, it is not its logical component that makes Russell's reasoning paradoxical, but the *id est* rules encoding naive comprehension. Propositional logic alone is too weak to allow for the formulation of paradoxical expression and thereby there cannot be anything paradoxical about a derivation in $\mathtt{NI}^{\supset}$.[5]

6.3 A Solution to Ekman's Overgeneration

In almost all presentations of natural deduction, reductions are presented as means to get rid of redundancies within proofs. This is also the background of Tennant's analysis, who writes:

> The reduction procedures for the logical operators are designed to eliminate such unnecessary detours within proofs.
>
> So are other abbreviatory procedures σ, which have the general form of 'shrinking' to a single occurrence of A, any logically circular segments of branches (within the proof) of the form shown below to the left:
>
> $$\begin{array}{c} A \\ \hline B_1 \\ \vdots \\ B_n \\ \hline A \end{array} \quad \overset{\sigma}{\triangleright} \quad A$$
>
> One thereby identifies the top occurrence of A with the bottom occurrence of A, and gets rid of the intervening occurrences of $B_1, \ldots, B_n$, that form the filling of this unwanted sandwich. Logically, one should live by bread alone.
>
> (Tennant [111], pp. 199–200)

Given this, Tennant should have nothing to object against the reduction `Ekman` as it is a variant of σ. However, the understanding of reduction as "abbreviatory procedures" is not the only possible one. We actually claim that this understanding is not appropriate for meaning-theoretical investigations, and take Ekman's paradox to be a striking phenomenon that points to this fact.

As we detailed in Chap. 2, from a PTS standpoint proofs should be viewed as abstract entities linguistically represented by natural deduction derivations and reduction procedures for derivations can then be viewed as yielding a criterion of identity between proofs.

When one considers conversions besides β-reductions and η-expansions, a minimal requirement for the acceptability of a new conversion should be that of not trivializing identity of proof, in the sense that is should always be possible to exhibit at least one proposition A and two derivations having A as conclusion that belong to two distinct equivalence classes (see, for details, Sect. 2.4). If this requirement is not met, the intensional conception of PTS we advocated in Sect. 2.6 would collapse into a merely extensional picture: for every proposition there would be either a (single) proof or there would be no proof at all. On such an understanding, the notion of reduction is much narrower than the one arising from taking reductions as "abbreviation procedures". On this narrower conception Ekman's alleged conversion turns out to be no conversion at all.

As we recalled in Sect. 2.4, it is well-known that any equivalence relation extending $\beta\eta$-equivalence trivializes the identity of proofs in $\mathtt{NI}^{\supset}$. As we will now show, Ekman's conversion is actually sufficient to trivialize the identity of proofs induced by $\supset\beta$ alone.

To begin with, instead of `Ekman` we actually consider Ekman's reduction in the more general form

$$\dfrac{\begin{matrix}\mathscr{D}'' \\ B \supset A\end{matrix} \qquad \dfrac{\begin{matrix}\mathscr{D}' \\ A \supset B\end{matrix} \quad \begin{matrix}\mathscr{D} \\ A\end{matrix}}{B}\supset\mathrm{E}}{A}\supset\mathrm{E} \qquad \overset{\texttt{Ekman}^*}{\rhd} \qquad \begin{matrix}\mathscr{D} \\ A\end{matrix}$$

i.e., we allow for $A \supset B$ and $B \supset A$ to be obtained by derivations $\mathscr{D}'$ and $\mathscr{D}''$. This means that we assume, as it is natural to do, that Ekman's reduction is closed under substitution of derivations for undischarged assumptions.[6]

For simplicity of exposition, we reason in the extension $\mathtt{NI}^{\wedge\supset}$ of $\mathtt{NI}^{\supset}$. A corresponding, but less well-readable example could be given in $\mathtt{NI}^{\supset}$ as well. Consider the formulas $A \wedge A$ and A and the following proofs of their mutual implications[7]:

$$\langle 1\rangle\ \dfrac{\dfrac{\overset{1}{A} \quad \overset{1}{A}}{A \wedge A}\wedge\mathrm{I}}{A \supset (A \wedge A)}\supset\mathrm{I} \qquad\qquad \langle 2\rangle\ \dfrac{\dfrac{\overset{1}{A \wedge A}}{A}\wedge\mathrm{E}_1}{(A \wedge A) \supset A}\supset\mathrm{I}$$

Given an arbitrary derivation $\mathscr{D}$ of $A\wedge A$, consider the following derivation $\mathscr{D}'$:

$$\mathscr{D}' = \left\{ \dfrac{\langle 1\rangle\ \dfrac{\dfrac{\overset{1}{A} \quad \overset{1}{A}}{A \wedge A}\wedge\mathrm{I}}{A \supset (A \wedge A)}\supset\mathrm{I} \qquad \dfrac{\langle 2\rangle\ \dfrac{\dfrac{\overset{2}{A \wedge A}}{A}\wedge\mathrm{E}_1}{(A \wedge A) \supset A}\supset\mathrm{I} \qquad \begin{matrix}\mathscr{D} \\ A \wedge A\end{matrix}}{A}\supset\mathrm{E}}{A \wedge A}\supset\mathrm{E} \right.$$

Observe that $\mathscr{D}'$ reduces both to $\mathscr{D}$ (with an application of $\texttt{Ekman}^*$) and to the following derivation (with two applications of $\supset\beta$):

$$\dfrac{\dfrac{\begin{matrix}\mathscr{D} \\ A \wedge A\end{matrix}}{A}\wedge\mathrm{E}_1 \qquad \dfrac{\begin{matrix}\mathscr{D} \\ A \wedge A\end{matrix}}{A}\wedge\mathrm{E}_1}{A\wedge A}$$

If we suppose that $\mathscr{D}$ ends with an introduction rule, i.e. that the form of $\mathscr{D}$ is the following (for some arbitrary derivations $\mathscr{D}_1$ and $\mathscr{D}_2$ of A):

$$\dfrac{\begin{matrix}\mathscr{D}_1 \\ A\end{matrix} \qquad \begin{matrix}\mathscr{D}_2 \\ A\end{matrix}}{A\wedge A}\wedge\mathrm{I}$$

we thus have that

$$\begin{matrix}\mathscr{D}' \\ A\wedge A\end{matrix} \qquad \overset{\texttt{Ekman}^*}{\rhd} \qquad \dfrac{\begin{matrix}\mathscr{D}_1 \\ A\end{matrix} \qquad \begin{matrix}\mathscr{D}_2 \\ A\end{matrix}}{A \wedge A}\wedge\mathrm{I}$$

and

$$\begin{array}{c}\mathscr{D}'\\ A\wedge A\end{array} \quad \overset{\supset\beta}{\rhd} \quad \dfrac{\dfrac{\dfrac{\begin{array}{cc}\mathscr{D}_1 & \mathscr{D}_2\\ A & A\end{array}}{A\wedge A}\wedge\mathrm{I}}{A}\wedge\mathrm{E}_1 \qquad \dfrac{\dfrac{\begin{array}{cc}\mathscr{D}_1 & \mathscr{D}_2\\ A & A\end{array}}{A\wedge A}\wedge\mathrm{I}}{A}\wedge\mathrm{E}1}{A\wedge A}\wedge\mathrm{I} \quad \overset{\wedge\beta_1}{\rhd} \quad \dfrac{\begin{array}{cc}\mathscr{D}_1 & \mathscr{D}_1\\ A & A\end{array}}{A\wedge A}\wedge\mathrm{I}$$

respectively. Therefore the adoption of Ekman's (starred) reduction implies that the following two derivations

$$\dfrac{\begin{array}{cc}\mathscr{D}_1 & \mathscr{D}_1\\ A & A\end{array}}{A\wedge A} \qquad\qquad \dfrac{\begin{array}{cc}\mathscr{D}_1 & \mathscr{D}_2\\ A & A\end{array}}{A\wedge A}$$

are equivalent with respect to reducibility, i.e. that they represent the same proof. This means that also the following two derivations, which result from the previous ones by extending each of them with an application of $\wedge\mathrm{E}_2$:

$$\dfrac{\dfrac{\begin{array}{cc}\mathscr{D}_1 & \mathscr{D}_1\\ A & A\end{array}}{A\wedge A}}{A}\wedge\mathrm{E}_2 \qquad\qquad \dfrac{\dfrac{\begin{array}{cc}\mathscr{D}_1 & \mathscr{D}_2\\ A & A\end{array}}{A\wedge A}}{A}\wedge\mathrm{E}_2$$

are equivalent with respect to reducibility. If we apply to each of them the reduction $\wedge\beta_2$, we obtain the two derivations

$$\begin{array}{c}\mathscr{D}_1\\ A\end{array} \qquad\qquad \begin{array}{c}\mathscr{D}_2\\ A\end{array}$$

meaning that $\mathscr{D}_1$ and $\mathscr{D}_2$ belong to the same equivalence class induced by $\supset\beta$, $\wedge\beta_1$, $\wedge\beta_2$ and `Ekman`*. Therefore by using Ekman's (starred) reduction in addition to the standard reductions, we can show that any two derivations $\mathscr{D}_1$ and $\mathscr{D}_2$ of a formula A represent the same proof. As argued above, this is a devastating consequence. If we require that reductions do not trivialize the notion of identity of proofs, Ekman's transformation does not count as a reduction.

We thus propose to amend Tennant's paradoxicality criterion by requiring that reductions do not trivialize identity of proofs. In this way the problem posed by Ekman's result for the Prawitz-Tennant test for paradoxicality is resolved in that Ekman's derivation E now fails to count as a paradox.[8]

Ekman's "paradox" not only teaches us the importance of an appropriate notion of reduction for formulating a proof-theoretic criterion of paradoxicality, but also tells us something about the nature of paradoxical propositions. What triggers a genuine paradox is not simply the assumption that a proposition is *interderivable* with its own negation, as in Ekman's derivation. A genuine paradox is a proposition A such that there are proofs from A to $\neg A$ and from $\neg A$ to A that composed with each other give us the identity proof A (i.e., the formula A considered as proof of A from A), that is a genuine paradox is a proposition which is *isomorphic* (in the sense of Sect. 2.5) with its own negation.

In conclusion, if one requires that reductions must not trivialize proof identity, there are strong reasons to reject Ekman's reduction. Unfortunately, there does not seem to be any immediate way of applying this strategy to solve the other kind of overgeneration cases observed by Crabbé. From the perspective of identity of proof, there is a strong asymmetry between the two cases, and the overgeneration cases observed by Crabbé show a particular resilience.

6.4 Von Plato's Solution to Ekman

A different kind of solution to the overgeneration phenomenon observed by Ekman was put forward by von Plato [62] and recently reinstated by Tennant [112], who showed how it could be used to overcome also the other kind of overgeneration cases. For Tennant, this alternative solution is preferable not only because it allows one to solve both issues at once, but also because it does not require one to introduce criteria to select what counts as an appropriate reduction. The aim of what follows is a critical discussion of this alternative solution, which shows, at least, that the question of what is to count as an appropriate reduction cannot be evaded so quickly as Tennant apparently supposes.

To clarify our position, we are strongly sympathetic to the Tennant-Prawitz analysis of paradoxes, and we do not take the kind of overgeneration observed by Ekman as being a real threat, provided the criterion for paradoxality is based on a qualified notion of reduction procedure. On the other hand, we do regard the kind of overgeneration observed by Crabbé as problematic (even on our refined formulation of the Prawitz-Tennant criterion) and calling for further investigations.

What we are not at all sympathetic with is the "solution" to both kinds of overgeneration proposed by von Plato and Tennant, which will be shown in the remaining part of the present chapter to be, in fact, no solution at all, being flawed by the same problems of Tennant's original proposal. The line of argument developed in the remaining part of the chapter is thereby meant as a further—though indirect—reason to adopt our solution to the Ekman kind of overgeneration, and to further investigate the exact nature of the Crabbé kind of overgeneration.

According to von Plato [62] the source of Ekman's problem[9] is the form of the elimination rule for implication, and he suggested that the problem could be solved by replacing $\supset$E with its general version:

$$\dfrac{A \supset B \qquad A \qquad \begin{matrix}[B] \\ C\end{matrix}}{C}\supset\mathrm{E}_g$$

We call $\mathtt{NI}^{\supset}_{g}$ the system obtained from $\mathtt{NI}^{\supset}$ by replacing $\supset$E with $\supset\mathrm{E}_g$.

Consecutive applications of the introduction and of the general elimination rule for implication also constitute a redundancy that can be eliminated according to the following reduction (we call the formula of the form $A \supset B$ eliminated by the reduction an $\supset_g$*-maximal formula occurrence*):

$$\dfrac{\langle n\rangle\;\dfrac{\begin{matrix}\overset{n}{[A]}\\ \mathscr{D}\\ B\end{matrix}}{A \supset B} \qquad \begin{matrix}\mathscr{D}'\\ A\end{matrix} \qquad \begin{matrix}\overset{m}{[B]}\\ \mathscr{D}''\\ C\end{matrix}}{C}\,\langle m\rangle \quad \overset{\supset_g\beta}{\rhd} \quad \begin{matrix}\mathscr{D}'\\ [A]\\ \mathscr{D}\\ [B]\\ \mathscr{D}''\\ \mathrm{C}\end{matrix}$$

In $\mathtt{NI}_g^{\supset}$ the derivation of Ekman's paradox can be recast as follows:

$$\langle 6\rangle\,\dfrac{\neg A \supset A \qquad \langle 3\rangle\,\dfrac{\langle 2\rangle\,\dfrac{A \supset \neg A \qquad \overset{3}{A} \qquad \langle 1\rangle\,\dfrac{\overset{2}{\neg A} \quad \overset{3}{A} \quad \overset{1}{\bot}}{\bot}}{\bot}}{\neg A} \qquad \langle 5\rangle\,\dfrac{A \supset \neg A \qquad \overset{6}{A} \qquad \langle 4\rangle\,\dfrac{\overset{5}{\neg A} \quad \overset{6}{A} \quad \overset{4}{\bot}}{\bot}}{\bot}}{\bot} \qquad (\mathbf{E}'_{\mathbf{g}})$$

The reduction $\supset_g\beta$ does not apply to $\mathbf{E}'_{\mathbf{g}}$. Moreover, neither does `Ekman` (obviously, since $\mathbf{E}'_{\mathbf{g}}$ is formulated with the general elimination rule and not with modus ponens) nor any generalization thereof,

> which have the general form of 'shrinking' to a single occurrence of A, any logically circular segments of branches (within the proof) of the form shown below to the left
>
> $$\begin{matrix}\dfrac{A}{B_1}\\ \vdots\\ \dfrac{B_n}{A}\end{matrix} \quad \rhd A$$
>
> (Tennant [111], pp. 199–200)

which Tennant [112] calls *subproof compactification*. Note that, as Ekman [14] already observed, $\in$-`Red` and $\in^z$-`Red` are also instances of subproof compactification (and so are the standard reductions for conjunction of Prawitz [65]), though neither $\supset\beta$ nor $\supset_g\beta$ are.

On these grounds, von Plato concludes that

> the problem about normal form in Ekman [15] is solved by a derivation using the general $\supset$E rule. (von Plato [62], p. 123)

6.5 Another "Safe Version" of Russell's Paradox

Independently of Crabbé, Tennant [108, 109] proposed a weakening of naive comprehension, but in the context of a (negative) free logic. By free logic, one means a logic which is free from the assumption that singular terms denote. Using Zermelo's comprehension one wishes to neutralize Russell's paradox by recasting Russell's reasoning as showing that no set contains its Russell subset as element. Similarly, Tennant wishes to recast Russell's reasoning as showing that Russell's term lacks a denotation.

That a term t does possess a denotation is expressed by the formula $\exists! t =_{def} \exists x(t = x)$, and accordingly in (negative) free logic the introduction rule for identity is weakened to the effect that $t = t$ can be derived only if one has previously shown that t denotes[10]:

$$\frac{\exists! t}{t = t}$$

In this setting, Tennant proposes to replace the rules for naive comprehension with rules to introduce and eliminate set terms in the context of identity statements. As our focus here is not a discussion of set-theoretic paradoxes in the context of free logics, but rather of the proof-theoretic analysis of paradoxical derivations, we omit some premises and dischargeable assumptions of the form $\exists! t$ from Tennant's rule. In this way it is easier to highlight the analogy with von Plato's solution to Ekman's paradox discussed in the previous section. Observe however that each derivation using the rules here discussed should be understood as an abbreviation of a derivation using Tennant's original rules. The interested reader can easily reconstruct full derivations by adding the (in most cases trivial) subderivations of each of the missing premises of each rule application.

Here is the simplified version of Tennant's rules:

$$\frac{\begin{matrix}[A(y/x)] & & [y \in s] \\ y \in s & & A(y/x)\end{matrix}}{\{x : A\} = s}\ \{\}^{=}\mathrm{I} \qquad \frac{\{x : A\} = s \qquad A(t/x)}{t \in s}\ \{\}^{=}\mathrm{E}_1$$

with y *eigenvariable*

$$\frac{\{x : A\} = s \qquad t \in s}{A(t/x)}\ \{\}^{=}\mathrm{E}_2$$

We call $\mathtt{NI}^{\supset\in=}$ the system that results by adding these rules to $\mathtt{NI}^{\supset\in}$.[11]

It is important to observe that in Tennant's reformulation of the rules of comprehension we have two elimination rules for set terms, and the two elimination rules of Tennant correspond respectively to Prawitz's $\in$I and $\in$E rules. By taking s to be $\{x : A\}$ we have that Tennant's $\{\}^{=}\mathrm{E}_1$ allows one to infer $t \in \{x : A\}$ from $A(t/x)$ *together with* the premise $\{x : A\} = \{x : A\}$, and that $\{\}^{=}\mathrm{E}_2$ allows one to infer $A(t/x)$ from $t \in \{x : A\}$ *together with* the premise $\{x : A\} = \{x : A\}$. (The extra premises can be regarded as expressing the requirement that $\{x : A\}$ is a denoting set term: are remarked above, in contrast to predicate logic in which $t = t$ is always derivable for any term t, in free logic such premises need to be derived from the assumption $\exists!\{x : A\}$.)

