Chapter 2
Basic Properties of Plasma in Fluid Model

Abstract If the spatial variation of plasma is longer than the particle mean free path and the time variation is sufficiently longer than the plasma Coulomb collision time, the plasma can be approximated as being in local thermal equilibrium (LTE) at any point (t, **r**). Then the velocity distribution functions of the particles become Maxwellian. In addition, assuming Maxwellian is also a good assumption in many cases even for collisionless plasmas such as high-temperature fusion plasmas. In the fluid model of plasmas, The plasmas can be described in terms of five variables characterizing local Maxwellian: the density n(t, **r**), flow velocity vector **u**(t,**r**), and temperature T(t, **r**). So, the mathematics used in fluid physics is widely applicable to studying plasma phenomena.

Although conventional fluids are neutral, plasma fluids of electrons and ions couple with electromagnetic fields. It is, therefore, necessary to solve Maxwell's equations simultaneously. It is also possible to approximate electrons and ions as two different fluids or as a single fluid in case-by-case. This requires an insight into what kind of physics is important in our problem.

After reviewing the basic equation of fluids, several fluid models for plasmas are shown. Especially, a variety of waves appears because of charged particle fluids are derived to know why waves are fundamental to knowing the plasma dynamics. The mathematical method to obtain the wave solutions as an initial value problem is explained as well as the meaning of the resultant dispersion relations.

Magneto-hydrodynamic equations (MHD) are derived to explain the effects of the Biermann battery, magnetic dynamo, etc. The relationship of magnetic field and vortex flow is studied. Resistive MHD is derived including the Nernst effect, which becomes important for the magnetic field in strong electron heat flux.

Finally, electromagnetic (EM) waves in magnetized plasmas are derived to see how to use for diagnostics in the laboratory and observation of wide range of electromagnetic waves from the Universe.

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2.1 Introduction

The plasma is characterized by the collective motion of many charged particles through the Coulomb interaction. It is usually impossible to use the particle-in-cell (PIC) simulation to study a long-time and large-scale evolution of plasmas with many orders of magnitude differences in density and temperature in a system. It is more convenient to use the fluid approximation of plasmas by introducing macroscopic physical quantities, which are statistical averages of velocity moments of plasma particle velocity (momentum) distribution function.

In the history of the fluid model, Euler and Lagrange derived a mathematical model of fluid dynamics. By the middle of the eighteenth century, "Bernoulli's theorem," which is said to be the first fundamental law of hydrodynamics, was proposed, followed by "Euler's equation of motion" and "Lagrange's equation of motion," which are said to have given birth to modern fluid dynamics.

The fluid model has been applied to gas dynamics. In most cases, it is a good approximation to assume that the relatively high-density plasma such as laser plasma is neutral fluid in the local thermodynamic equilibrium (LTE). Note that the fluid approximation assumes the distribution function of electrons and ions are Maxwellian around their flow velocities, consequently, the so-called kinetic effects to be studied in Volume 4 can be neglected because they are not so important.

In this chapter, it is explained how plasma is approximated as fluid and what kind of basic equations are used for plasmas as fluids. It is not necessary to assume that the plasma is completely ionized gas and that the fluid equations are applicable to the plasmas with the ionization process progressing. The fluid equations are widely used for cases to study the dynamics of the plasmas in any state such as neutral gas, fluid, and even solid.

They are also collectively referred to as "continuum media". Then the image of plasma in this textbook will be very wide. In assuming the plasma as continuum media, another physical state such as gas, liquid, solid, strongly coupled plasma, etc. are also expressed as plasmas in the present textbook, case by case.

2.1.1 Coulomb Collision Relaxation Times

Ideal plasma consists of many ions and electrons, and they are interacting with the Coulomb force in collisional plasma or freely moving by thermal motion in the collisionless plasma case. If the distribution function of the ions and electrons is not and far from Maxwellian distributions, it is required to study the plasma based on the kinetic theory starting with the Vlasov equation. In the present Volume 2, however, the plasma is assumed to consist of the ions and electrons characterized by shifted Maxwellian distributions.

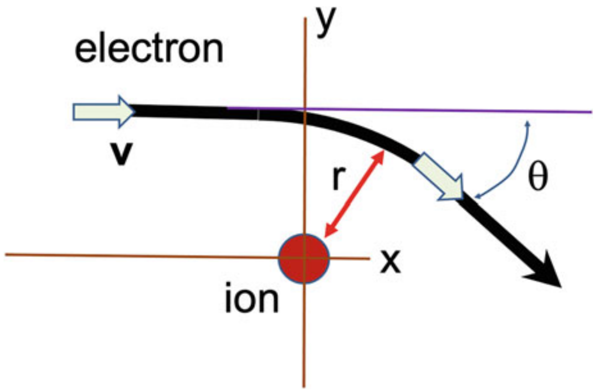


Fig. 2.1 Schematic of an electron scattering by Coulomb force of a heavy ion at the center

As shown in Fig. 2.1, the Coulomb collision cross-section for an electron with velocity **v** by the ion Coulomb force in plasma is intuitively derived as follows.

$$|\Delta(m\boldsymbol{v})|_y \approx \frac{Ze^2}{4\pi\varepsilon_0 r^2}\Delta t, \quad \Delta t \approx \frac{r}{v}, \tag{2.1}$$

where the strong interaction becomes effective when the electron kinetic energy of the electron is comparable to the Coulomb force by the ion. Set its radius r_c. Since this radius r_c is the effective distance for strong interaction, the Coulomb collision cross-section of the electron is evaluated as follows.

$$\sigma_c \sim \pi r_c^2 \propto \frac{Z^2}{v^4} \tag{2.2}$$

With use of this cross-section, the collision frequency, the inverse of collision time τ_c, is given as

$$\nu_c(v) = \frac{1}{\tau_c(v)} = n_i \sigma_c v \propto \frac{Z}{v^3} n_e \tag{2.3}$$

Note that the Coulomb collision time strongly depends on the electron velocity and it vanishes for high-velocity electrons. This is the case of the collisionless plasma for high-temperature and low-density plasmas, such as plasmas in magnetically confined fusion devices.

In evaluating the average collision time for the Maxwellian distribution of electrons, the velocity in (2.3) is replaced with the thermal velocity $v_e = (T_e/m)^{1/2}$. This is roughly equal to the distribution function relaxation time τ_e for the electron group. The derivation from (2.1) can also apply to the ion-ion Coulomb collision and the velocity dependence is the same as (2.3). So, roughly speaking the ion relaxation time $\tau_i \sim (v_e/v_i)^3 >> \tau_e$, where v_i is the ion thermal velocity.

However, the electro-ion energy (temperature) relaxation time τ_T (*has* roughly the following relation [1].

$$\tau_e < < \tau_i < < \tau_T \tag{2.4}$$

The temperature relaxation time τ_T is roughly given as

$$\tau_T = \frac{m_i}{m_e} \tau_e \tag{2.5}$$

Since τ_e is almost equal to the classical laser heating time as shown in [1], the temperature relaxation time is almost three orders of magnitude longer than the laser heating time, consequently, the ion temperature is much lower than the electron temperature in the laser heating region.

In the high-density and low-temperature regions such as near and in solid-density plasma, however, it is reasonable to assume a single temperature, $T_e = T_i = T$, and both electron and ion temperatures are equal and given as T. In assuming the plasma is a single fluid, the fluid model of the plasma can be developed by use of conventional fluid mechanics to neutral fluids like air, water, and even solids. Most of the regions of laser-driven plasmas such as shock waves and implosion dynamics in over the solid density are well described by the single temperature and neutral fluid model. Let us start with the basic property of such neutral fluids.

2.1.2 Fluid Model for Laser-Plasma

In this chapter, a variety of fluid assumptions and fluid equations are introduced to describe the fluid dynamics of laser-produced plasma. The models explained in this chapter are listed as.

One Fluid and One Temperature Fluid Model This is the traditional fluid equation to be applicable for the simplest case. The equation is the same as normal fluid, while the equation of state (EOS) from a wide range of temperature and density should be modeled in another way. This is because the pressure should be modeled from solids at room temperature where quantum physics is essential to the state of ionization and to the high-density state of strongly coupling plasma. In addition, electrons and radiation mainly x-rays, transfer the local energy in space. Such transports are modeled in general by heat conduction, while the conduction coefficient is a function of the temperature. In the case where the heat conductivity is proportional to the power of the temperature, it is possible to solve the heat wave analytically in some cases.

One Fluid and Two Temperature Fluid Model In laser-produced plasma, the plasma density and temperature change in time and space over many orders of magnitude. In the ablating and expanding region, the plasma density is relatively low, so that it is better to assume the ions and electrons have different temperatures. In addition, absorbed laser energy near the critical density is carried to an over-dense region via electrons and radiation heat conductions. The equation of both temperatures, especially electron one should include such heat conduction terms. The first step is to model them with diffusion approximation, and so-called **Spitzer's heat conductivity** is included in the electron temperature equation. However, the diffusion model is not valid in any case of laser plasma and **non-local heat conduction** will be discussed in Chap. 6.

Two Fluid Equations It is hard to extend the above model to two fluids for electrons and ions. It is, however, general that two-fluid models are required to study the phenomena over the Debye length much shorter than the mean free path. It is not so bad to neglect to solve the energy equations and use some simple relation like isothermal or adiabatic relations. Since the attractive force by charge separation for two fluids is strong and the distance is about Debye length, the two-fluid model is applied to study plasma waves induced by charge separations. Laser propagation in plasma is affected only by electron motion and ions can be assumed at rest in high-frequency motions.

Mathematics for Wave Analysis For small perturbations in any type of fluid equations, it is general to obtain coupled partial differential equations providing wave phenomena. The basic equations are modified to coupled, algebraic equations after linearization, Fourier transformation, and Laplace transformation. It becomes clear by solving the wave equation as an initial value problem that why finding the poles of the **dispersion relation** is enough to discuss the waves in plasma. The waves in plasma are important to know how fast the energy of a local disturbance diffuses or propagates around via wave propagation. If there are many waves in a complicated plasma, how fast the plasma confinement breaks is predicted by knowing the fastest wave.

Magneto-hydrodynamic (MHD) Equations It is usual to use a strong magnetic field to confine plasmas. Even in laser-produced plasma, strong magnetic fields are produced. In most cases, macroscopic fluid dynamics is controlled by ion motion, but electrons are easily run in the plasma by electric and magnetic fields in the plasma. The fluid model for plasma in external and internal magnetic fields is the MHD model, where additionally the equations to the electric current and magnetic field should be coupled. In most case, the equation to the current is replaced with a generalized **Ohm's law**. Ideal MHD model for collisionless plasma and resistive MHD model including magnetic diffusion by resistivity and heat flux are derived.

MHD waves In strongly magnetized plasma, new waves due to the magnetic tension and pressure become important as the fastest waves. The former is called the Alfven wave and the latter is called the compressible **Alfven wave**. Of course, it is more complicated because the thermal pressure force couples with the compressional Alfven wave. In addition, circularly polarized-Alfven waves can carry the angular momentum of the plasma particles. So-called Alfven breaking by this torsional Alfven wave plays an important role to carry out the angular momentum of accretion discs around the stellar objects in the Universe.

Electromagnetic Waves Electromagnetic waves are widely used in our life and also used for plasma experiments for measurement and probing plasmas. External magnetic fields modify dramatically the dispersion relation of the waves. In astrophysics, they are widely used to study the dynamics in the Universe by observing radio waves to gamma rays. It is also used to study the magnetic field in plasmas. The laser is electromagnetic wave with strong electric and magnetic fields. To know the properties of lasers in plasmas is a fundamental as studied in the book as well as well reviewed in Volume 1.

2.2 Neutral and Single Fluid Approximation of Plasma

2.2.1 Fluid Assumption

What is the fluid approximation? Consider a local fluid element, say in a volume with a unit mass, out of all fluid composed of many particles as one mass point. First, consider the equation of motion of this local fluid called a **fluid element**. This is the governing equation on the average flow velocity of the fluid element. At the same time, it is considered that the fluid element has internal energy as a small thermodynamic system, and a governing equation relating to its temperature is derived from the first law of the thermodynamics. In addition, the mass of the fluid element should be governed by the conservation relation.

It is useful to know the intuitive image of the fluid approximation. Since fluids such as water are packed closely with molecules, it is natural for the readers to imagine that the molecules of water in a certain volume move together like a large mass point. However, if this is the case of gas with a mean free path much longer than the intermolecular distance, the molecules that make up the unit mass volume of fluid will interchange with external particles from time to time. Then, the image of moving together is wrong. However, since fluid equations can be mathematically derived from the Boltzmann equation, it is enough to think that a continuous group of fluid fragments like a point mass is a mathematical concept.

However, the fluid approximation is reasonable only when the assumption of local thermal equilibrium is satisfied so that the velocity distribution function of particles made of fluid is close to Maxwellian. The same is true for the plasma. The mean free path of high-temperature plasma easily becomes longer than the plasma size. Still, the fluid equations are indeed appropriate mathematical models as long as the velocity distribution function of the plasma is close to the Maxwell distribution locally. This mathematics is shown in Appendix A.

Consider the thermodynamics of the fluid element. With the internal energy density set to ε, the first law of thermodynamics should be satisfied.

$$d\varepsilon = -PdV + dQ \tag{2.6}$$

Here, P is the pressure, V = 1/ρ is the volume occupied by a unit mass (referred to as a **specific volume**), ρ is the mass density, and dQ indicates the amount of energy that flows into this volume as thermal energy by thermal conduction, heating, cooling, etc., and also the heat flowing out from the volume.

The fluid of the specific volume moves at the average flow velocity vector **u**. The equation of motion is inferred from the Newton equation,

$$\rho \frac{d}{dt}\boldsymbol{u} = \rho \boldsymbol{F} - \nabla P \tag{2.7}$$

Here, **F** is an external force. The difference from the Newton equation is that the force due to the pressure appears in (2.7). When molecules cannot move freely like in water and move as a whole locally, the pressure can be regarded as the average force by the surrounding fluid molecules acting on the surface of the specific volume. It is hard to understand intuitively the physics image, however, when the particles are freely moving in the volume like in the case of collisionless plasma. As explained later, the pressure is the momentum flux of the microscopic particles passing through the unit surface per unit time. The reason why this is added to the Newton equation as a slope of pressure will be explained after deriving it mathematically.

To analyze any fluid phenomenon, two other equations are necessary. One is the equation for mass density, the other is the form of the pressure P as a function of density and internal energy. The former is called the **equation of continuity** and the latter is called the **equation of state**, respectively. A fluid whose density hardly changes like water in our daily life is called an **incompressible fluid**. In this case, the density is unchanged, and the equation of continuity is not necessary. In studying the plasma produced by the high-intensity laser, however, it is easy to create the pressure of millions of atmospheres in the matters and the compression of even water is easy. In many cases in the laser-plasmas, it is necessary to consider the phenomena of compressive fluids whose density also changes dramatically.

The governing equation for the mass density is called the **equation of continuity** (the continuity equation) and is given by the following relation,

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \boldsymbol{u}) = 0 \tag{2.8}$$

This is also called the **mass conservation law** of fluid. It is clear that (2.8) is written in the same partial differential equation as the energy conservation equation of the electromagnetic wave in Vol. 1. In (2.8), $\rho \boldsymbol{u}$ is the **mass density flux**.

Integrating (2.8) over an arbitrary volume V, the following relations are obtained.

$$\oint \frac{\partial \rho}{\partial t} dV = \frac{\partial}{\partial t}\left(\oint \rho dV\right) \tag{2.9}$$

$$\oint \nabla(\rho \boldsymbol{u}) dV = \oint \rho \boldsymbol{u} \cdot d\boldsymbol{S} \tag{2.10}$$

Here, (2.10) is the **Gauss theorem**. Then, (2.8) has the following relation in the integrated expression.

$$\frac{\partial}{\partial t}\left(\oint \rho dV\right) = -\oint \rho \boldsymbol{u} \cdot d\boldsymbol{S} \tag{2.11}$$

This indicates that the time change of the mass in any volume V is determined by the difference between the mass escaping from the volume surface and the mass flowing in. Therefore, total mass is kept constant when the volume V is taken to be the volume of the total system without any external mass flux.

As mathematically proved with (2.8), it can be shown in a general form.

$$\frac{\partial W}{\partial t} + \nabla \boldsymbol{Q} = 0 \tag{2.12}$$

This is a partial differential equation showing the **conservation relation** to a physical quantity W and its flux **Q**.

2.2.2 Basic Equations of Fluid Dynamics

There are two ways of expression of fluid equations. They are mathematically the same as explained later, while the concept of physical quantities is different:

1. Lagrange fluid equation
2. Euler fluid equation

These differences appear with difference on time derivative. **Lagrange** type is written with **total derivative** (*d*/*dt*), while **Euler** type is expressed with partial derivative ($\partial/\partial t$). Both are directly related as follows.

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla \tag{2.13}$$

Here, $\boldsymbol{u} \cdot \nabla$is called the **convection term** of fluid, and it means the change of variables due to convective motion of fluid elements.

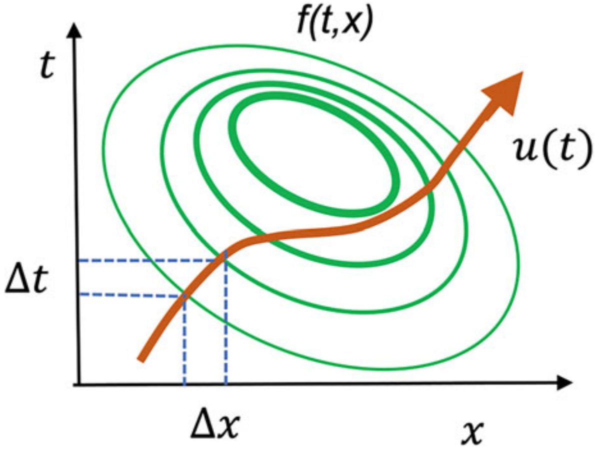


Fig. 2.2 Equi-contours of a fluid field *f*(t, x) in one dimension in space. The solid line is trajectory of a fluid element. The total derivative, Lagrange derivative, has the relation with the fluid field in Euler coordinate. This figure shows the relation between Lagrangian and Euler time derivative intuitively

In Newton mechanics, the equation of motion is governed by the force applied to a point mass. On the other hand, the Maxwell equation is that governing physical quantities of electric and magnetic fields defined in a given space. There are two ways to formulate hydrodynamic equations as the former image, Lagrange type, and Euler type that defines the field variables such as velocity fields like the electric field. One can choose one of two types of fluid equations that is convenient for one's problem and mathematically easier to solve that problem.

The relationship between the two types can be understood from Fig. 2.2. The contour lines are equi-contours of a certain physical quantity (e.g.; the velocity in x-direction) obtained by solving a fluid equation, where the brown line is the x-trajectory of a fluid element. For the sake of simplicity, assume that space is only one dimension on the x axis and consider the physical quantity shown in Fig. 2.2 as f (t, x). Then, the following relation can be obtained for small amounts Δx and Δt.

$$\Delta f = \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial x}\Delta x \tag{2.14}$$

This is rewritten to be

$$\frac{\Delta f}{\Delta t} = \frac{\partial f}{\partial t} + \frac{\Delta x}{\Delta t}\frac{\partial f}{\partial x} \tag{2.15}$$

As can be seen in Fig. 2.2, since the local fluid moves in space, at the limit of Δt → 0, (2.15) becomes,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + u_x \frac{\partial f}{\partial x} \tag{2.16}$$

Therefore, it is clear that in the three-dimensional space (2.16) becomes (2.13).

When analyzing a fluid phenomenon, the Lagrange type of equation is easy to solve mathematically, and the image of physical results is also easy to grasp in some cases. For example, a one-dimensional compressible fluid is good example. However, when analyzing complicated flow or carrying out multidimensional computer simulation, the Euler type is generally easier to solve numerically.

In the fluid equation of the Lagrange type, (2.7) can be regarded as the equation of motion for the point mass, ρ. Since it also carries the thermodynamic quantities, from the expression (2.6) applied to the internal energy per unit mass of the fluid, the following energy equation can be derived.

$$\frac{d\varepsilon}{dt} = -P\frac{dV}{dt} + \frac{dQ}{dt} \tag{2.17}$$

Here, the specific volume

$$V = \frac{1}{\rho} \tag{2.18}$$

It is noted that (2.17) is rewritten after inserting (2.18) into (2.17) and using (2.8),

$$\frac{d\varepsilon}{dt} = -\frac{P}{\rho}\nabla \boldsymbol{u} + \frac{dQ}{dt} \tag{2.19}$$

As a result, the following coupled equations are derived for the fluid equation in Lagrange type

$$\frac{d\rho}{dt} = -\rho\nabla \cdot \boldsymbol{u} \text{ (Equation of continuity)} \tag{2.20}$$

$$\rho\frac{d\boldsymbol{u}}{dt} = -\nabla P + \rho\boldsymbol{F} \text{ (Equation of motion)} \tag{2.21}$$

$$\frac{d\varepsilon}{dt} = -\frac{P}{\rho}\nabla \boldsymbol{u} + \frac{dQ}{dt} \text{ (Energy equation)} \tag{2.22}$$

On the other hand, in the Euler type, the hydrodynamic variables are considered as the field quantities in the time and space (t, **r**). As like the Maxwell equations, the equation governing such field quantities is a partial differential equation. Therefore, (2.20) can be in the Euler type as follows.

$$\frac{\partial\rho}{\partial t} + \nabla(\rho\boldsymbol{u}) = 0 \tag{2.23}$$

$$\rho\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\boldsymbol{u} = -\nabla P + \rho\boldsymbol{F} \tag{2.24}$$

$$\left(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\nabla\right)\varepsilon=-\frac{P}{\rho}\nabla\boldsymbol{u}+\frac{dQ}{dt} \tag{2.25}$$

It should be noted that the equations of (2.23, 2.24, and 2.25) differ significantly from that of the Maxwell equation; namely, the fluid equations are nonlinear equations. For example, the **nonlinearity** of fluid dynamics stems from the convection term:

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u} \tag{2.26}$$

As the fluid velocity increases, the convective term cannot be neglected and it plays an important role in the formation of a unique structure like shock waves to be discussed in Chap. 3.

2.2.3 Conservation Relations

Three conservation equations can be derived from the fluid equations. Consider the case without the external source force **F** in (2.14) and heat Q in (2.25). The conservation of mass has already been shown in (2.8). Next is the fluid momentum density conservation law, which is obtained by adding two equations after multiplying (2.24) and (2.23) by the flow vector **u**.

$$\frac{\partial}{\partial t}(\rho\boldsymbol{u})+\nabla\left(\rho\boldsymbol{u}\otimes\boldsymbol{u}+\overleftrightarrow{P}\right)=0 \tag{2.27}$$

Here, the mathematical symbol "$\otimes$" means in this case to create a three-dimensional tensor from three-dimensional vectors, whose (i, j) component is $u_i u_j$ in (2.27). This is called a **dyadic product** (tensor). The pressure was generally indicated in the form of a tensor in three-dimension. For the ideal gas, the pressure is a scalar and it can be considered that a scalar pressure is multiplied by a unit tensor in (2.27).

The third and final one is the energy conservation equation. The sum of the kinetic energy of the fluid flow and the thermodynamic internal energy should be conserved. By deriving an equation for each energy density for both, the total conservation form can be obtained after adding both equations. Specifically, multiplying (2.23) by $1/2u^2$, multiplying (2.24) by **u**, and multiplying them by (2.25) multiplied by density ρ give

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2+\rho\varepsilon\right)+\nabla\left\{\left(\frac{1}{2}\rho u^2+\rho\varepsilon+P\right)\boldsymbol{u}\right\}=0 \tag{2.28}$$

Here, P is a scalar for the sake of simplicity. Even if the pressure is tensor, the tensor pressure can be easily written in the same form as (2.28). In gas dynamics, the **enthalpy** in the form is also introduced.

$$h = \varepsilon + \frac{P}{\rho} \tag{2.29}$$

As a result, it is found that the conservation form of the fluid equation consists of (2.8) for mass, (2.27) for momentum, and (2.28) for total energy. It is easily found that these three equations are in the form of standard conservation Eqs. (2.12).

