A Linear Perspective on Cut-Elimination for Non-wellfounded Sequent Calculi with Least and Greatest Fixed-Points

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Abstract. This paper establishes cut-elimination for $\mu\mathsf{LL}^\infty$, $\mu\mathsf{LK}^\infty$ and $\mu\mathsf{LJ}^\infty$, that are non-wellfounded sequent calculi with least and greatest fixed-points, by expanding on prior works by Santocanale and Fortier [20] as well as Baelde et al. [3,4]. The paper studies a fixed-point encoding of LL exponentials in order to deduce those cut-elimination results from that of $\mu\mathsf{MALL}^\infty$. Cut-elimination for $\mu\mathsf{LK}^\infty$ and $\mu\mathsf{LJ}^\infty$ is obtained by developing appropriate linear decorations for those logics.

Keywords: LL · μ-calculus · Non-wellfounded proofs · cut elimination

1 Introduction

On the Non-Wellfounded Proof-Theory of Fixed-Point Logics. In the context of logics with induction and coinduction (such as logics with inductive definitions *à la* Martin Löf [6,9,10,25], or variants of the μ-calculus [11,22,23]), the need for a (co)inductive invariant (in the form of the Park's rule for induction) is replaced by the ability to pursue the proof infinitely, admitting non-wellfounded branches, when considering non-wellfounded and circular proofs (also called cyclic, or regular proofs, since the proof tree is a regular tree, with finitely many distinct subtrees). In such frameworks, sequent proofs may be finitely branching but non-wellfounded derivation trees and infinite branches shall satisfy some validity condition. (Otherwise one could derive any judgement, see Fig. 1(a).) Various validity conditions have been considered in the literature [3].

The non-wellfounded and circular proof-theory of fixed-points attracted a growing attention first motivated by proof-search [1,7,8,16–18,28] and more recently by a Curry-Howard perspective, studying the dynamics of the cut-elimination in those logics [4,20,29] where formulas correspond to (co)inductive types. Notice also that when interested in the computational content of proofs, we will not focus solely on the regular fragment as we expect, for instance, that we can write a regular program that computes a non-ultimately periodic stream.

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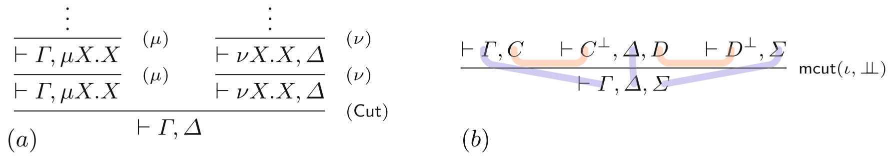

Fig. 1. (a) Example of an invalid circular pre-proof (b) Schema of the multicut rule

Cut-Elimination and LL*.* When studying the structure of proofs and their cut-elimination properties, LL, Girard's Linear Logic [21], is a logic of choice: the careful treatment of structural rules gives access to a lot of information and a fine-grained control over cut-reduction. The constrained use of structural rules indeed renders the cut-elimination theorem more informative than in LJ and of course LK. Interestingly it provided a positive feedback on the understanding of LJ and LK: by decorating intuitionistic and classical proofs with enough exponential modalities $(!, ?)$, they can become LL proofs and one can therefore refine the original cut-elimination relations [12,21]. This approach impacted the understanding of evaluation strategies of programming languages such as call-by-name and call-by-value notably. Another way to view this is by noting that, in LK, the additive and multiplicative presentations of conjunction (resp. disjunction) can be shown to be interderivable thanks to structural rules. This fails in LL and it is the reason why LL has well-established additive – $\oplus, \&, \top, 0$ – (resp. multiplicative $⅋, \otimes, \bot, 1$) *fragments*. It is the role of the exponential fragment to relate the additive and multiplicative worlds, by mean of the fundamental equivalence: $!A \otimes !B \dashv\vdash !(A \& B)$ (and its dual, $?A ⅋ ?B \dashv\vdash ?(A \oplus B)$). The exponential modalities are precisely introduced where structural rules are needed to restore the equivalence between the additive and multiplicative conjunctions; in categorical models of LL [26], this principle is referred to as Seely isomorphisms.

Cut-Elimination for Non-Wellfounded Proofs. Proving cut-elimination results for non-wellfounded proofs in the presence of least and greatest fixed-points requires to use reasoning techniques coping with the non-inductive structure of the considered formulas (fixed-points formulas regenerate) and proof objects (which are non-wellfounded). For instance, Santocanale and Fortier [20] proved cut-elimination for the regular fragment of non-wellfounded proofs of purely additive linear logic with fixed points, $\mu\mathsf{ALL}^\infty$, while Baelde *et al.* [4] proved cut-elimination for non-wellfounded proofs with additive and multiplicative connectives, $\mu\mathsf{MALL}^\infty$. In both cases, the proof relies on a generalization of the cut-rule, the *multicut* rule (which abstracts a portion of a proof tree constituted only of cut inferences see Fig. 1(b)) and on a reasoning by contradiction to prove that one can eliminate cuts at the limit of an infinite cut reduction sequence, while preserving the validity condition. Baelde *et al.* [3,4] use a so-called "locative" approach by modelling sequents as sets of formulas paired with addresses which determines uniquely the formula occurrence in a sequent and makes explicit the ancestor relation used to trace the progress along branches.

Moreover, the cut-elimination proof proceeds by a rather complex semantical, roundabout, argument relying on a soundness theorem.

In a slightly different direction, Das and Pous [15] proved a cut-elimination result for Kleene algebras and their variants. This can be viewed as a non-commutative version of intuitionistic MALL with a particular form of inductive construction, Kleene's star. Kuperberg et al [24] and more specifically Pinault's PhD thesis [27] as well as Das [13] examine non-wellfounded versions of System T based on [15], exploring the computational content of non-wellfounded proofs.

Neither Santocanale and Fortier's [20,29], nor Baelde et al. [3,4] works captured full linear logic: the exponentials are missing and the proofs cannot deal with them in a simple way. Indeed, the proof for $\mu\mathsf{ALL}$ strongly relies on the assumption the sequents are pairs of formulas ($A \vdash B$) while in $\mu\mathsf{MALL}$, the locative approach taken by Baelde et al. is not well-suited to work with structural rules: the extension of the proof would be possible though highly technical. In contrast, our motto in the present work is to work with traditional sequents as lists of formulas and to exploit the (co)inductive nature of LL exponentials.