Redundancies constituted by consecutive applications of the introduction rule followed immediately by the corresponding elimination rule can be eliminated using the obvious reductions. Moreover, consecutive applications of the two elimination rules give rise to Ekmanesque redundancies of which one can get rid using the following reduction (we call respectively $\mathtt{Ekman}^{=}$ *and* $\mathtt{Ekman}^{=}$*-maximal formula* *occurrence* this transformation and the occurrence of $t \in s$ in the schematic derivation on the left-hand side):

$$\dfrac{\{x : A\} = s \qquad \dfrac{\{x : A\} = s \qquad \begin{array}{c}\mathscr{D}\\ A(t/x)\end{array}}{t \in s}\{\}^{=}\mathrm{E}_1}{A(t/x)}\{\}^{=}\mathrm{E}_2 \qquad \overset{\texttt{Ekman}^{=}}{\rhd} \qquad \begin{array}{c}\mathscr{D}\\ A(t/x)\end{array}$$

To reconstruct Russell's reasoning in this further setting Tennant suggests to choose both t and s to be some variable y and to take A to be the formula $\neg(x \in x)$. One thereby obtains the following instances of $\{\}^{=}\mathrm{E}_1$, $\{\}^{=}\mathrm{E}_2$ (as before we abbreviate Russell's term $\{x : \neg(x \in x)\}$ with r):

$$\dfrac{r = y \qquad \neg(y \in y)}{y \in y}\{\}^{=}\mathrm{E}_1 \qquad\qquad \dfrac{r = y \qquad y \in y}{\neg(y \in y)}\{\}^{=}\mathrm{E}_2$$

By abbreviating $y \in y$ with υ, we can reason as in Ekman's derivation **E** and thereby construct a derivation of $\bot$ depending on the assumption $r = y$:

$$\dfrac{\langle 1\rangle\,\dfrac{\dfrac{\dfrac{r = y \qquad \overset{1}{\upsilon}}{\neg\upsilon}\{\}^{=}\mathrm{E}_2 \qquad \overset{1}{\upsilon}}{\bot}\supset\mathrm{E}}{\neg\upsilon}\supset\mathrm{I} \qquad \dfrac{r = y \qquad \langle 1\rangle\,\dfrac{\dfrac{\dfrac{r = y \qquad \overset{1}{\upsilon}}{\neg\upsilon}\{\}^{=}\mathrm{E}_2 \qquad \overset{1}{\upsilon}}{\bot}\supset\mathrm{E}}{\neg\upsilon}\supset\mathrm{I}}{\upsilon}\{\}^{=}\mathrm{E}_1}{\bot}\supset\mathrm{E} \qquad (\mathbf{R}^{=})$$

As in $\mathbf{R}^{\mathbf{z}}$, the variable y in $\mathbf{R}^{=}$ occurs free neither in the conclusion nor in any undischarged assumption other than $r = y$. In the presence of the rules for the existential quantifier, the derivation $\mathbf{R}^{=}$ can thus be extended by $\exists$E and $\supset$I to a closed derivation of $\neg\exists x(r = x)$ that establishes that r has no denotation.

However, as in Ekman's **E**, the encircled occurrence of $\neg\upsilon$ is a maximal formula occurrence. The reader can easily check that by getting rid of it using $\supset\beta$, an $\texttt{Ekman}^{=}$-redundant formula occurrence is introduced. By getting rid of it using the following instance of $\texttt{Ekman}^{=}$:

$$\dfrac{r = y \qquad \dfrac{r = y \qquad \begin{array}{c}\mathscr{D}\\ \neg\upsilon\end{array}}{\upsilon}\{\}^{=}\mathrm{E}_1}{\neg\upsilon}\{\}^{=}\mathrm{E}_2 \qquad \overset{\texttt{Ekman}^{=}}{\rhd} \qquad \begin{array}{c}\mathscr{D}\\ \neg\upsilon\end{array}$$

one gets back to $\mathbf{R}^{=}$. As in the previous cases, in spite of its innocuous character the derivation fails to normalize. This overgeneration case seems a perfect blend of the two previously discussed, and Tennant [112] showed how the (purported) solution of von Plato to Ekman's case can be applied also to this one. Let us replace the elimination rules $\{\}^{=}\mathrm{E}_1$ and $\{\}^{=}\mathrm{E}_2$ with their general versions (we call the resulting system $\texttt{NI}_g^{\supset\in=}$):

$$\dfrac{\{x : A\} = u \qquad A(t/x) \qquad \begin{matrix}[t \in u] \\ C\end{matrix}}{C}\,\{\}^{=}\mathrm{E}_{1g}$$

$$\dfrac{\{x : A\} = u \qquad t \in u \qquad \begin{matrix}[A(t/x)] \\ C\end{matrix}}{C}\,\{\}^{=}\mathrm{E}_{2g}$$

Taking as before t and s to be y and A to be $\neg(x \in x)$, one thereby obtains the following instances of $\{\}^{=}\mathrm{E}_{1g}$, and $\{\}^{=}\mathrm{E}_{2g}$ (as before r abbreviates Russell's term $\{x : \neg(x \in x)\}$):

$$\dfrac{r = y \qquad \neg(y \in y) \qquad \begin{matrix}[y \in y] \\ C\end{matrix}}{C}\,\{\}^{=}\mathrm{E}_{1g} \qquad\qquad \dfrac{r = y \qquad y \in y \qquad \begin{matrix}[\neg(y \in y)] \\ C\end{matrix}}{C}\,\{\}^{=}\mathrm{E}_{2}$$

Using them one can give the following (apparently) redundancy-free derivation of $\bot$ from $r = y$ (as before υ abbreviates $y \in y$)[12]:

$$\langle 6\rangle\,\dfrac{r = y \qquad \langle 3\rangle\,\dfrac{\langle 2\rangle\,\dfrac{r = y \qquad \overset{3}{\upsilon} \qquad \langle 1\rangle\,\dfrac{\overset{2}{\neg\upsilon} \quad \overset{3}{\upsilon} \quad \overset{1}{\bot}}{\bot}}{\bot}}{\neg\upsilon} \qquad \langle 5\rangle\,\dfrac{r = y \qquad \overset{6}{\upsilon} \qquad \langle 4\rangle\,\dfrac{\overset{5}{\neg\upsilon} \quad \overset{6}{\upsilon} \quad \overset{4}{\bot}}{\bot}}{\bot}}{\bot} \qquad (\mathbf{R}_{\mathbf{g}}^{=\prime})$$

An application of $\exists$E followed by one of $\supset$I yields a closed normal derivation of $\neg\exists x(r = x)$.

6.6 Ekman on Decomposing Inferences

Although we believe that the Prawitz-Tennant analysis undoubtedly provides the basis for a proof-theoretic clarification of the phenomenon of paradoxes, we do not find the way out of the overgeneration cases proposed by von Plato and Tennant satisfactory.

It is true that in the derivations $\mathbf{E}_{\mathbf{g}}'$ and $\mathbf{R}_{\mathbf{g}}^{=\prime}$ no subproof compactification is possible. However, as we will now show, it is still possible to detect some redundancies which are *hidden* by the more involved shape of derivations constructed with general elimination rules. By defining procedures to get rid of these hidden redundancies, Ekmanesque loops will crop up again. In the remaining part of the chapter this suggestion will be made precise.

The possibility of reformulating his "paradox" using general elimination rules was clearly envisaged by Ekman in his doctoral thesis, where he introduces the notion of 'decomposing inference':

> Let Π and A designate the premise deductions and conclusion of a rule R respectively. That is, R is the inference schema:
>
> $$\frac{\Pi}{A}\,R$$
>
> We obtain the corresponding decomposing inference schema R_D as follows.
>
> $$\dfrac{\Pi \qquad \begin{matrix}[A]\\ \mathscr{E}\\ C\end{matrix}}{C}\,R_D$$
>
> We obtain the premise deductions of the inference schema R_D by adding one deduction $\mathscr{E}$ to the premise deductions of the R schema, where $\mathscr{E}$ designates a deduction in which occurrences of the conclusion A of the R schema, as open assumptions in $\mathscr{E}$ may be cancelled at the R_D inference. If, in the R schema, B designates an open assumption in any of the premise deductions Π and B may be cancelled at the R inference, then in the R_D schema, B also designates an open assumption of the same premise deduction and B may be cancelled at the R_D inference. (Ekman [14], pp. 9–10)

Obviously, in the case of $\supset$E, the decomposing inference $\supset\mathrm{E}_D$ is just the general rule $\supset\mathrm{E}_g$.[13]

Ekman [14, p. 10] introduces the notion of a *simple deduction corresponding to one with decomposing inferences* by giving an informal, though precise description of a procedure for translating derivations with decomposing inferences into derivations with the corresponding "simple" inferences. When restricted to the systems $\mathtt{NI}^{\supset}$ and $\mathtt{NI}^{\supset}_{g}$, Ekman's translation amounts to the following (the definition is by induction on the number of inference rules applied in a derivation)[14]:

1. If $\mathscr{D}$ is an assumption, then $\mathscr{D}^s = \mathscr{D}$
2. If $\mathscr{D}$ ends with an application of $\supset\mathrm{E}_g$, i.e. it is of the following form:

$$\langle n\rangle\;\dfrac{\begin{matrix}\mathscr{D}_1\\ A\supset B\end{matrix} \qquad \begin{matrix}\mathscr{D}_2\\ A\end{matrix} \qquad \begin{matrix}\overset{n}{[B]}\\ \mathscr{D}_3\\ C\end{matrix}}{C}\,\supset\mathrm{E}_g$$

then $\mathscr{D}^s$ has the following form:

$$\begin{matrix}\dfrac{\begin{matrix}\mathscr{D}_1^s\\ A\supset B\end{matrix} \qquad \begin{matrix}\mathscr{D}_2^s\\ A\end{matrix}}{[B]}\,\supset\mathrm{E}\\ \mathscr{D}_3^s\\ C\end{matrix}$$

3. If $\mathscr{D}$ ends with an application of $\supset$I, then $\mathscr{D}^s$ is obtained by applying $\supset$I to the translation $\mathscr{D}_1^s$ of the immediate subderivation $\mathscr{D}_1$ of $\mathscr{D}$.

   At this point Ekman writes:

> Let $\mathscr{H}$ and $\mathscr{H}'$ be a deduction with decomposing inferences and its corresponding simple deduction, respectively. Then indeed, $\mathscr{H}$ and $\mathscr{H}'$ both represent the same informal argument. The difference is only a matter of the display of the inferences. Therefore it ought to be the case that $\mathscr{H}$ is normal if and only if $\mathscr{H}'$ is normal. (Ekman [14], p. 13)

The translation $(\mathbf{E}'_\mathbf{g})^s$ of von Plato's derivation $\mathbf{E}'_\mathbf{g}$ into $\mathtt{NI}^{\supset}$ is indeed $\mathbf{E}'$. It is beyond doubt that the quoted passage hints at the possibility of extending the set of conversions of $\mathtt{NI}^{\supset}_g$ so that on the extended set of conversions von Plato's derivation $\mathbf{E}'_\mathbf{g}$ fails to normalize as well.

To this we now turn.

6.7 Implication-as-Link and General Ekman-Reductions

As a starting point, we recall Schroeder-Heister's [88] proposal to distinguish between two ways in which the assumption of an implication can be interpreted: Implication-as-rule and implication-as-link. In natural deduction the two interpretations correspond to the two distinct forms that the rule of implication elimination may take (see also [92]).

The adoption of $\supset$E yields the implication-as-rule interpretation. Suppose we have a derivation $\mathscr{D}$ of conclusion A. By assuming the implication $A \supset B$ we can extend $\mathscr{D}$ as if we had at our disposal a rule R allowing one to pass over from A to B:

$$\dfrac{A \supset B \qquad \begin{matrix}\mathscr{D}\\ A\end{matrix}}{B}\supset\text{E} \qquad \dfrac{\begin{matrix}\mathscr{D}\\ A\end{matrix}}{B}\, R : A \Rightarrow B$$

On the other hand, the adoption of $\supset$E$_g$ does not amount to assuming only the rule to pass over from A to B, but rather to assuming also the existence of a *link* connecting two distinct derivations:

$$\langle n\rangle\ \dfrac{A \supset B \qquad \begin{matrix}\mathscr{D}\\ A\end{matrix} \qquad \begin{matrix}\overset{n}{[B]}\\ \mathscr{D}'\\ C\end{matrix}}{C}\supset\text{E}_g \qquad \begin{matrix}\mathscr{D}\\ \dfrac{A}{B}\, R : A \Rightarrow B\\ \vdots\\ [B]\\ \mathscr{D}'\\ C\end{matrix}$$

Applications of the rule R correspond—even graphically—to the application of $\supset$E. This is not so in $\supset$E$_g$, where there is nothing in the structure of the rule which can be said to correspond to the application of the rule to pass from A to B. The transition from A to B remains implicit.

The implicit link in $\supset\mathrm{E}_g$ between the two subderivations $\mathscr{D}$ and $\mathscr{D}'$ is a form of transitivity: if B can be derived by means of $\mathscr{D}$ from a set of assumptions Γ (among which the rule R that allows one to pass over from A to B), and C can be derived by means of $\mathscr{D}'$ from some other set of assumptions Δ together with (a certain number of copies of) B, then C can be derived from Γ and Δ alone.

We wish to defend the claim that the transitivity principle encoded by $\supset\mathrm{E}_g$ *hides* a redundancy in the derivation $\mathbf{E}'_{\mathbf{g}}$. In fact, Ekman [14] himself refers to decomposing rules as 'cut-hiding'.

To state this intuition in a more explicit manner we take seriously the idea that in $\supset\mathrm{E}_g$ the minor premise A is linked with the assumptions of form B which are discharged by the application of the rule.

Certain configurations of two consecutive applications of $\supset\mathrm{E}_g$ may thus be viewed as constituting a redundancy. Consider for instance situations of the following kind:

$$\langle m\rangle\,\dfrac{A \supset B \qquad \begin{array}{c}\mathscr{D}\\ A\end{array} \qquad \langle n\rangle\,\dfrac{B \supset A \qquad \overset{m}{B} \qquad \begin{array}{c}\overset{n}{[A]}\\ \mathscr{D}'\\ C\end{array}}{C}}{C}$$

The formula A which is the conclusion of $\mathscr{D}$ is linked by $A \supset B$ to the discharged occurrence of B marked with m. This in turn is linked by $B \supset A$ to the discharged assumptions A marked by n. In other words, the two applications of the general elimination rule make one jump from A to B and back in a quite unnecessary way. This intuition, which is essentially Ekman's, can be spelled out by defining a new conversion to get rid of redundancies of this kind.

By directly linking together $\mathscr{D}$ and $\mathscr{D}'$, both applications of $\supset\mathrm{E}_g$ could be eliminated as follows:

$$\begin{array}{c}\mathscr{D}\\ [A]\\ \mathscr{D}'\\ C\end{array}$$

However, this is only possible if in the original derivation no other occurrence of B is discharged in $\mathscr{D}'$ by the application of $\supset\mathrm{E}_g$ marked with m.