2.2.4 Equation of State

In general, the external force **F** and heat exchange dQ are given in solving (2.20, 2.21, 2.22, 2.23, 2.24, and 2.25). Still, however, one another relation or equation is necessary in order to make the fluid equations in closed form. It is called the **equation of state** (**EOS**) and is a relation between internal energy ε and pressure P. The equation of state is usually given with a new thermodynamic quantity, temperature T, in the form

$$\begin{aligned} P &= P(\rho, T) \\ \varepsilon &= \varepsilon(\rho, T) \end{aligned} \tag{2.30}$$

In (2.22), the equation of energy can be transformed into the equation for temperature T, and the fluid equation can be solvable as a closed system of three equations for density ρ and flow velocity **u** (three-dimensional vector), and temperature T.

Except for the case of the ideal gas, how to calculate the equation of state (2.30) itself is a major research topic in the laser plasma. The matter changes from solid to liquid, neutral gas to plasma. In laser-plasma experiments, it is usually required to study strong shock waves and related phenomena, where the shock waves pass through the solid and change the overall states. In such a case, it is necessary to introduce an appropriate formula for the equation of state within the range of density and temperature over many orders of magnitude. It is also required to model the phase transitions, although the fluid approximation is still valid.

For the time being, the following EOS is assumed as the ideal gas for the fluid consisting of fully ionized ions with charge Z and electrons.

$$P = (Z+1) n_i T \tag{2.31}$$

$$\varepsilon = \frac{(Z+1)}{\gamma - 1} \frac{T}{M} \tag{2.32}$$

Here, T is expressed in units of energy including Boltzmann constant k_B. The unit of the electron volt (eV) is used in the present book for the temperature. In (2.31), n_i is the number density of ions and M is approximated by the ion mass, because the electron mass can be neglected. In addition, γ in (2.32) is the **specific heat ratio**. Note that γ is given simply as a function of the degree of freedom of the particle dynamics, and given number of the freedom, N, the following relation is satisfied.

$$\gamma = \frac{N+2}{N} \tag{2.33}$$

If there are no internal degrees of freedom in all particles of the fluid and the particles freely fly in the x, y, and z spatial three dimensions, N = 3 and it is enough to set $\gamma = 5/3$.

2.2.5 Thermodynamic Consistency

If the laser intensity is not so high, the electrons on the solid surface are heated, the inside of the solid is heated by the heat conduction of electrons, and the temperature increases from the surface. When the temperature is high, the solid melts and undergoes a phase transition to a liquid state. Furthermore, the laser intensity is strong, and vaporization occurs from the liquid state when the temperature becomes high enough. Depending on the material, the phase transition proceeds while the gas and the liquid are mixed, and the state becomes a neutral gas state. Furthermore, as neutral gas absorbs laser photons, dissociation progresses and ionization starts. If the laser intensity is sufficiently strong and the Z-value of the solid is not so high, the material becomes in a completely ionized plasma state, and the temperature increases further. The microphysics of ionization and related processes will be discussed in Chap. 5.

This book must be useful for analyzing the plasma process, and the plasma in stars, including non-ideal plasma with a wide range of temperature and density regions. If the fluid equations are used to analyze and simulate from a solid state to an ideal plasma with reasonable mathematical model, the ideal gas equation of state alone is not sufficient. When a star is born from a molecular cloud and evolves in time, non-ideal plasma should be studied about molecular dissociation, ionization processes etc. Therefore, the thermodynamics and statistical mechanics for understanding general properties of the equation of state should be studied including phase transition.

When the system is in thermodynamic equilibrium, the equation of state is determined only by two state functions of the system, for example, temperature T and density ρ like (2.30). In general, the equation of state can be derived from the thermodynamic potential. It is known that **Helmholtz free energy** F (T, ρ) per unit mass of the system is defined with a function of only (T, ρ), and it satisfies the following relations.

$$S = -\frac{\partial F}{\partial T} \tag{2.34}$$

$$\varepsilon = F + TS = -T^2 \frac{\partial}{\partial T}\left(\frac{F}{T}\right) \tag{2.35}$$

$$P = -\frac{\partial F}{\partial V}, \qquad V = \frac{1}{\rho} \tag{2.36}$$

Here, S is an entropy per unit mass.

If F (T, ρ) is given, the equation of state is uniquely determined. At the same time, the following **thermodynamic consistency** is satisfied automatically.

$$\frac{\partial \varepsilon}{\partial V} = -P + T\frac{\partial P}{\partial T} \tag{2.37}$$

The relation (2.37) is obtained by partial differentiation of (2.35) with V, partial differentiation of (2.34) with V, and the change of the order of V and T. As can be seen from (2.37), it is not allowed to define P and ε independently. Given a function of pressure P, the internal energy ε must be uniquely determined by (2.37). When modeling the pressure P with some approximation, ε is necessary to be obtained from (2.37) by integrating with V.

What happens if the equation of state does not satisfy (2.37)? When in the Newton equation, a point mass moves from the initial point to the endpoint in a mechanical potential, the energy obtained by the point mass is given by the difference between the potential energies at both points. As same as this, if the equation of state does not satisfy (2.37), the amount of change in internal energy depends on which path in the two-dimensional space of (V, T) the system took to change. This contradicts the assumption that the physical quantity changes in the thermal equilibrium state. In the ideal gas equation of state (2.31, 2.32, and 2.33), both sides of (2.37) disappear, and it turns out that (2.37) is satisfied for arbitrary constant γ.

2.2.6 Cold Pressure

Intense lasers are also used to study the properties of matter at extremely high pressure. High-pressure physics by the use of lasers is now an important research field. For example, the physics of the insulator-metal phase transitions is intensively studied these days as will be discussed in Chap. 8. The pressure and internal energy of the matter compressed more than its solid density are mainly determined by the pressure due to the exchange interaction of electrons and the Coulomb repulsive force among electrons and ions for covariant bonding matter and electron degenerate pressure due to **Pauli exclusive principle** for the matter with many free electrons like metals. Such pressure at T = 0 is called the **cold pressure** P_c and it is only a function of the density ρ.

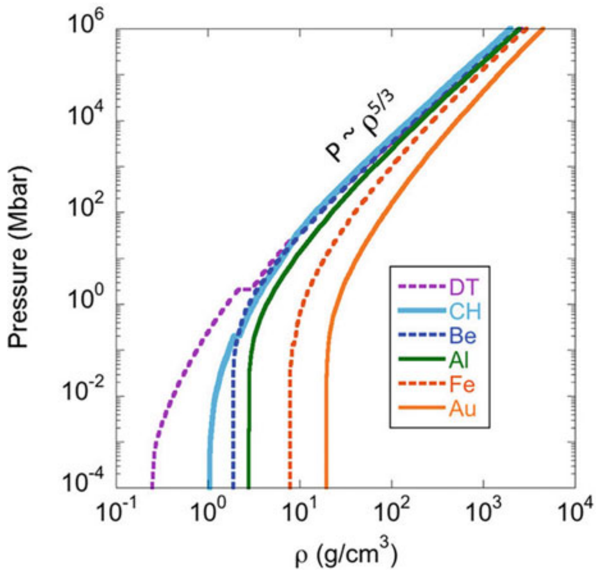


Fig. 2.3 The cold pressures of several materials commonly used as targets of laser plasma are plotted as function of density. Phase transition is clearly seen in the case of solid DT fusion fuel. Reprint with permission from Ref. [2]. Copyright 1998 by American Institute of Physics

The functional form of the pressure of matter at high density can be written in the form.

$$P(\rho, T) = P_c(\rho) + P_t(\rho, T) \tag{2.38}$$

where P_t is the **thermal component** and $P_t = 0$ at $T = 0$. The cold pressures of several materials are plotted in Fig. 2.3 [2], where the pressure is equal to zero at the nominal solid density at $T = 0$. The cold pressure is plotted for DT fusion fuel, CH plastic, Be, Al aluminum, Fe iron, and Au gold, widely used for solid targets irradiated by intense lasers. The cold pressure is due to molecular bonding pressure and the electron **Fermi pressure** at density near and higher than the solid density.

Then, it is clear that the cold component of the pressure should be calculated with (2.37) as

$$\varepsilon_c(\rho) = \int \frac{P_c(\rho)}{\rho^2} d\rho \tag{2.39}$$

Of course, if one has the function of the cold internal energy, the cold pressure is obtained by the density derivative of the cold energy by (2.39).

In solving the hydrodynamic equations with such a cold pressure, it is important to modify the equation of energy (2.22) as follows. Then, the total internal energy should be in the form.

$$\varepsilon(\rho, T) = \varepsilon_c(\rho) + \varepsilon_t(\rho, T) \tag{2.40}$$

where ε_t is the thermal component derived from (2.37) with P_t in (2.38). Note then that the cold component satisfies the relation.

$$d\varepsilon_c = -P_c dV \tag{2.41}$$

(2.41) indicates that the cold components of pressure and internal energy in (2.17) cancel to disappear from the equation of energy. It is enough to insert only the thermal component of (2.38) and (2.40) into (2.17).

The energy equation is converted to the equation for the temperature as

$$\frac{d}{dt}\varepsilon_t(\rho, T) = \frac{\partial \varepsilon_t}{\partial T}\frac{dT}{dt} + \frac{\partial \varepsilon_t}{\partial V}\frac{dV}{dt} \tag{2.42}$$

Inserting (2.42) to (2.17) the equation to the temperature is obtained from (2.25) as

$$\frac{\partial \varepsilon_t}{\partial T}\frac{dT}{dt} = -\left(P_t + \frac{\partial \varepsilon_t}{\partial V}\right)\frac{dV}{dt} + \frac{dQ}{dt} = -T\frac{\partial P_t}{\partial T}\nabla \boldsymbol{u} + \frac{dQ}{dt} \tag{2.43}$$

In the case where the thermal component can be neglected and the pressure is only the function of the density, fluid equations are closed only with (2.23) and (2.24) and it is not necessary to solve (2.25). This is the case, for example, where the electron Fermi pressure is dominant in compression. It is also the case where the cold pressure by the ion Coulomb force is dominant in higher-density.

On the other hand, the adiabatic compression without the heat term $Q = 0$ in (2.43) provides the temperature as a function of only the density, consequently, it is not necessary to solve the energy equation explicitly. In some cases, it is assumed that the pressure does not depend on the temperature in hydrodynamic model such as star formation in astrophysics or the formation of a large-scale structure in the Universe. Such a simplified EOS is used to model that "entropy decreases in the process of formation of stars" taking account of heating by compression and energy loss due to radiation emission. Such a case is called a **polytrophic process** in thermodynamics. That is,

$$P = A\rho^n \tag{2.44}$$

where A is a constant coefficient and the pressure is proportional to the n-th power of the density. In general, the "n" is called a **polytrophic exponent.**

2.3 Sound Waves

To know the fundamental property of the fluids, the linear response of small perturbations in the fluid Eqs. (2.23, 2.24, and 2.25) is studied. It is well known that they give the relation of sound waves propagating in any continuous media.

Now, when a very weak disturbance is generated externally in steady state fluid whose physical quantities are described with a subscript “0”. The physical quantities due to the disturbance are shown with a suffix “1”, where they change as a function of space and time. The linearized continuity equation and momentum equation are derived as follows.

$$\frac{\partial}{\partial t}\rho_1 + \rho_0 \nabla \boldsymbol{u}_1 = 0 \tag{2.45}$$

$$\rho_0 \frac{\partial}{\partial t}\boldsymbol{u}_1 = -\left(\frac{\partial P}{\partial \rho}\right)_0 \nabla \rho_1 \tag{2.46}$$

where $\mathbf{u}_0 = 0$ has been assumed. The external force is also neglected here. In (2.46), it is assumed that P = P (ρ).

Partial differentiation due to the density of pressure has a dimension of velocity squared, and the velocity (**sound velocity**) V_s of the sound wave propagating in the fluid can be defined as follows.

$$V_s = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)}\Bigg|_{\rho=\rho_0} \tag{2.47}$$

where RHS is a constant value calculated with ρ_0 and P_0. By substituting (2.47) into (2.46), a partial differential equation of the second order for the density disturbance is obtained.

$$\left(\frac{\partial^2}{\partial t^2} - {V_s}^2 \nabla^2\right)\rho_1 = 0 \tag{2.48}$$

For simplicity, if the density perturbation is assumed to propagate in the x direction and there is no spatial change in the y and z directions, (2.48) becomes the form.

$$\left(\frac{\partial}{\partial t} + V_s \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - V_s \frac{\partial}{\partial x}\right)\rho_1 = 0 \tag{2.49}$$

Equation (2.49) gives the waves of the first term which propagates in the x direction at the velocity Vs and the wave which the second term propagates in the -x direction. The dispersion relation to the frequency ω and wavenumber **k** is easily calculated as

$$\omega^2 = V_s^2 k^2 \tag{2.50}$$

In our everyday conversation, the density and pressure disturbance of sound is sufficiently small, and the sound wave propagates at the sound velocity obtained by substituting the air density and pressure in (2.47). In fact, we open and close the vocal cord membrane in the throat of the mouth and adjust the waveform of the pressure coming out of the mouth. This generates sound waves, and since the propagation speed of the sound waves is constant and independent of the wavelength as derived in (2.50), the pressure wave of the same waveform goes all the way. Therefore, as a vibration of the eardrum of the listener, the time change of the pressure wave is sensed, and the conversation sound is recognized.

This fact can be applied to the sound waves in solids, liquids, and gases. What is understood from (2.47) is that the sound velocity is related to the **bulk modulus B** defined as follows.

$$B = \rho_0 {V_s}^2 \tag{2.51}$$

The bulk modulus is the pressure to compress the matter to the density two times from the normal condition. It can be seen that the propagation speed of the sound wave is slower as it is easy to compress. In other words, in a continuous body regarded as **incompressible** such as a solid or a liquid which is difficult to compress in the everyday life. Since the density change due to pressure disturbance is almost as close as zero, the speed of sound is fast. That is, it can be seen that the incompressible fluid approximation is the limit of the infinite sound speed.

A typical example of the data of Bulk modulus is listed below. Shock waves produced by laser irradiation generate the pressure of more than Mega bar (10^6 atm.) and the physical property of highly compressed matter is studied.

water: 0.022×10^6 atm.,	carbon: 0.18×10^6 atm.,
aluminum: 0.75×10^6 atm.,	iron: 1.1×10^6 atm.,
polyethylene (CH): 0.04×10^6 atm.,	gold: 2.2×10^6 atm.,
air: 1 atm.,	solid hydrogen: 2000 atm.

The sound velocity is the most important physical parameter of any kind of fluid or continuous media. When the spontaneous release of high pressure happens due to some natural or artificial reason at a certain point in space, the sound waves play a role to relax the pressure in space. The sound waves also carry the energy around so that the pressure disturbance disappears as time goes on. Earthquake is due to the release or generation of huge energy underground and this energy is spread by the waves propagating in the ground with given velocities.

2.3.1 Wave Propagation in Spherical Geometry

It is useful to know how the waves propagate in the spherical geometry. At a far distance place from the initial disturbance, it can be assumed that the wave perturbation is spherically symmetric, and using the spherical Laplacian in (2.48), the wave equation becomes,

$$\frac{\partial^2}{\partial t^2}u_1 - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)u_1 = 0 \tag{2.52}$$

Using the fact that the wave energy is conserved in propagation, it is expected that the energy flux satisfies the relation,

$$\frac{1}{2}\rho_0 u_1^2 V_s \times 4\pi r^2 = const. \tag{2.53}$$

This energy conservation relation suggests the functional structure of the perturbations as

$$u_1 = \frac{A}{r}e^{ikr - i\omega t} + \frac{B}{r}e^{ikr + i\omega t} \tag{2.54}$$

where A and B are constants determined by the initial condition. The first one is a spherical wave propagating outward, while the second one is that coming from the out to the center. If the wave source is released at the center, B = 0 is satisfied. It is clear that (2.54) satisfies the wave equation in spherical geometry. If there is no damping of the perturbations, the amplitude of the waves decreases in proportion to 1/r due to the spherical geometry effect.

2.3.2 Importance of Wave Analysis

Assume that some fluid is at rest and in a stationary state. Suppose, for example, an energy is released at a certain point in the stationary fluid. In the case of air, it is clear what happens when a firework explodes in the air. Then, we hear strong sounds generated by the explosion, and of course we enjoy the fireworks. This is analogous to the case of confined plasma. When some energy is released in some point of plasma, the energy disperses to the surrounding by waves. Even with a slight perturbation in the plasma, waves are excited, and they disperse energy and momentum in the plasma. Because of this reason, it is very important to investigate the wave property in the equilibrium plasma beforehand. If the amplitude of the waves is large, the plasma itself is destroyed.

The role of the waves in such equilibrium state is experienced in the case of the propagation of the **seismic wave** due to the **earthquake**. As shown in Fig. 2.4, when energy generation suddenly occurs at the seismic source, the energy propagates to the surface of the earth through seismic waves, tsunamis, etc. The seismic waves are longitudinal and transverse waves propagating from the epicenter. The longitudinal wave propagates by compressing the soil with the velocity of about 10,000 km/h (6 km/s). This is the sound wave in the soil material. The transverse wave arises due to the viscosity of soil and propagates at 5000 km/h (3 km/s).

The former longitudinal waves are called the P waves, and the transverse waves are called the S waves. Furthermore, when both waves reach the surface of the earth,

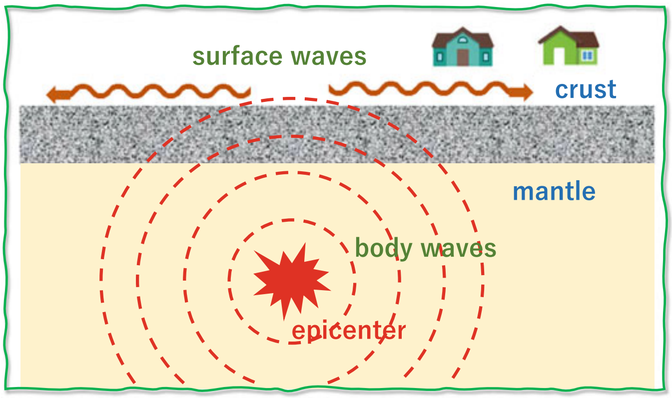


Fig. 2.4 Schematic of propagation of seismic waves from the energy source (epicenter). The longitudinal and transverse waves transport the energy of the earthquake to the surface of the earth. Once the waves arrive on the surface, the surface wave spread the energy over the wider area of the earth. To study the property of the waves of continuous media is important to know the energy transport by such a hydrodynamic motion

they will become surface waves propagating on the ground surface like the gravitational waves of the water surface explain in Volume 3. To issue an earthquake warning, the occurrence of an earthquake is detected by observing this P wave and S wave in advance. From the energy point of view, the waves act to disperse the energy soon after a huge energy such as a collapse of the ocean floor occurs locally.

It is the same in plasmas. Consider, for example, the case where the plasma ejected by the explosion of the Sun surface (solar flare) falls as the plasma energy from the outside into the plasma confined in the earth magnetosphere. The abruptly injected energy propagates in the Earth's magnetic field at high velocity by the **Alfven waves** discussed later. If the amplitude is too large, the plasma itself will be destroyed. When the plasma fluctuates greatly in the magnetic field, an electric field is generated, and electromagnetic waves are generated from the Maxwell equations. If this electric field is too strong, excessive current will flow in the circuit of the communication system and it will be destroyed. In order to predict such natural disasters, the research field of "space weather forecast" is promoted through collaboration between solar observation and numerical simulations.

2.3.3 Wave Optics and Metamaterial

For the readers, it is interesting to know more about another wave property of the sound waves. In Fig. 2.5, a **metamaterial** for sound wave is shown [3]. The scattering of a sound wave from the left on a rigid object at the center is shown on the left in Fig. 2.5. It is obvious that the wave is scattered due to the reflection by the object and interferometry makes the wave profile complex. The snapshot is the

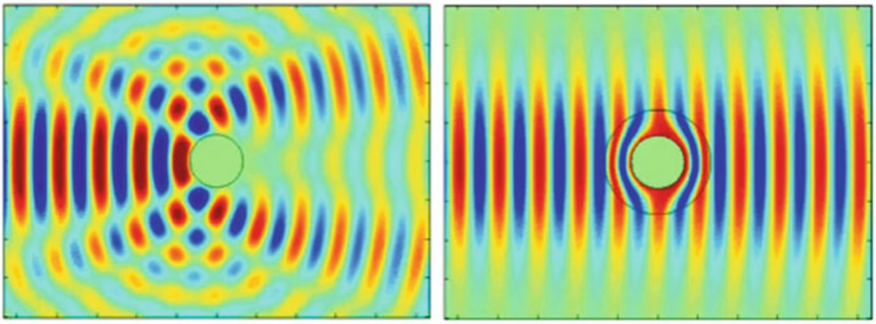


Fig. 2.5 A snapshot of the density contours of the sound wave scattered by an object. (**a**) is the case of scattering by a normal object, while (**b**) is by a meta-material with an additional shell around the object. Mathematical solution of the sound wave scattering allows almost no scattering object, and the object becomes transparent to the sound wave. Reprint with permission from Ref. [3]. Copyright 1998 by American Institute of Physics

density perturbation and quasi-specular reflection and deep shadow of the wave are observed in the right as the normal case. Surrounding the same object with an ideal cloaking shell, however, shows the absence of both reflection and shadow as shown on the right in Fig. 2.5. The wave power is transmitted around the metamaterial object with no losses and the existing of the object cannot be detected.

Such concept of metamaterial is also applicable to any kind of waves. This is the so-called stealth function, which can be applied to any wave, for example, the reflection of radio waves from an object can be avoided with the same idea to the electromagnetic waves.

2.4 Non-Ideal Fluid with Viscosity and Thermal Conduction

2.4.1 Viscosity and Reynolds Number

The ideal fluid has been assumed so far, and the effects such as molecular viscosity are ignored in the fluid equations. Also, we have ignored the heat conduction in the equation of energy. In any fluid dynamics with the non-uniformity of flow velocity and/or temperature, the viscosity and/or heat conduction should be evaluated at first whether they can be neglected.

Consider the fluid dynamics of weather and climate phenomena in our daily life. In summer, strong sunlight raises the temperature of the earth's surface. Heat conduction and convection phenomena in the air can cause the mirages. When this happens near the equator of the Pacific Ocean or the Atlantic Ocean, high temperature seawater evaporates, creating rising air flow, and typhoons and hurricanes are born by the Coriolis force stemming from the rotation of the Earth. In the typhoons and hurricanes, small eddies grow finally to one large vortex while being influenced by viscosity and nonlinear effects. It is well known that the lower the center pressure, the larger the size.

In order to study the actual fluid phenomena, there are cases where three coupled equations of ideal fluid (2.23, 2.24, and 2.25) are not appropriate. It is necessary to consider the viscosity effect accompanying the spatial variation of the flow velocity in the equation of motion. The viscosity relaxes the spatial variation of fluid velocity and converts the flow kinetic energy to the internal energy of fluid (heat). This is the viscosity heating of fluid same as the frictional heating of a body in a simple mechanics.

In the equation of energy, in addition, when the temperature changes spatially, it is necessary to consider the transport of the internal energy. This is a corresponding phenomenon of the diffusion of internal energy. The continuity equation is an equation of mass conservation and can be used as it is. Taking into account these effects, the equation of motion in (2.21) and the equation of energy in (2.22) should be modified.

Derivation of the details of that viscosity force is complicated, so the following equation of motion with a scalar viscosity is used in this book without any mathematical derivation.

$$\rho\frac{d\boldsymbol{u}}{dt} = -\nabla P + \rho\boldsymbol{F} + \frac{4}{3}\mu\nabla^2\boldsymbol{u} - \mu\nabla\times\nabla\times\boldsymbol{u} \tag{2.55}$$

This is **Navier-Stokes equation**. Here, μ is a **viscosity coefficient** and assumed to be constant. In (2.55), the first term of the viscosity is the diffusion of flow velocity with the **diffusion coefficient**

$$\nu = \frac{\mu}{\rho} \tag{2.56}$$

The ν in (2.56) is called the **kinematic viscosity coefficient**. The last term on RHS in (2.55) is the force that makes the flow velocity variation in space smooth when the flow velocity changes in perpendicular to the direction of flow vector.

In many cases in fluid dynamics, **incompressible assumption** is used. For example, fluid dynamics of climate change and subsonic aircraft can be studied by neglecting the change of the density. In the **subsonic flow** the flow velocity is lower than the sound velocity and the density perturbation due to the motion of a body can be smoothed out by the sound waves. In the **supersonic flow**, **shock waves** to be described in Chap. 3 are generated, then, the **compressibility** becomes essential in the fluid dynamics. The motion of a submarine in the sea water is studied with the assumption of incompressibility. The **Navier-Stokes** equation is used for complicated fluid calculations such as a change from laminar flow to turbulent flow of the wind flow in our life. The incompressible Navier-Stokes equation is directly derived from (2.55).