On the (Co)Inductive Nature of Exponential Modalities in Linear Logic. The original works by Baelde and Miller on fixed-points in linear logic [2,5] focus on $\mu\mathsf{MALL}$ only and present an encoding of the exponential modalities of LL using least and greatest fixed points. Indeed, the ? and ! modalities have an infinitary character which is well-known from the early days of linear logic (see Section V.5 of Girard's seminal paper [21]) and which is in fact respectively inductive for ? and coinductive for !; let us discuss it briefly here.

One can decide to contract a ?-statement any *finite* number of times before it is ultimately weakened or derelicted. It is therefore natural to represent $?A$ with formula $?^\bullet A = \mu X.A \oplus (\bot \oplus (X \parr X))$: A allows for dereliction, $\bot$ for weakening and $X \parr X$ will regenerate, by unfolding, two copies of $?^\bullet A$, making the contraction derivable. The $\oplus$ and μ connectives respectively provide the ability to choose either of those three inferences and to repeat finitely this process.

On the other hand, a !-formula is a formula which, during cut-elimination, shall maintain a proper interaction with *any number* of contractions, weakenings or derelictions: a proof concluded with a promotion shall be able to react to any number of duplications or erasure before the promotion actually interact with a dereliction to open the *exponential box*: from that follows the coinductive character of $!A$ modelled as $!^\bullet A = \nu X.A \& (1 \& (X \otimes X))$.

As discussed above and formally established by Baelde and Miller [5], the exponential rules can be derived in the finitary sequent calculus $\mu\mathsf{MALL}$: to any LL provable sequent can be associated a provable $\mu\mathsf{MALL}$ sequent via the above translations of the exponentials. However, until now one can hardly say more about this embedding for two deep reasons: (i) the fundamental Seely isomorphisms which relate the additive and multiplicative versions of conjunction (resp. disjunction) are still derivable through this encoding but they are no more isomorphisms and (ii) on the provability level as well, the encoding is not faithful: the $\mu\mathsf{MALL}$ provability of the translation of an LL sequent s does not entail the LL provability of s itself (counter-example due to Das [14]). A contribution of the present paper is to put to work Baelde and Miller's encoding, showing that, in the case of non-wellfounded proofs, its structure is faithful enough to extract information of the cut-reduction behaviour of the logic.

Contributions and Organization of the Paper. The main result of this paper is a cut-elimination theorem for $\mu\mathsf{LL}^\infty$, the non-wellfounded sequent calculus for linear logic extended with least and greatest fixed points. Our proof proceeds by encoding LL exponentials in $\mu\mathsf{MALL}^\infty$ and studying $\mu\mathsf{LL}^\infty$ cut-reduction sequences through their simulation in $\mu\mathsf{MALL}^\infty$ which may be a *transfinite* sequence. In Sect. 2, we introduce our logics, $\mu\mathsf{MALL}^\infty$, $\mu\mathsf{LL}^\infty$, $\mu\mathsf{LK}^\infty$ and $\mu\mathsf{LJ}^\infty$, altogether with their non-wellfounded proofs and validity conditions. We adapt $\mu\mathsf{MALL}^\infty$ cut-elimination theorem [4] to our setting where sequents are lists and prove a compression lemma for $\mu\mathsf{MALL}^\infty$ transfinite cut-reduction sequences. Section 3 constitutes the core of our paper: we define $\mu\mathsf{LL}^\infty$ cut-reduction rules, study the encoding of exponentials in $\mu\mathsf{MALL}^\infty$ and show that $\mu\mathsf{LL}^\infty$ cut-reduction steps can be simulated in $\mu\mathsf{MALL}^\infty$, before proving $\mu\mathsf{LL}^\infty$ cut-elimination theorem. We prove in Sect. 4, as corollaries, cut-elimination for $\mu\mathsf{LK}^\infty$ and $\mu\mathsf{LJ}^\infty$, the non-wellfounded sequent-calculi for classical and intuitionistic logic. While our result for $\mu\mathsf{LL}^\infty$ shows that any fair cut-reduction sequence produces a cut-free valid proof, our two other cut-elimination results are truly (infinitary) weak-normalization results. We finally conclude in Sect. 5 with perspectives. A major advantage of our approach is that $\mu\mathsf{MALL}^\infty$ cut-elimination proof and, to some extent, the validity conditions, are regarded as black boxes, simplifying the presentation of the proof and making it reusable wrt. other validity conditions or $\mu\mathsf{MALL}^\infty$ proof techniques. An additional by-product of our approach, to the theory of linear logic, is to illustrate the fact that Seely isomorphisms are not needed to reach a cut-free proof.

A companion technical report containing additional details on the definitions as well as full proofs is available online [30].

2 Non-Wellfounded Proofs: $\mu\mathsf{MALL}^\infty$, $\mu\mathsf{LL}^\infty$, $\mu\mathsf{LK}^\infty$, $\mu\mathsf{LJ}^\infty$

2.1 μ-Signatures and Formulas

Definition 1 (μ-signature). *A* **μ-signature** *is a set $\mathcal{C}$ of pairs (c, p) of a connective symbol c and a tuple p of elements of $\{+, -\}$. The arity of c, $\mathsf{ar}(c)$, is the length of p, while the elements of p indicate the mono/antitonicity of the connective in the given component. The empty tuple will be denoted as $()$*[1].

Example 2 (μ-signature associated with $\mu\mathsf{MALL}, \mu\mathsf{LL}, \mu\mathsf{LK}, \mu\mathsf{LJ}$). The μ-signatures associated with $\mu\mathsf{MALL}, \mu\mathsf{LL}, \mu\mathsf{LK}, \mu\mathsf{LJ}$ are:

[1] μ-signature can be enriched to consider quantifiers but we restrict to the propositional case here.