If such occurrences of B are present, then the lower application of $\supset\mathrm{E}_g$ is still needed in order to discharge them. This is perfectly reasonable, since these occurrences of B do not belong to the detour generated by the links of the two applications of $\supset\mathrm{E}_g$. We take the following reduction to be what in $\mathtt{NI}_g^{\supset}$ corresponds to $\mathtt{Ekman}$ (the occurrence of B in the leftmost derivation constituting the redundancy is encircled and will be called an $\mathtt{Ekman}_g$*-maximal formula occurrence*):

$$\langle n\rangle\ \dfrac{A \supset B \qquad \begin{matrix}\mathscr{D}\\ A\end{matrix} \qquad \langle m\rangle\ \dfrac{B \supset A \qquad \overset{n}{B} \qquad \begin{matrix}\overset{m}{[A]}\ \overset{n}{[B]}\\ \mathscr{D}'\\ C\end{matrix}}{C}\supset\mathrm{E}_g}{C}\supset\mathrm{E}_g \qquad \overset{\mathtt{Ekman}_g}{\rhd}$$

$$\overset{\mathtt{Ekman}_g}{\rhd} \qquad \langle n\rangle\ \dfrac{A \supset B \qquad \begin{matrix}\mathscr{D}\\ A\end{matrix} \qquad \begin{matrix}\mathscr{D}\\ [A]\ \overset{n}{[B]}\\ \mathscr{D}'\\ C\end{matrix}}{C}\supset\mathrm{E}_g$$

Observe now that von Plato's $\mathbf{E}_{\mathbf{g}}'$ contains an $\mathtt{Ekman}_g$-redundant formula occurrence (encircled):

$$\langle 6\rangle\ \dfrac{\neg A \supset A \qquad \langle 3\rangle\ \dfrac{\langle 2\rangle\ \dfrac{A \supset \neg A \qquad \overset{3}{A} \qquad \langle 1\rangle\ \dfrac{\overset{2}{\neg A} \quad \overset{3}{A} \quad \overset{1}{\bot}}{\bot}}{\bot}}{\neg A} \qquad \langle 5\rangle\ \dfrac{A \supset \neg A \qquad \overset{6}{A} \qquad \langle 4\rangle\ \dfrac{\overset{5}{\neg A} \quad \overset{6}{A} \quad \overset{4}{\bot}}{\bot}}{\bot}}{\bot} \qquad (\mathbf{E}_{\mathbf{g}}')$$

The redundancy can be eliminated using the following instance of $\mathtt{Ekman}_g$:

$$\langle n\rangle\ \dfrac{\neg A \supset A \qquad \begin{matrix}\mathscr{D}\\ \neg A\end{matrix} \qquad \langle m\rangle\ \dfrac{A \supset \neg A \qquad \overset{n}{A} \qquad \begin{matrix}\overset{m}{[\neg A]}\ \overset{n}{[A]}\\ \mathscr{D}'\\ \bot\end{matrix}}{\bot}}{\bot} \qquad \overset{\mathtt{Ekman}_g}{\rhd} \qquad \langle n\rangle\ \dfrac{\neg A \supset A \qquad \begin{matrix}\mathscr{D}\\ \neg A\end{matrix} \qquad \begin{matrix}\mathscr{D}\\ [\neg A]\ \overset{n}{[A]}\\ \mathscr{D}'\\ \bot\end{matrix}}{\bot}$$

By applying this instance of $\mathtt{Ekman}_g$ to $\mathbf{E}_{\mathbf{g}}'$ one obtains the following derivation:

$$\langle 8\rangle\ \dfrac{\neg A \supset A \qquad \langle 3\rangle\ \dfrac{\langle 2\rangle\ \dfrac{A \supset \neg A \qquad \overset{3}{A} \qquad \langle 1\rangle\ \dfrac{\overset{2}{\neg A} \quad \overset{3}{A} \quad \overset{1}{\bot}}{\bot}}{\bot}}{\neg A} \qquad \langle 7\rangle\ \dfrac{\langle 6\rangle\ \dfrac{\langle 5\rangle\ \dfrac{A \supset \neg A \qquad \overset{6}{A} \qquad \langle 4\rangle\ \dfrac{\overset{5}{\neg A} \quad \overset{6}{A} \quad \overset{4}{\bot}}{\bot}}{\bot}}{\neg A} \qquad \overset{8}{A} \quad \overset{7}{\bot}}{\bot}}{\bot} \qquad (\mathbf{E}_{\mathbf{g}})$$

The encircled occurrence of $\neg A$ is the conclusion of an application of $\supset$I and the major premise of an application of $\supset\mathrm{E}_g$ and thus it is an $\supset_g$-redundant formula occurrence. By applying $\supset_g\beta$ to this derivation, one gets back to $\mathbf{E}_{\mathbf{g}}'$. That is, by enriching the set of conversions with $\mathtt{Ekman}_g$, the process of normalizing the derivations $\mathbf{E}_{\mathbf{g}}$ and $\mathbf{E}_{\mathbf{g}}'$ gets stuck in a loop in the same way as that of the derivations $\mathbf{E}$ and $\mathbf{E}'$. As already observed in Sect. 6.6, $\mathbf{E}'$ is the image of $\mathbf{E}_{\mathbf{g}}'$ under the translation $()^s$ from $\mathtt{NI}_g^{\supset}$ to $\mathtt{NI}^{\supset}$, and as the reader can easily check the same is true of $\mathbf{E}$ and $\mathbf{E}_{\mathbf{g}}$.

It is easy to see that the foregoing line of reasoning can be extended in a straightforward manner to Tennant's derivation $\mathbf{R}_\mathbf{g}^{=}$ in $\mathtt{NI}_g^{\supset\in^=}$. In particular, the remarks on the cut-hiding nature of $\supset\mathrm{E}_g$ can be applied to Tennant's $\{\}^{=}\mathrm{E}_1$ and $\{\}^{=}\mathrm{E}_2$ as well. Hence, we can define the general version of the $\mathtt{Ekman}^{=}$-reduction depicted in Table 6.1.

The derivation $\mathbf{R}_\mathbf{g}^{=\prime}$, like $\mathbf{E}_\mathbf{g}^{\prime}$, contains a hidden redundancy that can be eliminated using $\mathtt{Ekman}^{=}$. As the reader can check, by applying the reduction one obtains a derivation that, like $\mathbf{E}_\mathbf{g}$, contains an $\supset_g$-redundant formula occurrence. By eliminating it using $\supset_g\beta$ one gets back to Tennant's $\mathbf{R}_\mathbf{g}^{=\prime}$. Moreover, the translation $()^s$, mapping $\mathtt{NI}_g^{\supset}$-derivations onto $\mathtt{NI}^{\supset}$-derivations, can be easily extended to a translation $()^{s^\in}$ mapping $\mathtt{NI}_g^{\supset\in^=}$-derivations onto $\mathtt{NI}^{\supset\in^=}$-derivations. The image of Tennant's derivation $\mathbf{R}_\mathbf{g}^{=\prime}$ and of the derivation to which $\mathbf{R}_\mathbf{g}^{=\prime}$ reduces via $\mathtt{Ekman}_g$ are Tennant's [108, 109] derivations $\mathbf{R}^{=\prime}$ and $\mathbf{R}^{=}$ respectively.

6.8 Copy-and-Paste Subproof Compactification

As observed by Tennant, both $\mathtt{Ekman}$ and $\mathtt{Ekman}^{=}$ are instances of the general reduction pattern called by Tennant subproof compactification. Crudely put, the adoption of general elimination rules has the result of chopping up derivations and scattering around their subderivations. As a consequence, it is natural to generalize subproof compactification to a reduction pattern that could be called *copy-and-paste subproof compactification*: if a derivation $\mathscr{D}$ contains a subderivation $\mathscr{D}'$ of A and some assumptions of the form A are discharged in $\mathscr{D}$, the result of replacing $\mathscr{D}'$ for the discharged assumptions of A may bring to light hidden possibilities of applying subproof compactification. Although some subderivations may have to be copied in the process, the overall result will be a derivation depending on less assumptions than the original one and containing less (explicit or implicit) redundancies.

Instances of copy-and-paste sub-proof compactification are not just the conversions $\mathtt{Ekman}_g$ and $\mathtt{Ekman}_g^{=}$, but also all other known reductions, in particular $\supset\beta$ (and $\supset_g\beta$), that could be analyzed as consisting of one (respectively two) step(s) of "copy-and-paste", where the "copy-and-paste" operation could be schematically depicted as follows:

$$\dfrac{\overset{n}{[A]} \qquad \begin{matrix}\mathscr{D}'\\ A\end{matrix}}{\begin{matrix}\mathscr{D}\\ C\end{matrix}} \quad \rhd \quad \dfrac{\begin{matrix}\mathscr{D}'\\ [A]\end{matrix} \qquad \begin{matrix}\mathscr{D}'\\ A\end{matrix}}{\begin{matrix}\mathscr{D}\\ C\end{matrix}}$$

followed by one step of subproof compactification. In the case of $\supset$-$\mathtt{Red}$, we would have:

Table 6.1 The reduction $\mathtt{Ekman}^{=}_{g}$

$$\langle n\rangle\,\dfrac{\{x:A\}=s \qquad \begin{matrix}\mathscr{D}\\ A(t/x)\end{matrix} \qquad \langle m\rangle\,\dfrac{\{x:A\}=s \qquad \overset{n}{t\in s} \qquad \begin{matrix}\overset{m}{[A(t/x)]}\ \overset{n}{[t\in s]}\\ \mathscr{D}'\\ C\end{matrix}}{C}\,\{\}^{=}\mathrm{E}_{2g}}{C}\,\{\}^{=}\mathrm{E}_{1g} \qquad \overset{\mathtt{Ekman}^{=}_{g}}{\triangleright} \qquad \langle n\rangle\,\dfrac{\{x:A\}=s \qquad \begin{matrix}\mathscr{D}\\ A(t/x)\end{matrix} \qquad \begin{matrix}\mathscr{D}\\ [A(t/x)]\ \overset{n}{[t\in s]}\\ \mathscr{D}'\\ C\end{matrix}}{C}\,\{\}^{=}\mathrm{E}_{1g}$$

$$\langle n\rangle\ \dfrac{\dfrac{\begin{matrix}\overset{n}{[A]}\\ \mathscr{D}\\ B\end{matrix}}{A\supset B}\supset\mathrm{I}\qquad \begin{matrix}\mathscr{D}'\\ A\end{matrix}}{B}\supset\mathrm{E}\quad\rhd\quad \dfrac{\dfrac{\begin{matrix}\mathscr{D}'\\ [A]\\ \mathscr{D}\\ B\end{matrix}}{A\supset B}\supset\mathrm{I}\qquad \begin{matrix}\mathscr{D}'\\ A\end{matrix}}{B}\supset\mathrm{E}\quad\rhd\quad \begin{matrix}\mathscr{D}'\\ [A]\\ \mathscr{D}\\ B\end{matrix}$$

6.9 General Introduction Rules and $\mathtt{Ekman}_g$

It may be retorted that, compared to Ekman's original conversion, the conversion $\mathtt{Ekman}_g$ is much less straightforward, and one may wonder whether in the end, it is not just artificial. We rebut this criticism by observing that $\mathtt{Ekman}_g$ is just as plausible as $\mathtt{Ekman}$. Or at least, that this is the case if one (like von Plato himself) is willing to accept not only general elimination rules but general introduction rules as well.

According to Negri and von Plato [53], not only elimination rules, but also introduction rules can be recast in general form, according to the following idea: "General introduction rules state that if a formula C follows from a formula A, then it already follows from the immediate grounds for A; general elimination rules state that if C follows from the immediate grounds for A, then it already follows from A." (*ibid.* 217).[15]

For example, the Prawitz-Gentzen introduction and elimination rules for conjunction are recast in general form as follows:

$$\dfrac{A \qquad B \qquad \begin{matrix}[A\wedge B]\\ C\end{matrix}}{C}\wedge\mathrm{I}_g \qquad\qquad \dfrac{A\wedge B \qquad \begin{matrix}[A][B]\\ C\end{matrix}}{C}\wedge\mathrm{E}_g$$

Milne [47] argued for the significance of these rules for the inferentialist project of characterizing the meaning of logical constants through the inference rules governing them. In this context he suggested reductions to eliminate consecutive applications of the general introduction and elimination rules for a connective. In the case of conjunction, Milne's proposal amounts to the following transformation:

$$\langle n\rangle\ \dfrac{\begin{matrix}\mathscr{D}_1\\ A\end{matrix}\qquad \begin{matrix}\mathscr{D}_2\\ B\end{matrix}\qquad \langle m_1,m_2\rangle\ \dfrac{\overset{n}{A\wedge B}\qquad \begin{matrix}\overset{m_1}{[A]}\ \overset{m_2}{[B]}\\ \mathscr{D}'\\ C\end{matrix}}{C}\wedge\mathrm{E}_g}{C}\wedge\mathrm{I}_g\quad\rhd\quad \begin{matrix}\mathscr{D}_1\ \ \mathscr{D}_2\\ [A][B]\\ \mathscr{D}'\\ C\end{matrix}$$

However, one cannot exclude that the application of the general introduction rule labeled with $\langle n\rangle$ discharges some occurrences of $A\wedge B$ in $\mathscr{D}'$ as well. Such further occurrences (if any) are not part of the redundancy, and the application of $\wedge\mathrm{I}_g$ would still be needed to discharge them. The solution consists in revising Milne's proposed reduction as follows[16]:

$$\langle n\rangle\ \dfrac{\begin{array}{c}\mathscr{D}_1\\ A\end{array}\quad \begin{array}{c}\mathscr{D}_2\\ B\end{array}\quad \langle m_1,m_2\rangle\ \dfrac{\overset{n}{A\wedge B}\qquad \begin{array}{c}[\overset{m_1}{A}]\ [\overset{m_2}{B}]\ [A\overset{n}{\wedge} B]\\ \mathscr{D}'\\ C\end{array}}{C}\,\wedge\mathrm{E}_g}{C}\,\wedge\mathrm{I}_g \quad \overset{\wedge_G\beta}{\rhd} \quad \langle n\rangle\ \dfrac{\begin{array}{c}\mathscr{D}_1\\ A\end{array}\quad \begin{array}{c}\mathscr{D}_2\\ B\end{array}\qquad \begin{array}{c}\mathscr{D}_1\ \ \mathscr{D}_2\\ [A]\,[B]\,[A\overset{n}{\wedge} B]\\ \mathscr{D}'\\ C\end{array}}{C}\,\wedge\mathrm{I}_g$$

The conversion $\wedge_G\beta$ certainly has the flavor of $\mathtt{Ekman}_g$. To spell out the analogy between reductions for general introduction-elimination patterns and $\mathtt{Ekman}_g$ in full, we consider the general version of the introduction and elimination rules for naive set theory of Prawitz:

$$\dfrac{A(t/x)\qquad \begin{array}{c}[t\in\{x:A\}]\\ C\end{array}}{C}\,\in\mathrm{I}_g \qquad\qquad \dfrac{t\in\{x:A\}\qquad \begin{array}{c}[A(t/x)]\\ C\end{array}}{C}\,\in\mathrm{E}_g$$

and the reduction $\in_G\beta$ associated with $\in\mathrm{I}_g$ and $\in\mathrm{E}_g$ (we call $\in_G$-*redundant formula occurrence* the encircled formula occurrence):

$$\langle n\rangle\ \dfrac{\begin{array}{c}\mathscr{D}\\ A(t/x)\end{array}\qquad \langle m\rangle\ \dfrac{\overset{n}{t\in\{x:A\}}\qquad \begin{array}{c}[A(\overset{m}{t}/x)]\ \ [t\in\{\overset{n}{x}:A\}]\\ \mathscr{D}'\\ C\end{array}}{C}\,\in\mathrm{E}_g}{C}\,\in\mathrm{I}_g \quad \overset{\in_G\text{-Red}}{\rhd}$$

$$\overset{\in_G\text{-Red}}{\rhd}\quad \langle n\rangle\ \dfrac{\begin{array}{c}\mathscr{D}\\ A(t/x)\end{array}\qquad \begin{array}{c}\mathscr{D}\\ [A(t/x)]\,[t\in\{\overset{n}{x}:A\}]\\ \mathscr{D}'\\ C\end{array}}{C}\,\in\mathrm{I}_g$$

By removing all occurrences of $\neg A\supset A$ and of $A\supset\neg A$ from von Plato's $\mathbf{E}'_g$ and replacing all occurrences of A with occurrences of ρ, all applications of $\supset\mathrm{E}_g$ with major premises $\neg A\supset A$ and of $A\supset\neg A$ in $\mathbf{E}'_g$ are turned into applications of the following instances of $\in\mathrm{I}_g$ and $\in\mathrm{E}_g$ respectively:

$$\dfrac{\neg\rho\qquad \begin{array}{c}[\rho]\\ C\end{array}}{C}\,\in\mathrm{I}_g \qquad\qquad \dfrac{\rho\qquad \begin{array}{c}[\neg\rho]\\ C\end{array}}{C}\,\in\mathrm{E}_g$$

Thus $\mathbf{E}'_g$ becomes the following derivation of $\bot$ in the system obtained by extending $\mathtt{NI}^{\supset}_g$ with $\in\mathrm{I}_g$ and $\in\mathrm{E}_g$ (we call it $\mathtt{NI}^{\supset\in}_G$)[17]:

$$\langle 6\rangle\,\dfrac{\langle 3\rangle\,\dfrac{\langle 2\rangle\,\dfrac{\overset{3}{\rho}\qquad \langle 1\rangle\,\dfrac{\overset{2}{\neg\rho}\quad \overset{3}{\rho}\quad \overset{1}{\bot}}{\bot}\,\supset\!\mathrm{E}_g}{\bot}\,\in\!\mathrm{E}_g}{\neg\rho}\,\supset\!\mathrm{I}\qquad \langle 5\rangle\,\dfrac{\overset{6}{\rho}\qquad \langle 4\rangle\,\dfrac{\overset{5}{\neg\rho}\quad \overset{6}{\rho}\quad \overset{4}{\bot}}{\bot}\,\supset\!\mathrm{E}_g}{\bot}\,\in\!\mathrm{E}_g}{\bot}\,\in\!\mathrm{I}_g \qquad (\mathbf{R}'_G)$$

The derivation $\mathbf{R}'_G$ in fact contains an $\in_G$-redundant formula occurrence (encircled). To eliminate this redundancy we can apply the following instance of $\in_G$-`Red`:

$$\langle n\rangle\,\dfrac{\begin{matrix}\mathscr{D}\\ \neg\rho\end{matrix}\qquad \langle m\rangle\,\dfrac{\overset{n}{\rho}\qquad \begin{matrix}[\overset{m}{\neg\rho}][\overset{n}{\rho}]\\ \mathscr{D}'\\ C\end{matrix}}{C}\,\in\!\mathrm{E}_g}{C}\,\in\!\mathrm{I}_g \quad \triangleright \quad \langle n\rangle\,\dfrac{\begin{matrix}\mathscr{D}\\ \neg\rho\end{matrix}\qquad \begin{matrix}\mathscr{D}\quad\ \\ [\neg\rho]\,[\overset{n}{\rho}]\\ \mathscr{D}'\\ C\end{matrix}}{C}\,\in\!\mathrm{I}_g$$

As the reader can check, one thereby introduces a new $\supset_g$-redundant formula occurrence. By getting rid of this redundancy using $\supset_g\beta$ one gets back the derivation $\mathbf{R}'_G$ from which one started.

The relation between the relevant instances of $\in_G$-`Red` and of $\texttt{Ekman}_g$ is exactly the same as that between the relevant instances of $\in$-`Red` and of `Ekman`. Thus, as Ekman's reduction can be seen as encoding Russell's paradox in $\texttt{NI}^{\supset}$, the general Ekman reduction we propose can be seen as encoding the version of Russell's paradox with general rules in $\texttt{NI}^{\supset}_g$.