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla\left(\frac{P}{\rho}\right) + \nu\nabla^2\boldsymbol{u}, \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \tag{2.57}$$

In the incompressible case, it is enough to solve (2.57) instead of (2.20, 2.21, 2.22, 2.23, 2.24, and 2.25).

Change the equation of (2.57) to a non-dimensional form. For example, consider the wind flowing around a building. Assume that L is the typical length of the building, U is the typical speed of the wind, ρ is the density of the air, and P is the atmospheric pressure. Then, with a hat "~" (tilde) to the dimensionless quantities, the physical variables can be transferred to non-dimensional variables as follows.

$$\tilde{r} = \frac{r}{L}, \quad \tilde{t} = \frac{tU}{L}, \quad \tilde{\boldsymbol{u}} = \frac{\boldsymbol{u}}{U}, \quad \tilde{p} = \frac{P}{\rho U^2} \tag{2.58}$$

Using these dimensionless variables, the Navier-Stokes Eq. (2.57) can be rewritten in non-dimensional form.

$$\frac{\partial \tilde{\boldsymbol{u}}}{\partial \tilde{t}} + \left(\tilde{\boldsymbol{u}} \cdot \tilde{\nabla}\right)\tilde{\boldsymbol{u}} = -\tilde{\nabla}\tilde{p} + \frac{1}{Re}\tilde{\nabla}^2\tilde{\boldsymbol{u}} \tag{2.59}$$

Whatever the scale of fluid phenomenon, the fluid dynamic phenomenon governed by (2.59) depends only on the dimensionless parameter

$$Re = \frac{UL}{\nu} = \frac{\rho UL}{\mu} \tag{2.60}$$

If the two fluid systems with different scales are similar, the fluids governed by (2.59) with the same value of "Re" has the same mathematical solution in the dimensionless form. This dimensionless quantity Re is an important value in discussing fluid turbulence and is called **Reynolds number**. For example, it is useful to calculate the value of Re in case of the wind around the building; for example, U = 10 m/s, L = 100 m, and air mean-free-path l $\left(\sim 20\ \text{Å}\right)$,

$$Re \approx \frac{UL}{\ell V_s} \approx \frac{10m/s \times 100m}{\left[\pi\left(2 \times 10^{-9}m\right)^2 \times 10^{25}m^{-3}\right]^{-1} \times 300m/s} \approx 10^7 \tag{2.61}$$

It is a very large number. Therefore, macroscopic fluid phenomena can be discussed by neglecting the effect of viscosity. It is noted, however, that the viscosity becomes essential in discussing fluid turbulences. The small vortexes in the turbulence disappear after transferring their kinetic energy to the thermal energy because of the viscosity.

The convection term $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$ of the Navier-Stokes Eq. (2.57) is called the **inertia term** in the analysis of fluid turbulence. The term $\nu\nabla^2\boldsymbol{u}$ is a **viscosity term**. By inserting typical values in these terms, the inertia term is U^2/L and the viscosity term is $\nu U/L^2$, so the following relationship is satisfied

$$Re = \frac{UL}{\nu} = \frac{(\text{inertia term})}{(\text{viscosity term})} \tag{2.62}$$

This is the definition of Reynold number, but it is not an exact definition. In gas dynamic phenomena in our daily life, the Reynold number is very large as (2.61). In the case of the wing of an airplane, it is as large as $\mathrm{Re} = 10^8$. Rather, it suggests that nonlinear effects (inertia term) often dominate phenomena in a variety of fluid and gas phenomena in our daily life.

In the case of non-dimensional Eq. (2.59), the solution depends only on Re number, if two systems are in a similar structure. In fluid turbulence it is known that turbulent energy flows from large vortices to small vortices in three-dimensional case due to inertia terms. The larger the Reynold number, the larger and smaller vortices coexist at the same location in the fluid turbulence.

Also, as the water or wind flows crossing a building of the cylinder structure changes its flow pattern according to the increase of the Reynold number due to the

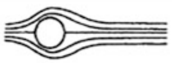
Reynolds number regime	Flow regime	Flow form	Flow characteristic
$\mathrm{Re} \to 0$	Creeping flow		Steady, no wake
$3-4 < \mathrm{Re} < 30-40$	Vortex pairs in wake		Steady, symmetric separation
$\substack{30\\40} < \mathrm{Re} < \substack{80\\90}$	Onset of Karman vortex street		Laminar, unstable wake
$\substack{80\\90} < \mathrm{Re} < \substack{150\\300}$	Pure Karman vortex street		Karman vortex street
$\substack{150\\300} < \mathrm{Re} < \substack{10^5\\1.3\cdot 10^5}$	Subcritical regime		Laminar, with vortex street instabilities
$\substack{10^5\\1.3\cdot 10^5} < \mathrm{Re} < 3.5 \cdot 10^6$	Critical regime		Laminar separation Turbulent reattachment Turbulent separation Turbulent wake
$3.5 \cdot 10^6 < \mathrm{Re}$	Supercritical regime (transcritical)		Turbulent separation

Fig. 2.6 The flow pattern changes depending on the Reynold number of the system. The figure shows the flow forms and flow characteristic for the case where a laminar flow comes to the cylindrical obstacle with different Reynold numbers. It changes dramatically from laminar flow to turbulent flow. Preprint from Ref. [4] with kind permission from Springer Science + Business Media

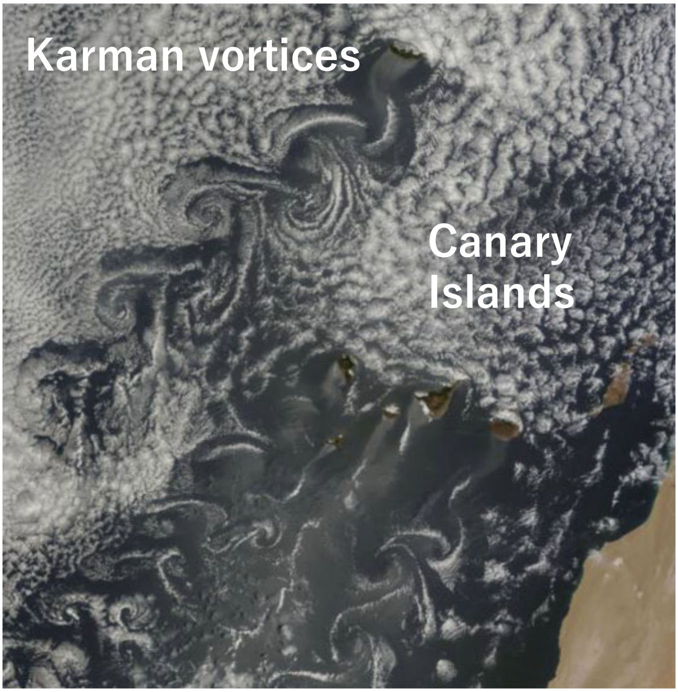


Fig. 2.7 Karman vortex in the nature. Canary Islands kick up Karman vortices, May 20, 2015. (NASA earth observatory)

increase of flow velocity. As shown in Fig. 2.6 [4], with increase of the flow velocity, the flow pattern is initially a laminar flow, peeling will occur next, the **Karman vortex** is formed, and it transits to turbulent flow. In Fig. 2.7, Karman vortices generated by the Canary Island are shown as a picture of clouds by a NASA weather satellite.

2.4.2 Wave Damping by Viscosity

It is clear intuitively that the viscosity plays a role to damp the amplitude of the sound waves. The wave equation including the viscosity in (2.48) is easily obtained to a Fourier component with k in the plane geometry.

$$\frac{\partial^2}{\partial t^2}u_1 + k^2\nu\frac{\partial}{\partial t}u_1 + k^2{V_s}^2 = 0 \tag{2.63}$$

Since (2.63) is an equation of a harmonic oscillation, the second term corresponds to a friction or damping term. The dispersion relation of (2.63) is

$$\omega \approx \pm kV_s - i\frac{\nu}{2}k^2 \tag{2.64}$$

So, mathematically it is easy to confirm that the solution is proportional to the form,

$$exp(\mp ikV_s t)\, exp\left(-\frac{\nu}{2}k^2 t\right) \tag{2.65}$$

The term with a positive ν always damps the amplitude of the waves. It is informative to evaluate how large this damping is for the sound waves in the air. The vocal sound “do” is 260 Hz and the wavelength is about 1 m. The length of wave propagation L_d until the exponent of the damping reduces a factor ½ is estimated roughly,

$$L_d \approx \frac{V_s}{\nu k^2} \approx \frac{\lambda}{\ell}\frac{\lambda}{(2\pi)^2} >> \lambda \tag{2.66}$$

where λ is the wavelength, l is the air mean-free-path, and approximate relation $\nu \sim l$Vs was used. Since the vocal sound wavelength is much longer than the molecular mean-free-path, viscos damping can be neglected for our vocal sound waves. Note that the spherical damping is dominant as shown in (2.54).

2.4.3 Thermal Conduction

Next, **thermal conduction** should be also included in the equation of energy in high temperature plasmas. This is a phenomenon in which internal energy is given from or deprived by the surrounding fluid to a local fluid due to heat transport. This is included in the term corresponding to dQ in (2.6). Therefore (2.22) is modified to the form

$$\frac{d\varepsilon}{dt} = -\frac{P}{\rho}\nabla \boldsymbol{u} + \frac{4}{3}\frac{\mu}{\rho}(\nabla \boldsymbol{u})^2 + \frac{1}{\rho}\nabla(\kappa \nabla T) \tag{2.67}$$

where κ is the **heat conduction coefficient** and the second term on RHS indicates that the flow kinetic energy is converted into heat energy by the viscosity. Thus, (2.67) shows a viscous term also appears in the energy eq. (2.22).

Under the condition that the density is constant and no flow in (2.67), it reduces to the equation of **temperature diffusion**. In plasmas without magnetic field, the electron thermal conduction is more important.

$$\frac{\partial T_e}{\partial t} = \nabla(\chi_e \nabla T_e) \tag{2.68}$$

Here, T_e is the electron temperature and χ_e is the **electron temperature diffusion coefficient**. In high-temperature plasmas, strong dependence of the Coulomb collision time to the electron velocity shown in (2.3) results the Maxwellian averaged diffusion coefficient is proportional to

$$\chi_e \sim \ell_e v_e \propto T_e^{5/2} \tag{2.69}$$

Therefore, the temperature diffusion Eq. (2.68) is a nonlinear equation and Fourier decomposition method is not applicable.

Fortunately, it is well-known that as long as the temperature dependence is power law and the initial and boundary conditions are not complicated, the following self-similar method helps to reduce the partial differential equation to an ordinary differential equation, which is in general one-dimensional eigen-value problem [5].

2.4.4 Self-Similar Solution

Discuss a general case where the diffusion coefficient is given in the form.

$$\chi = aT^n \tag{2.70}$$

Here, "a" is a constant and "n" is a numerical value indicating the degree of nonlinearity. In general, n is often an integer or half integer.

In the case where fluid is heated to a high temperature locally before fluid moves, the heat conduction becomes important than the effect of the sound waves We investigate one-dimensional plane given by the following equation.

$$\frac{\partial}{\partial t}T = a\frac{\partial}{\partial x}\left(T^n\frac{\partial}{\partial x}T\right) \tag{2.71}$$

Let's solve (2.71) while explaining the mathematics of **self-similar solution**.

From the dimensional analysis of the Eq. (2.71), with the coordinates of the characteristic front of the **heat conduction wave** taken as x_f and the average temperature as T_a at the time t, (2.71) should satisfy the following dimensional relation.

$$aT_a^n \sim \frac{x_f^2}{t} \tag{2.72}$$

To solve this, it is required to impose an initial or boundary condition.

As an example, suppose that energy E_0 is instantaneously generated at $x = 0$. Then the law of conservation of energy is

$$E_0 = \frac{1}{\gamma-1}\rho_0\int_{-\infty}^{+\infty}T(t,x)dx \sim \frac{2}{\gamma-1}\rho_0 T_a x_f = const. \tag{2.73}$$

where $1/(\gamma-1)$ is the heat capacity. Equation (2.73) suggests the relation.

$$T_a x_f \sim \frac{\gamma - 1}{2\rho_0} E_0 \equiv \alpha : \text{const.} \tag{2.74}$$

Inserting (2.74) to (2.72), the time dependence of the heat wave front is

$$x_f \sim (a\alpha^n t)^{\frac{1}{n+2}} \tag{2.75}$$

The average temperature is obtained by substituting (2.75) into (2.82),

$$T_a(t) \sim \left(\frac{\alpha^2}{at}\right)^{\frac{1}{n+2}} \tag{2.76}$$

The dimensionless variable ξ is introduced as follows.

$$\xi = \frac{x}{x_f(t)} \tag{2.77}$$

The temperature is given by introducing a dimensionless function g(ξ),

$$T(t, x) = T_a(t) g(\xi) \tag{2.78}$$

The time and space differentiations are modified as.

$$\frac{\partial}{\partial t} T = -\frac{1}{n+2} \frac{T_a(t)}{t} \left(g + \xi \frac{dg}{d\xi}\right) \tag{2.79}$$

$$a \frac{\partial}{\partial x}\left(T^n \frac{\partial}{\partial x} T\right) = a \frac{{T_a}^{n+1}}{{x_f}^2} \frac{d}{d\xi}\left(g^n \frac{dg}{d\xi}\right) \tag{2.80}$$

Equation (2.71) becomes an ordinary differential equation after one integration as follows.

$$(n+2) g^n \frac{dg}{d\xi} + \xi g = 0 \tag{2.81}$$

where the integral constant is zero evaluated as $g(\xi) = 0$ at ξ infinity. This can be easily integrated, by assuming the value of at the wave-front ($\xi = \xi_0$) constant,

$$g(\xi) = \begin{cases} \beta \left[1 - \left(\frac{\xi}{\xi_0}\right)^2\right]^{\frac{1}{n}} & |\xi| \le \xi_0 \\ 0 & |\xi| \ge \xi_0 \end{cases} \tag{2.82}$$

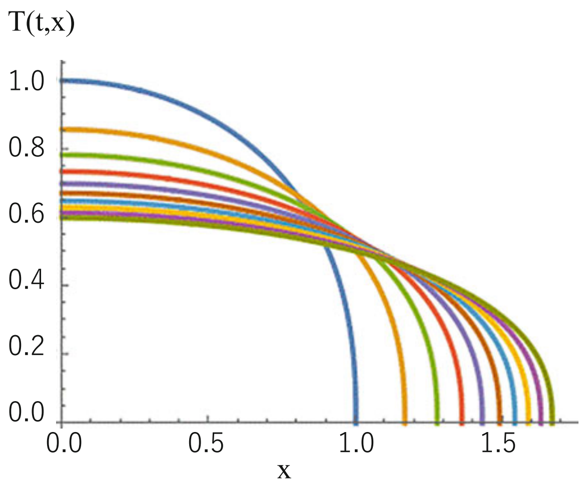


Fig. 2.8 Time evolution of the nonlinear heat wave with n = 5/2, electron heat wave, in non-dimensional space and time. The nonlinearity makes the temperature profile being flat at high temperature region because of better conduction and steep profile near the front due to less conduction. In plotting the non-dimensional profile, we set $a = \alpha = \beta = \xi_0 = 1$ in (2.75), (2.76), (2.82) and (2.84) for normalized time t = 1, 2, . . ., 10

where β is a constant and is given as:

$$\beta = \left\{ \frac{n}{2(n+2)} {\xi_0}^2 \right\}^{1/n} \tag{2.83}$$

For the case of electron heat conduction with n = 5/2, the time evolution of temperature is shown in Fig. 2.8. As can be seen from the functional form of (2.82) with a large n, the wave front of heat conduction can be clearly defined unlike the case of linear diffusion with n = 0. Furthermore, the higher the temperature, the larger the thermal conductivity coefficient. The temperature profile is a flat shape rather than Gaussian for n = 0 case.

The solution is

$$T(t,x) = \beta \left(\frac{\alpha^2}{at} \right)^{\frac{1}{n+2}} g(\xi) \tag{2.84}$$

Note that the non-dimensional constant ξ_0 is not obtained yet. It is obtained so that the total energy is conserved as (2.73). Inserting (2.84) into the second term in (2.73), the total energy becomes E_0. In the case where the resultant ordinary differential equation is not analytically integrated, the problem becomes an eigen-value problem with numerical integration, where ξ_0 becomes the eigen-value.

It is useful to solve for the case of n = 0, well-known linear diffusion problem, with this self-similar method. It is a standard way to solve (2.71) by using Fourier transformation. However, it can be applicable only for the linear diffusion. As seen above, the self-similar method can be applied to nonlinear diffusion equations, too. As clear in (2.82), it is not straight forward in n = 0 case. With this self-similar method, (2.81) becomes

$$2\frac{dg}{d\xi} + \xi g = 0 \tag{2.85}$$

This can be easily solved to obtain the well-known solution.

$$T(t, x) \propto \frac{1}{\sqrt{at}} e^{-x^2/(at)} \tag{2.86}$$

Finally, let me explain the qualitative relationship between fluid motion and heat conduction waves. Generally, when rapid heating occurs on the surface of a solid, the speed of the heat conduction wave is very high, and the heat conduction wave propagates through the solid without fluid motion. Even in an insulator, free electrons increase at once by rapid heating, and a heat conduction wave propagates while ionizing atoms in the inside.

However, the speed of the heat conduction wave suddenly decreases as seen in (2.75), and when the speed becomes about the sound speed of the heated region, the movement of the ions such as strong sound waves and shock waves will accompany. After that, a structure of density and temperature, in which heat conduction and fluid motion are combined as almost stationary state, is formed as deflagration wave to be explained in Chap. 3.

Here, we showed how the self-similar method is powerful mathematics in solving a partial differential equation. The self-similar method has been applied to find analytical solutions of spherical implosion and explosion of compressible fluids driven by strong spherical shock waves. This topic will be studied in Chap. 4.

2.5 Incompressibility and Vortex

2.5.1 Incompressible Fluid

The **compressibility** of fluids and gases such as water and air are defined as

$$\eta = \frac{1}{B} = -\frac{1}{V}\frac{dV}{dP} = \frac{1}{\rho}\frac{d\rho}{dP} \tag{2.87}$$

The “B” is the bulk modulus defined in (2.51) and corresponds to the pressure required to compress the nominal density twice.

For incompressible fluids, the equation of continuity (2.20) can be replaced with the follow simple relation.

$$\nabla \cdot \boldsymbol{u} = 0 \tag{2.88}$$

Here, we introduce the definition of **vorticity** vector $\boldsymbol{\omega}$.

$$\boldsymbol{\omega} \equiv \nabla \times \boldsymbol{u} \tag{2.89}$$

In the case of incompressible and vortex-free flow ($\boldsymbol{\omega} = 0$), the flow velocity is defined with a potential (**velocity potential**), ϕ.

$$\nabla \times \boldsymbol{u} = 0 \quad \Rightarrow \quad \boldsymbol{u} = \nabla \phi \tag{2.90}$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad \Rightarrow \quad \Delta \phi = 0 \tag{2.91}$$

Such flow is called the **potential flow**.

It is better to see the image of the relation between the vortex and velocity vector. Image the case where a vortex is located at the center in the cylindrical geometry. It is easy to see the structure of the flow velocity vector by use of the analogy between electric current and static magnetic field. For a static condition, The Ampere law in the Maxwell equations is

$$\nabla \times \boldsymbol{H} = \boldsymbol{j}, \qquad \boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} \tag{2.92}$$

Regarding the electric current as vorticity, the magnetic field vector corresponds to the fluid flow velocity. In the case where the vortex is localized at the center, the absolute value of the flow velocity decreases in proportion to 1/r. This is well-known magnetic field distribution in the electromagnetism.

With a mathematical formula, Lagrange derivative of the flow velocity is rewritten as

$$\frac{d\boldsymbol{u}}{dt} = \frac{\partial \boldsymbol{u}}{\partial t} + \nabla\left(\frac{1}{2}u^2\right) - \boldsymbol{u} \times \nabla \times \boldsymbol{u} \tag{2.93}$$

In the potential flow, from (2.93) and (2.90), the equation of motion (2.21) becomes.

$$\nabla\left(-\frac{\partial \phi}{\partial t} + \int \frac{dP}{\rho} + \frac{1}{2}|\nabla \phi|^2 + U\right) = 0 \tag{2.94}$$

Here, the external force is given as a potential force

$$\boldsymbol{F} = -\nabla U \tag{2.95}$$

From the Eq. (2.94), the following relation is found.

$$-\frac{\partial \phi}{\partial t} + \int \frac{dP}{\rho} + \frac{1}{2}|\nabla \phi|^2 + U = 0 \tag{2.96}$$

Here, RHS of (2.96) can be an arbitral function of time, f (t), but if we redefine the velocity potential like $\phi' = \phi - \int f(t)dt$, we can generalize that f (t) = 0. In a steady

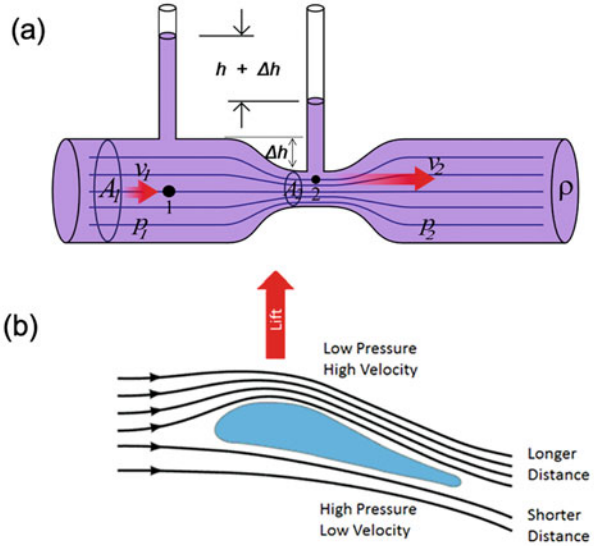


Fig. 2.9 Engineering application of Bernoulli's theorem. (**a**) The difference of the height indicated the difference of velocity. This is used to measure the speed of an airplane flying at subsonic velocity where incompressibility is good assumption. (**b**) is the principle of lift force of airplanes. Design the main wing structure so that the top is longer than the bottom, and the pressure of the air passing over the top is lower than the pressure below. When the airplane takes off, this pressure difference provides lift and helps the aircraft to climb

flow of incompressible and vortex-free fluids, therefore, the following relation should be satisfied.

$$\int \frac{dP}{\rho} + \frac{1}{2}u^2 + U = (\text{constant in space}) \tag{2.97}$$

The relation of (2.96) and (2.97) is called **Bernoulli's theorem**.

Using (2.97), a mechanical device for measuring the speed of flow velocity is designed as in Fig. 2.9a. This principle has been used to measure the speed of subsonic-aircrafts. As seen in Fig. 2.9b, the wing of an aircraft is designed so that the upper length is longer than the lower length, then the flow velocity is higher in the top of the wings and the pressure is lower. The difference of the pressure pushes the wings upward. This force is called the **lift force**.

2.5.2 *Incompressibility Assumption*

In most of the hydrodynamic phenomena, the **incompressible assumption** can be used and the problems become simpler than the compressible fluids. Then, the equation of continuity and equation of energy can be replaced by the incompressible condition (2.88). In general, the incompressible Navier-Stokes Eq. (2.55) is only one equation to be solved by coupling with (2.88).