- $\mu\mathsf{MALL}$ signature: $\mathcal{C}_{\mu\mathsf{MALL}} = \{⅋, \otimes, \oplus, \&\} \times \{(+,+)\} \cup \{0, 1, \top, \bot\} \times \{()\}$;
- one-sided $\mu\mathsf{LL}$ signature: $\mathcal{C}_{\mu\mathsf{LL}_1} = \mathcal{C}_{\mu\mathsf{MALL}} \cup \{!, ?\} \times \{(+)\}$;
- two-sided $\mu\mathsf{LL}$ signature: $\mathcal{C}_{\mu\mathsf{LL}_2} = \mathcal{C}_{\mu\mathsf{LL}_1} \cup \{(\multimap, (-,+)), (\cdot^\perp, (-))\}$;
- $\mu\mathsf{LK}$ signature: $\mathcal{C}_{\mu\mathsf{LK}} = \{\wedge, \vee\} \times \{(+,+)\} \cup \{(\Rightarrow, (-,+))\} \cup \{\top, \mathsf{F}\} \times \{()\}$;
- $\mu\mathsf{LJ}$ signature: $\mathcal{C}_{\mu\mathsf{LJ}} = \mathcal{C}_{\mu\mathsf{LK}}$.

Definition 3 (Pre-formulas). *Given a μ-signature $\mathcal{C}$, a countable set $\mathcal{V}$ of fixed-point variables and a set of atomic formulas $\mathcal{A}$, the set of* **pre-formulas** *over $\mathcal{S}$ is defined as the least set $\mathcal{F}_\mathcal{S}$ such that: (α) $\mathcal{A} \cup \mathcal{V} \subseteq \mathcal{F}_\mathcal{S}$; (β) for every c of arity n in $\mathcal{C}$ and $F_1, \ldots, F_n \in \mathcal{F}_\mathcal{S}$, $c(F_1, \ldots, F_n) \in \mathcal{F}_\mathcal{S}$; (γ) for every $X \in \mathcal{V}$ and pre-formula $F \in \mathcal{F}_\mathcal{S}$, $\mu X.F \in \mathcal{F}_\mathcal{S}$ and $\nu X.F \in \mathcal{F}_\mathcal{S}$.*

Definition 4 (Positive and negative occurrences of a variable). *Given a μ-signature $\mathcal{C}$ and a fixed-point variable $X \in \mathcal{V}$, one defines by induction on pre-formulas the fact, for X, to occur positively (resp. negatively) in a pre-formula : (α) X occurs positively in X; (β) X occurs positively (resp. negatively) in $c(F_1, \ldots, F_n)$, for $(c, p) \in \mathcal{C}$, if there is some $1 \leq i \leq n$ such that X occurs positively (resp. negatively) in F_i and $p_i = +$ or there is some $1 \leq i \leq n$ such that X occurs negatively (resp. positively) in F_i and $p_i = -$; (γ) X occurs positively (resp. negatively) in $\sigma Y.F$, for $\sigma \in \{\mu, \nu\}$, if $Y \neq X$ and X occurs positively (resp. negatively) in F.*

Definition 5 (μ-formula). *A μ-formula F over a signature $\mathcal{S}$ is a pre-formula containing no free fixed-point variable and such that for any sub-pre-formula of F of the form $\sigma X.G$, all occurrences of X in G are positive.*

Definition 6. *One-sided $\mu\mathsf{LL}$ formulas are those formulas defined over the signature $\mathcal{C}_{\mu\mathsf{LL}_1}$ together with a set of atomic formulas $\{a, a^\perp \mid a \in \mathcal{A}\}$ for a countable set $\mathcal{A}$. Negation $(\_)^\perp$ is the involution on pre-formulas defined by:*
$(a^\perp)^\perp = a$; $\bot^\perp = 1$; $\top^\perp = 0$; $(F ⅋ G)^\perp = F^\perp \otimes G^\perp$; $(F \oplus G)^\perp = F^\perp \& G^\perp$; $(?\,F)^\perp = !\,F^\perp$; $X^\perp = X$; $(\nu X.F)^\perp = \mu X.F^\perp$.

Definition 7 (μ-Fischer-Ladner subformulas). *Given a μ-signature $\mathcal{C}$ and a μ-formula F, $FL(F)$ is the least set of formulas such that:*

- *$F \in FL(F)$;*
- *$c(F_1, \ldots, F_n) \in FL(F) \Rightarrow F_1, \ldots, F_n \in FL(F)$ for $c \in \mathcal{C}$;*
- *$\sigma X.B \in FL(F) \Rightarrow B[\sigma X.B/X] \in FL(F)$ for $\sigma \in \{\mu, \nu\}$.*

Example 8. Let us consider $F = \nu X.((a ⅋ a^\perp) \otimes (!X \otimes \mu Y.X))$. $FL(F)$ is the set $\{F, (a ⅋ a^\perp) \otimes (!\,F \otimes \mu Y.F), a ⅋ a^\perp, a, a^\perp, !\,F \otimes \mu Y.F, !\,F, \mu Y.F\}$.

The finiteness of $FL(F)$ makes it an adequate notion of subformula:

Proposition 9. *For any μ-signature $\mathcal{S}$ and μ-formula F, $FL(F)$ is finite.*

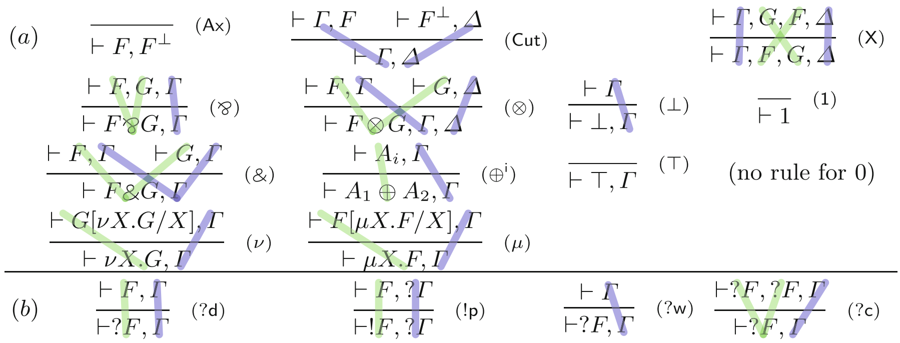

Fig. 2. (a) $\mu\mathsf{MALL}^\infty$ Inferences (b)$\mu\mathsf{LL}^\infty$ Exponential Inferences

2.2 μMALL$^\infty$, μLL$^\infty$, μLK$^\infty$ & μLJ$^\infty$ Inference Rules

Now, we define the inference rules associated with the above μ-signatures.