6.10 Conclusions and Outlook

The addition of the conversion `Ekman` to the standard set of reductions for $\texttt{NI}^{\supset}$ (consisting of $\supset\beta$ alone) results in counterexamples to normalization. These can be viewed as simulations in the propositional setting of the counterexamples to normalization in the extension of `NM` with Prawitz's rules for naive set theory. The "safe" version of Russell's paradox proposed by Tennant [108, 109] faces the same problem as soon as one considers—besides reductions to get rid of introduction-elimination redundancies—the further reduction $\texttt{Ekman}^{=}$.

Replacing standard elimination rules with their general versions does not help. As we have shown, it is possible to define general versions of the Ekmanesque reductions, that can be seen as simulating the reduction for general introduction and elimination rules for naive set theory. Using these reductions, Ekman's paradox and Tennant's safe version of Russell's paradox fail to normalize even when formulated using general elimination rules.

As we pointed out in the first part of the chapter, we take these phenomena to call for a thorough investigation of criteria of acceptability for reduction procedures. In particular, we proposed as a (minimal) criterion that reductions must not trivialize the notion of identity of proofs induced by the standard reductions.

On such an understanding of reductions, neither Ekman nor its variants are acceptable, but only reductions to get rid of introduction-elimination patterns. Thus, Ekman's derivations do not qualify as paradoxical, nor does Tennant's safe version of Russell's paradox, independently of whether standard or general rules are adopted.

As remarked, the phenomenon observed by Crabbé is however unaffected by our proposed constraint on reductions, thus showing that further work is required for a thorough analysis of paradoxes along the lines of the Prawitz-Tennant analysis.[18]

Notes to This Chapter

1. Since we will make no use of $\bot$E in this chapter, we assume to be working in extensions of $\mathtt{NI}^{\supset}$, rather than of $\mathtt{NI}^{\supset\bot}$ as we did in the previous chapters, thereby taking $\bot$ to simply be a distinguished atomic proposition.
2. Although this is undoubtedly a case of overgeneration for the Prawitz-Tennant analysis of paradoxes, it is worth stressing that, if we judge whether the derivation $\mathbf{R^z}$ denotes a proof using the notions of validity developed in the previous chapter, we have that although the derivation turns out not to be valid** according to Definition 5.5 it would qualify as valid* according to Definition 5.2. In the present chapter, the issue of validity is left on the background, as the goal is that of sharpening the original Prawitz-Tennant analysis according to which a paradox is a non-normalizing derivation of $\bot$.
3. We observe already now that there is a fundamental distinction between Ekman and, say, $\supset\beta$ in that the latter is a means of getting rid of an application of an introduction rule followed by an application of the corresponding elimination and it thus an immediate consequence of the harmony governing the two rules. Not so for Ekman, that may thus be seen as lacking a *prima facie* plausible meaning-theoretical justification. This remark will be fully exploited in Sect. 6.3 below to untrigger the kind of overgeneration discussed in this section.
4. Observe that the loop is not only a feature of the particular derivations E and E'. Ekman demonstrated that in $\mathtt{NI}^{\supset}$ there is no derivation of $\bot$ from $A \supset \neg A$ and $\neg A \supset A$, which is normal with respect to $\supset\beta$ and Ekman.
5. Elia Zardini observes that the derivations $\mathbf{E}$ and $\mathbf{E}'$ are paradoxical because there are instances of them which are paradoxical. Observe however, that $\mathbf{R}$ and $\mathbf{R}'$ are not simply instances of $\mathbf{E}$ and $\mathbf{E}'$, as they do not arise by simply instantiating A with ρ, but moreover by replacing the assumptions $\neg A \supset A$ and $A \supset \neg A$ with genuine inferential steps, and it is to these steps that the source paradoxicality is—in Tennant's intentions—to be ascribed.
6. As we show in Note 8 even without this generalization a corresponding counterexample can be given.
7. A substantially equivalent counterexample in the purely implicational fragment can be obtained using the formulas $A \supset (A \supset B)$ and $A \supset B$.

8. In defense of Ekman, one might argue that he formulates his reduction with $A \supset B$ and $B \supset A$ in assumption position according to `Ekman`, whereas to show that his reduction trivializes identity of proofs we considered the generalized form `Ekman`*. This generalized form is closed under substitution of derivations for open assumptions. Now it is hard to make sense of a notion of reduction *not* closed under substitution in this sense. However, the following example demonstrates our trivialization result even without this assumption, on the basis of Ekman's reduction in the form `Ekman`. The following derivation (encircled is an Ekman redundant formula):

$$\dfrac{\langle 2\rangle\,\dfrac{\langle 1\rangle\,\dfrac{\overset{1}{A\supset(A\wedge A)}\qquad \dfrac{\overset{2}{(A\wedge A)\supset A}\qquad \dfrac{\begin{matrix}\mathscr{D}_1\\ A\end{matrix}\qquad\begin{matrix}\mathscr{D}_2\\ A\end{matrix}}{A\wedge A}\,{\wedge}\mathrm{I}}{A}\,{\supset}\mathrm{E}}{\dfrac{A\wedge A}{(A\supset(A\wedge A))\supset(A\wedge A)}\,{\supset}\mathrm{I}}\,{\supset}\mathrm{E}}{((A\wedge A)\supset A)\supset(A\supset(A\wedge A))\supset(A\wedge A))}\,{\supset}\mathrm{I}\qquad \langle 3\rangle\,\dfrac{\dfrac{\overset{3}{A\wedge A}}{A}\,{\wedge}\mathrm{E}_1}{(A\wedge A)\supset A}\,{\supset}\mathrm{I}}{\dfrac{(A\supset(A\wedge A))\supset(A\wedge A)\qquad \langle 4\rangle\,\dfrac{\dfrac{\overset{4}{A}\qquad\overset{4}{A}}{A\wedge A}\,{\wedge}\mathrm{I}}{A\supset(A\wedge A)}\,{\supset}\mathrm{I}}{A\wedge A}\,{\supset}\mathrm{E}}\,{\supset}\mathrm{E}$$

reduces via `Ekman` to the following (in which the applications of $\supset$I without numeral do not discharge anything):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\mathscr{D}_1\\ A\end{matrix}\qquad\begin{matrix}\mathscr{D}_2\\ A\end{matrix}}{A\wedge A}}{(A\supset(A\wedge A))\supset(A\wedge A)}\,{\supset}\mathrm{I}}{((A\wedge A)\supset A)\supset(A\supset(A\wedge A))\supset(A\wedge A))}\,{\supset}\mathrm{I}\qquad \langle 3\rangle\,\dfrac{\dfrac{\overset{3}{A\wedge A}}{A}\,{\wedge}\mathrm{E}_1}{(A\wedge A)\supset A}\,{\supset}\mathrm{I}}{(A\supset(A\wedge A))\supset(A\wedge A)}\qquad \langle 4\rangle\,\dfrac{\dfrac{\overset{4}{A}\qquad\overset{4}{A}}{A\wedge A}}{A\supset(A\wedge A)}\,{\supset}\mathrm{I}}{A\wedge A}$$

which in turn reduces via two applications of $\supset\beta$ to

$$\dfrac{\begin{matrix}\mathscr{D}_1\\ A\end{matrix}\qquad\begin{matrix}\mathscr{D}_2\\ A\end{matrix}}{A\wedge A}$$

On the other hand, by applying first $\supset\beta$ (for four times) and then $\wedge\beta_1$ (twice) to the first derivation one obtains

$$\dfrac{\begin{matrix}\mathscr{D}_1\\ A\end{matrix}\qquad\begin{matrix}\mathscr{D}_1\\ A\end{matrix}}{A\wedge A}$$

In other words, we have that the two derivations

$$\dfrac{\begin{matrix}\mathscr{D}_1\\ A\end{matrix}\qquad\begin{matrix}\mathscr{D}_2\\ A\end{matrix}}{A\wedge A}\qquad\qquad \dfrac{\begin{matrix}\mathscr{D}_1\\ A\end{matrix}\qquad\begin{matrix}\mathscr{D}_1\\ A\end{matrix}}{A\wedge A}$$

are equivalent with respect to reducibility even when one adopts the restricted form of Ekman's reduction. Thus the restricted form of Ekman's reduction is sufficient to trivialize identity of proofs (by the argument given in the main text).

9. It should be observed that von Plato [62] is not in the least interested in the issue of paradoxes, and regards Ekman's phenomenon as a problem for normalization in minimal propositional logic.
10. Similar modifications of the rules of the quantifiers are required as well, see, e.g., Tennant [108, Sect. 7.10]. Note that the qualification "negative" is essential, as in positive free logics $t = t$ holds also when t is a non-denoting term, see for details.
11. For the present purposes, no substantial use we will made of the rules for the existential quantifier.
12. Alternatively, by taking A to be $\neg x \in x$ and both u and t to be Russell's term r, one obtain the following pair of instances of $\{\}^{=}\mathrm{E}_{1g}$, and $\{\}^{=}\mathrm{E}_{2g}$:

$$\dfrac{r = r \qquad \neg(r \in r) \qquad \begin{array}{c}[r \in r]\\ C\end{array}}{C}\,\{\}^{=}\mathrm{E}_{1g} \qquad\qquad \dfrac{r = r \qquad r \in r \qquad \begin{array}{c}[\neg(r \in r)]\\ C\end{array}}{C}\,\{\}^{=}\mathrm{E}_{2}$$

from which one can construct a derivation of $\perp$ from $r = r$ having the same form of the derivation $\mathbf{R}_{\mathbf{g}}^{='}$. As observed above, in free logic $r = r$ does not come for free, but it rather has to be derived from $\exists! r$, i.e. from $\exists x(r = x)$. By an application of $\supset$I one therefore obtains another closed normal derivation of $\neg\exists x(r = x)$. This derivation is discussed in Tennant [115, Sect. 6] albeit in slightly different form. The consequence of applications of $\{\}^{=}\mathrm{E}_{1g}$ are atomic formulas, and hence cannot be themselves major premises of an application of an elimination rule. For this reason, the rule need not be put in general elimination form to guarantee that major premises of elimination rules stand proud as required in Core Logic (see Note 16 to Chap. 3). Similar considerations apply to applications of $\supset$E with major premise $\neg A$. In spite of the hybrid use of general and standard elimination rules, the considerations to be developed in the remaining part of the chapter apply, *mutatis mutandis*, to this derivation as well.

13. The two notions of *general rule* and *decomposing inference* do not in general coincide, since according to the schema given by Ekman the decomposing inferences associated with the conjunction elimination rules of Gentzen [20] and Prawitz [65] differ from the (more commonly adopted) single elimination rule considered by Schroeder-Heister [85] following Prawitz [69]:

$$\dfrac{A \wedge B \qquad \begin{array}{c}[A]\\ C\end{array}}{C}\,\wedge\mathrm{E}_{1D} \qquad \dfrac{A \wedge B \qquad \begin{array}{c}[B]\\ C\end{array}}{C}\,\wedge\mathrm{E}_{2D} \qquad \dfrac{A \wedge B \qquad \begin{array}{c}[A][B]\\ C\end{array}}{C}\,\wedge\mathrm{E}_{\mathrm{PSH}}$$

($\wedge\mathrm{E}_{1D}$ and $\wedge\mathrm{E}_{2D}$ are in fact the elimination rules for conjunction one would obtain by $\mathtt{JR}$-inversion, see Sect. 3.7.) It is finally worth observing that the notion of decomposing inference is not restricted to elimination rules only. When applied to introduction rules, it yields what Negri and von Plato [53, pp. 213ff.] called *general introduction rules*. More on this in Sect. 6.9.

14. Here we are not assuming derivations to be normal, as in Tennant's Core Logic (see Note 16 to Chap. 3). Hence, the major premises of $\supset\mathrm{E}_g$ need not be assumptions.
15. In fact, general introduction rules are nothing but the decomposing inference corresponding to the usual introduction rules according to the pattern proposed by Ekman given in Sect. 6.6.
16. Kürbis [39] is also aware of the difficulty in Milne's original reductions. He gives a proof of the normalization theorem for an intuitionistic natural deduction system with general introduction and elimination rules using slightly different reductions than the one here presented.
17. To obtain a derivation in a system in which all rules are in general form, one should have to add an extra discharged premise in correspondence of the application of $\supset$I so to turn it into an application of $\supset\mathrm{I}_g$:

$$\langle 1\rangle\ \dfrac{\begin{matrix}\overset{1}{[A]}\\ \mathscr{D}\\ \bot\end{matrix}}{\neg A}\supset\mathrm{I} \qquad \dashrightarrow \qquad \langle 1,2\rangle\ \dfrac{\begin{matrix}\overset{1}{[A]}\\ \mathscr{D}\\ \bot\end{matrix} \qquad \overset{2}{\neg A}}{\neg A}\supset\mathrm{I}_g$$

18. Further investigation is also needed to clarify the exact relationship between the Prawitz-Tennant analysis of paradoxes based on normalization failure and the solution to paradoxes consisting in restricting the use of the cut rule in sequent calculus, a solution which goes back at least to Hallnäs [29] and that has been recently brought up again by several authors, notably Ripley [79]. Given the close correspondence between normalization in natural deduction and cut elimination in sequent calculus, the solution to paradoxes arising from restricting to normalisable derivations can certainly be seen as anticipating current non-transitive sequent-calculus-based solutions (see also Note 8 to Chap. 5). The adoption of general elimination rules called for by Tennant brings the two approaches even closer, given that general elimination rules more directly correspond to sequent calculus left rules than standard elimination rules. The results presented in this chapter, however, suggest that the relationship between two two approaches is not as obvious as one may assume. Von Plato's and Tennant's derivations correspond to cut-free derivations, and thereby it is *prima facie* unclear to which sort of transformation on sequent calculus derivations, the reductions we proposed correspond. Moreover whereas in natural deduction we have two kinds of derivations (normalisable and non-normalisable ones) in sequent calculus, by ruling out the cut rule from the outset (as Ripley, but also Tennant in his most recent work, recommend to do) no such distinction is available, and hence the original Tennant-Prawitz criterion for paradoxicality based on looping reduction sequences cannot immediately be reformulated in a cut-free setting. Arguably, by allowing cut as a primitive rule, a distinction analogous to the one available in natural deduction can be formulated in sequent calculus as well (that is, between

derivations for which the cut-elimination procedure does or does not enter a loop) and the reductions for general elimination rules can find a counterpart in the sequent calculus setting as well. But a thorough investigation of these issues must be left for another occasion.

Concluding Remarks

In the first part of this work we introduced the idea of harmony starting from general considerations about the architecture of a theory of meaning. We then introduced the idea that in the natural deduction setting, the notion of harmony can be made precise in terms of certain transformation on the structure of derivations, reductions and expansions. We have shown how a cluster of formal results about reductions in natural deduction can be interpreted through the lens of a broadly Fregean conception of meaning. In the resulting picture formal derivations are viewed as linguistic representations of abstract proofs, and the transformations on derivations arising in connection with harmony are understood as preserving the identity of the proofs denoted by the derivations to which they are applied. We argued that the characteristic trait of this semantic picture is its intensional nature. Intensionality is here understood in at least two different senses: first, the proposed conception of proof-theoretic semantics delivers a finer grained analysis of consequence than traditional ones, in that it allows one to distinguish between different proofs of a formula or consequence claim; second, it allows to define a notion of isomorphism, a finer grained equivalence relation between propositions than mere interderivability.

After having successfully solved the problem posed by quantum disjunction for this account of harmony, we discussed several procedures—to which we referred with the term 'inversion principles'—to define, given a collection of introduction rules, a collection of elimination rules for which reductions and generalized expansions can be uniformly defined. Reflection on the structure of the elimination rules generated by inversion suggests that the fact that the introduction and elimination rules for a certain kind of propositions are in harmony means that the set of proofs for these propositions are defined by induction, and in particular the elimination rules obtained by inversion play the role of the final clause of inductive definitions. We concluded the first part of this work by discussing the fact that, although the rules obtained by means of an inversion principle can plausibly be seen as harmonious, inversion does not deliver an exhaustive characterization of harmony. We examined the proposal, implicit in the work of several authors, and explicitly advocated by Schroeder-Heister, that harmony should be defined by coupling inversion with intederivability: a pair of collections of introduction and elimination rules is harmonious iff the elimination rules are

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interderivable with those obtained from the introduction rules using inversion (here the choice of the inversion principle does not matter, as different inversion principles yield interderivable collections of elimination rules). We dubbed the resulting notion 'harmony by intederivability' and we showed that harmony by interderivability is not satisfactory from the intensional standpoint we advocated. In particular, we discussed a collection of rules in harmony by interderivability but such that the reductions and expansions associated with the rules yield a trivial notion of isomorphism.

The second part of the present work was devoted to an issue of an apparently rather remote nature: namely that of paradoxes. The discussion of paradoxes turns out however to be intimately connected with different elements of the proof-theoretic picture developed in the first part of the work.

First, our starting point is what we called the Prawitz-Tennant analysis of paradoxes, which identifies paradoxes with derivations of a contradiction that cannot be reduced to normal form. The process by means of which a derivation is reduced to normal form is, in semantic terms, the process of assigning to it its denotation. We thus propose to rephrase the Prawitz-Tennant analysis in semantic terms as the claim that paradoxical derivations are non-denoting expressions.

Second, on the traditional conception of proof-theoretic semantics, preservation of provability is taken as a definition of correctness of rules. This not only reverses the most natural order of conceptual dependency (according to which a proof is defined as what is obtained by applying correct inference rules). Moreover, it prevents the applicability of proof-theoretic semantic to languages equipped with paradoxical expressions. In such languages, inference rules which we intuitively acknowledge as correct fail to "preserve normalizability". That is, when these inferences are applied to derivations that normalize (i.e. that denote a proof) they yield derivations which need not normalize (i.e. that might fail to denote a proof). We therefore propose that the notion of correctness should be defined independently of preservation of provability by appealing to the notion of harmony. Harmonious rules are correct by definition, and the fact that they preserve provability is a welcome feature which however need not hold in all circumstances.