Let us consider the condition under which the assumption of incompressible fluid is appropriate. Evaluate the density perturbation $\delta\rho$ in a uniform fluid with density ρ_0. Induced velocity perturbation $\delta\mathbf{u}$, originated by the compressibility can be given as the same form in (2.45) and (2.46) as follows.

$$\frac{\partial}{\partial t}\delta\rho = -\rho_0 \nabla \delta\mathbf{u} \tag{2.98}$$

$$\frac{\partial}{\partial t}\delta\mathbf{u} = -\frac{1}{\rho_0}\nabla\delta P = -\frac{V_s^2}{\rho_0}\nabla\delta\rho \tag{2.99}$$

$$V_s^2 = \left(\frac{\partial P}{\partial\rho}\right) \tag{2.100}$$

where δP is an induced pressure perturbation and V_s is the sound velocity defined in (2.47). From (2.98) and (2.99), the following relation is obtained. Assuming $|\nabla| \approx 1/L$ *and* $|\partial/\partial t| \approx \tau$, the relation is obtained

$$\left|\frac{\delta\rho}{\rho_0}\right| \approx \frac{L}{\tau}\frac{|\delta\mathbf{u}|}{V_s^2} \tag{2.101}$$

It is clear from (2.101) that the density change by the compressibility can be neglected for the case with sufficiently high sound velocity V_s. For example, a car speed of 100 km/s is about $L/\tau \sim |\delta\mathbf{u}| \sim 0.1 V_s$ and the density perturbation is only 1%. In the case of a supersonic jet fighter with Mach number more than unity, the density perturbation roughly evaluated with (2.101) is more than unity, so that hydrodynamics should be solved as compressible fluid.

2.5.3 Vortex Equation

The following equation is obtained by taking the rotation of the equation of motion (2.57).

$$\frac{\partial}{\partial t}\boldsymbol{\omega} = \nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) + \frac{1}{\rho^2}\nabla\rho \times \nabla P + \frac{4}{3}\nu\nabla^2\boldsymbol{\omega} \tag{2.102}$$

This can be rewritten in the form.

$$\frac{\partial}{\partial t}\boldsymbol{\omega} + \boldsymbol{u}\cdot\nabla\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} - \boldsymbol{\omega}(\nabla\boldsymbol{u}) + \frac{1}{\rho^2}\nabla\rho \times \nabla P + \frac{4}{3}\nu\nabla^2\boldsymbol{\omega} \tag{2.103}$$

Here, $\boldsymbol{\omega}$ is the vorticity defined in (2.89).

The first term of RHS in (2.103) is a term that exists only in the case of three-dimensional flow, and a vortex localized in a certain region is stretched to the direction of vorticity vector by this term. In the three-dimensional fluid turbulence, the vortexes are cascaded to thin, long vortexes in the turbulence energy spectrum. This is called **cascade** in wavenumber space of the turbulence. As the vortex is elongated, the viscosity of the fourth term on RHS of (2.103) acts as a damping term of the vorticity. This is the effect of dissipation in which the kinetic energy of the vortex is converted into the thermal energy of the fluid.

The second term in (2.103) disappears in incompressible fluid. This term is the compressibility effect of amplifying the vorticity in the fluid as it is compressed. And, the third term is a source term for vortex generation. Note that it depends on the property of the **equation of state** (EOS). If pressure is only a function of the density, this term disappears.

Even with ideal fluid equation of state, vortexes are generated by this term. The third term in (2.103) is called the **baroclinic term**. In fluid dynamics, the **baroclinity** is a measure of the stratification in a fluid. A baroclinic atmosphere is one for which the density depends on both the temperature and the pressure; contrast this with **barotropic** atmosphere, for which the density depends only on the pressure. In atmospheric terms, the barotropic zones of the Earth are generally found in the central latitudes, or tropics, whereas the baroclinic areas are generally found in the mid-latitude/polar regions.

Let us see the difference of role of the first term of RHS of (2.103) in two- and three-dimensional vortices. The effect appears as the difference between a **tornado** and a **typhoon**. Our Earth's atmosphere is about 5 km as the thickness of the air fluid. Since a relatively small-scale vortex is subjected to the three-dimensional effect and becomes a thin and long vortex like a tornado. However, in the case of typhoon, the size of the vortex is as much as 1000 km compared to the atmosphere thickness of ~5 km, which is like a vortex in a very thin water in two-dimension. Therefore, the typhoon is a two-dimensional vortex and the first term on RHS of (2.103) does not work for the evolution of vortex. In fact, when examining the two-dimensional vortex turbulence, it turns out that the energy of the vortex moves in the wavenumber space from small to large wavenumber direction. This is called an **inverse cascade**.

To see explicitly the force to generate vortices on the surface of the earth, it is better to write the vortex equation in a rotating system with an angular momentum $\mathbf{\Omega}$. Then, Navier-Stokes Eq. (2.57) includes two new forces, since Newton equation is modified in this frame as

$$\frac{d\boldsymbol{v}}{dt} = \frac{\boldsymbol{F}}{m} + 2\boldsymbol{v} \times \mathbf{\Omega} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) \tag{2.104}$$

where the second and third terms in RHS are the **Coriolis** and centrifugal forces, respectively. Of course, $\mathbf{r}$ and $\mathbf{v}$ are those in the rotating coordinate. It is relatively simpler to solve Navier-Stokes equation in the rotating system with these two forces.

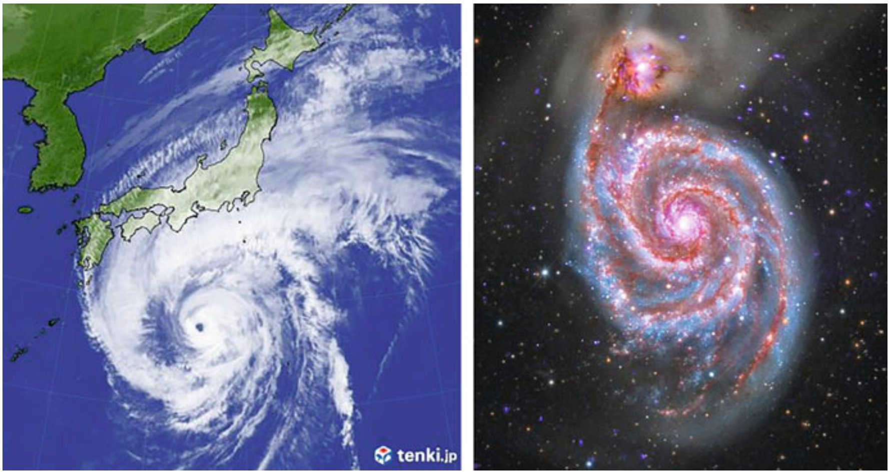


Fig. 2.10 A satellite picture of clouds by a typhoon attacking Japan islands (left, Courtesy of Tenki.jp), and X-ray image of the spiral Galaxy M51 taken by Chandra x-ray satellite by NASA (right by NASA/CXC/SAO)

It is easy to image the centrifugal force which is always perpendicular to the rotating axis (**Ω**-direction). The Coriolis force is easily understood with the analogy of Lorentz force. Regarding 2**Ω** is like an constant magnetic field vector in the z-direction in the cylindrical coordinate, the Coriolis force is the same mathematical form as the Lorentz force. For example, it is perpendicular force to the rising air in Summer to generate vortices due to Coriolis force.

A picture of clouds in a typhoon approaching to Japan and Korea is taken by a weather forecast satellite as shown in Fig. 2.10a. The vortex is enhanced on the way to the north by the Coriolis force. It is clearly seen the "eye" of the typhoon at the center.

It is interesting to compare the spiral motion of the typhoon to the spiral motion of a galaxy. A galaxy consists of about 100 billion starts, and it can be regarded as fluid. In general, galaxy is of pancake type structure and starts are rotating around the center of the galaxy with given angular momentum. In Fig. 2.10b, an observed x-ray image of the M51 galaxy is shown. The spiral motion of the stars in the galaxy is observed in the satellite image. The image suggests that the vorticity is localized at the center of the galaxy.

2.6 One-Fluid and Two-Temperature Fluid Model

It seems to be that a precise treatment of laser-plasma should be based on the two-fluid and two-temperature model identifying the ion and electron fluids. As shown in (2.4) and (2.5), the next better is the assumption of two temperatures but one fluid. Two fluid is required when the charge separation is important to affect the fluid phenomena. This is the case near low density and high-temperature region like laser expanding plasma to be discussed in Chap. 3. The other case is the situation where the charge neutrality is satisfied, but the electron current is strong to generate strong magnetic field affecting the fluid dynamics. This is modeled by magneto-hydro-dynamics (MHD) fluid discussed later.

As described in Chap. 2 in Vol. 1, the velocity distribution of heated electrons becomes Maxwellian with an electron temperature T_e due to collisions between electrons, and the ions tend to be a Maxwell distribution with T_i relatively slowly, if the time scale of phenomena is longer than τ_e and τ_i in (2.4). However, it takes even longer time for both temperatures to have the same value as suggested in (2.5). In the case of laser heating, electron cyclotron heating, etc., the electrons in the plasma are heated from outside, and the ions are heated through the temperature relaxation process by the electron-ion Coulomb collision. Therefore, it is generally better to assume that the temperatures of electrons and ions are different in such plasmas.

In addition, plasma shock waves such as blast waves of supernova remnants heat the ions at the shock front, and the electrons slowly receive energy from the ions due to temperature relaxation. In the magnetic field confined plasma, when plasma is heated by a microwave source whose frequency is adjusted to resonate the ion cyclotron frequency, the ion temperature rises but the electron is heated slowly via the ion-electron collision.

Of course, if the time scale of the fluid dynamics is faster than the electron and ion relaxation times, there is no guarantee that the distribution functions are in Maxwellian. However, the fluid equation is much easier to solve compared to the Boltzmann equations directly. It is better to start with solving a plasma dynamic with the fluid model at first. Then, if some violation of fluid assumption is found in the solution, it is better to consider how to take into account the kinetic effect in the basic equation.

The one-fluid and two-temperature fluid model is widely used as the basic equation for studying the whole dynamics from laser heating to fusion burn in laser nuclear fusion implosion dynamics. The basic equations are derived from the two fluid equations given, for example, in the Babinski's book [6]. To assume charge neutrality, the scale length of the plasma fluid variation in space should be sufficiently longer than the Debye length. Then, one fluid approximation is reasonable, and the equation of continuity and the equation of motion are same as (2.20) and (2.21), respectively.

$$\frac{d\rho}{dt} = -\rho\nabla\boldsymbol{u} \tag{2.105}$$

$$\rho\frac{d\boldsymbol{u}}{dt} = -\nabla P \tag{2.106}$$

However, the energy equation should be formulated for the ion and electron fluids as separate thermodynamic systems. As important terms, the heat conductions, temperature relaxation, and energy source terms should be included. For example, S_e is the energy source by laser and radiation, and S_i is a source by the collisional heating by fusion-product particles. Then, both energy equations are written as,

$$\rho \frac{d\varepsilon_i}{dt} = -P_i \nabla \boldsymbol{u} - \nabla \boldsymbol{q}_i + Q_{ei} + S_i \tag{2.107}$$

$$\rho \frac{d\varepsilon_e}{dt} = -P_e \nabla \boldsymbol{u} - \nabla \boldsymbol{q}_e - Q_{ei} + S_\mathrm{e} \tag{2.108}$$

Here, ε_i and ε_e are the internal energies per unit mass of ion and electron fluids, q_i and q_e are heat fluxes of ions and electrons, Q_{ei} is the **temperature relaxation** in unit time per unit mass from electrons to ions (or vice versa when Q_{ei} is negative). The S_i and S_e are terms of energy sources and losses due to other effects to ion and electron fluids, respectively. For example, when the thermal x-ray radiation exchanges energy with plasma, the term S_e should include the effects of heating and cooling by radiation. That term must be combined with an equation on radiation in a self-consistent way. In (2.107) and (2.108), the **thermal conduction coefficients** of ions and electrons, and the temperature relaxation term are given as follows [6].

$$\kappa_e = \alpha(Z) \frac{n_e T_e \tau_e}{m_e} \propto T_e^{\frac{5}{2}}, \quad \kappa_i = 3.9 \frac{n_i T_i \tau_i}{m_i} \propto T_i^{5/2} \tag{2.109}$$

$$Q_{ei} = \frac{m_e}{m_i} \frac{T_e - T_i}{\tau_e} \tag{2.110}$$

Here, $\tau_e$is the relaxation time given in (2.4). The collision time due to the Coulomb collision is subtly changed depending on the ionization degree Z of the ions, so it is applied as a correction term $\alpha(Z)$ thereof. For example, $\alpha(Z)= 3.16$ for $Z = 1$, others are in Ref [6]. Note that the thermal conduction coefficient roughly given as (2.69) for electrons and the same form $l_i v_i$ for ions except numerical factors. It is found that the ion thermal conduction is much weaker than the electron thermal conduction even for the same temperature profile.

Note that the one-fluid, two temperature fluid model is the basic equations for an **integrated-physics code** for laser fusion simulation. Depending on the difference of problems, additional physics are included in the basic equations.

2.7 Two Fluid Equation of Plasma

For analyzing short time and short scale plasma phenomena, two-fluid plasma model is widely used. In a short time, plasma is in general collisionless and there is no time for the ions and electrons to become Maxwell distributions via the Coulomb collision. Nevertheless, the ion and electron particle distributions are assumed to be in Maxwellian, and they are described with fluid models like the neutral fluids. The mathematical proof of the fluid approximation is shown in Appendix-A.

In general, the energy equations to the ions and electrons are neglected for simplicity by assuming constant temperature or adiabatic response. For simplicity, consider the case of fully ionized hydrogen plasma. Extension to partially ionized non-ideal plasma with other atomic and charge numbers is straightforward, if there is no need to be consistent with complicated atomic processes explained in Chap. 5.

Plasma consists of two kinds of charged particle groups whose masses are different by thousand times. Since the masses are very different, the ions behave separately as ion fluid and the electrons as electron fluid. Assume that both electrons and ions behave as the ideal fluid, and their degrees of freedom are only three-dimensional translational motions. Then, both equations of state are the same as (2.31) and (2.32) with N = 3 and γ = 5/3. Then, the following equations are obtained for the ion fluid, after setting the number density of ions n_i and setting its flow velocity and temperature to $\mathbf{u}_i$ and T_i.

$$\frac{\partial n_i}{\partial t} + \nabla(n_i \boldsymbol{u}_i) = 0 \tag{2.111}$$

$$m_i\left(\frac{\partial}{\partial t} + \boldsymbol{u}_i \cdot \nabla\right)\boldsymbol{u}_i = -\frac{1}{n_i}\nabla(n_i T_i) + e(\boldsymbol{E} + \boldsymbol{u}_i \times \boldsymbol{B}) - \frac{1}{n_i}\boldsymbol{R} \tag{2.112}$$

Both the ion and electron gases are the ideal gas and charged particle ions are affected by Lorentz force. The electron fluid follows the equations.

$$\frac{\partial n_e}{\partial t} + \nabla(n_e \boldsymbol{u}_e) = 0 \tag{2.113}$$

$$m_e\left(\frac{\partial}{\partial t} + \boldsymbol{u}_e \cdot \nabla\right)\boldsymbol{u}_e = -\frac{1}{n_e}\nabla(n_e T_e) - e(\boldsymbol{E} + \boldsymbol{u}_e \times \boldsymbol{B}) + \frac{1}{n_e}\boldsymbol{R} \tag{2.114}$$

Here, **R** is the force due to friction appearing when the velocities of the electron and the ion fluids are different, and is given as the following form [6].

$$\boldsymbol{R} = -\frac{m_e n_e}{\tau_{ei}}(\boldsymbol{u}_e - \boldsymbol{u}_i) \tag{2.115}$$

Note that both the viscosity and thermal conductions have been neglected above.

Electric and magnetic fields appearing in Lorentz force are not only imposed externally but also generated by the ion and electron fluid motions. The charge density ρ and current density $\boldsymbol{j}$ are defined as

$$\rho = \mathrm{e}n_i - \mathrm{e}n_e \quad \boldsymbol{j} = \mathrm{e}n_i\boldsymbol{u}_i - \mathrm{e}n_e\boldsymbol{u}_e \tag{2.116}$$

(2.116) should be used in solving the Maxwell equations.

$$\textbf{Faraday's Law}\ \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{2.117}$$

$$\textbf{Ampere's Law}\ \frac{1}{\mu_0}\nabla\times\boldsymbol{B}=\boldsymbol{j}+\varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t} \tag{2.118}$$

$$\textbf{Poisson Equation}\ \varepsilon_0\nabla\cdot\boldsymbol{E}=\rho \tag{2.119}$$

$$\textbf{Gauss law}\nabla\cdot\boldsymbol{B}=0 \tag{2.120}$$

It is necessary to solve the fluid equations by coupling with the Maxwell equations at the same time self-consistently. This point is essential difference compared to the conventional fluid or gas with no charge, and soling any plasma physics is more complicated mathematically. So, when the computer appeared in the late 1940's, the researchers in plasma physics became a pioneer in computational physics even with fluid assumption of plasmas.

The basic equations were given above; however, it is not always necessary to couple all equations. For example, to investigate the phenomenon which is too fast for the ion fluid to follow, it is reasonable to keep the ions fixed and consider only the motion of the electron fluid. In a phenomenon, on the other hand, in which ions move slowly and the electrons just accompany them, the electron fluid follows the motion of the ion fluid. In such case, it is found the charge neutrality is good assumption without solving the electron equation.

Plasma shock wave structures are studied precisely by soling complete equations of two fluid and two temperature fluid equation [7]. They have solved structure of stationary plasma shock waves and studied the effects of charge separation, electron and ion heat conduction, temperature relaxation, viscosity, etc. The same kind of research was done in the book [5], where the details are given in Chap. 7. It is too much here to discuss such precise calculation consistently by solving all above fluid equations, so the reader interested are recommended to read such references.

2.7.1 Electron Plasma Waves

Typical waves in two fluid plasma driven by the electric field are the **electron plasma wave** and **ion acoustic wave**. The electron plasma wave is sustained by the electric field due to charge separation by the electron motion. So, the electron inertial force balances the force by the electric field and electron pressure. In this case, the ions can be assumed to be at rest, namely, the ions cannot move because of their larger mass.

To know the linear response of any plasma from an equilibrium state, small perturbations of physical quantities are considered after neglecting all nonlinear terms. This mathematical prescription is called "linearization". In the electron plasma wave, the linear perturbations are the following electron density, electron velocity, and electrostatic field.

$$n_{e1}, \quad \boldsymbol{u}_{e1}, \quad \boldsymbol{E}_1 \tag{2.121}$$

Then, the continuity Eq. (2.113) is linearized as

$$\frac{\partial}{\partial t} n_{e1} + n_{e0} \nabla \boldsymbol{u}_{e1} = 0 \tag{2.122}$$

The equation of motion of electrons (2.114) is linearized as

$$m_e \frac{\partial}{\partial t} \boldsymbol{u}_{e1} = -\frac{1}{n_{e0}} \nabla P_{e1} - e\boldsymbol{E}_1 \tag{2.123}$$

The Poisson Eq. (2.119) is linearized as

$$\nabla \boldsymbol{E}_1 = -\frac{e}{\varepsilon_0} n_{e1} \tag{2.124}$$

Assuming that the pressure is given in the form (2.31), the linearized pressure is

$$P_{e1} = \left(\frac{\partial P_e}{\partial n_e} \right) n_{e1} = \gamma T_e n_{e1} \tag{2.125}$$

where T_e is assumed to be given. In (2.125), the γ is evaluated so that the electron motion is one-dimensional adiabatic and γ = 3 from (2.33).

The dispersion relation to the **electron plasma waves** is easily obtained as

$$\omega^2 = {\omega_{pe}}^2 + 3{v_e}^2 k^2, \quad v_e = \sqrt{T_e / m_e} \tag{2.126}$$

where ω_{pe} is the **electron plasma frequency**, or simply called **plasma frequency** defined as

$$\omega_{pe} = \sqrt{\frac{e^2 n_e}{\varepsilon_0 m_e}} \tag{2.127}$$

In the cold plasma limit, the plasma waves tend to a simple oscillation

$$\omega^2 = {\omega_{pe}}^2 \tag{2.128}$$

This oscillation of electrons is called **plasma oscillation**. Note that ω_{pe} is a **resonance frequency** of the electrons in cold plasma.

2.7.2 Ion Acoustic Waves

If the time scale of the oscillation of perturbations is long enough, it is necessary to take account of the motion of ion fluid. The electric field due to the charge separation in slow time scale attracts the electrons, so that the electron fluid almost follows the motion of the ion fluid. Consider the electron and ion density perturbations are slightly different, while the electrons follow the ion fluid motion. Both fluids are coupled by the electrostatic field. Derive the equations for linear perturbations of the following four quantities.

$$n_{i1}, \quad \boldsymbol{u}_{i1}, \quad n_{e1}, \quad \boldsymbol{E}_1 \tag{2.129}$$

The linearized equation of (2.111) is given as

$$\frac{\partial}{\partial t} n_{i1} + n_{i0} \nabla \boldsymbol{u}_{i1} = 0 \tag{2.130}$$

Eq. (2.112) becomes

$$m_i \frac{\partial}{\partial t} \boldsymbol{u}_{i1} = -\frac{1}{n_{i0}} \nabla P_{i1} - e\boldsymbol{E}_1 \tag{2.131}$$

Then, in (2.114) it is possible to neglect the inertial term. Assume that the electric field should balance with the electron pressure force.

$$-\frac{1}{n_{e0}} \nabla P_{e1} - e\boldsymbol{E}_1 = 0 \tag{2.132}$$

The final relation is the Poisson Eq. (2.119).

$$\nabla \boldsymbol{E}_1 = \frac{e}{\varepsilon_0} (n_{i1} - n_{e1}) \tag{2.133}$$

After Fourier transformation, the ion density perturbation is found to have the following relation with the electron density perturbation.

$$n_{i1} = \left(1 + k^2 {\lambda_{De}}^2\right) n_{e1} \tag{2.134}$$

Note that both density perturbations are almost the same for small k, while the electron density does not follow the ion density for large k.

By solving the above coupled equations after Fourier-Laplace transformation, the dispersion relation of the **ion acoustic wave** or simply the **ion wave** is obtained.

$$\omega^2 = \frac{\gamma_i T_i}{m_i} + \frac{\gamma_e T_e}{m_i} \frac{1}{\left(1 + k^2 \lambda_{De}{}^2\right)}, \quad \gamma_i = \frac{5}{3}, \quad \gamma_e = 1 \tag{2.135}$$

where the ion fluid is adiabatic and the electron fluid is assumed to keep a constant temperature because of the dominant electron thermal conduction. It is usual that the most of plasmas have higher electron temperature compared to the ion one. (2.135) is usually written to be

$$\omega^2 = C_s{}^2 k^2 \frac{1}{1 + k^2 \lambda_{De}{}^2} \tag{2.136}$$

where C_s is the **ion acoustic velocity** defined by

$$C_s = \sqrt{\frac{\gamma_e T_e}{m_i}}, \tag{2.137}$$

The dispersion relation (2.136) is plotted in Fig. 2.11. It is noted that the ion acoustic wave phase velocity satisfies the relation.

$$v_i < \frac{\omega}{k} < < v_e \tag{2.138}$$

Once this relation is not satisfied, the ion waves are damped by the kinetic effect, so-called **Landau damping**. The physics of Landau damping will be discussed in Volume 4.

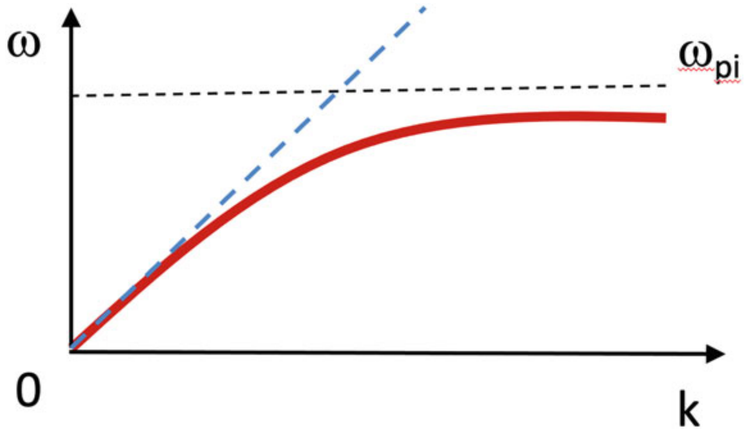


Fig. 2.11 The dispersion relation of the ion acoustic wave is plotted by the red line. It starts with a constant velocity at small k, while saturates at large k region

2.8 Mathematics for Wave Analysis

The waves in plasma produced by the motion of ions and electrons coupled with Maxwell equation have the same role as the seismic waves, the sound wave, etc. So, if there is a disturbance somewhere in the plasma, the induced waves carry energy so as to disperse the energy spatially and temporally. As the result, plasma confinement is prohibited in some cases. When the amplitude of the waves is sufficiently small, it can be analyzed as weak deviation from the equilibrium state. Then, the governing equations can be linearized, and it is enough to solve the linearized wave equations. **Fourier-Laplace transformation** has been used to obtain the wave dispersion relation, but any precise mathematics has not been explained, yet. To proof the mathematics, we start with the small vibration of an oscillator before the wave analysis. In the wave theory, it is standard to analyze using Fourier-Laplace expansion, and it is strait forward to use the mathematics of the analysis of such an oscillator.