Definition 10 (Sequents and inferences). *A* **sequent** $s = \Gamma \vdash \Delta$ *over a* μ*-signature* $\mathcal{S}$ *is a pair of finite lists* Γ, Δ *of* $\mathcal{S}$*-formulas:* Γ *is the* **antecedent** *and* Δ *the* **succedent**. *An* **inference rule** r*, usually presented by a schema, is the data of a* **conclusion sequent**, **premise sequents**, *together with an* **ancestor relation** *relating formulas of the conclusion with formulas of the premises. A rule has a subset of distinguished* **principal formulas** *of the conclusion.*

Convention 1. *In the following, the ancestor relation will be depicted as colored lines joining related formulas. The* **principal** *formulas of an inference are the formulas which are explicitly spelled out in the conclusion sequent of an inference, not described via a context meta-variable. A formula occurrence of an inference is said to be* **active** *if it is principal or related to a principal formula by the ancestor relation. We will freely use the derived rules obtained by* **pre- and post-composition with the exchange rule**, *adapting the ancestry relation accordingly. Finally, for one-sided sequent calculi with an involutive negation* $\cdot^\perp$*, we may write* $\Gamma \vdash \Delta$ *for sequents* $\vdash \Gamma^\perp, \Delta$ *to clarify the computational behaviour of our examples (keeping the rule names unchanged).*

Definition 11 ($\mu\mathsf{MALL}^\infty, \mu\mathsf{LL}^\infty, \mu\mathsf{LK}^\infty, \mu\mathsf{LJ}^\infty$)**.** $\mu\mathsf{MALL}^\infty$ *inferences are given in Fig. 2. Those for one-sided* $\mu\mathsf{LL}^\infty$ *in Fig. 2(a) and 2(b). Those for* $\mu\mathsf{LK}^\infty$ *in Fig. 3. Those for* $\mu\mathsf{LJ}^\infty$ *by considering only inference from Fig. 3 where the succedent of both premises and conclusion sequents are singletons.*

In the above sequent calculi, every inference but the cut satisfies the subformula property *wrt.* FL-subformulae. The 2-sided $\mu\mathsf{LL}^\infty$ sequent calculus, over $\mathcal{C}_{\mu\mathsf{LL}_2}$, is defined as usual and not recalled here for space constraints.

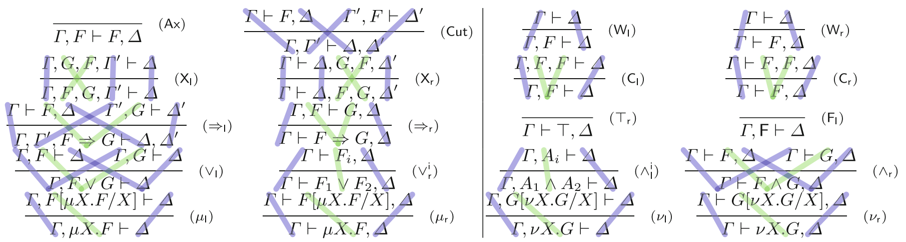

Fig. 3. μLK$^{\infty}$ Two-sided Inferences

2.3 Pre-proofs and Validity Conditions

Definition 12 (Pre-proofs). *The set* $\mathsf{P}_{\mathcal{S},\mathcal{I}}$ *of* $\mathcal{I}$-**pre-proofs** *associated to some of the above* μ*-signatures* $\mathcal{S}$ *and sets of inferences* $\mathcal{I}$ *is the set of* **finite or infinite** *trees whose nodes are correctly labelled with inferences and sequents.*

Pre-proofs are equipped with a metric structure as follows: we define a ***distance*** $\mathsf{d} : \mathsf{P}_{\mathcal{S},\mathcal{I}} \times \mathsf{P}_{\mathcal{S},\mathcal{I}} \to \mathbb{R}$ as: $\mathsf{d}(\pi, \pi') = 0$ if $\pi = \pi'$ and $\mathsf{d}(\pi, \pi') = 2^{-k}$ where k is the length of the shortest position where π and π' differ otherwise.

Example 13. Consider μLJ formulas $N = \mu X.\top \vee X$ and $S = \nu X.N \wedge X$. They represent nats and streams of nats. The μLJ$^{\infty}$ derivations of Fig. 4 respectively represent natural numbers, successor function, $n{::}n+1{::}n+2{::}\ldots$, the double functions and the function that builds a stream enumerating the natural numbers from its input: the cut-elimination process considered below will ensure that cutting π_{k} with π_{enum} will infinitarily reduce to $\pi^{\mathsf{k}}_{\mathsf{from}}$. Figure 5 shows other examples of μLL$^{\infty}$ pre-proofs, discussed with the validity condition.

The back-edge arrow to a lower sequent is notation to describe a fixed-point definition of the proof object: the subproof rooted in the source is equal to the proof rooted in the target. Trivially there is a unique solution.

In the following, we assume given a μ-signature $\mathcal{S}$ and a sequent calculus S for this signature and we shall define the valid S-proofs as a subset of S-pre-proofs, by introduction a ***thread-based validity condition***.

Definition 14 (Thread and validity). *Given a pre-proof* π *and an infinite branch* $\beta = (s_i)_{i\in\omega}$ *in* π*, a* **thread** *for* β *is an infinite sequence* θ *of formula occurrences such that* $\forall i \in \omega$, θ_i *is a formula occurrence of* s_i *and* θ_i *and* θ_{i+1} *are ancestor of each other.* θ *is said to* **support** β.

A formula F *is* **recurring** *in a thread* θ *of* β *if there are infinitely many i such that* θ_i *is an occurrence of* F.