Finally, the question of which transformations on derivations should be regarded as acceptable from an intensional point of view is extensively discussed. On the one hand, it is shown that on too coarse a notion of reduction the Prawitz-Tennant criterion overgenerates (i.e. it ascribes paradoxicality also to derivations about which there is nothing paradoxical). On the other hand, it is shown that different choices of criteria of identity for open derivations (which represent functions from proofs of their assumptions to proofs of their conclusions) yield different results concerning the conservativity of paradoxical expressions.

The present work leaves several issues open. One is that of giving an exhaustive account of harmony compatible with an intensional conception of proofs (for a recent fully-fledged proposal see [61]). Another is that of which are the most appropriate criteria of identity for open derivations. The answers to further questions depend on this, notably that of which derivations should be regarded as those that denote proofs in the most direct way possible.

Moreover, the investigations in the present work were restricted to calculi of natural deduction for propositional languages. A challenge for the intensional account of proof-theoretic semantics is the possibility of being applied beyond the scope of purely propositional languages, by considering not only language containing first- and second-order quantifiers, but also specific theories, such those for arithmetic. The latter in particular seem to offer a "real-world" setting to test not only the general tenability of the proof-theoretic conception of meaning, but also of the account of paradoxes developed in the second part of the present work.

Another interesting line of research is that of more thoroughly investigating the possibility of applying the ideas here developed to other proof-theoretic formalisms, and in particular to sequent calculus. In recent years, the adoption of sequent calculus has become predominant in the philosophical literature, due to its greater flexibility over natural deduction. As is well known, standard natural deduction assigns a distinguished role to intuitionistic logic, and hence it is often associated with "revisionary" theses concerning logic and meaning. Sequent calculus may thus be the instrument for bringing the conceptual tools developed by proof-theoretic semantics (and in particular by its intensional variant that was here advocated) to the realm of classical logic.

We hope that the results obtained in the present work—and the lines of further investigation here envisaged—show how questions related to identity of proofs are a key to disclose the intimate connection between meaning and proof.

Appendix A
The Calculus of Higher-Level Rules

Abstract In this technical appendix we spell out the formal details of the calculus of higher-level rule first introduced by Schroeder-Heister. The presentation follows that of [85] rather than that of [86], with some differences that will be highlighted in due course. We will make fully precise some technical notions concerning the derivability of rules and one of the inversion principles discussed in Chap. 3.

A.1 Some Preliminary Remarks

The calculus of higher-level rules introduced by Schroeder-Heister [85, 86] is a proof-theoretic framework which generalizes the natural deduction calculi of Gentzen [20] and Prawitz [65] in two respects: (i) not only formulas but also rules can be assumed in the course of derivations; (ii) when applying a rule in a derivation, not only formulas but also (previously assumed) rules can be discharged.

This yields a hierarchy of different rule-levels at the base of which we have the limit case of formulas (rules of level 0), and production rules (rules of level 1, such as $\wedge$I, $\wedge\text{E}_1$, $\wedge\text{E}_2$ and $\supset$E).

A typical example of a rule of level 2 is $\supset$I, which allows the discharge of formulas (i.e. of rules of level 0). Informally, the content of this rule is that in order to establish $A \supset B$ one need not be able to infer B outright, but it is enough to be able to infer B from A. As this possibility is exactly what is expressed by the rule allowing the inference of B from A, we adopt the terminological convention that the premise of $\supset$I is not B, but rather the rule that allows one to pass over from A to B (for the role played by B in $\supset$I we will use the term 'immediate premise').

In general, the premises of rules of level $l \geq 1$ will be rules of level $l - 1$ and for each level $l \geq 2$, the application of a rule of level l in a derivation will allow the discharge of rules of level $l - 2$.

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The presentation of the calculus of higher-level rules given below follows quite closely that of Schroeder-Heister [85], but with a major difference, namely the handling of discharge. In [85] the details of discharge are left implicit (the definition of 'derivation of A depending on M', on pp. 50–51, implicitly adopts the so-called complete discharge convention, see Troelstra and Schwichtemberg [122], Sect. 2.1.9, pp. 43–45). Schroeder-Heister [86] treats discharge in a more detailed way, generalizing the notion of discharge function introduced by Prawitz [65]. The way in which discharge will be dealt here follows rather the handling of variables in the λ-calculus (as done, for instance, by Girard et al. [26], Troelstra and Schwichtemberg, [122]). Although this induces certain complications (essentially, the need of identifying derivations up to renaming of discharge indexes, see Sect. A.6), it provides a fine-grained treatment of discharge that is essential for the investigation of the intensional aspects of proofs which is the goal of the present work.

We finally remark that, in the main text we stuck to the usual understanding of rules as metalinguistic schemata. Though intuitively appealing, this choice is not very well-suited for a rigorous formulation of the calculus of higher-level rules. When the rules of a calculus based on a language, say $\mathcal{L}$, are handled as belonging to the meta-language of $\mathcal{L}$, the natural setting to define the notion of rule seems to be the meta-meta-language of $\mathcal{L}$. This meta-linguistic ascent is not necessary provided that the object-language is equipped with some form of quantification over proposition, as in [94]. In such a setting, rule schemata are just object-language rules containing some universally quantification. For consistency with the main text, we will however avoid the explicit introduction of universal quantification, and thereby stick to a distinction between (concrete) rules and rule schemata.

A.2 Concrete Rules

We consider a propositional language $\mathcal{L}$ whose formulas are built from denumerably many *atomic formulas* $\alpha_1, \alpha_2, \ldots$ using denumerably many *connectives* of different arities, among which we have the standard intuitionistic ones ($\wedge, \supset, \vee, \ldots$) as well as less standard ones (such as $\texttt{tonk}$, $\odot$, $\sharp$, $\flat$, $\natural$, and other *ad hoc* symbols). We will use $\dagger$ (possibly with primes) as a metavariable for connectives of arbitrary arity.

Capital letters $A, B, \ldots$ (possibly with sub-scripts) will be used as metavariables for formulas of $\mathcal{L}$, and will be referred to as *schematic letters*. We further assume the metalanguage to be an extension of the object language $\mathcal{L}$ and the metalinguistic expressions obtained from atomic formulas of $\mathcal{L}$ and metalinguistic schematic letters using the connectives of $\mathcal{L}$ and the metavariables for them will be called *schematic formulas*.

As we did so far, we will use $\mathscr{D}$ (possibly with subscripts and primes) as a metavariable for derivations. By a *schematic derivation* we will understand the result of replacing in derivations formulas with schematic formulas and subderivations with metavariables for derivations.

We remark that concrete (as opposed to schematic) derivations in standard natural deduction calculi depend on object-language formulas, and not on metalinguistic formula schemata. In the same way, as it will made clear in Definitions A.3 and A.5, we will define derivations in the calculus of higher-level rules as depending not on rules (understood as metalinguistic schemata) but rather on the object-language instances of these rules, which we will call *concrete rules*. (On the other hand, introduction and elimination rules will be taken, as we implicitly did in the main text, as metalinguistic schemata (see Sect. A.4 below.)[1]

Following Schroeder-Heister [86] we adopt a tree-like notation for concrete rules (and thus, in the metalanguage, for rules as well). We will use R, $R_1, \ldots$ as metavariables for concrete rules:

Definition A.1 (*Concrete rule, consequence, premise, immediate premise*)

- Every formula A is a concrete rule of level 0.
- If A is a formula and $R_1, \ldots, R_n$ are concrete rules of maximum level l, then

$$\frac{R_1 \qquad \ldots \qquad R_n}{A}$$

  is a concrete rule of level $l + 1$.

The root of the tree constituting a concrete rule R is called the *consequence* of R. Let R be the concrete rule of level ≥ 1:

$$\frac{R_1 \qquad \ldots \qquad R_n}{A}$$

$R_1, \ldots, R_n$ are called the *premises* of R. The consequences of $R_1, \ldots, R_n$ are called the *immediate premises* of R.

The premises of each premise of a concrete rule R of level $l + 2$ (if any) are concrete rules of level l. As the definitions in the next section will make clear, these can be discharged by applications of R in a derivation. In the literature, it is common to use a "bracketed notation" for rules to make explicit which assumptions can be discharged by their applications. We will adapt this notation to the present context so that a concrete rule R of the form indicated to the left below will be sometimes written as on the right below (if R has n premises and the ith premise has in turn m_i premises, for all $1 \leq i \leq n$ and $1 \leq j \leq m_i$, we indicate the jth premise of the ith premise of R with R_{ij}):

$$\frac{\dfrac{R_{11} \ \ldots \ R_{1m_1}}{B_1} \qquad \ldots \qquad \dfrac{R_{n1} \ \ldots \ R_{nm_n}}{B_n}}{A} \qquad\qquad \frac{\begin{matrix}[R_{11}]\ldots[R_{1m_1}] \\ B_1\end{matrix} \qquad \ldots \qquad \begin{matrix}[R_{n1}]\ldots[R_{nm_n}] \\ B_n\end{matrix}}{A}$$

We also use a linear notation for concrete rules, so that a concrete rule of level $l + 1$ is written $(R_1; \ldots; R_n \Rightarrow A)$, where the outermost brackets will be often omitted.

A.3 Structural Derivations

We treat assumptions as partitioned into classes, where an assumption class in a derivation is a collection of spatially located occurrences of the same concrete rule within a derivation. The partitioning is achieved by associating to each concrete rule used as assumption a (not necessarily distinct) natural number. Roughly speaking, in a given derivation, two occurrences of the same rule that are marked by the same number belong to the same assumption class if and only if they are both undischarged or they are discharged by the same rule application.[2] Thus, properly speaking an assumption is not just a concrete rule, but a pair consisting of a concrete rule and number. These informal remarks are made precise by the following three definitions and by the examples following them.

Definition A.2 (*Assumption*) For any concrete rule R of level l and natural number n, $\overset{n}{R}$ is an *assumption of level* l. An assumption $\overset{u}{R}$ will be sometimes called an *assumption of* R.

Definition A.3 (*Structural derivations*)

- Any assumption $\overset{n}{A}$ of level 0 is a *structural derivation of conclusion* A.
- If
  - R is the rule $\dfrac{\begin{matrix}[R_{11}]\ldots[R_{1m_1}] & & [R_{n1}]\ldots[R_{nm_n}] \\ B_1 & \ldots & B_n\end{matrix}}{A}$
  - and for all $1 \leq i \leq n$, $\mathscr{D}_i$ is a derivation of conclusion B_i;
  - and for all $1 \leq i \leq n$ and $1 \leq j \leq m_i$, all u_{ij}s are natural numbers, such that for all $1 \leq h \leq m_i$, if $R_{ij} = R_{ih}$ then $u_{ij} \neq u_{ih}$;
  - and u is a natural number,

  then the following:

$$\langle u_{11}, \ldots, u_{1m_1}, \ldots, u_{n1}, \ldots, u_{nm_n}\rangle \; \dfrac{\begin{matrix}\mathscr{D}_1 & & \mathscr{D}_n \\ B_1 & \ldots & B_n\end{matrix}}{A} \; \overset{u}{R}$$

  is a *structural derivation of conclusion* A.

For readability, empty lists of discharge indexes to the left of inference lines will be omitted.

The set $UA(\mathscr{D})$ of undischarged assumptions of a structural derivation $\mathscr{D}$ is defined by recursion as follows:

Definition A.4 (*Undischarged assumptions*)

- If $\mathscr{D} = \overset{u}{A}$ then $UA(\mathscr{D}) = \left\{\overset{u}{A}\right\}$.

- If

$$\mathscr{D} = \langle u_{11}, \ldots, u_{1m_1}, \ldots, u_{n1}, \ldots, u_{nm_n} \rangle \; \dfrac{\begin{matrix} \mathscr{D}_1 & & \mathscr{D}_n \\ B_1 & \ldots & B_n \end{matrix}}{A} \; \overset{u}{R}$$

with

$$R = \dfrac{\begin{matrix} [R_{11}] \ldots [R_{1m_1}] & & [R_{n1}] \ldots [R_{nm_n}] \\ B_1 & \ldots & B_n \end{matrix}}{A}$$

then

$$UA(\mathscr{D}) \; = \; \bigcup_{i=1}^{n} \left(UA(\mathscr{D}_i) \setminus \bigcup_{j=1}^{m_i} \left\{ \overset{u_{ij}}{R_{ij}} \right\} \right) \; \cup \; \left\{ \overset{u}{R} \right\}$$

If $\mathscr{D}$ is a derivation of conclusion A and, for some u, $\overset{u}{R} \in UA(\mathscr{D})$ we say that A *depends on* R *in* $\mathscr{D}$*, or simply that* $\mathscr{D}$ *depends on* R.

As the previous definition makes clear, by an assumption we understand a rule *together* with a numeric label. The set of undischarged assumptions of a derivation thus registers not only whether a derivation depends on some rule R but on how many distinct assumptions of R (that is, many distinct assumptions of the same rule R, distinguished by their labels) $\mathscr{D}$ depends.

Example A.1 The set of assumptions of the following structural derivation

$$\dfrac{\dfrac{\overset{12}{\alpha_3 \supset \alpha_4} \qquad \overset{31}{\alpha_3 \supset \alpha_4}}{\alpha_1} \; \overset{2}{\alpha_3 \supset \alpha_4; \alpha_3 \supset \alpha_4 \Rightarrow \alpha_1} \qquad \overset{31}{\alpha_3 \supset \alpha_4}}{\alpha_9 \wedge \alpha_5} \; \overset{142}{\alpha_1; \alpha_3 \supset \alpha_4 \Rightarrow \alpha_9 \wedge \alpha_5}$$

is the set

$$\{ \overset{12}{\alpha_3 \supset \alpha_4} \, , \; \overset{31}{\alpha_3 \supset \alpha_4} \, , \; \overset{2}{\alpha_3 \supset \alpha_4; \alpha_3 \supset \alpha_4 \Rightarrow \alpha_1} \, , \; \overset{142}{\alpha_1; \alpha_3 \supset \alpha_4 \Rightarrow \alpha_9 \wedge \alpha_5} \}$$

containing two assumptions of the same formula (i.e. concrete rule of level 0) $\alpha_3 \supset \alpha_4$, one labeled by 12, the other labeled by 31 (the latter assumption occurs twice in the derivation). The set contains two further assumptions of two concrete rules of level 1 (each occurring only once in the derivation). In tree-like notation these are written as follows:

$$\dfrac{\alpha_3 \supset \alpha_4 \qquad \alpha_3 \supset \alpha_4}{\alpha_1} \qquad\qquad \dfrac{\alpha_1 \qquad \alpha_3 \supset \alpha_4}{\alpha_9 \wedge \alpha_5}$$

Example A.2 In the following structural derivation

$$\langle 2,3\rangle\ \dfrac{\langle 1\rangle\ \dfrac{\overset{3}{\alpha_1} \qquad \dfrac{\overset{1}{(\alpha_2 \wedge \alpha_3)}}{\alpha_4}\ \overset{2}{(\alpha_2 \wedge \alpha_3) \Rightarrow \alpha_4}}{\alpha_5}\ \overset{4}{\alpha_1; ((\alpha_2 \wedge \alpha_3) \Rightarrow \alpha_4) \Rightarrow \alpha_5}}{\alpha_6}\ \overset{5}{(\alpha_1; ((\alpha_2 \wedge \alpha_3) \Rightarrow \alpha_4) \Rightarrow \alpha_5) \Rightarrow \alpha_6}$$

the conclusion α_6 depends on the two concrete rules (of level 2 and 3 respectively): $(\alpha_1; ((\alpha_2 \wedge \alpha_3) \Rightarrow \alpha_4) \Rightarrow \alpha_5)$ and $((\alpha_1; ((\alpha_2 \wedge \alpha_3) \Rightarrow \alpha_4) \Rightarrow \alpha_5) \Rightarrow \alpha_6)$. In tree-like (bracketed) notation, the two concrete rules look respectively as follows:

$$\dfrac{\alpha_1 \qquad \dfrac{\alpha_2 \wedge \alpha_3}{\alpha_4}}{\alpha_5} \left(= \dfrac{\alpha_1 \qquad \overset{[\alpha_2 \wedge \alpha_3]}{\alpha_4}}{\alpha_5} \right) \qquad \dfrac{\dfrac{\alpha_1 \qquad \dfrac{\alpha_2 \wedge \alpha_3}{\alpha_4}}{\alpha_5}}{\alpha_6} \left(= \dfrac{\overset{[\alpha_1][(\alpha_2 \wedge \alpha_3) \Rightarrow \alpha_4]}{\alpha_5}}{\alpha_6} \right)$$

A.4 Rules and K-derivations

As anticipated, inference rules governing connectives cannot[3] be identified with concrete rules. The reason is twofold and is analogous to the reason why, in a natural deduction formulation of some theory, axiom schemata are fundamentally different from assumptions. Whereas an assumption is always the assumption of a specific sentence, an axiom schema is used in a derivation by instantiating it on some (object-language) sentence which, by analogy with concrete rules, one may call "concrete" axiom. Moreover, although both concrete axioms and assumptions represent the starting point of derivations, the conclusions of the derivations depend only on the latter and not on the former ones. Analogously, the rules governing logical connectives are schemata whose instances are concrete rules. For example, the two (distinct) concrete rules:

$$\dfrac{\alpha_1 \qquad \alpha_2}{\alpha_1 \wedge \alpha_2} \qquad\qquad \dfrac{(\alpha_3 \supset \alpha_4) \qquad \alpha_2}{(\alpha_3 \supset \alpha_4) \wedge \alpha_2}$$

are instances of the same rule (schema), namely:

$$\dfrac{A \qquad B}{A \wedge B}$$

The rule $\wedge$I can thus be identified with the (metalinguistic) schema $A, B \Rightarrow A \wedge B$ all of whose instances are (different) concrete rules.[4,5] Moreover, contrary to arbitrary concrete rules, concrete rules which are instances of $\wedge$I are to be considered as primitive (at least if the calculus is equipped with the rule $\wedge$I), and thus, as for concrete axioms, the conclusion of a derivation should not depend on them.