2.8.1 Initial Value Problem of an Equation of Oscillation

First, let's solve exactly the initial value problem of the equation for a harmonic oscillator with a damping term by the Laplace's method. The equation of the harmonic oscillator with the eigen-frequency ω_0 and damping coefficient γ can be written as follows.

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = 0 \tag{2.139}$$

Multiplying (2.139) by $e^{i\omega t}$ and introducing time-integrated variables, Laplace transformation is carried out with the definition.

$$X(\omega) = \int_0^\infty x(t)e^{i\omega t}dt \tag{2.140}$$

It should be noted that even if $x(t)$ grows exponentially in time, $Im\ (\omega)$ should be a positive value so that this integral (2.140) must not diverge. Carrying out the Laplace transform of (2.139), the second term becomes

$$\int_0^\infty \frac{dx}{dt}e^{i\omega t}dt = xe^{i\omega t}\big|_0^\infty - i\omega X(\omega) = -x(0) - i\omega X(\omega) \tag{2.141}$$

The Laplace transform of the first derivative includes the initial value x (0). The second derivative is

$$\int_0^\infty \frac{d^2x}{dt^2}e^{i\omega t}dt = \frac{dx}{dt}e^{i\omega t}\Big|_0^\infty - i\omega\int_0^\infty \frac{dx}{dt}e^{i\omega t}dt = -\dot{x}(0) + i\omega x(0) - \omega^2 X(\omega) \quad (2.142)$$

This includes the first derivative at t = 0, $\dot{x}(0)$ = dx / dt (t = 0).

If Laplace transform is performed accurately as mentioned above, Laplace transformed equation of (2.139) is given as

$$\left(\omega^2 + 2i\gamma\omega - \omega_0^2\right)X(\omega) = -\dot{x}(0) + (i\omega - 2\gamma)x(0) \quad (2.143)$$

LHS of (2.143) is factorized.

$$\left(\omega^2 + 2i\gamma\omega - \omega_0^2\right) = (\omega - \omega_1)(\omega - \omega_2) \quad (2.144)$$

Here,

$$\omega_1 = \sqrt{\omega_0^2 - \gamma^2} - i\gamma, \quad \omega_2 = -\sqrt{\omega_0^2 - \gamma^2} - i\gamma \quad (2.145)$$

Since $X(\omega)$ is given in (2.143), the Laplace inverse transformation is performed to give

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{-i\omega t}d\omega \\ &= -\frac{\dot{x}(0)}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-i\omega t}}{(\omega-\omega_1)(\omega-\omega_2)}d\omega \\ &\quad + \frac{x(0)}{2\pi}\int_{-\infty}^{+\infty}\frac{(i\omega - 2\gamma)e^{-i\omega t}}{(\omega-\omega_1)(\omega-\omega_2)}d\omega \end{aligned} \quad (2.146)$$

Here, 1/2π of (2.146) is a normalization constant.

The integration of (2.146) is easily carried out with the **residue theorem** and **Cauchy's theorem**. For the sake of simplicity, the real part of the Eq. (2.145) is rewritten to be,

$$\Omega = \sqrt{\omega_0^2 - \gamma^2} \quad (2.147)$$

The integral of (2.146) has singular points$\omega = \omega_1, \omega_2$. In the Laplace transform defined in (2.140), it was required that the imaginary part of ω should be positive and large enough so that the integral of (2.140) does not diverge. Now, in the integral of (2.146), take the value of Im (ω) is sufficiently large negative value and extend the integral to a closed curve (red) like Fig. 2.12. Then, according to the Cauchy's theorem, this line integral (2.140) is obtained by adding negative signs to the

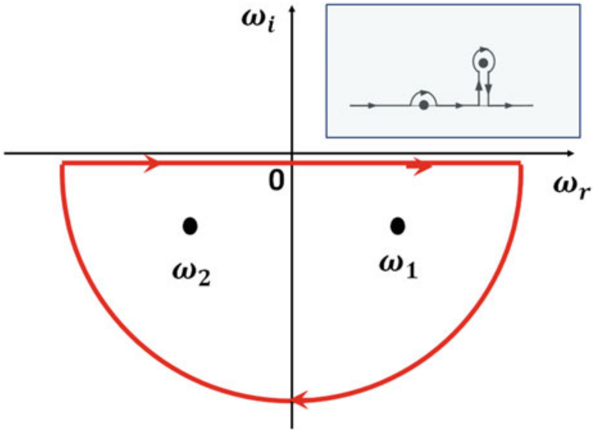


Fig. 2.12 The integration loop in the complex space for the inverse-integration of Laplace transformation. In the case of inverse-Laplace integration, it is necessary to down the integration line to the negative infinite circle for convergence of the integration. Then, it is found that Cauchy's theorem indicates that only the contribution by the pole remains in the loop integration. If there are roots with positive imaginary part like shown in the small box above, then the integration pass should be modified as shown in the small box. Then, we obtain exponentially growing solutions

residues from the two poles like in Fig. 2.12. It should be the direction of the clockwise. The first term on RHS of (2.146) is integrated.

$$-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-i\omega t}}{(\omega-\omega_1)(\omega-\omega_2)}d\omega=-\frac{ie^{-\gamma t}}{2\Omega}\left(e^{i\Omega t}-e^{-i\Omega t}\right)$$
$$=\frac{e^{-\gamma t}}{\Omega}\,sin(\Omega t) \tag{2.148}$$

The second term is

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{(i\omega-2\gamma)e^{-i\omega t}}{(\omega-\omega_1)(\omega-\omega_2)}d\omega=\frac{e^{-\gamma t}}{2}\left(e^{i\Omega t}+e^{-i\Omega t}\right)+\frac{i\gamma e^{-\gamma t}}{2\Omega}$$
$$\times\left(e^{i\Omega t}-e^{-i\Omega t}\right)=e^{-\gamma t}\left(cos(\Omega t)+\frac{\gamma}{\omega_3}\,sin(\Omega t)\right) \tag{2.149}$$

Finally, the solutions are obtained by substituting (2.148) and (2.149) into (2.146) as follows.

$$x(t)=e^{-\gamma t}\left[\frac{sin(\omega_3 t)}{\Omega}\dot{x}(0)+\left(cos(\Omega t)+\frac{\gamma}{\Omega}\,sin(\Omega t)\right)x(0)\right] \tag{2.150}$$

It is easy to confirm that the solution obtained in this way satisfies the initial condition.

In the absence of the damping term ($\gamma = 0$) the solution is simplified from (2.150).

$$x(t) = \dot{x}(0)\,\frac{sin(\omega_0 t)}{\omega_0} + x(0)\,cos(\omega_0 t) \tag{2.151}$$

The form of (2.151) is the general solution of the harmonic oscillator in the form.

$$x(t) = Asin(\omega_0 t) + Bcos(\omega_0 t) \tag{2.152}$$

The constants of A and B in (2.152) should be determined from initial conditions as (2.151).

The case of (2.139) is easy to solve even as an initial value problem as seen above. However, if the equation becomes higher order, third or fourth order differential one, it is hard to solve as above. Then, if the Laplace transform is used, the differential equations become algebraic equations, eventually resulting in a problem of finding poles in the Laplace inverse transformation. This is easy and useful as a general theory. This advantage is very powerful.

It is useful to know the case where the relation (2.144) has solutions with positive imaginary. Then, it is necessary to down the integration contour from above to below by avoiding the singular point as shown in the inlet at the top right in Fig. 2.12. In this case, the solution has a term exponentially growing in time. So, the change of the contour from the Laplace to inverse-Laplace transformation should be carried out by paying attention to the assumption for the convergence of the integral for $t > 0$.

2.8.2 Solving with Fourier-Laplace Method

Apply the Fourier decomposition to the equation for the electromagnetic waves in vacuum. The mathematics are the same for the sound waves, plasma waves and any other waves. Solve the initial value problem for the Fourier component of wavenumber **k**. The Laplace transform same as the harmonic oscillator is used for this Fourier mode. For simplicity, try to solve the one-dimensional problem of space with the x direction. The basic equation is

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)E = 0 \tag{2.153}$$

where c is the speed of light in vacuum. (2.153) is the same type of equation as (2.49) and expanded as,

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)E = 0 \tag{2.154}$$

It is clear (2.154) has the general solution.

$$E(t, x) = f(x - ct) + g(x + ct) \tag{2.155}$$

where f and g are arbitrary functions. The first term of (2.155) is the wave propagating to the right in the x axis, and the second term is the wave propagating to the left. Here, f and g are determined by the initial condition. Since (2.153) is a linear partial differential equation, the principle of superposition can be used. Then, the solution can be given in the form with the sum of the Fourier components.

$$E(t, x) = \sum\nolimits_k E_k(t) e^{ikx} \tag{2.156}$$

Inserting (2.156) into (2.153), the following ordinary differential equations are obtained for each Fourier component.

$$\frac{d^2 E_k}{dt^2} + c^2 k^2 E_k = 0 \tag{2.157}$$

Assuming $\gamma = 0$ in (2.139) and $\omega_0^2 = c^2 k^2$ in (2.139), (2.157) is of the same form as the harmonic oscillator. Therefore, from (2.151) the solution to the initial value problem is obtained.

$$E_k(t) = \frac{dE_k(0)}{dt} \frac{sin(\omega_0 t)}{\omega_0} + E_k(0)\, cos(\omega_0 t) \tag{2.158}$$

The solution can be obtained with the Fourier decomposition of the initial condition.

Inserting (2.158) to (2.156), the following form is obtained as the solution.

$$E(t, x) = \sum\nolimits_k A_k e^{-ik(ct - x)} + \sum\nolimits_k B_k e^{ik(ct + x)} \tag{2.159}$$

where A_k and B_k are given by the Fourier transformation of the initial condition.

It is useful to know that partial differential equations can be solved as ordinary differential equations in the case of linear perturbations. Furthermore, solving the initial value problem of the Eq. (2.154) is nothing without finding the poles in closed curve of the Cauchy integral in the two-dimensional complex space. The solution has the form proportional to exp.(−iωt), and its frequency and growth rate (or damping rate) are the real part and the imaginary part of the singular points, respectively.

Therefore, the solution of the algebraic equation corresponding to the singular points can be symbolically expressed

$$\omega = \omega(\boldsymbol{k}) \tag{2.160}$$

The relation (2.160) is generally called a **dispersion relation**. For wavenumber **k**, the number of waves is equal to the number of singularities of the denominator of the Laplace inverse transformation. The number of singularities increases as the basic equations become more complicated. Electromagnetic waves are simple, second-order equations, but there are numerous waves in the plasma. Therefore, rather than directly solving the differential equation, it is better to use the Fourier-Laplace transform to obtain the algebraic equation of dispersion relation, for example, the dispersion relation of electromagnetic wave in the vacuum is simple as

$$\omega^2 = c^2 k^2 \tag{2.161}$$

In the case that the dispersion relation is a real function and has roots of complex, there is always a solution of wave growing in time. In such a case, the wave is said to be unstable. To find the instabilities in plasma is very common subject even in laser-produced plasmas as will be seen later.

2.9 Magneto-Hydrodynamic Equation of Plasma

An ion is much heavier than an electron. Therefore, the relatively slow change in the plasma dynamics is often determined by the inertial of the ions. In this case, electrons move in association with ions so as to avoid charge separation to form a strong electric field. However, since the high temperature plasma has a high electric conductivity, the electron flow keeps electric current even in weak electric field. Then, while maintaining charge neutrality, an electron current is generated, and it is better to regard that the ions move slowly with strong magnetic field due to the electron current.

In such a case, there is no need to solve the above two fluid equations separately. In general, the behavior of plasma is approximated by **Magneto-Hydro-Dynamics (MHD)** equation derived below. For example, in magnetic confined plasmas, we first study the confinement condition of plasma with use of the MHD equation. This MHD equation was derived by **H. Alfven**, awarded the Nobel Prize in Physics in 1970. His achievement is stated in the citation for this award, “fundamental research and discovery with magneto-hydrodynamics as meaningful application to various parts of plasma physics”.

In recent years, observation technology has been advanced rapidly to provide details of the plasmas in the Universe. As the result, there is a movement to reconstruct astrophysics based on plasma physics, for example, the explosive phenomenon in the Universe. In the laser-plasma, the generation of magnetic field or coupling of external magnetic field has become an important topic mainly relating to l**aboratory astrophysics**, such as magnetic reconnection [8].

The MHD equation have been introduced as basic equation to describe space plasmas and magnetically confined plasma. The magnetic field is ubiquitous in the Universe. It is important to understand the approximation in obtaining MHD equation and the property of the equation. The MHD equation is derived while explaining the derivation procedure and approximation.

Multiply (2.111) by m_e, multiply (2.113) by m_i, take the sum of both, and divide it by (m_i + m_e). In derivation, charge neutrality $n_e/Z = n_i = n$ is assumed as explained above. Furthermore, the mass density and flow rate of the MHD fluid are introduced as

$$\rho = (m_i + m_e)n, \quad \boldsymbol{v} = \frac{m_i \boldsymbol{u}_i + m_e \boldsymbol{u}_e}{m_i + m_e} \tag{2.162}$$

Then, the mathematical process above gives

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} = -\nabla P + \boldsymbol{j} \times \boldsymbol{B} \tag{2.163}$$

In obtaining (2.163), the convective term was neglected. MHD equation is applicable only when the flow velocity is sufficiently slow

$$|\boldsymbol{v} \cdot \nabla \boldsymbol{v}| < < \left| \frac{\partial \boldsymbol{v}}{\partial t} \right| \tag{2.164}$$

Next, the following equation is obtained by multiplying (2.112) by m_e, multiplying (2.114) by m_i, taking a difference and approximating $m_e < < m_i$.

$$\frac{\partial \boldsymbol{j}}{\partial t} = \frac{e^2 \rho}{m_i m_e} \left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} - \frac{\nu_{ei} m_e}{n e^2} \boldsymbol{j} \right) - \frac{e}{m_e} \boldsymbol{j} \times \boldsymbol{B} - \frac{e}{m_i} \nabla P_i + \frac{e}{m_e} \nabla P_e \tag{2.165}$$

Since the phenomenon is slow because of the heavy ions, it is reasonable to neglect LHS of (2.165) in what follows. Because of large mass ratio the term of P_i on the right side can also be ignored relative to the term with P_e. Then, (2.165) reduces to a **generalized Ohm's law**.

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \frac{1}{\sigma_{ei}} \boldsymbol{j} + \frac{1}{en} (\boldsymbol{j} \times \boldsymbol{B} - \nabla P_e) \tag{2.166}$$

Here, σ_{ei} is the electric conductivity. The resistivity is $1/\sigma_{ei}$ and it stems from the Coulomb scattering of electrons by ions in plasma.

The first term of the parenthesis in the second term on RHS of (2.166) is called the **Hall effect**. This means if there is current flow under an external magnetic field, a potential difference appears in the vertical direction. The second term of the bracket on RHS of (2.166) shows the effect of **ambipolar electric field** which can be generated by electrons with large mobility to escape by the pressure gradient of electrons. Without magnetic field and pressure gradient in (2.166), it reduces to the well-known **Ohm's law** in the form.

$$\boldsymbol{j} = \sigma (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{2.167}$$

2.9.1 Biermann Battery Effect

Now, assume that (2.166) is an equation giving the electric field. It is necessary to formulate governing equations for the magnetic field and density for the completion of the coupled equations for MHD phenomena. It is clear that the equation for density is a continuity equation of (2.111) for the density (2.162).

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \boldsymbol{v}) = 0 \tag{2.168}$$

The equation governing the magnetic field is obtained by taking the rotation of (2.166) and using Maxwell Eq. (2.117).

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) - \nabla \times \left(\frac{1}{\mu_0 \sigma_{ei}} \nabla \times \boldsymbol{B}\right) - \nabla \times \left(\frac{\boldsymbol{j} \times \boldsymbol{B}}{en}\right) + \nabla \times \left(\frac{1}{en} \nabla P_e\right) \tag{2.169}$$

This equation is the governing equation of the magnetic field and can be rewritten as a combination of three terms with **B** and one source term.

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) - \nabla \times \left(\frac{1}{\mu_0 \sigma_{ei}} \nabla \times \boldsymbol{B}\right) - \nabla \times \left(\frac{\boldsymbol{j} \times \boldsymbol{B}}{en}\right) - \frac{1}{en} \nabla n \times \nabla T_e \tag{2.170}$$

The last term in (2.170) plays a role of source and sink of magnetic field. This term is called **Biermann battery effect** [9].

In laser plasma experiment, Biermann battery effect is used to generate magnetic fields to study, for example, **magnetic reconnection** physics [10, 11]. When a single intense laser, shown with the yellow arrow, irradiates a foil as shown in Fig. 2.13a, the produced plasma expands to the laser direction. Since the thermal conduction by electron is dominant and the electrons spread almost uniformly in the hemi-sphere, while the ions expand dominantly in the normal direction. Then, $\nabla n \times \nabla T_e$ term in (2.170) is produced like a torus (doughnut) shape as shown by blue in Fig. 2.13a. The surrounding arrows show charge current vector by expanding electrons.

With use of a short-pulse proton beam (E = 32.8 MeV) generated by an ultra-short laser pulse, a snapshot of the proton beam bending image is obtained as shown in Fig. 2.13b, where the dark image shows the region that the proton beams are bended by the magnetic field. The maximum strength of magnetic field is reported about 2 MG [10]. Note that the spatial size of Fig. 2.13b is about 1 mm and the life time of magnetic field is of the order of ns.

By use of such strong magnetic field, dynamics of magnetic reconnection has been studied. Two intense lasers are focused on an aluminum plate with separation distance of ~1 mm to produce the same two magnetic field structure. The bending of the proton particles is measured to evaluate the magnetic field profiles as shown in

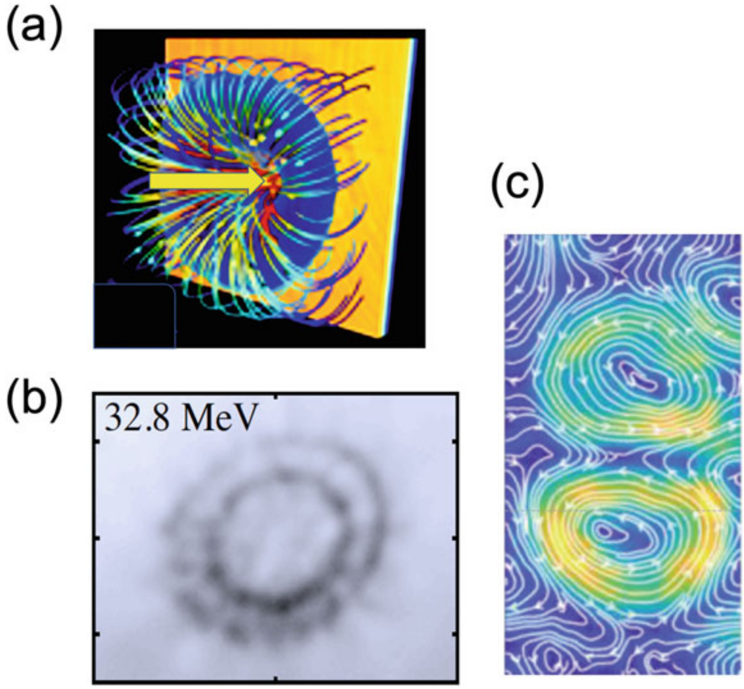


Fig. 2.13 Magnetic field generation in laser produced plasmas via Biermann battery effect. (**a**) Schematic of generation mechanism of magnetic field. A laser irradiated as yellow arrow at the center. (**b**) Proton back-light image of laser-produced magnetic field. (**c**) Magnetic lines overlapped on the proton back-light image to study magnetic reconnection in irradiating two lasers from the same direction. Reprint with permission from Refs. [10, 11]. Copyright 1998 by American Institute of Physics

Fig. 2.13c [11]. The lines are magnetic field line speculated with the proton image. Several snap shots are obtained to study the time evolution of topology of magnetic field.

2.9.2 Similarity of Vortex and Magnetic Fields

Let us discuss about the similarity of (2.169) to the equation of the vortex in neutral fluid (2.102). Except for the Hall effects, it is clear that both are mathematically same. In other words, if any vortexes seen in neutral fluid are generated in plasma, the plasma has electric current along the vortex flow, (see Fig. 2.14). Any vortex in plasma accompanies electric current, a relative motion of the electrons with respect to the ions, consequently, the magnetic field is generated. The vortex is a very important concept such as turbulence and turbulent transport in neutral fluid. In

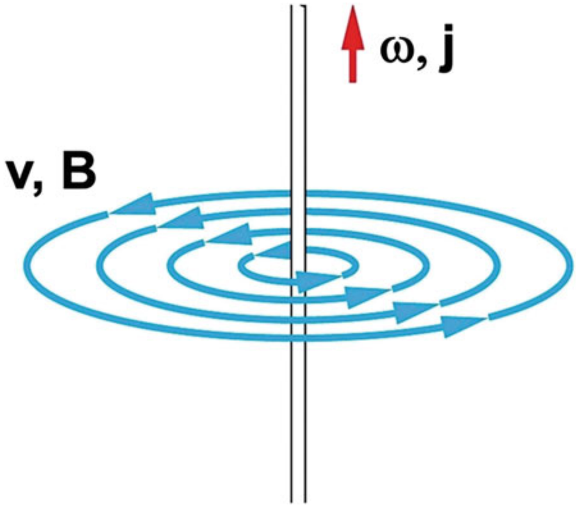


Fig. 2.14 The relation of flow velocity and magnetic field induced by the vortex and electron current in two-dimensional space, respectively. In the neutral fluid, the vortex is generated by the baroclinic term. This means that if the fluid is conducting plasma, the generation of vorticity means the generation of magnetic field in plasma

plasma, the transport of charged particles is strongly affected by magnetic field, as magnetic field turbulence is developed by vortexes.

Since MHD phenomena are generally discussed after neglecting the Hall effect or Biermann battery effect, the equation for the magnetic field is solved including the first two terms of (2.169). If the electric conductivity is also constant, the equation of (2.169) reduces to

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \frac{1}{\sigma_{ei}\mu_0} \nabla^2 \boldsymbol{B} \tag{2.171}$$

The first term on RHS is the convection term and the magnetic field winds around the plasma flow. The electric resistance of the plasma appears in the second term. In the case where the plasma resistivity cannot be neglected, the magnetic field diffuses in space, consequently, charged particles diffuse across the magnetic field. The diffusion of the charged particles is equivalent to the magnetic field diffusion through the plasma. The diffusion term disappears if the plasma is a perfect conductor, namely collisionless plasma.

In general, the diffusion term of magnetic field is regarded same as the Reynolds number (2.62) of the neutral fluid. The diffusion of magnetic field also plays a role in converting the magnetic field energy into the thermal energy of the plasma. Therefore, the **magnetic Reynolds number** in the MHD can be defined as the dimensionless quantity corresponding to the Reynolds number as follows.

$$R_m = \frac{(inertial\ term)}{(magnetic\ diffusion)} = \mu_0 \sigma_{ei} U L \tag{2.172}$$

Here, U and L are a characteristic flow velocity and a size of plasma. For $R_m >> 1$ the plasma can be described with the **ideal MHD** equation to be explained below. In many of laser plasmas in the laboratory, R_m is not so large. On the other hand, the plasmas in the Universe have very large L and/or very low density, therefore, in either case $R_m \rightarrow \infty$ can be assumed. It is good enough to assume the ideal MHD for study of such plasmas.

2.9.3 Ideal MHD Plasma

When the diffusion coefficient of the magnetic field is dominated by the Coulomb scattering, the diffusion is not important relatively in the laboratory plasmas aiming for nuclear fusion at high temperature or in space plasmas with large scale. Therefore, the ideal plasma approximation in which the magnetic Reynolds number R_m is a very large means that the diffusion term can be neglected. However, in a phenomenon that is governed by dissipation such as magnetic reconnection on the solar surface, it is difficult to explain the observed dynamics by the classical diffusion only due to the Coulomb scattering. In such a case, be aware that the resistivity due to magnetic turbulence induced by plasma wave instabilities becomes dominant. Such resistivity is called **anomalous resistivity** and will be discussed in Vol. 4.