A thread θ *is* **valid** *if it contains infinitely often the principal formula (occurrence) of a* ν *or* μ *rule and if the set of recurring formulas of* θ *has a least element (for the usual subformula ordering) which is (i) a* ν *formula when the least element occurs in the succedents or (ii) a* μ *formula if it occurs in the antecedents. A pre-proof is* **valid** *if all its infinite branches have a suffix supported by a valid thread.*

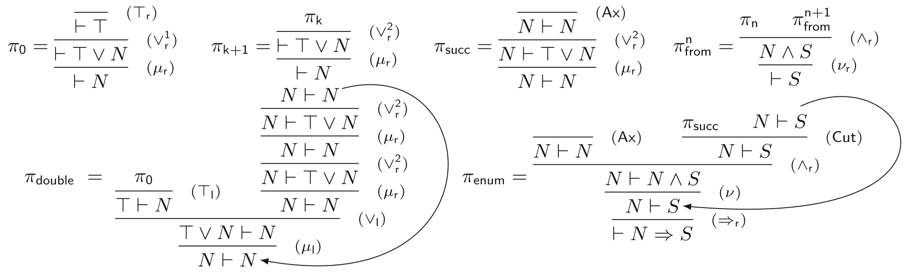

Fig. 4. Examples of $\mu\mathsf{LJ}^\infty$ pre-proofs

Example 15 ((Non-)valid pre-proofs). Consider the pre-proof in Fig. 5(a), with $F = \nu X.((a \parr a^\perp) \otimes (!X \otimes \mu Y.X))$ and $G = \mu Y.F$. The rightmost branch is supported by the green thread for which the least recurring formula is F, a ν-formula. All other branches are valid: this pre-proof is valid. Consider now the same pre-proof but with $F = \nu X.((a \parr a^\perp) \otimes (!X \otimes G))$ and $G = \mu Y.\nu X.((a \parr a^\perp) \otimes (!X \otimes Y))$. G is now a subformula of F and G, a μ-formula, and becomes the least recurring formula of all threads along the right-most infinite branch. This branch is invalid: the pre-proof is not a proof. Examples of $\mu\mathsf{LL}^\infty$ invalid pre-proofs are given in Fig. 1(a),5(b–c). In Fig. 4, π_{double} has a left thread on N while $\pi^{\mathsf{n}}_{\mathsf{from}}, \pi_{\mathsf{enum}}$ have right threads on S: they are valid.

2.4 Non-Locative $\mu\mathsf{MALL}^\infty$ Cut-Elimination Theorem

The validity condition defines a subset of pre-proofs, ensuring good properties for those non-wellfounded derivations that satisfy the validity condition. In this paper, we will mainly be interested in cut-elimination theorem, which was proved for $\mu\mathsf{MALL}^\infty$ [4] and that we review in this subsection. In [4], a somehow stronger result than cut-elimination is proved: infinitary strong normalization with respect to the class of *fair* reduction sequences.

The only new result developed in this subsection is the lifting of the occurrence-based cut-elimination result of [4] to our setting system, for which we first introduce the multicut inference and review the main multicut-reduction steps for $\mu\mathsf{MALL}^\infty$ before defining fair reductions. The cut-elimination results of [4, 20] do not rewrite cuts, *per se*, but subtrees of cuts in the form of an abstraction called ***multicut*** which is a variable arity inference defined as follows:

Definition 16. *The* **multicut inference** *is given by the data of (i) a conclusion sequent s, (ii) a non-empty list of premises $(s_1, \dots, s_n), n \geq 1$, (iii) an* **ancestor relation** *ι which is an injective map from the conclusion formulas to the premise formulas and relates identical formulas and additionally (iv) a* **cut-connectedness relation** *$\perp\!\!\!\perp$ which is a total, symmetric, binary relation among the formula occurrences of the premises which are not ancestor of a conclusion*

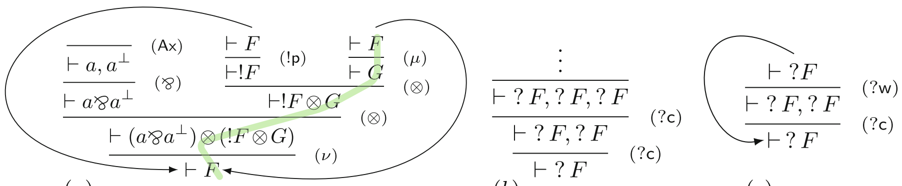

Fig. 5. Examples of valid and invalid pre-proofs

formula, which relates dual formulas[2] *and which satisfies a connectedness and acyclicity condition (see [3,4]). The multicut inference has no principal formula.*

We write this multicut rule as: $\dfrac{s_1 \quad \dots \quad s_n}{s}\ \mathsf{mcut}(\iota, \perp\!\!\!\perp)$.

In the following, we only consider $\mu\mathsf{MALL}^\infty$ pre-proofs with specific multicuts:

Definition 17 ($\mu\mathsf{MALL}^\infty_\mathsf{m}$). *$\mu\mathsf{MALL}^\infty_\mathsf{m}$ (pre)proofs are those (pre)proofs built from $\mu\mathsf{MALL}^\infty$ inferences and the multicut, such that (i) any branch contains at most one multicut and (ii) any occurrence of a cut is above a multicut inference.*

In the following, we shall always assume, even without mentioning it, that we consider proofs in $\mu\mathsf{MALL}^\infty_\mathsf{m}$ (as well as $\mu\mathsf{LL}^\infty_\mathsf{m}$, $\mu\mathsf{LJ}^\infty_\mathsf{m}$, $\mu\mathsf{LK}^\infty_\mathsf{m}$). We need the following definition (from [4]), identifying the premises of an mcut which are cut-connected to a given formula occurrence:

Definition 18 (Restriction of a mcut-context). *Consider an occurrence of a mcut $\dfrac{s_1 \quad \dots \quad s_n}{s}\ \mathsf{mcut}(\iota, \perp\!\!\!\perp)$ and assume s_i to be $\vdash F_1, \dots, F_k$. We define $\mathcal{C}_{F_j}, 1 \leq j \leq k$, to be the least set of sequent occurrences contained in $\{s_1, \dots, s_n\}$ such that:*
(i) If $\exists k, l$ such that $(k,l) \perp\!\!\!\perp (i,j)$, then $s_k \in \mathcal{C}_{F_j}$;
(ii) for any $k, k' \neq i$, if $s_k \in \mathcal{C}_{F_j}$ and $\exists l, l'$ such that $(k,l) \perp\!\!\!\perp (k', l')$, then $s_{k'} \in \mathcal{C}_{F_j}$.
We define $\mathcal{C}_\emptyset = \emptyset$ and $\mathcal{C}_{F,\Gamma} = \mathcal{C}_F \cup \mathcal{C}_\Gamma$.

When relating $\mu\mathsf{LL}^\infty$ and $\mu\mathsf{MALL}^\infty$ mcut-sequences below, we shall consider not only finite sequence nor ω-indexed sequences but also transfinite sequences. Those are sequences of triples of a proof, a redex and the position of the redex in the proof tree. A position p has a ***depth*** $\mathsf{dpth}(p)$ which is its length.