These remarks can be made precise by defining the notion of derivation in a calculus K, where a calculus is a collection of rule schemata whose instances are to be taken as primitive in the construction of derivations. Since the rules of K are metalinguistic schemata, the definition should be understood as given in the meta-metalanguage of $\mathcal{L}$, which we assume to be an extension of the metalanguage in which we use capital bold letters $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$ (resp. $\boldsymbol{R}_1, \boldsymbol{R}_2, \ldots$) as meta-metalinguistic variables for metalinguistic schematic formulas (rep. rules). Every (metalinguistic) function σ from schematic letters to formulas of $\mathcal{L}$ determines a particular instance of each rule $\boldsymbol{R}$, that we call the σ-instance of $\boldsymbol{R}$ and that we indicate with $\sigma(\boldsymbol{R})$.

Definition A.5 (K-derivation)

- Any assumption $\overset{n}{A}$ of level 0 is a K-derivation of conclusion A.
- If the formula A is the σ-instance of a primitive rule (of level 0) $\boldsymbol{R}$ of K, then

$$\frac{}{A}\,\boldsymbol{R}\,\sigma$$

  is a K-derivation of conclusion A.
- If
  - R is the concrete rule $\dfrac{\begin{matrix}[R_{11}]\ldots[R_{1m_1}] \\ B_1\end{matrix} \quad \ldots \quad \begin{matrix}[R_{n1}]\ldots[R_{nm_n}] \\ B_n\end{matrix}}{A}$
  - and for all $1 \leq i \leq n$, $\mathscr{D}_i$ is a K-derivation of conclusion B_i;
  - and for all $1 \leq i \leq n$ and $1 \leq j \leq m_i$, all u_{ij}s are natural numbers, such that for all $1 \leq h \leq m_i$, if $R_{ij} = R_{ih}$ then $u_{ij} \neq u_{ih}$;
  - and u is a natural number,

  then the following:

$$\langle u_{11}, \ldots, u_{1m_1}, \ldots, u_{n1}, \ldots, u_{nm_n}\rangle\ \dfrac{\begin{matrix}\mathscr{D}_1 \\ B_1\end{matrix} \quad \ldots \quad \begin{matrix}\mathscr{D}_n \\ B_n\end{matrix}}{A}\,\overset{u}{R}$$

  is a K-derivation of conclusion A.
- If
  - the concrete rule $R = \dfrac{\begin{matrix}[R_{11}]\ldots[R_{1m_1}] \\ B_1\end{matrix} \quad \ldots \quad \begin{matrix}[R_{n1}]\ldots[R_{nm_n}] \\ B_n\end{matrix}}{A}$ is the σ-instance of a primitive rule $\boldsymbol{R}$ of K;
  - and for all $1 \leq i \leq n$ $\mathscr{D}_i$ is a K-derivation of conclusion B_i;
  - and for all $1 \leq i \leq n$ and $1 \leq j \leq m_i$, all u_{ij} are natural numbers, such that for all $1 \leq h \leq m_i$, if $R_{ij} = R_{ih}$ then $u_{ij} \neq u_{ih}$;

  then the following:

$$\langle u_{11}, \ldots, u_{1m_1}, \ldots, u_{n1}, \ldots, u_{nm_n} \rangle \; \dfrac{\begin{matrix} \mathscr{D}_1 & & \mathscr{D}_n \\ B_1 & \ldots & B_n \end{matrix}}{A} \; \boldsymbol{R}\,\sigma$$

is a K-derivation of conclusion A.

The definition of the set of undischarged assumptions of a K-derivation $\mathscr{D}$ is obtained by adding the following clauses to Definition A.4.

Definition A.4 (*Undischarged assumptions [continued]*)

- If $\mathscr{D} = \dfrac{}{A}\,\boldsymbol{R}\,\sigma$ then $UA(\mathscr{D}) = \emptyset$
- If

$$\mathscr{D} = \langle u_{11}, \ldots, u_{1m_1}, \ldots, u_{n1}, \ldots, u_{nm_n} \rangle \; \dfrac{\begin{matrix} \mathscr{D}_1 & & \mathscr{D}_n \\ B_1 & \ldots & B_n \end{matrix}}{A} \; \boldsymbol{R}\,\sigma$$

with

$$\boldsymbol{R} = \dfrac{\begin{matrix} [\boldsymbol{R}_{11}] \ldots [\boldsymbol{R}_{1m_1}] & & [\boldsymbol{R}_{n1}] \ldots [\boldsymbol{R}_{nm_n}] \\ \boldsymbol{B}_1 & \ldots & \boldsymbol{B}_n \end{matrix}}{\boldsymbol{A}}$$

then

$$UA(\mathscr{D}) = \bigcup_{i=1}^{n} \left(UA(\mathscr{D}_i) \setminus \bigcup_{j=1}^{m_i} \left\{ \sigma(\overset{u_{ij}}{\boldsymbol{R}_{ij}}) \right\} \right)$$

A.5 Derivation of Rules and Derivability

According to the definition of structural (respectively K-)derivation, the conclusion of a structural (resp. K-)derivation is always a formula (i.e. a concrete rule of level 0). The derivation of a formula A however "can also be regarded" as the derivation of a concrete rule R having A as consequence. We spell out this intuition in the next definition and clarify it with an example:

Definition A.6 (*Derivation of a rule of level* > 0) If $\mathscr{D}$ is a structural (resp. K-) derivation of A then for any sequence of assumptions $\langle \overset{u_1}{R_1}, \ldots, \overset{u_n}{R_n} \rangle$, we say that the pair $\langle \mathscr{D}, \langle \overset{u_1}{R_1}, \ldots, \overset{u_n}{R_n} \rangle \rangle$ is a structural (resp. K-) derivation of conclusion $R_1; \ldots; R_n \Rightarrow A$ relative to $\langle u_1 \ldots u_n \rangle$.

The set of undischarged assumptions of $\langle \mathscr{D}, \langle \overset{u_1}{R_1}, \ldots, \overset{u_n}{R_n} \rangle \rangle$ is defined as follows:

$$UA(\langle \mathscr{D}, \langle \overset{u_1}{R_1}, \ldots, \overset{u_n}{R_n} \rangle \rangle) = UA(\mathscr{D}) \setminus \bigcup_{i=1}^{n} \left\{ \overset{u_i}{R_i} \right\}$$

When the context clearly specifies a sequence of assumptions $\overset{u_1}{R}_1, \ldots, \overset{u_n}{R}_n$ such that $\langle \mathscr{D}, \langle \overset{u_1}{R}_1, \ldots, \overset{u_n}{R}_n \rangle\rangle$ is a derivation of R relative to $\langle u_1, \ldots, u_n \rangle$, we will use $\dfrac{\mathscr{D}}{R}$ to abbreviate $\langle \mathscr{D}, \langle \overset{u_1}{R}_1, \ldots, \overset{u_n}{R}_n \rangle\rangle$.

Example A.3 Let $\mathscr{D}$ be the following structural derivation:

$$\dfrac{\overset{1}{\alpha} \qquad \overset{1}{\alpha}}{\beta}\ \alpha; \alpha \overset{2}{\Rightarrow} \beta$$

of conclusion β depending on α and $\alpha; \alpha \Rightarrow \beta$ (more precisely, $UA(\mathscr{D}) = \{\overset{1}{\alpha},\ (\alpha; \alpha \overset{2}{\Rightarrow} \beta)\}$).

According to Definition A.6, we have that

- $\langle \mathscr{D}, \langle \alpha; \alpha \overset{2}{\Rightarrow} \beta \rangle\rangle$ is a structural derivation of $(\alpha; \alpha \Rightarrow \beta)\ \Rightarrow\ \beta$ depending on α (more precisely, $UA(\langle \mathscr{D}, \langle \alpha; \alpha \overset{2}{\Rightarrow} \beta \rangle\rangle) = \{\overset{1}{\alpha}\}$);
- $\langle \mathscr{D}, \langle \overset{1}{\alpha} \rangle\rangle$ is a structural derivation of $\alpha \Rightarrow \beta$ depending on $\alpha; \alpha \Rightarrow \beta$ (more precisely, $UA(\langle \mathscr{D}, \langle \overset{1}{\alpha} \rangle\rangle) = \{\alpha; \alpha \overset{2}{\Rightarrow} \beta\}$);
- $\langle \mathscr{D}, \langle \overset{1}{\alpha}, (\alpha; \alpha \overset{2}{\Rightarrow} \beta) \rangle\rangle$ is a structural derivation of $\alpha; (\alpha; \alpha \Rightarrow \beta) \Rightarrow \beta$ depending on the empty set of assumptions.
- $\langle \mathscr{D}, \langle \overset{3}{\alpha}, (\alpha; \alpha \overset{4}{\Rightarrow} \beta) \rangle\rangle$ is a structural derivation of $\alpha; (\alpha; \alpha \Rightarrow \beta) \Rightarrow \beta$ depending on α and $\alpha; \alpha \Rightarrow \beta$ (more precisely, $UA(\langle \mathscr{D}, \langle \overset{3}{\alpha}, (\alpha; \alpha \overset{4}{\Rightarrow} \beta) \rangle\rangle) = \{\overset{1}{\alpha},\ (\alpha; \alpha \overset{2}{\Rightarrow} \beta)\}$).

Observe that if in $\mathscr{D}$ the label of one of the two assumptions of the form α had been 3, then $\langle \mathscr{D}, \langle \overset{1}{\alpha} \rangle\rangle$ would have been a structural derivation of $\alpha \Rightarrow \beta$ depending on both α and $\alpha; \alpha \Rightarrow \beta$ (more precisely, $UA(\langle \mathscr{D}, \langle \overset{1}{\alpha} \rangle\rangle)$ would have been $\{\overset{3}{\alpha}, (\alpha; \alpha \overset{2}{\Rightarrow} \beta)\}$).

We conclude this section by introducing the notion of derivability. We use Γ and Δ (resp. $\boldsymbol{\Gamma}, \boldsymbol{\Delta}$) possibly with subscripts and primes) as metavariables for sets of concrete rules (sets of rule schemata) . With Γ, Δ and Γ, R we abbreviate respectively $\Gamma \cup \Delta$ and $\Gamma \cup \{R\}$ (and similarly for $\boldsymbol{\Gamma}, \boldsymbol{\Delta}$ and $\boldsymbol{\Gamma}, \boldsymbol{R}$).

Definition A.7 (Derivability of rules) We say that a concrete rule R with consequence A is *structurally (resp.* $\mathtt{K}$ *-)derivable from* Γ (notation $\Gamma \vdash_{(\mathtt{K})} R$) iff there is a structural (resp. $\mathtt{K}$-)derivation $\mathscr{D}$ of conclusion A such that

- if $R = A$ then for all $\overset{v}{R'} \in UA(\mathscr{D})$, $R' \in \Gamma$;
- if $R = (R_1; \ldots; R_n \Rightarrow A)$ then there are natural numbers $u_1 \ldots u_n$ such that $\langle \mathscr{D}, \langle \overset{u_1}{R}_1, \ldots \overset{u_n}{R}_n \rangle\rangle$ is a structural (resp. $\mathtt{K}$-)derivation of R relative to $\langle u_1, \ldots, u_n \rangle$ and such that for all R', if $\overset{v}{R'} \in UA(\langle \mathscr{D}, \langle \overset{u_1}{R}_1, \ldots, \overset{u_n}{R}_n \rangle\rangle)$ for some v, then $R' \in \Gamma$.

We say that *a rule* $\boldsymbol{R}$ *is structurally (resp.* K*-)derivable from* $\boldsymbol{\Gamma}$ (notation $\boldsymbol{\Gamma} \vdash_{(\mathtt{K})} \boldsymbol{R}$) iff each instance R of $\boldsymbol{R}$ is structurally (resp. K-)derivable from instances of the rules in $\boldsymbol{\Gamma}$.

More in general, we may introduce the following notion.

Definition A.8 (Interderivability of collection of rules) Two collections of rules are *interderivable* (we indicate interderivability with $\dashv\vdash$) if and only if each rule in one collection is structurally derivable from the rules in other collection and vice versa.

Example A.4 Let

$$\begin{aligned}
\natural\mathrm{E}_1 &= A\natural B;\ A \Rightarrow B\\
\natural\mathrm{E}_2 &= A\natural B;\ B \Rightarrow A\\
\natural\mathrm{E}_3 &= A\natural B \Rightarrow A\\
\natural\mathrm{E}_3^* &= A\natural B \Rightarrow B
\end{aligned}$$

It is easily seen that the two collections of rules consisting of $\natural\mathrm{E}_1$, $\natural\mathrm{E}_2$, $\natural\mathrm{E}_3$ and $\natural\mathrm{E}_1$, $\natural\mathrm{E}_2$, $\natural\mathrm{E}_3^*$ respectively (See Sect. 3.10) are interderivable. Since both collections of rules share $\natural\mathrm{E}_1$ and $\natural\mathrm{E}_2$, to show their interderivability it is enough to show that any instance of $\natural\mathrm{E}_3$ is structurally derivable from some instances of $\natural\mathrm{E}_1$, $\natural\mathrm{E}_2$ and $\natural\mathrm{E}_3^*$ and that any instance of $\natural\mathrm{E}_3^*$ is structurally derivable from some instances of $\natural\mathrm{E}_1$, $\natural\mathrm{E}_2$ and $\natural\mathrm{E}_3$:

$$\frac{A\natural B \qquad \dfrac{A\natural B}{A}\,(A\natural B) \Rightarrow A}{B}\,(A\natural B);\ A \Rightarrow B \qquad\qquad \frac{A\natural B \qquad \dfrac{A\natural B}{B}\,(A\natural B) \Rightarrow B}{A}\,(A\natural B);\ B \Rightarrow A$$

Henceforth we will use 'derivation' for both 'structural derivation' and 'derivations in K', the context making clear whether we are referring to structural derivations, to derivations in a specific calculus K or to derivations in any calculus.

A.6 Identifications up to Renaming Discharge Indexes

In the next section two basic properties of derivability will be reproved for the formulation of the calculus of higher-level rules we have just given: reflexivity and transitivity. (Analogous proofs of these properties have been given for the two other formulations of the calculus in [85, 86].)

The proofs will based on the two notions of 'composition of derivations' and 'identity derivations' which will play an essential role in presenting the inversion principle in the final section of the appendix.

In order to simplify the definition of these notions, we will identify derivations up to renaming of discharge indexes. That is, we will treat, for instance, the following:

$$\langle 1 \rangle \; \frac{\overset{1}{\alpha}}{\alpha \supset \alpha} \supset\mathrm{I} \qquad \langle 2 \rangle \; \frac{\overset{2}{\alpha}}{\alpha \supset \alpha} \supset\mathrm{I}$$

as "the same" derivation. To cash out this idea in a formally precise way would have required to call the previously defined derivations "pre-derivations" and to introduce derivations as equivalence classes of pre-derivations modulo an equivalence relation, to be defined after the model of α-equivalence in the λ-calculus. We avoid giving the details of the equivalence as it would require the introduction of further baroque elements to the already heavy formal apparatus. We only remark that analogous identifications will be tacitly made for derivations of rules of level > 0. So, for instance, we identify the derivation $\langle \mathscr{D}, \langle \overset{1}{\alpha} \rangle\rangle$ of Example A.3, with the derivation $\langle \mathscr{D}', \langle \overset{3}{\alpha} \rangle\rangle$, where

$$\mathscr{D}' = \frac{\overset{3}{\alpha} \qquad \overset{3}{\alpha}}{\beta} \; \alpha; \overset{2}{\alpha} \Rightarrow \beta$$

A.7 Composition of Derivations and Transitivity

Definition A.9 Let $\mathscr{D}$ be a derivation of A such that $\langle \mathscr{D}, \langle \overset{v_1}{R_1} \ldots \overset{v_o}{R_o} \rangle\rangle$ is a derivation of $R_1; \ldots; R_o \Rightarrow A$ relative to $\langle v_1, \ldots, v_o \rangle$, and for all $1 \leq k \leq o$, let $\mathscr{D}_k$ be a derivation of B_k such that $\langle \mathscr{D}_k, \langle \overset{w_{k1}}{R_{k1}}, \ldots, \overset{w_{kp_k}}{R_{kp_k}} \rangle\rangle$ is a derivation of R_k relative to $\langle w_{k1}, \ldots, w_{kp_k} \rangle$.