The basic equation for the magnetic field of the **ideal MHD** is (2.171) without resistivity.

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) \tag{2.173}$$

Use the following mathematical relation to the convection term.

$$\nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{v} - \boldsymbol{v} \cdot \nabla \boldsymbol{B} - \boldsymbol{B} \nabla \boldsymbol{v} \tag{2.174}$$

where the relation $\nabla \cdot \boldsymbol{B} = 0$ has been used. From the equation of continuity (2.168),

$$\nabla \cdot \boldsymbol{v} = -\frac{1}{\rho}\frac{d\rho}{dt} \tag{2.175}$$

Inserting (2.174) to (2.173) and replacing the second term on RHS of (2.174) to LHS of (2.173), a new relation is obtained.

$$\frac{d\boldsymbol{B}}{dt} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{v} + \frac{\boldsymbol{B}}{\rho}\frac{d\rho}{dt} \tag{2.176}$$

This can be rewritten to be

$$\frac{d}{dt}\left(\frac{\boldsymbol{B}}{\rho}\right) = \left(\frac{\boldsymbol{B}}{\rho} \cdot \nabla\right)\boldsymbol{v} \tag{2.177}$$

It is found from (2.177) that when the flow is perpendicular to the magnetic field, RHS of (2.177) disappears and the quantity $\boldsymbol{B}/\rho$ is preserved along the plasma flow.

In addition, an equation of motion (2.163) is

$$\rho \frac{d\boldsymbol{v}}{dt} = \boldsymbol{j} \times \boldsymbol{B} - \nabla (P_e + P_i) \tag{2.178}$$

In (2.178), the first term on RHS is modified from Ampere's eq. (2.118) by neglecting the displacement current.

$$\boldsymbol{j} \times \boldsymbol{B} = \frac{1}{\mu_0} (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = -\nabla \left(\frac{B^2}{2\mu_0} \right) + \frac{1}{\mu_0} (\boldsymbol{B} \cdot \nabla) \boldsymbol{B} \tag{2.179}$$

This means that the force due to the magnetic field acts on the plasma as the **magnetic pressure** with the first term of RHS in (2.179) and the **magnetic tension** with the second term.

Here, the ideal MHD equation is closed with the three equations; namely, the equation of continuity (2.168), the equation for motion (2.178), and the equation for magnetic field (2.177). It is also necessary to give EOS for the pressure in (2.178). From the equation of motion, the ratio between the pressure due to the particles and that due to the magnetic field is a dimensionless quantity called **plasma β** value and is defined as.

$$\beta = \frac{(\text{plasma pressure})}{(\text{magnetic pressure})} = \frac{P_i + P_e}{\frac{B^2}{2\mu_0}} \tag{2.180}$$

Magnetic field confinement fusion machine such as **Tokamak** has β value of 1–2 percent. In order to extract energy by nuclear fusion and to put it into practical use, it is said that from the viewpoint of various losses, any fusion machine is necessary to have the β value more than 10%. Therefore, researches on **spherical Tokamak** with high β values are actively studied. Also, in the solar surface, the magnetic field is very strong, and plasma research focuses on physical phenomena in the so-called low beta (low-β) plasma. However, the laser produced plasma and the various plasmas in the Universe are in general high-beta (high-β) plasma. In high-β plasmas, the magnetic field influence on the charged particle transport becomes more important than the magnetic pressure.

It is useful to see the pressure form acting on MHD explicitly. Using the following relation to (2.179),

$$(\boldsymbol{B} \cdot \nabla)\boldsymbol{B} = \nabla (\boldsymbol{B} \otimes \boldsymbol{B}) \tag{2.181}$$

$$[\boldsymbol{B} \otimes \boldsymbol{B}]_{ij} \equiv B_i B_j$$

The conservation form of the momentum density is given from (2.178) in the form:

$$\frac{\partial}{\partial t}(\rho \boldsymbol{v}) + \nabla\left(\rho \boldsymbol{u} \otimes \boldsymbol{u} + \overleftrightarrow{\boldsymbol{T}}\right) = 0 \tag{2.182}$$

Here, the tensor $\overleftrightarrow{\boldsymbol{T}}$ is given to be

$$\overleftrightarrow{\boldsymbol{T}} = \left(P + \frac{B^2}{2\mu_0}\right)\overleftrightarrow{\boldsymbol{I}} - \frac{1}{\mu_0}\boldsymbol{B} \otimes \boldsymbol{B} \tag{2.183}$$

where $\overleftrightarrow{\boldsymbol{I}}$ is the unit tensor and P is the total pressure, $P = P_i + P_e$. The tensor of (2.183) is the total tensor pressure acting on the MHD fluid.

It is useful to show explicitly the component of the tensor:

$$T_{ik} = \left(P + \frac{B^2}{2\mu_0}\right)\delta_{ik} - \frac{B_i B_k}{\mu_0} \tag{2.184}$$

In the local frame in which the direction of the magnetic field is in the z-direction, $\overleftrightarrow{\boldsymbol{T}}$ can be given in the form.

$$\overleftrightarrow{\boldsymbol{T}} = \begin{bmatrix} P + \dfrac{B^2}{2\mu_0} & 0 & 0 \\ 0 & P + \dfrac{B^2}{2\mu_0} & 0 \\ 0 & 0 & P - \dfrac{B^2}{2\mu_0} \end{bmatrix} \tag{2.185}$$

As is clear from (2.185), the pressure by the magnetic field is in the perpendicular direction to the magnetic field vector. On the other hand, the tension works in the magnetic field direction as negative pressure. The magnetic field component in the z direction is physically.

(z-component by B)

$$= \left(\text{magnetic pressure} : \frac{B^2}{2\mu_0}\right) + \left(\text{magnetic tension} : -\frac{B^2}{\mu_0}\right) \tag{2.186}$$

Finally, the energy conservation equation of the MHD fluid is in the form.

$$\frac{\partial}{\partial t}U + \nabla \boldsymbol{S} = 0 \tag{2.187}$$

where the energy density U and the energy flux density **S** are

$$U = \frac{1}{2}\rho v^2 + \frac{1}{\gamma - 1}P + \frac{B^2}{2\mu_0} \tag{2.188}$$

$$\boldsymbol{S} = \left(\frac{1}{2}\rho v^2 + \frac{\gamma}{\gamma - 1}P\right)\boldsymbol{v} + \frac{1}{\mu_0}\boldsymbol{B} \times (\boldsymbol{v} \times \boldsymbol{B}) \tag{2.189}$$

In deriving (2.188) and (2.189), the ideal EOS for both particles have been assumed in the forms in (2.31) and (2.32) with the same specific heat γ, say $\gamma = 5/3$ for the fully ionized plasma.

2.9.4 Magnetic Dynamo Effect

Magnetic field grows even for the case without the source term like the Biermann battery effect in (2.170). Given fluid velocity field $\boldsymbol{v}(\boldsymbol{r})$ in (2.171), it has an eigen function $\boldsymbol{B}_0(\boldsymbol{r})$ in the form: $\boldsymbol{B}(\boldsymbol{r}, t) = \boldsymbol{B}_0(\boldsymbol{r}) \exp(\gamma t)$, where γ is the eigen value representing the growth rate of the magnetic field.

The principle of the growth of magnetic energy is explained intuitively like this. As explained in (2.186), the magnetic field has tension force and one need a work to stretch the magnetic field line in the direction of the magnetic vector. When the topology of flow field $\boldsymbol{v}(\boldsymbol{r})$ is complicated due to the convective motion in rotating plasma fluid system, for example, the conducting fluid inside the earth, plasma in the Sun, etc., the length of magnetic field line is possibly stretched by the convective motion, if the resistivity term in (2.171) is small enough, namely large R_m case.

In the case when the plasma pressure is much larger than the magnetic pressure, it is a good approximation to solve Navier-Stokes Eq. (2.57) independently from (2.171).

After solving NS equation and find almost stational convective motion, the eigen-value problem with reasonable boundary condition is solve to obtain the form $\boldsymbol{B}_0(\boldsymbol{r}) \exp(\gamma t)$ for the linear stability analysis. Then, the nonlinear evolution can be studied by solving numerically (2.171). We may find the nonlinear saturation profile of the magnetic field, where magnetic field is always enhanced by the first term in (2.171) to balance the dissipation of the second term. This is the case of magnetic field of the earth and the Sun.

In Fig. 2.15, the magnetic field near the surface of the Sun observed via radiation emission by electrons in their cyclotron motions is shown [12]. Such strong magnetic field is originally produced by the magnetic dynamo effect in the deep inside of the Sun. It is clear that since the magnetized region has lower density than non-magnetized neighbor plasma, the magnetic field rises by the buoyancy. Figure 2.15 is a snap shot of such magnetic field appeared on the surface and will disappear later via the magnetic reconnection.

Fig. 2.15 The magnetic field near the surface of the Sun observed via radiation emission by electrons in cyclotron motions [12]. Credit: NASA NASA/TRACE

2.9.5 Plasma Confinement by Magnetic Field

Eq. (2.178) gives the condition of plasma confined in magnetic field. The basic equation to solve configurations of plasma and magnetic field in the state of force balance is given as,

$$\boldsymbol{j} \times \boldsymbol{B} = \nabla P, \quad \boldsymbol{j} = \frac{\nabla \times \boldsymbol{B}}{\mu_0}, \quad \nabla \cdot \boldsymbol{B} = 0 \tag{2.190}$$

To find a configuration of magnetic confinement device for collisionless fusion plasma, this ideal MHD equation should be solved at first. It is clear from the divergence-free property of magnetic field ($\nabla \cdot \boldsymbol{B} = 0$) that the solution should have **torus** topology as shown in Fig. 2.16. as represented by **Tokamak** machine.

One of mathematically simple solution in an ideal one-dimension is the pinch plasma. To generate strong x-ray flux like that by lasers, **Z-pinch** machine driven by pulse power has been used [13]. The Z-machine has a solution of (2.190) with assuming one-dimensional cylindrical symmetric geometry, where **j** in the z-direction and **B** in the azimuthal direction. Then, (2.178) reduces to the.

$$\frac{dP(r)}{dr} + \frac{B_\theta(r)}{\mu_0 r}\frac{d}{dr}[rB_\theta(r)] = 0 \tag{2.191}$$

Then, (2.191) can be rewritten to be the force balance relation,

$$\frac{d}{dr}\left(P + \frac{B_0^2}{2\mu_0}\right) + \frac{B_0^2}{\mu_0 r} = 0 \tag{2.192}$$

Equation (2.192) represents that the pressure force by plasma and magnetic field balances with the tension force of magnetic field in (2.184). Solving (2.191), the

Fig. 2.16 Optimum structure of MHD solution, torus

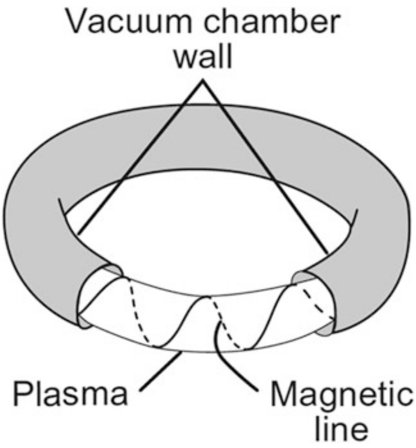
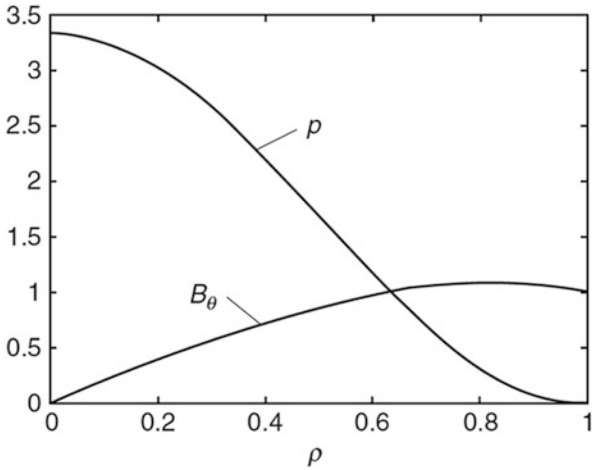


Fig. 2.17 Normalized pressure and magnetic field profiles of ideal one-dimensional Z-pinch solution



normalized profile of the magnetic field $B_\theta(r)$ and pressure P(r) of the Z-pinch is shown in Fig. 2.17. The magnetic field is normalized by the value at the outer radius ($\rho = 1$) and the pressure is normalized by the magnetic pressure at $\rho = 1$. The size of the radius is arbitrary as shown with the normalized radius ρ.

The wire-array Z-machine is used to study the possibility of **MagLIF** (magnetic laser inertial fusion) with combination of Z-pinch compression and laser heating [14]. Combining the magnetic field in the compression phase, the particle heat conduction can be reduced to relax the fuel ignition condition. It is well known, however, that the Z-pinch plasma confinement is unstable to perturbation from the cylindrical symmetry of the plasma and magnetic field, and MHD stability has to be studied.

2.9.6 Resistive MHD in Strong Heat Flux

In high-density plasmas produced by lasers or Z-pinch, the ideal MHD is not appropriate to describe the dynamics of magnetic field and fluid phenomena. In the case where strong heat flow in proportion to -∇T is important to the fluid dynamics, (2.179) is not an appropriate relation. This is because in deriving (2.179), we have assumed that the distribution function is Maxwellian with electron flow velocity $\mathbf{u}_e$ defined in (2.116).

In laser produced plasma, the heat flow carries the absorbed laser energy to the over-dense region, and the temperature is non-uniform in space. In such a case, the distribution function is not isotropic and it deforms in the direction of the heat flow. In general, it is enough to consider the heat flux by the electrons and the following discussion is done for the electron distribution function $f_e(\mathbf{v}, \mathbf{x}, t)$.

As will be seen in Chap. 6, the heat flux formula like (2.109) is derived by starting with Boltzmann equation. We follow the formulation given in [15]. The Boltzmann equation with a simplified **Krook collision operator** is given in the form.

$$\frac{\partial f_e}{\partial t} + \boldsymbol{v} \cdot \nabla f_e - \frac{e}{m}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \frac{\partial f_e}{\partial \boldsymbol{v}} = -\nu_{ei}(f_e - f_M) \tag{2.193}$$

where f_M is the local Maxwell distribution with n_e and T_e. Consider that (2.190) is in the local frame of the ion motion. The collision frequency ν_{ei} due to Coulomb collision of electrons by ions is given in (2.3).

Note that the collision frequency is a function of the electron velocity. In the standard way to solve (2.193) is the perturbation method, where the gradient length of T_e is assumed much longer than the electron mean-free-path. Then, it is assumed that

$$f_e = f_0 + \boldsymbol{f}_1 \cdot \frac{\boldsymbol{v}}{v} \tag{2.194}$$

where $\boldsymbol{f}_1$ is a vector function and small enough compared to f_0. Assuming f_0 is Maxwellian f_M and inserting (2.194) into (2.193), the equation to the perturbed distribution function is obtained.

$$\frac{\partial f_1}{\partial t} + \boldsymbol{v} \cdot \nabla f_0 - \frac{e}{m}\boldsymbol{E} \cdot \frac{\partial f_0}{\partial \boldsymbol{v}} - \frac{e}{m}\boldsymbol{B} \times f_1 = -\nu_{ei} f_1 \tag{2.195}$$

In general, the perturbation of the distribution consists of the two terms due to the mean flow and the temperature gradient. When both are in the x-direction, the $\boldsymbol{f}_1$ has only x-component and it can be expressed in the form:

$$f_1 = a_1 \boldsymbol{j} + a_2 \boldsymbol{q}_T \tag{2.196}$$

where **j** is the electric current and $\mathbf{q}_T$ is the heat flux by electrons. In (2.195), a_1 and a_2 are constants. When the heat flux is neglected and in addition the velocity dependence in ν_{ei} of (2.3) is neglected in (2.195), (2.166) is obtained by taking the velocity moment of (2.195).

However, when the heat flux term is included, the v^3 moment of ν_{ei} should be considered. Then, the generalized Ohm's law is obtained after neglecting the time dependence in the form.

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \frac{\boldsymbol{j}}{\sigma *} + \frac{1}{en_e}(\boldsymbol{j} \times \boldsymbol{B} - \nabla P_e) - \frac{1}{e}\nabla T_e - \frac{2}{5}\frac{\boldsymbol{q}_e \times \boldsymbol{B}}{P_e} \tag{2.197}$$

where $\sigma^* = 5/2\sigma$ with σ in (2.166). The factor 5/2 stems from the v^3 dependence of the collision frequency ν_{ei}. The last two terms on RHS in (2.197) appear due to the heat flux proportional to∇T_e. Note that the heat flux $\mathbf{q}_e$ in (2.197) is not equal to the $\mathbf{q}_T$ in (2.192). Since the energy is also carried by the plasma flow and $\mathbf{q}_e$ is purely heat flux remaining only for $\mathbf{j} = 0$. It is shown in [15]

$$\boldsymbol{q}_e \approx \boldsymbol{q}_T - \frac{5}{2}\frac{T_e}{e}\boldsymbol{j} = -\kappa_e \nabla T_e \tag{2.198}$$

In order to keep the fundamental structure of the Ohm's law as (2.166), it is required to derive the structure of $\mathbf{j}/\sigma$ term. In the real case, the Coulomb collision frequency is proportional to v^{-3} and σ in (2.166) should be replaced with $\sigma^* = 5/2\sigma$. With inclusion of v-dependence of the Coulomb collision frequency, the Hall term is found to have two terms. One is proportional to the current and the other is proportional to the heat flux. So, consistently, the Hall term is given as the total convection flow velocity as

$$\boldsymbol{u}_B = -\frac{\boldsymbol{j}}{en_e} + \frac{2}{5}\frac{\boldsymbol{q}_e}{P_e} \tag{2.199}$$

This term is called the **Nernst effect**. The importance of the Nernst effect in laser ablation plasma was pointed out in [15].

2.10 MHD Waves

Consider linear perturbations of the ideal MHD equation. The underlying formula is the equation of motion for the velocity of the magnetized fluid (2.163) and the eq. (2.173) for the magnetic field. Suppose that a stationary plasma is confined by an external magnetic field $\mathbf{B}_0$. For example, consider the plasma trapped in the earth's magnetic field. Before linearizing, the following operation is applied to (2.173).

$$\nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{v} - (\boldsymbol{v} \cdot \nabla)\boldsymbol{B} + \boldsymbol{v}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{v}) \tag{2.200}$$

Since from the Maxwell equation$\nabla \cdot \boldsymbol{B} = 0$, three terms remain in (2.200). Then, the basic equations are

$$\rho \frac{d}{dt}\boldsymbol{v} = \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} \tag{2.201}$$

$$\frac{\partial}{\partial t}\boldsymbol{B} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{v} - (\boldsymbol{v} \cdot \nabla)\boldsymbol{B} - \boldsymbol{B}(\nabla \cdot \boldsymbol{v}) \tag{2.202}$$

Assume the form of the linear perturbations of the magnetic field and velocity as

$$\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1 \tag{2.203}$$

$$\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_1 \tag{2.204}$$

Linearize Eqs. (2.201) and (2.202), and assume the plasma is at rest, namely $\boldsymbol{v}_0 = 0$. Consider that the perturbation is assumed to be incompressible $\nabla \cdot \boldsymbol{v} = 0$. Then, the basic equations for the linear components are

$$\rho_0 \frac{\partial}{\partial t}\boldsymbol{v}_1 = \frac{1}{\mu_0}\{(\nabla \times \boldsymbol{B}_1) \times \boldsymbol{B}_0 + (\nabla \times \boldsymbol{B}_0) \times \boldsymbol{B}_1\} \tag{2.205}$$

$$\frac{\partial}{\partial t}\boldsymbol{B}_1 = (\boldsymbol{B}_0 \cdot \nabla)\boldsymbol{v}_1 - (\boldsymbol{v}_1 \cdot \nabla)\boldsymbol{B}_0 \tag{2.206}$$

Since the current producing the external magnetic field $\mathbf{B}_0$ is outside the plasma, the second term of the parenthesis in (2.205) does not exist in the plasma.

Consider two cases separately, namely the wave propagates parallel or perpendicular to the external magnetic field. Waves propagating to the parallel direction are called **Alfven waves**, and in the perpendicular case they are called **magnetic sonic waves** or **compressible Alfven waves**.

2.10.1 Alfven Waves

Let's use the Fourier decomposition to the linear perturbations and find the dispersion relation of the wave with wave number **k**. First, the incompressibility is assumed for the case where the vibration is perpendicular to the magnetic field and the wave number **k** is in the direction of the magnetic field. The direction of the external magnetic field is the z-axis direction as shown in Fig. 2.18. First of all, in the simple case, assuming that the wave number is in the z direction ($\theta = 0$) and the wave oscillation is in the x direction, both (2.205) and (2.206) remain only the x component as follows.

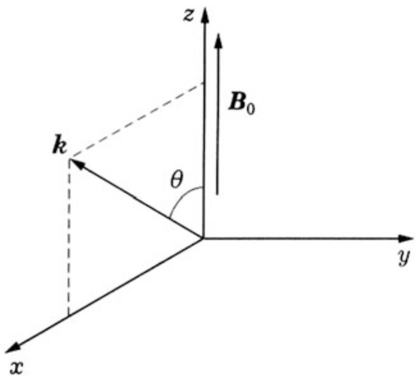


Fig. 2.18 The definition of the coordinate to study the waves in the constant external magnetic field in the z-direction

$$\rho_0 \frac{\partial}{\partial t} v_1 = \frac{B_0}{\mu_0} \frac{\partial}{\partial z} B_1 \tag{2.207}$$

$$\frac{\partial}{\partial t} B_1 = B_0 \frac{\partial}{\partial z} v_1 \tag{2.208}$$

By taking $\partial/\partial t$ for (2.207) and substituting (2.208) into (2.207), the following wave equation is obtained.

$$\frac{\partial^2}{\partial t^2} v_1 - {V_A}^2 \frac{\partial}{\partial z} v_1 = 0 \tag{2.209}$$

Here, V_A is called the **Alfven velocity**. The Alfven velocity is defined as follows.

$$V_A = \sqrt{\frac{B_0^2}{\mu_0 \rho_0}} \tag{2.210}$$

This velocity is the value obtained by dividing the tension of the magnetic field of (2.186) by the mass density. As in the image shown in Fig. 2.19, it is a wave caused by the ions wound around the magnetic field vibrating due to the tension of the magnetic field. It is the same as the acoustic of the strings of a guitar. The acoustic sound becomes higher when the string is strongly tensioned (strong magnetic field), the thicker the string (the higher the ion density), the lower the acoustic sound is.

Since the propagation velocity of the Alfven wave is constant, the dispersion relation of the Alfven waves is

$$\omega^2 = {V_A}^2 k^2 \tag{2.211}$$

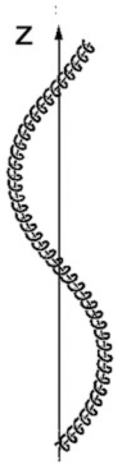


Fig. 2.19 A schematic of the perturbed magnetic field and ions in cyclotron motion following the deformed magnetic field line

It is important to note that if the plasma is low-β plasma confined by some external magnetic field, the Alfven speed is faster than the sound waves in (2.47) and ion acoustic waves (2.137). Therefore, energy spontaneously generated in the plasma is dominantly carried by the Alfven waves.

It is informative to obtain (2.211) by the energy principle. Let us find the change of the magnetic energy due to the sinusoidal distortion of the magnetic field δW_B and the kinetic energy for the ions around the magnetic field δW_k over one wavelength $\lambda = 2\pi/k$. As displacing $\xi(x,t) = \xi_0(t)\ sin\ (kz)$ in the perpendicular direction of the background magnetic field, the following energies are defined.

$$\delta W_B = (tension) \times (elongated\ length\ of\ the\ magnetic\ field)$$
$$\delta W_k = (kinetic\ energy\ of\ oscillation.)$$

Both are easily calculated to be the following forms per one wavelength

$$\delta W_B = \frac{B_0^2}{\mu_0}\left\{\int_0^\lambda \sqrt{1+\left(\frac{\partial\xi}{\partial z}\right)^2}dz - \lambda\right\} = \frac{\lambda}{4}(k\xi_0)^2\frac{B_0^2}{\mu_0} \tag{2.212}$$

$$\delta W_k = \int_0^\lambda \frac{1}{2}\rho_0\left(\frac{d\xi}{dt}\right)^2 dz = \frac{\lambda}{4}\rho_0\left(\frac{d\xi_0}{dt}\right)^2 \tag{2.213}$$

Here, **Lagrangian** is defined by considering ξ_0 as the generalized coordinate. Then, by solving the Euler-Lagrange equation, a simple oscillator equation can be derived. The frequency is easily obtained

$$\omega^2 = k^2 {V_A}^2 \tag{2.214}$$

2.10.2 Compressive Alfven Wave (Magneto Acoustic Waves)

Consider longitudinal waves propagating perpendicularly to the magnetic field. In this case, of course, since it is compressible, the pressure term of the eq. (2.178) also remains as the effect of finite temperature. The external magnetic field is the z direction as shown in Fig. 2.18, the oscillation is the x direction, and the wave number **k** is also in the x direction ($\theta = \pi/2$). Then, the compressibility comes out, so the basic equations are (2.168), (2.178), and (2.173).