Definition 19 (mcut-reduction rules, transfinite sequences). *$\mu\mathsf{MALL}^\infty$ mcut-reduction sequences are directly adapted from [3,4]. Given an ordinal λ, a* **transfinite reduction sequence** *of length λ, or $\lambda\mathsf{TRS}$, is a λ-indexed sequence $(\pi_i, r_i, p_i)_{i \in \lambda}$ such that $\pi_i \longrightarrow^{p_i}_{r_i} \pi_{i+1}$, for any i such that $i+1 \in \lambda$, where the reduction occurs at position p_i reducing mcut-redex r_i.*

[2] When working with two-sided sequents, $\perp\!\!\!\perp$ will relate identical formulas, one in a succedent, the other in an antecedent.

Definition 20 (Weak and strong convergence). *A (transfinite) mcut reduction sequence* $(\pi_i, r_i, p_i)_{i\in\alpha}$ *is* **weakly converging** *if for any limit ordinal* $\beta \in \alpha$, $lim(\pi_i)_{i\in\beta} = \pi_\beta$. $(\pi_i, r_i, p_i)_{i\in\alpha}$ *is* **strongly converging** *if it is weakly converging and moreover for any limit ordinal* $\beta \in \alpha$, $lim(\mathsf{dpth}(p_i))_{i\in\beta} = +\infty$.

Remark 21. The cut-reduction rules preserve the property that every branch of a proof has at most one multicut inference: $\mu\mathsf{MALL}^\infty_\mathsf{m}$ is closed by cut-reduction.

A $\mu\mathsf{MALL}^\infty_\mathsf{m}$ pre-proof π may contain multiple cut-redexes: $\pi \longrightarrow^{p_1}_{r_1} \pi_1$ and $\pi \longrightarrow^{p_2}_{r_2} \pi_2$. As usual, a notion of residual associates to (r_1, p_1), a set of redexes of π_2, $(r_1, p_1)/(r_2, p_2)$ which is generalized to reduction sequences: $(r_1, p_1)/\sigma$.

Definition 22 (Fair reduction sequences). *A reduction sequence* $(\pi_i, r_i, p_i)_{i\in\omega}$ *is* **fair** *if for all* $i \in \omega$ *and* r, p *such that* $\pi_i \longrightarrow^p_r \pi'$ *there is some* $j \geq i$ *such that* π_j *does not contain a residual of* (r, p) *anymore.*

Theorem 23. *Every fair mcut-reduction sequence of* $\mu\mathsf{MALL}^\infty$ *valid proofs of* $\vdash \Gamma$ *(strongly) converges to a cut-free valid proof of* $\vdash \Gamma$.

2.5 Compressing Transfinite $\mu\mathsf{MALL}^\infty$ Cut-Reduction Sequences

In the previous paragraph, we introduced not only ω-indexed sequences, but transfinite $\mu\mathsf{MALL}^\infty$ cut-reduction sequences as we shall need reduction beyond ω when simulating $\mu\mathsf{LL}^\infty$ cut-elimination in $\mu\mathsf{MALL}^\infty$. We shall now prove that a class of transfinite $\mu\mathsf{MALL}^\infty$ mcut-reduction sequences can be compressed to $\omega\mathsf{TRS}$. This result can be viewed as adapting to our setting the compression lemma from infinitary rewriting [31], even though we require more on the structure of the compressed sequences as it will be useful to establish $\mu\mathsf{LL}^\infty$ cut-elimination.

Definition 24 (Depth-increasing). *A* $\mu\mathsf{MALL}^\infty$ *cut reduction sequence* $\sigma = (\pi_i, r_i, p_i)_{i\in\omega}$ *is* **depth-increasing** *if* $(\mathsf{dpth}(p_i))_{i\in\omega}$ *is (weakly) increasing.*

Definition 25 (Reordering). *An mcut reduction sequence* $\sigma = (\pi_i, r_i, p_i)_{i\in\alpha}$ *is a reordering of* $\sigma' = (\pi'_i, r'_i, p'_i)_{i\in\beta}$ *if there is a bijection* o *between* α *and* β *such that for any* $i \in \alpha$, $(r'_{o(i)}, p'_{o(i)}) = (r_i, p_i)$.

Proposition 26 (Compression lemma). *Let* $\sigma = (\pi_i, r_i, p_i)_{i\in\alpha}$ *be a strongly converging* $\mu\mathsf{MALL}^\infty$ *transfinite cut-reduction sequence. There exists a* $\mu\mathsf{MALL}^\infty$ *cut-reduction sequence* $\mathsf{Comp}(\sigma) = (\pi'_i, r'_i, p'_i)_{i\in\beta}$ *which is a reordering of* σ, *depth-increasing, strongly converging with the same limit as* σ *and such that* $\beta = \alpha$ *if* α *is finite and* $\beta = \omega$ *otherwise.*

3 Cut-Elimination Theorem for $\mu\mathsf{LL}^\infty$

The aim of this section is to prove the following theorem:

Theorem 27. *For any valid $\mu\mathsf{LL}^\infty$ proof π, fair $\mu\mathsf{LL}^\infty$ mcut-sequences from π converge to cut-free $\mu\mathsf{LL}^\infty$ proofs.*

The idea of the proof and outline of the present section are as follows:

1. We shall first define the cut-reduction rules for $\mu\mathsf{LL}^\infty$ by extending $\mu\mathsf{MALL}^\infty$ *multicut*-reduction with rules for reducing exponential cuts.
2. We then encode exponentials with fixed-points and translate $\mu\mathsf{LL}^\infty$ sequents (resp. pre-proofs) into $\mu\mathsf{MALL}^\infty$, preserving validity both ways.
3. We will then simulate $\mu\mathsf{LL}^\infty$ reductions in $\mu\mathsf{MALL}^\infty$: a single $\mu\mathsf{LL}^\infty$ step may require an infinite, or even transfinite, $\mu\mathsf{MALL}^\infty$ mcut-reduction sequence.
4. Finally, we will study the simulation of fair $\mu\mathsf{LL}^\infty$ cut-reduction sequences. Even though the simulation of $\mu\mathsf{LL}^\infty$ sequences builds transfinite sequences, we shall see that one can associate a(n almost) fair $\mu\mathsf{MALL}^\infty$ mcut-reduction sequence to any fair $\mu\mathsf{LL}^\infty$ mcut-reduction sequence, and conclude.
The next four subsections will closely follow the above pathway.