The *composition of* $\mathscr{D}$ *with* $\mathscr{D}_1, \ldots \mathscr{D}_o$ *relative to* $\overset{v_1}{R_1}, \ldots, \overset{v_o}{R_o}$, to be indicated either as $\mathscr{D}[{}^{\mathscr{D}_1}/_{\overset{v_1}{R_1}}, \ldots, {}^{\mathscr{D}_o}/_{\overset{v_o}{R_o}}]$ (shortly: $\mathscr{D}[\ldots {}^{\mathscr{D}_k}/_{\overset{v_k}{R_k}} \ldots]$), or in tree-like notation and leaving the $v_1, \ldots, v_o$ implicit, as:

$$\begin{array}{ccc} \mathscr{D}_1 & & \mathscr{D}_o \\ [R_1] & \ldots & [R_o] \\ & \mathscr{D} & \\ & A & \end{array}$$

is defined by recursion on the sum $\Sigma_{k=1}^{o} l_k$ of the levels l_k of the R_ks, with a subrecursion on the structure of $\mathscr{D}$ as follows:

1. Base case: $\Sigma_{k=1}^{o} l_k = 0$:

   a. First base sub-case: $\mathscr{D} = \overset{u}{A}$:
      * If $v_k = u$ and $R_k = B_k = A$ for some $1 \leq k \leq o$, then $\mathscr{D}[\ldots {}^{\mathscr{D}_k}/_{\overset{v_k}{R_k}} \ldots] = \mathscr{D}_k$.
      ** Otherwise $\mathscr{D}[{}^{\mathscr{D}_1}/_{\overset{v_1}{R_1}}, \ldots, {}^{\mathscr{D}_o}/_{\overset{v_o}{R_o}}] = \mathscr{D}$.

   b. Second base sub-case: $\mathscr{D} = \frac{}{A} \, \boldsymbol{R}\,\sigma$: as 1.a.**.

c. First recursive sub-case: $\mathscr{D} = \langle u_{11}, \ldots, u_{1m_1}, \ldots, u_{n1}, \ldots, u_{nm_n} \rangle \dfrac{\begin{matrix} \mathscr{D}'_1 & & \mathscr{D}'_n \\ C_1 & \ldots & C_n \end{matrix}}{A} \overset{u}{R}$:

Assuming that all u_{ij} are different from all v_k, and that none of the $\mathscr{D}_k$ depends on any of $\overset{u_{ij}}{R_{ij}}$ (otherwise the u_{ij}s can be renamed), we define

$$\mathscr{D}[\ldots {}^{\mathscr{D}_k}/_{\underset{R_k}{v_k}} \ldots] = \langle u_{11}, \ldots, u_{1m_1}, \ldots, u_{n1}, \ldots, u_{nm_n} \rangle \dfrac{\begin{matrix} \begin{matrix} \mathscr{D}_1 & & \mathscr{D}_o \\ [R_1] & \ldots & [R_o] \end{matrix} & & \begin{matrix} \mathscr{D}_1 & & \mathscr{D}_o \\ [R_1] & \ldots & [R_o] \end{matrix} \\ \mathscr{D}'_1 & & \mathscr{D}'_n \\ C_1 & \ldots & C_n \end{matrix}}{A} \overset{u}{R}$$

d. Second recursive sub-case: $\mathscr{D} = \langle u_{11}, \ldots, u_{1m_1}, \ldots, u_{n1}, \ldots, u_{nm_n} \rangle \dfrac{\begin{matrix} \mathscr{D}'_1 & & \mathscr{D}'_n \\ C_1 & \ldots & C_n \end{matrix}}{A} \mathbf{R}\,\sigma$: as 1.c.

2. Recursive case: $\Sigma_{k=1}^{o} l_k > 0$:

a. First base sub-case: $\mathscr{D} = \overset{u}{A}$: As 1.a.

b. Second base sub-case: $\mathscr{D} = \dfrac{}{A}\, \boldsymbol{R}\,\sigma$: As 1.b.

c. First recursive sub-case: $\mathscr{D} = \langle u_{11}, \ldots, u_{1m_1}, \ldots, u_{n1}, \ldots, u_{nm_n} \rangle \dfrac{\begin{matrix} \mathscr{D}'_1 & & \mathscr{D}'_n \\ C_1 & \ldots & C_n \end{matrix}}{A} \overset{u}{R}$

As in 1.c., we assume that all u_{ij} are different from all v_k (otherwise they can be renamed).

* If $v_k = u$ and $R_k = R$ for some $1 \le k \le o$, then

$$R = R_k = \dfrac{\begin{matrix} [R'_{11}] \ldots [R'_{1m_1}] & & [R'_{n1}] \ldots [R'_{nm_n}] \\ C_1 & \ldots & C_n \end{matrix}}{A}$$

and $B_k = A$, $\mathscr{D}_k$ is a derivation of A, $p_k = n$ and $\langle \mathscr{D}_k, \langle \overset{w_{k1}}{R'_1}, \ldots, \overset{w_{kn}}{R'_n} \rangle\rangle$ is a derivation of $R'_1; \ldots R'_n \Rightarrow A$ relative to $\langle w_{k1}, \ldots, w_{kn} \rangle$, where for all $1 \le i \le n$, $R'_i = (R_{i1}; \ldots R_{in_i} \Rightarrow C_i)$.

Observe that for all $1 \le i \le n$, $\langle \mathscr{D}'_i, \langle \overset{u_{i1}}{R_{i1}}, \ldots, \overset{u_{im_i}}{R_{in_i}} \rangle\rangle$ is a derivation of R'_i relative to $\langle u_{i1}, \ldots, u_{im_i} \rangle$.

We define the composition of $\mathscr{D}$ with $\mathscr{D}_1 \ldots \mathscr{D}_o$ relative to $\overset{v_1}{R_1}, \ldots \overset{v_o}{R_o}$ as the result of composing (relative to $\overset{w_{k1}}{R'_1} \ldots \overset{w_{kn}}{R'_n}$) the derivation $\mathscr{D}_k$ with the result of composing each $\mathscr{D}'_i$ with the derivations $\mathscr{D}_1, \ldots, \mathscr{D}_o$ relative to $\overset{v_1}{R_1}, \ldots \overset{v_o}{R_o}$, i.e.

$$\mathscr{D}[\ldots {}^{\mathscr{D}_k}/^{v_k}_{R_k} \ldots] \;=\; \mathscr{D}_k \left[{}^{\mathscr{D}'_1\left[\ldots {}^{\mathscr{D}_k}/^{v_k}_{R_k} \ldots\right]}/^{w_{k1}}_{R'_1} \;,\; \ldots \;,\; {}^{\mathscr{D}'_n\left[\ldots {}^{\mathscr{D}_k}/^{v_k}_{R_k} \ldots\right]}/^{w_{kn}}_{R'_n} \right]$$

Graphically:

$$\mathscr{D}[\ldots {}^{\mathscr{D}_k}/^{v_k}_{R_k} \ldots] \;=\; \begin{array}{ccc} \begin{array}{ccc} \mathscr{D}_1 & & \mathscr{D}_o \\ [R_1] & \ldots & [R_o] \end{array} & & \begin{array}{ccc} \mathscr{D}_1 & & \mathscr{D}_o \\ [R_1] & \ldots & [R_o] \end{array} \\ \mathscr{D}'_1 & & \mathscr{D}'_n \\ [R'_1] & \ldots & [R'_n] \\ & \mathscr{D}_k & \\ & A & \end{array}$$

** Otherwise, as 1.c.

d. Second recursive sub-case: as 1.d.

Lemma A.1 *For all derivations $\mathscr{D}, \mathscr{D}_1, \ldots, \mathscr{D}_o$ as in Definition A.9:*

$$UA(\mathscr{D}[{}^{\mathscr{D}_1}/^{v_1}_{R_1}, \ldots, {}^{\mathscr{D}_o}/^{v_o}_{R_o}]) \;\subseteq\; \left(UA(\mathscr{D}) \;\setminus\; \bigcup_{k=1}^{o} \left\{ \overset{v_k}{R_k} \right\} \right) \cup \bigcup_{k=1}^{o} UA(\mathscr{D}_k)$$

Proof We prove the lemma by induction on the sum $\Sigma_{k=1}^{o} l_k$ of the levels l_k of the R_ks with a sub-induction on the structure of $\mathscr{D}$.

1. Base case: $\Sigma_{k=1}^{o} l_k = 0$:

   a. First base sub-case: $\mathscr{D} = \overset{u}{A}$

   * If $v_k = u$ and $R_k = B_k = A$ for some $1 \leq k \leq o$,

$$\begin{aligned} UA(\mathscr{D}[{}^{\mathscr{D}_1}/^{v_1}_{R_1}, \ldots, {}^{\mathscr{D}_o}/^{v_o}_{R_o}]) &= UA(\mathscr{D}_k) \\ &= \left(\{\overset{u}{A}\} \;\setminus\; \{\overset{u}{A}\} \right) \cup\; UA(\mathscr{D}_k) \\ &\subseteq \left(UA(\mathscr{D}) \;\setminus\; \textstyle\bigcup_{k=1}^{o} \left\{ \overset{v_k}{R_k} \right\} \right) \cup\; \textstyle\bigcup_{k=1}^{o} UA(\mathscr{D}_k) \end{aligned}$$

   ** Otherwise

$$UA(\mathscr{D}[{}^{\mathscr{D}_1}/^{v_1}_{R_1}, \ldots, {}^{\mathscr{D}_o}/^{v_o}_{R_o}]) \;=\; \{\overset{u}{A}\} \;\subseteq\; \left(UA(\mathscr{D}) \;\setminus\; \bigcup_{k=1}^{o} \left\{ \overset{v_k}{R_k} \right\} \right) \cup \bigcup_{k=1}^{o} UA(\mathscr{D}_k)$$

   b. Second base sub-case: $\mathscr{D} = \dfrac{}{A}\,\boldsymbol{R}\,\sigma$. Obvious, since $UA(\mathscr{D}) = \emptyset$.

c. First inductive sub-case: $\mathscr{D} = {}_{\langle u_{11},\ldots,u_{1m_1},\ldots,u_{n1},\ldots,u_{nm_n}\rangle}\dfrac{\begin{array}{c}\mathscr{D}'_1\\ C_1\end{array}\quad\ldots\quad\begin{array}{c}\mathscr{D}'_n\\ C_n\end{array}}{A}\,\overset{u}{R}$

assuming $R = \dfrac{\begin{array}{c}[R'_{11}]\ldots[R'_{1m_1}]\\ D_1\end{array}\quad\ldots\quad\begin{array}{c}[R'_{n1}]\ldots[R'_{nm_n}]\\ D_n\end{array}}{A}$ we have that

$$UA(\mathscr{D}[{}^{\mathscr{D}_1}/_{\underset{R_1}{v_1}},\ldots,{}^{\mathscr{D}_o}/_{\underset{R_o}{v_o}}]) = \bigcup_{i=1}^{n}\left(UA(\mathscr{D}'_i[{}^{\mathscr{D}_1}/_{\underset{R_1}{v_1}},\ldots,{}^{\mathscr{D}_o}/_{\underset{R_o}{v_o}}]) \setminus \bigcup_{j=1}^{m_i}\left\{\overset{u_{ij}}{R'_{ij}}\right\}\right) \cup \{\overset{u}{R}\}$$

which, by applying the induction hypothesis and performing an elementary calculation, is easily seen to be included into $\left(UA(\mathscr{D}) \setminus \bigcup_{k=1}^{o}\left\{\overset{v_k}{R_k}\right\}\right) \cup \bigcup_{k=1}^{o} UA(\mathscr{D}_k)$.

d. Second inductive sub-case: $\mathscr{D} = {}_{\langle u_{11},\ldots,u_{1m_1},\ldots,u_{n1},\ldots,u_{nm_n}\rangle}\dfrac{\begin{array}{c}\mathscr{D}'_1\\ C_1\end{array}\quad\ldots\quad\begin{array}{c}\mathscr{D}'_n\\ C_n\end{array}}{A}\,\mathbf{R}\,\sigma$:

the case is similar to 1.c.

2. Inductive case: $\Sigma_{k=1}^{o} l_k > 0$:

a. First base sub-case: $\mathscr{D} = \overset{u}{A}$: As 1.a.

b. Second base sub-case: $\mathscr{D} = \dfrac{}{A}\,\boldsymbol{R}\,\sigma$: As 1.b.

c. First inductive sub-case: $\mathscr{D} = {}_{\langle u_{11},\ldots,u_{1m_1},\ldots,u_{n1},\ldots,u_{nm_n}\rangle}\dfrac{\begin{array}{c}\mathscr{D}'_1\\ C_1\end{array}\quad\ldots\quad\begin{array}{c}\mathscr{D}'_n\\ C_n\end{array}}{A}\,\overset{u}{R}$

- If $v_k = u$ and $R_k = R$ for some $1 \leq k \leq o$, then

$$R = R_k = \dfrac{\begin{array}{c}[R'_{11}]\ldots[R'_{1m_1}]\\ C_1\end{array}\quad\ldots\quad\begin{array}{c}[R'_{n1}]\ldots[R'_{nm_n}]\\ C_n\end{array}}{A}$$

and we have that by induction hypothesis for each $1 \leq i \leq n$

$$UA(\mathscr{D}'_i[\ldots {}^{\mathscr{D}_k}/_{\underset{R_k}{v_k}} \ldots]) \subseteq \left(UA(\mathscr{D}_i) \setminus \bigcup_{k=1}^{o}\left\{\overset{v_k}{R_k}\right\}\right) \cup \bigcup_{k=1}^{o} UA(\mathscr{D}_k)$$

and that

$$UA(\mathscr{D}[\ldots {}^{\mathscr{D}_k}/_{\underset{R_k}{v_k}} \ldots]) \subseteq \left(UA(\mathscr{D}_k) \setminus \bigcup_{i=1}^{n}\left\{\overset{w_{ki}}{R'_i}\right\}\right) \cup \bigcup_{i=1}^{n} UA(\mathscr{D}'_i[\ldots {}^{\mathscr{D}_k}/_{\underset{R_k}{v_k}} \ldots])$$

which can be seen to be included into $\left(UA(\mathscr{D}) \setminus \bigcup_{k=1}^{o} \left\{\overset{v_k}{R_k}\right\}\right) \cup \bigcup_{k=1}^{o} UA(\mathscr{D}_k)$ by

- Otherwise as in 1.c.

d. Second inductive sub-case: as 1.d.

□

Corollary A.1 (Transitivity) *If* $\Gamma_1 \vdash R_1, \ldots, \Gamma_n \vdash R_n$ *and* $R_1, \ldots, R_n, \Delta \vdash R$ *then* $\Gamma_1, \ldots, \Gamma_n, \Delta \vdash R$.

Proof For all $1 \leq i \leq n$, if $\Gamma_i \vdash R_i$, then there is a derivation $\mathscr{D}_i$ of the consequence of R_i satisfying the conditions of Definition A.7; and similarly for $R_1, \ldots, R_n, \Delta \vdash R$ there is a derivation $\mathscr{D}$ of the consequence of R satisfying the conditions of Definition A.7.

If $R = A$, then $\mathscr{D}$ is a derivation of A such that for all R' if $\overset{u}{R'} \in UA(\mathscr{D})$ for some u, then $R' \in \Gamma$.

For all i, let m_i be the number of distinct assumptions $\overset{u_{ih}}{R_i} \in UA(\mathscr{D})$ (with $1 \leq h \leq m_i$). Let $\mathscr{D}^*$ be the derivation

$$\mathscr{D}^* = \mathscr{D}\left[\overset{w_1}{R_1}\Big/\overset{u_{11}}{R_1},\ \ldots\ ,\ \overset{w_1}{R_1}\Big/\overset{u_{1m_1}}{R_1}\ , \ldots\ ,\ \overset{w_n}{R_n}\Big/\overset{u_{n1}}{R_n},\ \ldots\ ,\ \overset{w_n}{R_n}\Big/\overset{u_{nm_n}}{R_n}\right]$$

for fresh $w_1 \ldots w_n$. By Definition A.6, $\langle \mathscr{D}^*, \langle \overset{w_1}{R_1}, \ldots, \overset{w_n}{R_n} \rangle\rangle$ is a derivation of $R_1; \ldots; R_n \Rightarrow A$ relative to $\langle w_1, \ldots, w_n \rangle$ and by Lemma A.1 the composition of $\mathscr{D}^*$ with $\mathscr{D}_1 \ldots \mathscr{D}_n$ relative to $\overset{w_1}{R_1}, \ldots, \overset{w_n}{R_n}$, we call it $\mathscr{D}^{**}$, is such that

$$UA(\mathscr{D}^{**}) \subseteq \left(UA(\mathscr{D}^*) \setminus \bigcup_{i=1}^{n} \left\{\overset{w_i}{R'_i}\right\}\right) \cup \bigcup_{i=1}^{n} UA(\mathscr{D}_i)$$

and thus for every $\overset{v}{R^{**}} \in UA(\mathscr{D}^{**})$, we have that $R^{**} \in \Gamma_1, \ldots, \Gamma_n, \Delta$ and thus that $\Gamma_1, \ldots, \Gamma_n, \Delta \vdash A = R$.

If R is of level higher then 0, the corollary is established by an analogous reasoning. □

We conclude this section by observing that the common way of indicating an application of, say, $\supset$I in a derivation:

$$\begin{array}{c} \overset{n}{[A]} \\ \mathscr{D} \\ \langle n \rangle\ \dfrac{B}{A \supset B}\ \supset\mathrm{I} \end{array}$$

can be understood in accordance with the definition of composition, that is the schema above indicates a derivation obtained by appending an application of ⊃I with discharge index n to the composition of $\mathscr{D}$ with $\overset{n}{A}$ relative to some (underdetermined by the context) $\overset{m}{A}$.