As a new perturbation, density perturbation arises from the compressibility.

$$\rho = \rho_0 + \rho_1 \tag{2.215}$$

By inserting (2.215), (2.203), and (2.204) into the three basic equations and linearizing them, the following linearized equations are obtained.

$$\frac{\partial}{\partial t}\rho_1 + \rho_0 \frac{\partial}{\partial x} v_1 = 0 \tag{2.216}$$

$$\rho_0 \frac{\partial}{\partial t} v_1 = -{C_S}^2 \frac{\partial}{\partial x}\rho_1 - \frac{B_0}{\mu_0}\frac{\partial}{\partial x} B_1 \tag{2.217}$$

$$\frac{\partial}{\partial t} B_1 = -B_0 \frac{\partial}{\partial x} v_1 \tag{2.218}$$

By taking the time differentiation of (2.217) and using (2.216) and (2.218), a partial differential equation of the second order is obtained.

$$\frac{\partial^2}{\partial t^2} v_1 - {V_S}^2 \frac{\partial^2}{\partial x^2} v_1 - {V_A}^2 \frac{\partial^2}{\partial x^2} v_1 = 0 \tag{2.219}$$

Here, V_s is the sound velocity defined in (2.47). The dispersion relationship is easily obtained from (2.219) as

$$\omega^2 = k^2\left({V_A}^2 + {V_S}^2\right) \tag{2.220}$$

This is a wave called the **magneto acoustic wave**. When a compressional wave is generated in the direction perpendicular to the magnetic field, the density perturbation is oscillated by not only the magnetic pressure but also the pressure of the plasma. This is the reason of the name, magneto acoustic waves. For the case without thermal pressure, this wave is called the **compressional Alfven wave**.

The difference of the magnetic field displacement of the wave of (2.214) and that of (2.220) is clear. Although the Alfven waves are transverse wave and the displacement of the magnetic field is perpendicular to the propagation direction of the wave, the compressive Alfven wave is the same as the ion acoustic wave and it is a longitudinal wave. Since the magnetic pressure also contributes to the restoring force, the propagation velocity is faster than the ion acoustic waves.

2.10.3 Ion Acoustic Wave and Three Waves

We investigated the longitudinal and transverse waves, but there is a wave which receives restoring force by the pressure propagating in the magnetic field direction. Since the motion is parallel to the magnetic field, the force due to the magnetic field can be neglected (same as in the case without the magnetic field). Neglecting the magnetic field in (2.71), the following wave equation is obtained.

$$\frac{\partial^2}{\partial t^2} v_1 - {V_S}^2 \frac{\partial^2}{\partial x^2} \mathrm{v}_1 = 0 \tag{2.221}$$

This is the same as the acoustic wave given at (2.48). Note that inclusion of charge separation effect, the dispersion relation of the ion acoustic waves (2.135) is reproduced.

Therefore, we had already three waves. The waves propagating along the magnetic field are the **Alfven wave** (transverse wave) and **ion acoustic wave** (longitudinal wave). The wave propagating perpendicular to the magnetic field is the **magneto acoustic wave** (longitudinal wave).

Then, what kind of waves can propagate obliquely to the magnetic field? Does the ion acoustic wave change continuously to the magneto acoustic wave? Or may it be a mixed wave of longitudinal and transverse waves?

The basic equations are (2.168), (2.173), and (2.178). Assume that the arbitrary perturbations are given in the linearized forms and the angle that the wave number **k** forms with the magnetic field is θ as shown in Fig. 2.20. The oscillation component in the y-direction is transverse wave and the dispersion relation is

$$\omega^2 = k^2 {V_A}^2 \cos^2\theta \tag{2.222}$$

This is an obliquely propagating Alfven wave. At the same time there are two waves oscillating in the (x, z) plane, and after a bit messy calculation the dispersion relation can be found in the form.

$$\omega^4 - k^2\left({V_S}^2 + {V_A}^2\right)\omega^2 + k^4 {V_S}^2 {V_A}^2 \cos^2\theta = 0 \tag{2.223}$$

This can be easily solved and the dispersion relation is obtained as follows

$$\frac{\omega^2}{k^2} = \frac{\left({V_S}^2 + {V_A}^2\right)}{2} \pm \sqrt{\left({V_S}^2 + {V_A}^2\right)^2 - 4{V_S}^2 {V_A}^2 \cos^2\theta} \tag{2.224}$$

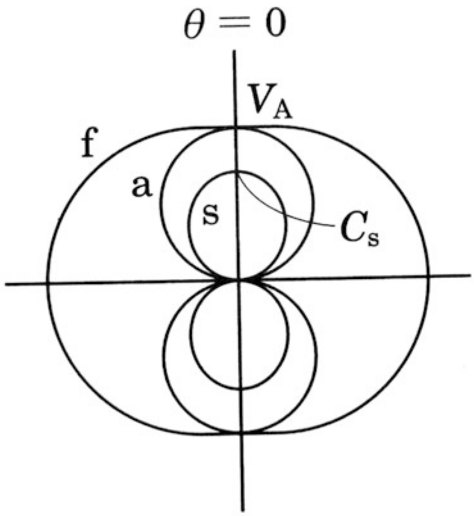


Fig. 2.20 The phase velocities of three waves induced by ion oscillations in an external magnetic field, called Friedrichs diagram. The "f" represents fast magneto-acoustic wave (fast wave), "a" Alfven wave, and "s" slow magneto-acoustic wave (slow wave)

In the case where the magnetic pressure is higher than the plasma pressure (lowβ) such as the earth's magnetosphere, the sun, and the magnetic confinement fusion device $V_A > V_S$ (= C_S in Fig. 2.20), the angular dependence of the phase velocity is shown in Fig. 2.20. As can be seen from (2.222) and (2.224), there are three waves at an arbitrary angle, two waves degenerate at $\theta = 0$, and at $\theta = \pi/2$ the waves are only the magneto acoustic waves. In Fig. 2.20, the "f" represents the fast magneto acoustic wave (**fast mode**), "a" the Alfven wave, and "s" the slow magnetic acoustic wave (**slow mode**). This diagram is refereed to **Friedrichs diagram**.

2.10.4 Torsional Alfven Wave

The circularly polarized Alfven waves couple with the angular momentum of plasma. This is important as a physical mechanism for releasing the angular momentum of **the accretion disk** in baby stars or planets. Consider an accretion disk with magnetic field as shown in Fig. 2.21. The accretion disk is modeled with a pan cake structure where the plasma is differentially rotating.

$$\boldsymbol{v} = r\Omega(r, z, t)\boldsymbol{e}_\phi \tag{2.225}$$

The magnetic field is assumed axially symmetric and is considered to be composed of two components: a poloidal component (z-direction) and a toroidal component (ϕ-direction).

$$\boldsymbol{B} = \boldsymbol{B}_p(r, z, t) + B_\phi(r, z, t)\boldsymbol{e}_\phi \tag{2.226}$$

The magnetic field is substituted into (2.202) to find the following relations.

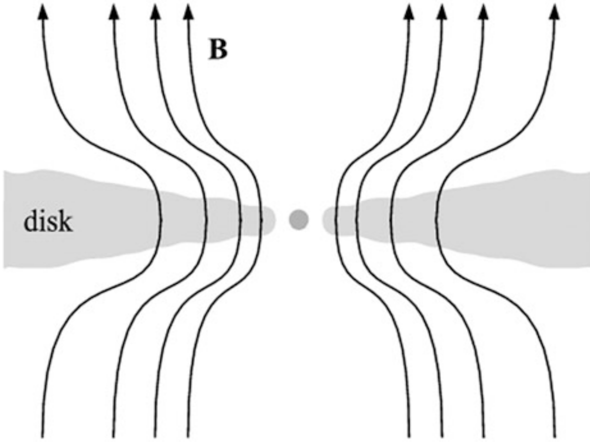


Fig. 2.21 A schematic image of cut-view of accretion disk and external magnetic field. The accretion disk is plasma and the strong interaction between the plasma and magnetic field is expected

$$\frac{\partial \boldsymbol{B}_p}{\partial t} = 0 \tag{2.227}$$

$$\frac{\partial B_\phi}{\partial t} = r\boldsymbol{B}_p \cdot \nabla \Omega \tag{2.228}$$

Here, on RHS of (2.228), only the first term on the right side of (2.202) remains, and the second two terms disappear.The condition to keep stationary rotation (2.228) required the relation.

$$\boldsymbol{B}_p \cdot \nabla \Omega = 0 \tag{2.229}$$

This is called **Ferraro's theorem** for a homogeneous rotation. If the magnetic field rotates at different angular velocities in the z-direction, the magnetic field twists and the rotation energy of plasma, that is, the angular momentum of plasma is converted into the energy of the magnetic field. However, since there is tension in the magnetic field, it should attempt to extract its twist outside the disc and to become a uniform magnetic field in the z-direction. The twist of the magnetic field is due to the angular momentum of the plasma of the disk, and the tension of the magnetic field transports the angular momentum by the **torsional Alfven wave** (explained below) outside the disk.

Furthermore, inserting (2.225) and (2.226) into the equation of motion (2.201) leads

$$(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{B} \tag{2.230}$$

Then, (2.201) becomes the following equation

$$\rho r \frac{\partial}{\partial t} \Omega = \frac{1}{\mu_0} B_p \frac{\partial}{\partial z} B_\phi \tag{2.231}$$

Here, the convection term of (2.201) automatically disappears as follows.

$$\boldsymbol{v} \cdot \nabla \boldsymbol{v} = 0 \tag{2.232}$$

Substituting (2.228) into (2.230) leads the following wave equation

$$\frac{\partial^2}{\partial t^2} \Omega - {V_{A,p}}^2 \frac{\partial^2}{\partial z^2} \Omega = 0 \tag{2.233}$$

Here, $V_{A,p}$ is the Alfven velocity due to the poloidal magnetic field. As can be seen from the derivation above, there is no linearization, therefore, the displacement in the z direction with respect to an arbitrary amplitude is transported outside the accretion disk at the Alfven velocity. As a result, the poloidal component of the magnetic field tries to keep the linear shape. (2.233) is the wave equation for the "**torsional Alfven wave**".

The accretion disk shown in Fig. 2.21 is formed by the plasmas falling to the gravitational center with rotating motion. The rotation motion is not of a constant angular momentum Ω in radial direction. Such differential rotation is known to induce the **magneto-rotational instability (MRI)** [16], and turbulent magnetic field is generated. The turbulent magnetic field enhances the transport of the matter falling to the central gravity, namely angular momentum of the matter transport. It is interesting to point out that a large-scale experiment plans to be carried out with a cylinder box filled with high-temperature liquid sodium (liquid metal) under differential rotation as shown in Fig. 2.22 [17]. Since the normal fluid in the differential rotating system, called **Taylor-Couette flow**, is unstable to fast rotating condition, the magnetic field is amplified by the dynamo effect as shown in Sect. 2.9.

2.11 Electromagnetic Wave in Magnetic Field

The hydrodynamic equations are the most useful ones to find the dynamical physics in many kinds of plasmas from the laboratory to the Universe. The physics of electromagnetic (EM) waves discussed here are usually used to measure, diagnose, or observe different kinds and different scales of plasmas. Of course, intense-lasers have been used to generate plasmas as shown in Volume 1. Strong microwaves are also used to heat magnetically confined plasma and processing plasmas [18]. In general, however, the electromagnetic waves due to electron current in plasmas are relatively high-frequency and the ions with larger mass cannot follow the electron motions. Since the most of fluid motions of plasmas are driven by the ion motions and the electromagnetic waves stemming from the electron motions do not couple with the fluid motions explained above.

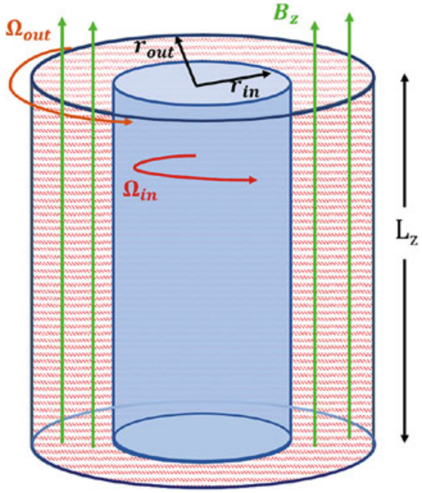


Fig. 2.22 A structure of sodium liquid experiment to study MRI by differential rotations. Reprint with permission from Ref. [17]. Copyright 1998 by American Physical Society

However, EM waves propagates not only in plasmas but also in vacuum. Therefore, EM waves are convenient waves for observing and investigating any plasmas. It is useful to know the fundamental property of the electromagnetic waves in plasmas. Some examples of applications for measurement and observation of plasmas are discussed here.

2.11.1 EM Waves in Plasmas

Electromagnetic waves are widely used for diagnostics of plasmas in the laboratory and observation of the Universe. In astronomy, the electromagnetic waves of wide range of wavelength have been observed to study energetic dynamics in the Universe. Since magnetic field is ubiquitous in Universe, it is also important to know the property of the electromagnetic waves in external magnetic field.

Maxwell equations provides the propagation of the electromagnetic waves in plasmas with the following simple equation as shown in Chap. 2.2.1 in Volume 1.

$$\left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right)\boldsymbol{E} = -\frac{1}{\varepsilon_0}\frac{\partial}{\partial t}\boldsymbol{j} \tag{2.234}$$

where $\boldsymbol{E}$ is the electric field of the electromagnetic waves and $\boldsymbol{j}$ is an plasma current induced by $\boldsymbol{E}$. It was already shown in Volume 1 that the dispersion relation of the electromagnetic waves in plasma is given as

$$\omega^2 = \omega_{pe}^2 + c^2k^2 \tag{2.235}$$

where ω_{pe} is the plasma frequency defined as

$$\omega_{pe}^2 = \frac{e^2 n_e}{\varepsilon_0 m} \tag{2.236}$$

Here, n_e is the electron density. Note that $\omega_{pe}^2 \propto m^{-1}$, the inverse of an electron mass. In Fig. 2.23 the dispersion relation is plotted with the solid line and with the dotted line of the light in vacuum.

The dispersion relation (2.235) indicates that the density of plasma with a size L can be measured from the phase shift of laser beams after passing through the plasma. By use of **holographic interferometry** technique, the density profile of an exploding foil heated by the other intense laser irradiated from the left is observed as shown in Fig. 2.24 [19]. The black-and-white stripe pattern shows the phase change due to the different densities of the measured light propagating in the expanding plasma.

The **refraction index** N is a function of density.

$$N \equiv \frac{c}{\frac{\omega}{k}} \leq 1 \tag{2.237}$$

This is used to obtain shadow image of plasmas. It is clear that the sharp density change reflects laser light impinging with a shallow angle. This property can be used to measure the spatial density structure of plasma shock waves etc. In Fig. 2.25, double exposure shadow images of laser-produced blast waves and the turbulence behind are shown [20].

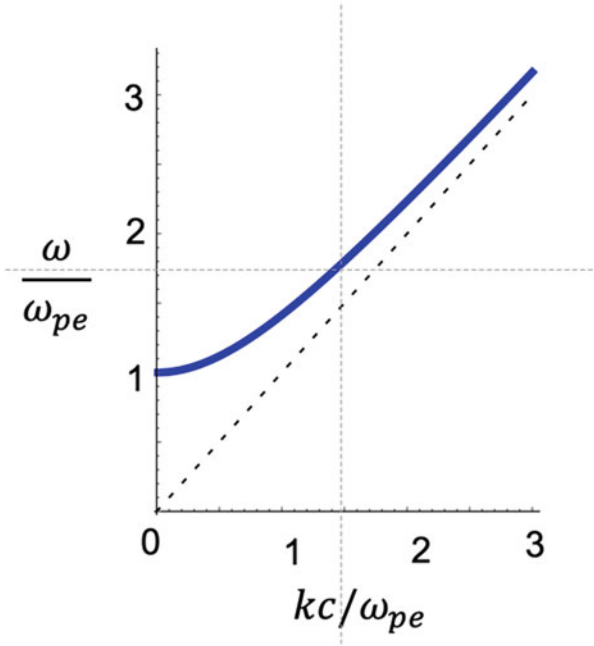


Fig. 2.23 Dispersion relation of electromagnetic field in plasma

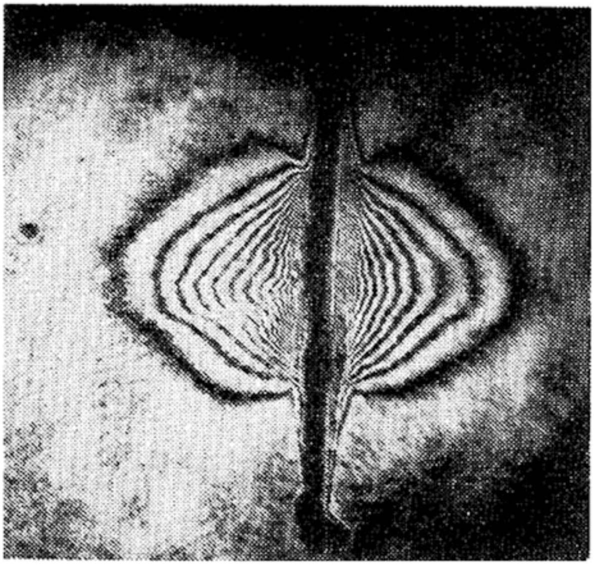


Fig. 2.24 A snapshot of holographic interferometry image of exploding foil. Reprint with permission from Ref. [19]. Copyright 1998 by American Physical Society

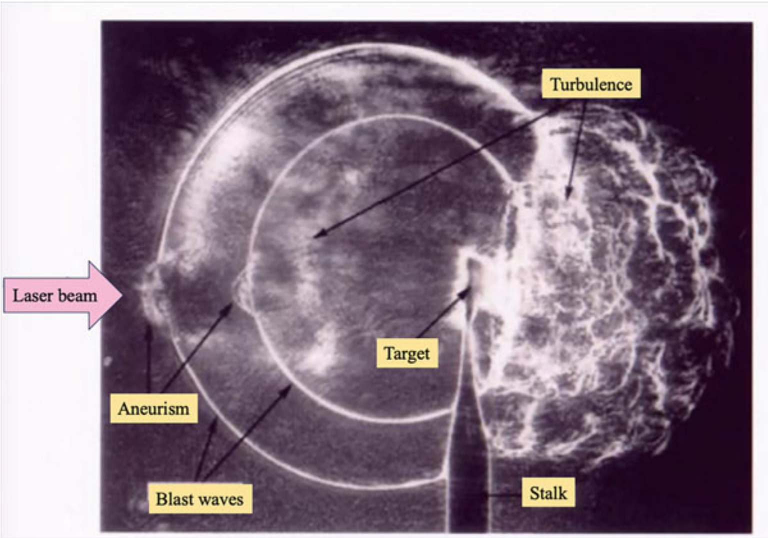


Fig. 2.25 Double exposure image of laser produced blast wave in nitrogen gas. Laser irradiates from left on aluminum target. Reprint from Ref. [20] with kind permission from Springer Science + Business Media. (Courtesy of B. Ripin.)

The **dispersion measure (DM)** defined as follows is also used to speculate the distance of a radio pulse source from a far distant space at L.

$$DM = \int_0^L n_e dx \tag{2.238}$$

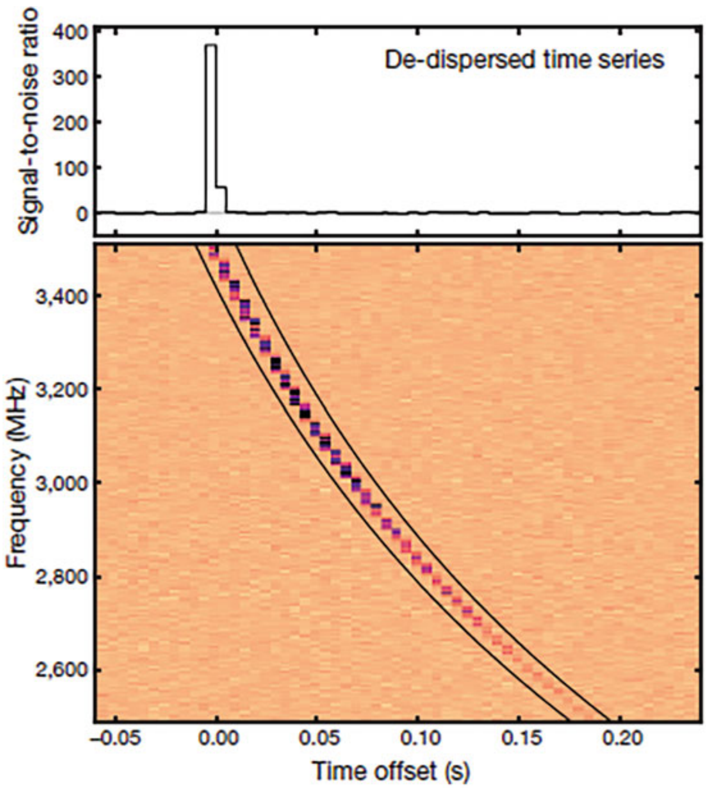


Fig. 2.26 Fast radio burst (FRB) signal observed in 2016. Time evolution of frequency. Reprinted by permission from Macmillan Publisher Ltd: Ref. [22], copyright 1993

For example, a radio pulse with high energy flux was observed near GHz radio wave at first in 2007. After this discovery of such a short radio pulse, the events are now called **FRB (Fast Radio Burst)** [21]. In Fig. 2.26, the signal of FRB 121102 observed in 2016 is shown [22]. Time–frequency data extracted from phased VLA visibilities at the burst location shows the ν^{-2} dispersive sweep of the burst. The solid black lines illustrate the expected sweep for DM = 558 pc cm^{-3}. The de-dispersed light curve is projected to the upper panel. The colour scale indicates the flux density.

The group velocity v_g of the electromagnetic waves in plasma with electron density n_e is

$$v_g = c\left(1 - \frac{\omega_{pe}^2}{\omega^2}\right)^{1/2} \tag{2.239}$$

This relation explains the reason of the delay of low frequency part. The time-delay of low frequency to high frequency ($\omega_L - \omega_H$) is obtained approximately for low-density plasma as

$$v_g \approx c\left(1 - \frac{1}{2}\frac{e^2}{\varepsilon_0 m}\frac{n_e}{\omega^2}\right) \tag{2.240}$$

$$\Delta t = \frac{L}{\Delta v_g} = \frac{1}{2}\frac{e^2}{\varepsilon_0 m}\left(\frac{1}{\omega_L^2} - \frac{1}{\omega_H^2}\right)DM \tag{2.241}$$

The pulse delay in Fig. 2.26 is used to evaluate the distance of the energy source of the burst, and it is found that DM is 558 cm^{-3}pc, which is about 12 times higher than the DM of the Milky Way galaxy. It is concluded that the energy source, which is not explained theoretically yet, is located at cosmological distance.

2.11.2 Electromagnetic Waves from Magnetized Plasmas

The dispersion relation is modified depending on how the induced current is related to the electric field of the electromagnetic waves. When the external magnetic field is applied or exists in plasmas, the electron motion is affected by the Lorentz force and the electric current is modified from the case without B-field given in (2.234). We have already studied the case of ion fluid motions driving MHD waves in external magnetic field in Sect. 2.10. It is in general the electromagnetic waves don't affect the MHD dynamics.