3.1 Cut-Elimination Rules for $\mu\mathsf{LL}^\infty$

$\mu\mathsf{LL}^\infty$ mcut-reduction is defined by extending $\mu\mathsf{MALL}^\infty$ multicut-reduction with the steps given in Fig. 6. The reduction rules for the exponentials assume a condition on the premisses of the multi-cut rule: all the proofs (hereditarily) cut-connected to some distinguished formula must have promotions as last inferences.

Definition 28 ((!p)-ready contexts). *A subset of the subproofs of a multicut is said to be* (!p)*-ready if all its elements are concluded with an* (!p) *rule.* $\mathcal{C}^!$ *will denote a* (!p)*-ready context and* $\mathcal{C}^!_\Gamma$ *a context restriction which is* (!p)*-ready.*

Remark 29. The condition for triggering the exponential key reductions (?w)/(!p) and (?c)/(!p) as well as the (!p)-commutation rule is expressed in terms of (!p)-readiness: for every ?-formula $?G$ in the context of a promotion which shall either commute or cut-reduce with a ?-rule, we require that $\mathcal{C}_{?G}$ is (!p)-ready.

3.2 Embedding $\mu\mathsf{LL}^\infty$ in $\mu\mathsf{MALL}^\infty$

To extend the cut-elimination result from $\mu\mathsf{MALL}^\infty$ to $\mu\mathsf{LL}^\infty$, we encode the exponential connectives using fixed points as follows, following Baelde [2]:

Definition 30. $?^\bullet(F) = \mu X.F \oplus (\bot \oplus (X \parr X))$; $!^\bullet(F) = \nu X.F \& (1 \& (X \otimes X))$

This straightforwardly induces an embedding of $\mu\mathsf{LL}^\infty$ into $\mu\mathsf{MALL}^\infty$:

Definition 31 (Embedding of $\mu\mathsf{LL}^\infty$sequents into $\mu\mathsf{MALL}^\infty$).

$(a)^\bullet$	$= a$	*if a is an atom*	$(\sigma X.F)^\bullet$	$= \sigma X.(F)^\bullet$	$, \sigma \in \{\mu, \nu\}$
$(u)^\bullet$	$= u$	*if* $u \in \{1, \bot, \top, 0\}$	$(?F)^\bullet$	$= ?^\bullet(F^\bullet)$	
$(A \star B)^\bullet$	$= (A)^\bullet \star (B)^\bullet$	*if* $\star \in \{\&, \oplus, \parr, \otimes\}$	$(!F)^\bullet$	$= !^\bullet(F^\bullet)$	

$$\dfrac{\mathcal{C} \qquad \dfrac{\vdash \Delta, F}{\vdash \Delta, ?F}\,(?\mathsf{d})}{\vdash \Sigma, ?F}\,\mathsf{mcut}(\iota, \perp\!\!\!\perp) \quad \underset{r}{\longrightarrow} \quad \dfrac{\dfrac{\mathcal{C} \qquad \vdash \Delta, F}{\vdash \Sigma, F}\,\mathsf{mcut}(\iota', \perp\!\!\!\perp)}{\vdash \Sigma, ?F}\,(?\mathsf{d})$$

$$\dfrac{\mathcal{C} \qquad \dfrac{\vdash \Delta, ?F, ?F}{\vdash \Delta, ?F}\,(?\mathsf{c})}{\vdash \Sigma, ?F}\,\mathsf{mcut}(\iota, \perp\!\!\!\perp) \quad \underset{r}{\longrightarrow} \quad \dfrac{\dfrac{\mathcal{C} \qquad \vdash \Delta, ?F, ?F}{\vdash \Sigma, ?F, ?F}\,\mathsf{mcut}(\iota', \perp\!\!\!\perp)}{\vdash \Sigma, ?F}\,(?\mathsf{c})$$

$$\dfrac{\mathcal{C} \qquad \dfrac{\vdash \Delta}{\vdash \Delta, ?F}\,(?\mathsf{w})}{\vdash \Sigma, ?F}\,\mathsf{mcut}(\iota, \perp\!\!\!\perp) \quad \underset{r}{\longrightarrow} \quad \dfrac{\dfrac{\mathcal{C} \qquad \vdash \Delta}{\vdash \Sigma}\,\mathsf{mcut}(\iota', \perp\!\!\!\perp)}{\vdash \Sigma, ?F}\,(?\mathsf{w})$$

$$\dfrac{\dfrac{\vdash F, ?\Gamma}{\vdash !F, ?\Gamma}\,(!\mathsf{p}) \qquad \mathcal{C}^!}{\vdash !F, ?\Sigma}\,\mathsf{mcut}(\iota, \perp\!\!\!\perp) \quad \underset{r}{\longrightarrow} \quad \dfrac{\dfrac{\vdash F, ?\Gamma \qquad \mathcal{C}^!}{\vdash F, ?\Sigma}\,\mathsf{mcut}(\iota', \perp\!\!\!\perp)}{\vdash !F, ?\Sigma}\,(!\mathsf{p})$$

$$\dfrac{\mathcal{C} \qquad \dfrac{\vdash F, \Gamma}{\vdash ?F, \Gamma}\,(?\mathsf{d}) \qquad \dfrac{\vdash F^\perp, ?\Delta}{\vdash !F^\perp, ?\Delta}\,(!\mathsf{p})}{\vdash \Sigma}\,\mathsf{mcut}(\iota, \perp\!\!\!\perp) \quad \underset{r}{\longrightarrow} \quad \dfrac{\mathcal{C} \qquad \vdash F, \Gamma \qquad \vdash F^\perp, ?\Delta}{\vdash \Sigma}\,\mathsf{mcut}(\iota, \perp\!\!\!\perp')$$

where $?F \perp\!\!\!\perp !F^\perp$ and $\perp\!\!\!\perp'$ coincides with $\perp\!\!\!\perp$ except for $F \perp\!\!\!\perp' F^\perp$.