A.8 Identity Derivations and Reflexivity

Definition A.10 (*Structural identity derivation*) Let R be a concrete rule of level l with n premises. For all natural numbers $u_1, \ldots, u_n, u$, the structural identity derivation of R relative to $\langle u_1, \ldots, u_n, u\rangle$, to be indicated as $\mathscr{I}(R)^{\langle u_1,\ldots,u_n,u\rangle}$ is defined by recursion on the level l of R as follows:

- If $l = 0$, then $R = A$ and $\mathscr{I}(R)^{\langle u\rangle} = \overset{u}{A}$.
- If $l = 1$, then $R = \dfrac{A_1 \quad \ldots \quad A_n}{A}$. We define $\mathscr{I}(R)^{\langle u_1,\ldots,u_n,u\rangle}$ to be $\dfrac{\overset{u_1}{A_1} \quad \ldots \quad \overset{u_n}{A_n}}{A}\overset{u}{R}$.
- If $l \geq 2$, then $R = \dfrac{R_1 \quad \ldots \quad R_n}{A}$, where $R_i = \dfrac{R_{i1} \quad \ldots \quad R_{im_i}}{B_i}$ for all $1 \leq i \leq n$. We define $\mathscr{I}(R)^{\langle u_1,\ldots,u_n,u\rangle}$ to be

$$(u_{11}, \ldots, u_{1m_1}, \ldots, u_{n1}, \ldots, u_{nm_n})\ \dfrac{\dfrac{\mathscr{I}(R_1)^{\langle u_{11},\ldots,u_{1m_1},u_1\rangle}}{B_1} \qquad \ldots \qquad \dfrac{\mathscr{I}(R_n)^{\langle u_{n1},\ldots,u_{nm_n},u_n\rangle}}{B_n}}{A}\overset{u}{R}.$$

Lemma A.2 *For all R with premises $R_1, \ldots R_n$ and $u_1, \ldots u_n, u$ (with n possibly 0),*

$$UA(\mathscr{I}(R)^{\langle u_1,\ldots u_n,u\rangle}) = \bigcup_{i=1}^{n}\left\{\overset{u_i}{R_i}\right\} \cup \left\{\overset{u}{R}\right\}$$

Proof The proof is by a simple induction on the level l of R. □

Corollary A.2 (Reflexivity) *For all R, $R \vdash R$*

Proof If the level of R is 0, then the lemma obviously holds. Otherwise, $R = R_1; \ldots; R_n \Rightarrow A$ and according to Definition A.10, Lemma A.2 and Definition A.6, for all $u_1, \ldots, u_n, u$, $\langle \mathscr{I}(R)^{\langle u_1,\ldots u_n,u\rangle}, \langle \overset{u_1}{R_1}, \ldots, \overset{u_n}{R_n}\rangle\rangle$ is a derivation of R relative to $\langle u_1, \ldots, u_n\rangle$ and $UA(\langle \mathscr{I}(R)^{\langle u_1,\ldots u_n,u\rangle}, \langle \overset{u_1}{R_1}, \ldots, \overset{u_n}{R_n}\rangle\rangle) = \{\overset{u}{R}\}$. □

In defining the notion of harmony in the next chapter, we will need the following variation of the notion of structural identity derivation:

Definition A.11 (K-identity derivation) Let R be the σ-instance of a rule $\boldsymbol{R}$ of level l with n premises belonging to some calculus K. For all natural numbers $u_1 \dots u_n$, the K-structural identity derivation of R relative to $\langle u_1 \dots u_n \rangle$, to be indicated as $\overline{\mathscr{I}(R)}^{\langle u_1,\dots,u_n\rangle}$, is defined by recursion on the level l of $\boldsymbol{R}$ as follows:

- If $l = 0$, then $R = A$ and $\overline{\mathscr{I}(R)}^{\langle\rangle} = \dfrac{}{A}\,\boldsymbol{R}\,\sigma$.
- If $l = 1$, then $R = \dfrac{A_1 \quad \dots \quad A_n}{A}$. We define $\overline{\mathscr{I}(R)}^{\langle u_1,\dots,u_n\rangle}$ to be $\dfrac{\overset{u_1}{A_1} \quad \dots \quad \overset{u_n}{A_n}}{A}\,\boldsymbol{R}\,\sigma$.
- If $l \geq 2$, then $R = \dfrac{R_1 \quad \dots \quad R_n}{A}$, where $R_i = \dfrac{R_{i1} \quad \dots \quad R_{im_i}}{B_i}$ for all $1 \leq i \leq n$. We define $\overline{\mathscr{I}(R)}^{\langle u_1,\dots,u_n\rangle}$ to be

$$(u_{11},\dots,u_{1m_1},\dots,u_{n1},\dots,u_{nm_n})\ \dfrac{\dfrac{\mathscr{I}(R_1)^{\langle u_{11},\dots,u_{1m_1},u_1\rangle}}{B_1} \qquad \dots \qquad \dfrac{\mathscr{I}(R_n)^{\langle u_{n1},\dots,u_{nm_n},u_n\rangle}}{B_n}}{A}\,\boldsymbol{R}\,\sigma$$

Lemma A.3 *For all R with premises $R_1,\dots,R_n$ and $u_1,\dots u_n$, (with n possibly 0)*

$$UA(\overline{\mathscr{I}(R)}^{\langle u_1,\dots u_n\rangle}) = \bigcup_{i=1}^{n}\left\{\overset{u_i}{R_i}\right\}$$

Proof The proof is by a simple induction on the level l of R. □

Corollary A.3 *If R is an instance of a primitive rule of* K*, then* $\vdash_{\mathrm{K}} R$.

Proof The proof follows the same pattern of the proof of Corollary A.2, only using the notion of K-identity derivation instead of that of structural identity derivation. □

When the $u_1 \dots u_n$ are clear from the context, we write $\overline{\mathscr{I}(R)}$ for $\overline{\mathscr{I}(R)}^{\langle u_1,\dots,u_n}$.

Given a rule $\boldsymbol{R}$ belonging to K, we indicate with $\mathscr{I}(\boldsymbol{R})$ the schematic derivation of which all $\mathscr{I}(\overline{R})$s (with R instance of $\boldsymbol{R}$) are instances.

Example A.5

If $R = \dfrac{\alpha_1 \supset \alpha_2 \qquad \dfrac{\dfrac{\alpha_1}{\alpha_2}}{\alpha_3}}{\alpha_3}$ then $\mathscr{I}(R)^{\langle 3,4\rangle} =$

$$\langle 2\rangle\ \dfrac{\alpha_1 \supset \alpha_2 \qquad \langle 1\rangle\ \dfrac{\dfrac{\overset{1}{\alpha_1}}{\alpha_2}\ \alpha_1 \overset{2}{\Rightarrow} \alpha_2}{\alpha_3}\ \overset{3}{(\alpha_1 \Rightarrow \alpha_2) \Rightarrow \alpha_3}}{\alpha_3}\ \overset{4}{R}$$

$$\text{If } \boldsymbol{R} = \dfrac{A \supset B \qquad \dfrac{\dfrac{A}{B}}{C}}{C} \quad \text{then} \quad \mathscr{I}(\boldsymbol{R})^{\langle 3 \rangle} = \quad \dfrac{A \supset B \qquad \langle 1 \rangle\, \dfrac{\dfrac{\overset{1}{A}}{B}\; A \overset{2}{\Rightarrow} B}{C}\; (A \Rightarrow \overset{3}{B}) \Rightarrow C}{C}\,\langle 2 \rangle\; \boldsymbol{R}$$

Note that while the conclusion α_3 of $\mathscr{I}(R)$ depends on $\alpha_1 \supset \alpha_2$, $(\alpha_1 \Rightarrow \alpha_2) \Rightarrow \alpha_3$ *and* on R itself, i.e. $((\alpha_1 \supset \alpha_2), ((\alpha_1 \Rightarrow \alpha_2) \Rightarrow \alpha_3) \Rightarrow \alpha_3)$, in all instances $\mathscr{I}(\overline{R})$ of $\mathscr{I}(\boldsymbol{R})$, the conclusion C depends on $A \supset B$ and $(A \Rightarrow B) \Rightarrow C$ only.

A.9 PSH-Inversion and Harmony

Assuming $\dagger$ to be an n-ary connective, we say that:

Definition A.12 (Introduction and elimination rules) A rule of the form

$$\frac{\boldsymbol{R}_1 \qquad \dots \qquad \boldsymbol{R}_m}{\dagger(A_1, \dots, A_n)} \qquad \texttt{(INTRO)}$$

is an *introduction rule for* $\dagger$ provided that all schematic letters occurring in the rules $\boldsymbol{R}_j$ $(1 \leq j \leq m)$ are among the A_is $(1 \leq i \leq n)$.

An *elimination rule for* $\dagger$ is any rule of the form

$$\frac{\dagger(A_1, \dots, A_n) \qquad \boldsymbol{R}_1 \qquad \dots \qquad \boldsymbol{R}_m}{C} \qquad \texttt{(ELIM)}$$

(This time no restriction is imposed on the schematic letters occurring in the rule.) The first premise $\dagger(A_1, \dots, A_n)$ of the elimination rules is called *major* premise.[6]

Particular collections of introduction (respectively elimination) rules for some connective $\dagger$ will be indicated with $\dagger\boldsymbol{I}$ (resp. $\dagger\boldsymbol{E}$), possibly with primes.

As explained in Sect. 3.4, by an inversion principle we understand a recipe to associate with any given collection of introduction rules a specific collection of elimination rules which is in harmony with it. In the context of the calculus of higher-level rule, PSH-inversion can be formulated as follows:

Definition A.13 (PSH-inversion) Given a collection of introduction rules $\dagger\boldsymbol{I}$ for $\dagger$, we indicate with PSH($\dagger$**I**) the collection of elimination rules containing only the following rule:

$$\frac{\dagger(A_1, \dots, A_n) \qquad \begin{matrix}[\boldsymbol{R}_{11}] \ \dots \ [\boldsymbol{R}_{1m_1}] \\ C\end{matrix} \qquad \dots \qquad \begin{matrix}[\boldsymbol{R}_{r1}] \ \dots \ [\boldsymbol{R}_{rm_r}] \\ C\end{matrix}}{C}\,\dagger\mathrm{E}_{\texttt{PSH}} \qquad \texttt{(PSH)}$$

in which C is a schematic letter different from all A_i (for all $1 \leq i \leq n$), and each of the minor premises corresponds to one of the introduction rules of $\dagger$, in the sense that the jth premise of the kth introduction rule $\boldsymbol{R}_{kj}$ (with $1 \leq k \leq r$, where r is the number of introduction rules; and $1 \leq j \leq m_k$ where m_k is the number of premises of the kth introduction rule) is identical to the jth premise of the kth minor premises of the elimination rule.

Example A.6 If $\vee \boldsymbol{I}$ consists of the two rules on the left hand side below, then $\texttt{PSH}(\vee \boldsymbol{I})$ consists of the rule on the right hand side below:

$$\frac{A}{A \vee B}\vee\mathrm{I}_1 \qquad \frac{B}{A \vee B}\vee\mathrm{I}_2 \qquad \frac{A \vee B \qquad \begin{matrix}[A]\\ C\end{matrix} \qquad \begin{matrix}[B]\\ C\end{matrix}}{C}\vee\mathrm{E}_{\texttt{PSH}}$$

Example A.7 If $\wedge \boldsymbol{I}$ consists of the rule on the left hand side below, then $\texttt{PSH}(\wedge \boldsymbol{I})$ consists of the rule on the right hand side below:

$$\frac{A \qquad B}{A \wedge B}\wedge\mathrm{I} \qquad \frac{A \wedge B \qquad \begin{matrix}[A]\,[B]\\ C\end{matrix}}{C}\wedge\mathrm{E}_{\texttt{PSH}}$$

Example A.8 If $\supset \boldsymbol{I}$ consists of the rule on the left hand side below, then $\texttt{PSH}(\supset \boldsymbol{I})$ consists of the rule on the right hand side below:

$$\frac{\begin{matrix}[A]\\ B\end{matrix}}{A \supset B}\supset\mathrm{I} \qquad \frac{A \supset B \qquad \begin{matrix}[A \Rightarrow B]\\ C\end{matrix}}{C}\supset\mathrm{E}_{\texttt{PSH}}$$

Let $\texttt{K}$ be a calculus consisting of primitive rules all of which have either the form of an introduction or of an elimination rule, and in which $\dagger\boldsymbol{E} = \texttt{PSH}(\dagger\boldsymbol{I})$. We can define $\dagger\beta$-reductions and a generalized $\dagger\eta$-expansions on derivation in $\texttt{K}$ as follows (in the expansion we abbreviate $\dagger(A_1, \dots, A_n)$ with $\dagger$):

$$\frac{\dfrac{\begin{matrix}\mathscr{D}_1 & & \mathscr{D}_{m_k}\\ R_{k1} & \dots & R_{km_k}\end{matrix}}{\dagger(A_1, \dots, A_n)}\dagger\mathrm{I}_k \qquad \begin{matrix}[R_{11}]\ \dots\ [R_{1m}]\\ \mathscr{D}'_1\\ C\end{matrix} \qquad \dots \qquad \begin{matrix}[R_{r1}]\ \dots\ [R_{rm_r}]\\ \mathscr{D}'_r\\ C\end{matrix}}{C}\dagger\mathrm{E}_{\texttt{PSH}} \quad \overset{\dagger\beta_k}{\rhd} \quad \begin{matrix}\mathscr{D}_1 & & \mathscr{D}_{m_k}\\ [R_{k1}] & \dots & [R_{km_k}]\\ & \mathscr{D}'_k & \\ & C & \end{matrix}$$

for $1 \leq k \leq r$

$$\begin{matrix}\mathscr{D}\\ [\dagger]\\ \mathscr{D}'\\ C\end{matrix} \quad \overset{\dagger\eta}{\lhd} \quad (u_{11}, \dots, u_{1m_1}, \dots, u_{r1}, \dots, u_{rm_r}) \; \frac{\begin{matrix}\mathscr{D}\\ \dagger\end{matrix} \qquad \begin{matrix}\overset{u_{11}}{R_{11}}\ \dots\ \overset{u_{1m_1}}{R_{1m_1}}\\ \mathscr{I}(\overline{\dagger\mathrm{I}_1})\\ [\dagger]\\ \mathscr{D}'\\ C\end{matrix} \qquad \dots \qquad \begin{matrix}\overset{u_{r1}}{R_{r1}}\ \dots\ \overset{u_{rm_r}}{R_{rm_r}}\\ \mathscr{I}(\overline{\dagger\mathrm{I}_r})\\ [\dagger]\\ \mathscr{D}'\\ C\end{matrix}}{C}\dagger\mathrm{E}_{\texttt{PSH}}$$

with $u_{11} \dots u_{rm_r}$ fresh for $\mathscr{D}$

where the reduced and expanded derivations are defined as in the proofs of Lemmas A.1 and A.2 above.

We conclude with a few remarks.

First, using PSH-inversion we can find a collection of elimination rules which is in harmony with any collection of introduction rules, provided no rule in the collection involves any restriction of the kind discussed in connection with quantum disjunction and "quantum-like" implication in Chapter 3. In fact, it is not clear which is the collection of elimination rules matching the collection of introduction rules consisting only of the restricted $\overline{\supset}$I rule discussed in Sect. 3.2.

Second, [85] established a normalization theorem for a particular calculus K comprising only rules obeying PSH-inversion, and such that the occurrence of † in the consequence (respectively major premise) of the rules is the only occurrence of a connective in the introduction (resp. elimination) rules. The proof of the result uses the β-reductions as well as the permutative conversions discussed in Sect. 3.5.

Third, although the analysis of harmony we proposed is based on the possibility of performing *local* transformations on derivations and not on the possibility of *globally* transforming any derivation into normal form, normalization is much more tight to the inversion principle than recently argued by e.g. [77], p. 575, and [94], p. 1207. As shown in Sect. 3.5, the adoption of generalized expansions makes it possible to simulate the permutations (on the connection between expansions and permutations see also [121]). Thus, normalization *is* a consequence of inversion whenever introduction and elimination rule schemata are allowed to contain at most one occurrence of one connective. Conversely, when the rules of a calculus obey PSH-inversion, failure of normalization is essentially tight to the presence of more than one occurrence of a connective in the introduction and elimination rules (in particular to the presence of negative occurrences of the connective, see [13], Sect. 2), this being the feature that enables the formulation of paradoxical connectives discussed in the second part of the present work.

Finally, as the interested reader can easily check, the precise formulation of generalizations of JR- and T-inversion follows the same pattern of the generalization of PSH-inversion, and it is therefore omitted.

Notes to Appendix

1. As observed at the end of Sect. A.1, in presence of propositional quantification in the object language, the distinction between object-language and schematic formulas "vanishes" in that a schematic formula can be seen as an object-language formula containing some quantifications.
2. More precisely, if they are discharged by the same rule application *and* they occur in the derivation of the same premise of the rule application.
3. Unless the language is extended with variables for propositions and a form of structural quantification is allowed in the (concrete) rules, see Note 1 above.

4. The informal remarks at the beginning of this chapter should therefore be understood in the light of these observations. For instance, when we said that $\supset$I, being a rule of level of 2, discharges rules of level 0, we should have said that the concrete rules which are instances of the (metalinguistic) rule (schema) are concrete rules of level of 2 and these discharge concrete rules of level 0 (which are object-language formulas).
5. However, it should be obvious that a rule cannot be merely identified with a certain metalinguistic expression: the two (distinct) metalinguistic expressions $A; B \Rightarrow A \wedge B$ and $C; A \Rightarrow C \wedge A$ should be regarded as the same rule! In other word, a rule is identified as an equivalence class of metalinguistic expressions induced by the uniform renaming of their schematic letters.
6. Sometimes, it is required that in any introduction (respectively elimination) rule the occurrence of † in the consequence (resp. major premise) is the only occurrence of a connective figuring in the rule. The requirement can however be lifted, thereby allowing the introduction and elimination rules of a certain connective † to "make reference" to other connectives, or even to itself. This possibility, envisaged already by Schroeder-Heister [86] is typically needed in giving the rules for negation, which usually make reference either to $\bot$ or to itself, and for characterizing the paradoxical expressions discussed in the second part of the present work.

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