It is better to consider two idealistic cases; one is when the EM wave propagates along with the magnetic field (Fig. 2.27), and the other is when EM wave propagates perpendicular to the magnetic field. This knowledge can be applicable to the general

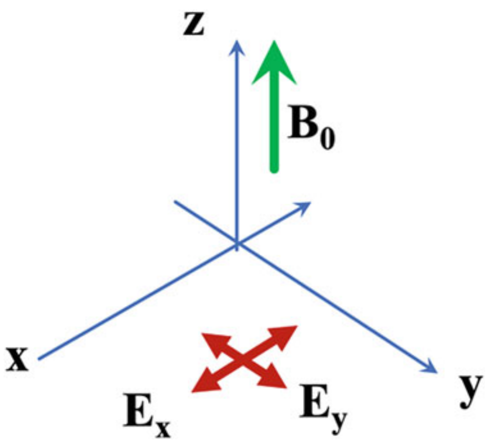


Fig. 2.27 Electric field of EM wave propagating to $\mathbf{B_0}$ or – $\mathbf{B_0}$ direction

case when EM wave propagates with an arbitrary angle to the magnetic field, although it is not discussed here.

Let us derive the induced current beginning with equation of motion of an electron.

$$m\frac{d}{dt}\boldsymbol{v} = -e(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{2.242}$$

Estimate the effect of magnetic field by assuming that the magnetic force is weak enough compared to the force by **E**. Then, the perturbation method gives a simple relation

$$\left|\frac{\boldsymbol{v} \times \boldsymbol{B}}{\boldsymbol{E}}\right| \sim \frac{\omega_{ce}}{\omega}, \quad \omega_{ce} = \frac{eB}{m} \tag{2.243}$$

where ω_{ce} is the electron cyclotron frequency. Namely, low frequency mode is strongly modified with ω near or lower than ω_{ce}. We consider here the case where EM wave propagates along the magnetic field, then, it is required to obtain coupled equations for the EM electric fields in x- and y-directions as in Fig. 2.27.

Here, we don't derive the dispersion relation because it needs a long calculation, and the readers wishing to know are recommended to refer to, e.g. [23]. There dispersion relation is the fourth order to ω in the form.

$$\left(\omega^2 - c^2k^2 - \alpha\right)^2 - \alpha^2\frac{\omega_{ce}^2}{\omega^2} = 0 \tag{2.244}$$

$$\alpha = \frac{\omega_{pe}^2}{1 - \omega_{ce}^2/\omega^2} \tag{2.245}$$

The dispersion relation (2.244) gives two independent modes. They are circularly polarized EM waves. The electric field of EM waves rotates to the right and left directions of magnetic field vector. Assuming that the magnetic field is in the z-direction and the rotating electric field with (k, ω) in (x, y) plane, two dispersion relations are obtained.

[R-wave] for the mode $E_x + iE_y$:

$$\omega^2 = c^2k^2 + \frac{\omega_{pe}^2}{1 - \omega_{ce}/\omega} \tag{2.246}$$

[L-wave] for the mode $E_x - iE_y$:

$$\omega^2 = c^2k^2 + \frac{\omega_{pe}^2}{1 + \omega_{ce}/\omega} \tag{2.247}$$

The real part of the electric field of the R-wave, $Re(E_x + iE_y)$, is given as

$$E = A\left[cos(kx - \omega t + \varphi)\boldsymbol{i}_x + \ sin(kx - \omega t + \varphi)\boldsymbol{i}_y\right] \tag{2.248}$$

where $\boldsymbol{i}_x$ and $\boldsymbol{i}_y$ are the unit vectors in x- and y- directions, and A and φ are constants.

It is clear that the L-wave has the negative sign for the second term (y-component) in (2.238). That is, the electric field vector of the R-wave rotates to the right, facing the z-direction, and the L-wave rotates to the left.

Note that the difference is only the sign of the denominators. Intuitively, we can image from (2.235) and (2.246) that the effect of magnetic field is regarded to assuming that the **effective mass** of electrons is given as

$$m_{eff} = \left(1 \mp \frac{\omega_{ce}}{\omega}\right)m \tag{2.249}$$

It is easy to know that the dispersion relation of the L-wave is given with that same as in Fig. 2.23 with the plasma frequency with the effective mass of "+" sign in (2.249). The **cut-off density** effectively decreases in the magnetic field. This means EM wave can propagates in the plasmas with less density than the nominal cut-off density, if there is a strong external magnetic field.

On the other hand, the R-wave has higher cut-off density for $\omega_{ce} < \omega$, and a new mode appears for low frequency EM wave with $\omega_{ce} < \omega$ as shown in Fig. 2.28 for the case of $\omega_{ce}/\omega_{pe} = 0.5$. In the case of $\omega = \omega_{ce}$, what happens is the **resonance of EM**

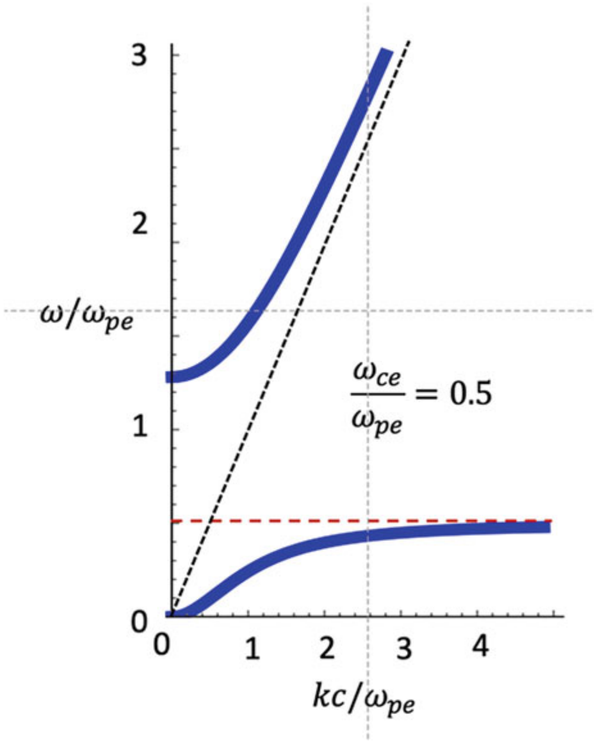


Fig. 2.28 Dispersion relation of the R-wave for the case with $\boldsymbol{\omega_{ce} = 0.5\omega_{pe}}$. The dashed lines are asymptotic of $\boldsymbol{\omega = \omega_{ce}}$ and $\boldsymbol{\omega = ck}$

wave and electron cyclotron motion. Then, the detail analysis gives the absorption of EM wave energy by the electron motion and the electron orbits continuously becomes larger in time. This resonance is used to heat electrons confined in strong magnetic field.

It is important to know that thanks to the electron cyclotron motion, the electric field of EM wave is maintained even in the density higher than the nominal cut-off density. The R-wave in the region $\omega < \omega_{ce}$ is called "**whistler wave**".

For the density n_e $[cm^{-3}]$ and magnetic field B [Gauss] units, both frequencies are.

$$\omega_{pe} = 5.6 \times 10^4 \sqrt{n_e} \left[s^{-1}\right]$$
$$\omega_{ce} = 1.8 \times 10^7 B \left[s^{-1}\right]$$

2.11.3 Faraday Rotation

It is well-known that when linearly polarized EM wave propagates along an external magnetic field, the polarization angle rotates because of the difference of dispersion relations of the R and L waves as shown in Fig. 2.29 [24]. Since the linearly polarized EM wave propagates as two circularly polarized waves of the R and L waves with different phase velocity, the combined EM with have a different angle of

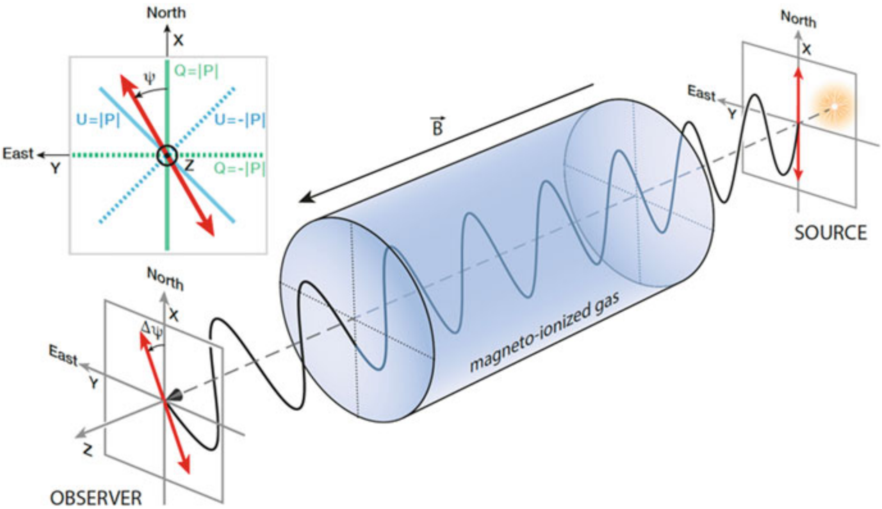


Fig. 2.29 The principle of Faraday rotation of linearly polarized EM wave traveling along magnetic field. The rotation angle is proportional to the Faraday rotation measure. Reprinted with permission from Ref. [24]. Copyright 1998 by Oxford University Press

polarization, when it goes out from the magnetized plasma. This phenomenon is called **Faraday rotation** in magnetized plasma. This effect was discovered by **M. Faraday** in 1845 with light propagating through magnetized glass.

The Faraday rotation has been used to measure the self-generated magnetic field in laser produced plasmas [25]. Irradiating a linearly polarized laser for diagnostic purpose like the case of Fig. 2.27, the shift of the polarization direction after the passage of magnetized plasma gives the information of the magnetic field in the plasma. The magnetic field is generated via **Biermann battery effect** in the laser-plasma as shown in Sect. 2.9. It was found that magnetic field of Mega Gauss is produced. It is noted, however, that the plasma β-value in (2.180) is still higher than unity, roughly $\beta \sim 100$. Since the laser plasma is small, but high-energy density, such strong magnetic field is produced during a short time of ns.

In Fig. 2.26, the polarization of the observed radio wave changes as a function of frequency. This fact can be used to evaluate the average magnetic field strength in the long path from the source. The principle is simple. For a give frequency ω, the difference of wavenumber, $\Delta k (\Delta k \ll k)$, of (2.246) and (2.247) is calculated to be,

$$\Delta k \approx \frac{\omega_{pe}^2}{\omega^2} \frac{\omega_{ce}}{c} \tag{2.250}$$

Integrating Δk over the propagation length L, the phase shift $\Delta\Phi$ is expressed in the form

$$\Delta\Phi = \int_0^L \Delta k dx = \frac{e^3}{\varepsilon_0 c m^2 \omega^2} \int_0^L n_e B_{\parallel} dx \equiv RM \lambda^2 \ [rad] \tag{2.251}$$

where $B_{\parallel}$ is the parallel magnetic field and λ is the wavelength. **RM** in (2.251) is called the **Faraday rotation measure**.

In the case where EM wave propagates perpendicular to external magnetic field, there are two modes depending on the polarization direction. For the case of E-field is parallel to the magnetic field, no magnetic effect appears because $\boldsymbol{v} \times \boldsymbol{B} = 0$ in (2.242) and the dispersion relation is given by (2.235). This mode is called the **ordinary wave**. On the other hand, when the polarization is perpendicular to the B-field, $\boldsymbol{v} \times \boldsymbol{B}$ term modify the EM propagation, and Lorentz force induces the plasma motion in **k** direction to couple with the electrostatic modes. This mode is called the **extraordinary wave**. The detail analysis is given, for example, in [23].

2.11.4 EM Waves from Magnetized Plasmas

A variety of EM waves from the radio waves to the γ-rays is generated by plasma electron motions in the Universe. **Synchrotron emission** of EM wave by highly relativistic electrons are strong EM sources. In Fig. 2.30, a schematic of the

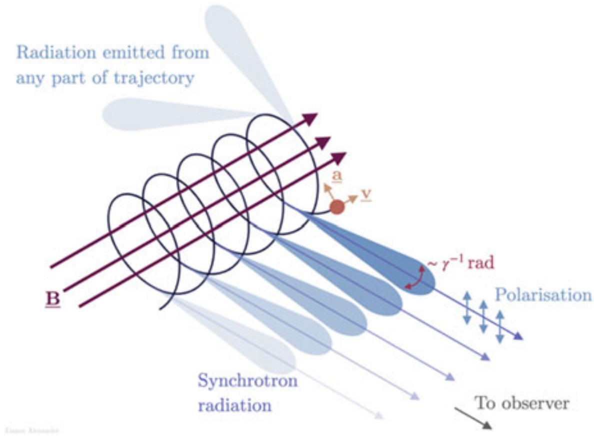


Fig. 2.30 Schematics of synchrotron radiation emission from a highly-relativistic electron rotating in magnetic field. The radiation is linearly polarized and the observed frequency is up-shifted by relativistic effect. Reprinted with permission from Ref. [26]. Copyright 1998 by Oxford University Press

mechanism of the emission is shown [26]. The radiation is emitted in the case where a charged particle is in accelerating motion as shown by Larmor. It is called the **Larmor emission** in non-relativistic electron case and the radiation is emitted dominantly in the direction perpendicular to the acceleration vector. In relativistic motion, it is called the **synchrotron emission**. The emission angle becomes narrow in proportion to $1/\gamma$, where γ is the Lorentz factor of a rotating electron. This is called **relativistic beaming** as discussed in Chap. 5 in Vol. 1. So, the emitted synchrotron frequency is up-shifted as $\omega = \omega_{ce}\gamma^2$. This means radio waves emitting from non-relativistic electrons becomes x-ray for highly relativistic electrons.

Supernova remnants (SNRs) are known to be a candidate where high-energy cosmic rays are generated by the shock waves (blast waves) produced by supernova explosion. Note that the detail physics of blast waves will be discussed in Chap. 4. The SNR of the supernova-1006 which exploded almost 1000 years ago is well studied with the radio to gamma-ray obserbation. The x-ray image shows a clear evidence of the particle acceleration in the vicinity of the blast wave front. With evaluated strength of magnetic field in μG range, it is speculated that highly relativistic electrons (up to 10^{15} eV) are emitting x-rays.

Since the synchrotron radiation is linearly polarized, the direction of magnetic field is speculated by the polarization measurement of radio wave. Assuming the magnetic field is given externally, and the structure is globally uniform in the SNR. The **degree of polarization** P is defined as

$$P \equiv \frac{I_{pol}}{I} \tag{2.252}$$

where I_{pol} is the intensity of polarized EM component and I is the total intensity. As seen in the distribution of the degree of polarization in Fig. 2.31, left-top and right-bottom are highly polarized [27]. From the distribution of the local polarization directions observed, it is concluded that the SN1006 has a large-scale magnetic field along the line from left-top to right-bottom [27].

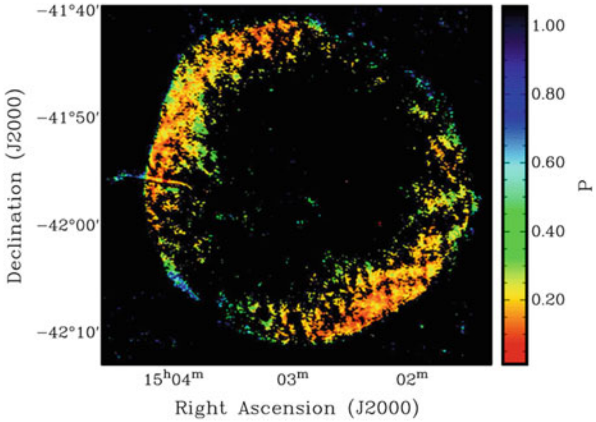


Fig. 2.31 Observed image of the degree of polarization of radio emission from SNR 1006 remnant. Reprinted with permission from Ref. [27]. Copyright by American Astronomical Society

Appendix-A: Fluid Approximation of Plasma

Basic equations to describe microscopic to macroscopic phenomena have been proposed in plasma physics so that they provide the essence of physics with the reduction of the degree of freedom. They leave only the degree of freedom as small as possible. Of course, it is better to solve the following **Vlasov equations** to ion and electron velocity distribution functions $f_\alpha(\mathbf{v})$ (α: ion or electron) as will be discussed in Volume 4 to study higher-freedom phenomena of laser-plasma.

$$\frac{\partial f_\alpha}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f_\alpha}{\partial \boldsymbol{r}} + \frac{q}{m}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \frac{\partial f_\alpha}{\partial \boldsymbol{v}} = 0 \tag{2.A-1}$$

where q and m are charge and mass of ion or electron, respectively.

Except for the case where the distribution functions are very far from shifted-Maxwellian, the fluid approximation of plasma is often adopted instead of Vlasov equation because of less freedom in the basic equations. Such a fluid is called "**electromagnetic fluid**" or "**magneto-hydrodynamic fluid**". As a matter of course, such modeling may cause loss of the physics that should originally appear. This point which cannot be derived by fluid model will be explained in relation to the **Landau damping** later.

In the case of neutral fluids, the mean free path is sufficiently shorter than the change length of the physical quantities and collision time is much shorter than the time scale of fluid changes. This means the distribution function in Boltzmann equation is well described with a local **Maxwell distribution**. Then, the collision term to appear in (2.A-1) in Boltzmann equation should also disappear mathematically. In such frequently colliding particle system like molecular gas, the velocity dependence of the distribution function is given with local Maxwell distribution and as seen below the fluid model is very reliable.

The same fluid model is used to describe plasmas regardless of collision dominant or not. Within the assumption of local Maxwellian, the velocity moment equations of

Vlasov equation give the fluid equations as show below. The fluid variables are defined by the velocity moments.

$$\text{Density}: n(t,\boldsymbol{r}) = \int_{-\infty}^{\infty} f d\boldsymbol{v} \tag{2.A-2}$$

$$\text{Flow rate}: \boldsymbol{u}(t,\boldsymbol{r}) = \frac{1}{n}\int_{-\infty}^{\infty} \boldsymbol{v} f d\boldsymbol{v} \tag{2.A-3}$$

$$\text{Temperature}: T(t,\boldsymbol{r}) = \frac{2}{3}\frac{1}{n}\int_{-\infty}^{\infty} \frac{1}{2}m(\boldsymbol{v}-\boldsymbol{u})^2 f d\boldsymbol{v} \tag{2.A-4}$$

$$\text{Heat flux}: \boldsymbol{q}(t,\boldsymbol{r}) = \int_{-\infty}^{\infty} \frac{1}{2}m(\boldsymbol{v}-\boldsymbol{u})^2(\boldsymbol{v}-\boldsymbol{u}) f d\boldsymbol{v} \tag{2.A-5}$$

Here, we assumed charged particles have no internal degrees of freedom and the specific heat ratio $C_p/C_v = \gamma = 5/3$ for simplicity. We defined the third moment of velocity in (2.A-5). In the case of the Maxwell distribution, the velocity distribution function is shifted by the mean flow velocity **u** and spreads with the width T of the temperature.

It is clear that (2.A-4) corresponds to the fact that the average kinetic energy per particle is 3/2 T. Maxwell distribution function is isotropic around $\mathbf{v} = \mathbf{u}$, and the heat flux $\mathbf{q} = 0$. This means if we can derive such moment equations for the unknown variables n, **u**, T, it is closed coupled equations and be principally possible to be solved. These equations are the basic equations for fluids.

There is no guarantee, however, that the plasma is collisionless and the velocity distribution function is close to the Maxwell distribution. The plasma distribution function may be determined by its production process, interaction with the confining wall, and so on. However, it is empirically proofed that in many cases the distribution can be approximated with the shifted Maxwell distribution. Although there is no theoretical validity to approximate plasmas as fluids, the fluid model is widely used instead of Vlasov kinetic model for more simplicity in mathematical treatment. If the deviation from Maxwellian is small enough, the heat flux **q**, viscosity etc. can be approximated proportional to the gradients of fluid quantities. The electron heat flux given in (2.A-5) will be derived later in Chap. 6.

Let's derive mathematically the equation of continuity and the equation of motion by taking the velocity moments of the Vlasov equation. First, the zeroth order moment is obtained by integrating (2.A-1) directly by **v.** The integral of the first term and the second term of Vlasov equation are simple. Although it seems that the integral value remains in the third term because of the force of $\boldsymbol{v} \times \boldsymbol{B}$, the actual integration disappears because v_i is not included in the i-component of $\boldsymbol{v} \times \boldsymbol{B}$ in actual calculation. Therefore, it reduces to

$$\frac{\partial n}{\partial t} + \nabla(n\boldsymbol{u}) = 0 \tag{2.A-6}$$

Next, integrate by multiplying (2.A-1) by the vector **v**. Calculate the i (= x, y, or z) component. The first term is simple. The following **v**' which is the velocity spread from the mean velocity **u** is defined for integrating the second term as (2.A-4).

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{v}' \tag{2.A-7}$$

The product with **v** is the vector and its i component of the second term of (2.A-1) is

$$\frac{\partial}{\partial x_j}\left(\int v_i v_j f d\boldsymbol{v}\right) = \frac{\partial}{\partial x_j}\left(n u_i u_j\right) + \frac{\partial}{\partial x_j}\left(\int {v_i}' {v_j}' f d\boldsymbol{v}\right) \tag{2.A-8}$$

Note that when a subscript that indicates a coordinate appears twice, such as j, it means to take the sum of the three components x, y, and z with respect to j. Such notation is called **Einstein notation**. The pressure is generated on the second term on the RHS of (2.A-8). Since the Maxwell distribution is isotropic in the velocity space around the mean velocity **u**, the second term in (2.A-8) reduces to

$$\int {v_i}' {v_j}' f d\boldsymbol{v} = nT\delta_{ij} \tag{2.A-9}$$

Here, δ_{ij} is the Kronecker delta, $\delta_{ij} = 1$ for i = j, and $\delta_{ij} = 0$ for otherwise. When the distribution function is isotropic, the pressure is a scalar. In general, the pressure is tensor unless the distribution function is isotropic.

The calculation of the force term of Vlasov equation is not so simple. Calculate the x component and calculate the y and z components in the same way. Multiply the third term of Vlasov equation by v_x and integrating it in v_x, v_y, and v_z space, the following is obtained.

$$\iiint dv_x dv_y dv_z v_x \frac{F_j}{m}\frac{\partial f}{\partial v_j} = \iint \frac{F_x}{m} dv_y dv_z \int v_x \frac{\partial f}{\partial v_x} dv_x \tag{2.A-10}$$

Here, we can put F_x out of the integral over v_x in (2.A-10). This is possible because Lorentz force of **F** = q (**E** + **v** × **B**), **E** does not depend on v_x and the x component of **v** × **B** does not depend on v_x. Carrying out the same mathematics for y and z components, it is clear the same logics works. The integral with respect to v_x in (2.A-10) is executed by using partial integral. With integration over v_y and v_z. Then, the x component of the velocity moments of the third term of Vlasov Eq. (2.A-1) is obtained. By performing the same calculation for v_y and v_z as well, we obtain all three components.

From the above mathematics, the flowing equation of motion is obtained.

$$m\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\boldsymbol{u} = -\frac{1}{n}\nabla P + q(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}) \tag{2.A-11}$$

Here, P is pressure, $P = nT$.

The second moment of velocity gives an equation for temperature T, but let's omit the derivation in the text. In the phenomenon that thermal conduction is important and it can be assumed that the temperature T is a constant, the fluid equation is closed only by (2.A-6) and (2.A-11).

The fluid approximation of plasma is the same as neutral fluid equations except for the force by the electromagnetic field Therefore, knowledge of fluid dynamics is fundamental for studying various phenomena of plasmas. In some cases, the plasma may be regarded as a neutral fluid and analysis becomes simpler.

Since the same procedure from Vlasov equation to fluid model is applicable to electron and ion distribution functions. Through such procedure, the basic equations for two fluid model can be obtained following the mathematics shown above.

Finally, it is noted that

1. Plasma phenomena of electron and ion particles can be well studied with two fluid model for electrons and ions by coupling with Maxwell equations. This is correct only when each particle is in thermodynamic equilibrium with Maxwell velocity distribution function.
2. The fluid model cannot provide the phenomenon of Landau damping. In **Landau damping**, only a small number of particles satisfying the resonant condition interact with electrostatic or electromagnetic waves. This is important physics appearing in plasmas. In addition, if the velocity distribution function is very far from a shifted-Maxwellian distribution in collisionless condition, a variety of new phenomena unpredicted with the fluid model appear in plasma as will be shown in Volume 4.

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