$$\dfrac{\mathcal{C}_\Gamma \qquad \mathcal{C}^!_{?F} \qquad \dfrac{\vdash ?F, ?F, \Gamma}{\vdash ?F, \Gamma}\,(?\mathsf{c})}{\vdash \Gamma', ?\Sigma}\,\mathsf{mcut}(\iota, \perp\!\!\!\perp) \quad \underset{r}{\longrightarrow} \quad \dfrac{\dfrac{\mathcal{C}_\Gamma \qquad \mathcal{C}^!_{?F} \qquad \mathcal{C}^!_{?F} \qquad \vdash ?F, ?F, \Gamma}{\vdash \Gamma', ?\Sigma, ?\Sigma}\,\mathsf{mcut}(\iota', \perp\!\!\!\perp')}{\vdash \Gamma', ?\Sigma}\,(?\mathsf{c})^\star$$

where $\mathcal{C}^!_{?F} \neq \emptyset$, $\perp\!\!\!\perp'$ corresponds to $\perp\!\!\!\perp$ on $\mathcal{C}_\Gamma$ and is a "duplication" of $\perp\!\!\!\perp$ on $\mathcal{C}^!_{?F}$ and each copy of $?F$ is in $\perp\!\!\!\perp'$-relation with the corresponding copy of $!F^\perp$ in $\mathcal{C}^!_{?F}$.

$$\dfrac{\mathcal{C}_\Gamma \qquad \mathcal{C}^!_{?F} \qquad \dfrac{\vdash \Gamma}{\vdash ?F, \Gamma}\,(?\mathsf{w})}{\vdash \Gamma', ?\Sigma}\,\mathsf{mcut}(\iota, \perp\!\!\!\perp) \quad \underset{r}{\longrightarrow} \quad \dfrac{\dfrac{\mathcal{C}_\Gamma \qquad \vdash \Gamma}{\vdash \Gamma'}\,\mathsf{mcut}(\iota', \perp\!\!\!\perp')}{\vdash \Gamma', ?\Sigma}\,(?\mathsf{w})^\star$$

where $\mathcal{C}^!_{?F} \neq \emptyset$ and $\perp\!\!\!\perp'$ corresponds to the restriction of $\perp\!\!\!\perp$ on $\mathcal{C}_\Gamma, \Gamma$.

Fig. 6. $\mu\mathsf{LL}^\infty$ mcut-reduction rules

Definition 32 ($\mu\mathsf{MALL}^\infty$derivability of the exponentials). *$\mu\mathsf{LL}^\infty$ exponential rules can be encoded in $\mu\mathsf{MALL}^\infty$ as shown in Fig. 7. We denote the derivable rules by $?\mathsf{d}^\bullet$, $?\mathsf{c}^\bullet$, $?\mathsf{w}^\bullet$ and $!\mathsf{p}^\bullet$ respectively. ($!\mathsf{p}^\bullet$ uses a circular proof.)*

Proposition 33 (Preservation of validity). *π is a valid $\mu\mathsf{LL}^\infty$ proof of $\vdash \Gamma$ iff $\pi^\bullet$ is a valid $\mu\mathsf{MALL}^\infty$ proof of $\vdash \Gamma^\bullet$.*

Proof (Proof sketch). We simply relate the infinite branches in both pre-proofs. Assuming that π is valid, consider the special case of an infinite branch β of $\pi^\bullet$ that, when entering the encoding of a promotion, follows the left-most premise of the $(\&)$ rule. To such an infinite branch it is easy to associate an infinite branch b of π. b is valid and supported by a thread t with least formula $\nu X.F$. $(\nu X.F)^\bullet$ is the least recurring formula in the thread θ associated with t in β: β is valid.

Dereliction :

$$\dfrac{\dfrac{\vdash F,\Delta}{\vdash F\oplus(\bot\oplus(?^\bullet F \parr ?^\bullet F)),\Delta}\,(\oplus^1)}{\vdash ?^\bullet F,\Delta}\,(\mu)$$

Contraction :

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash ?^\bullet F, ?^\bullet F \Delta}{\vdash ?^\bullet F \parr ?^\bullet F,\Delta}\,(\parr)}{\vdash \bot\oplus(?^\bullet F \parr ?^\bullet F),\Delta}\,(\oplus^2)}{\vdash F\oplus(\bot\oplus(?^\bullet F \parr ?^\bullet F)),\Delta}\,(\oplus^2)}{\vdash ?^\bullet F,\Delta}\,(\mu)$$

Weakening :

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash \Delta}{\vdash \bot,\Delta}\,(\bot)}{\vdash \bot\oplus(?^\bullet F \parr ?^\bullet F),\Delta}\,(\oplus^1)}{\vdash F\oplus(\bot\oplus(?^\bullet F \parr ?^\bullet F)),\Delta}\,(\oplus^2)}{\vdash ?^\bullet F,\Delta}\,(\mu)$$

Promotion :

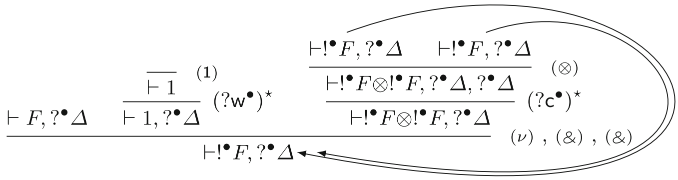

Fig. 7. μMALL$^\infty$ encoding of the exponential inferences

3.3 Simulation of μLL$^\infty$ Cut-Elimination Steps

Now we have to show that μLL$^\infty$ cut-elimination steps can be simulated by the previous encoding. *E.g.*, the commutation rule for dereliction is simulated by a $(\mu)/(\mathsf{Cut})$ commutation followed by a $(\oplus)/(\mathsf{Cut})$ commutation as follows:

$$\dfrac{\dfrac{\vdash F,G,\Gamma}{\vdash ?^\bullet F,G,\Gamma}\,(?\mathsf{d}^\bullet) \qquad \vdash G^\perp,\Delta}{\vdash ?^\bullet F,\Gamma,\Delta}\,(\mathsf{Cut}) \quad \longrightarrow^2 \quad \dfrac{\dfrac{\vdash F,G,\Gamma \qquad \vdash G^\perp,\Delta}{\vdash F,\Gamma,\Delta}\,(\mathsf{Cut})}{\vdash ?^\bullet F,\Gamma,\Delta}\,(?\mathsf{d}^\bullet)$$

The challenge is to show that the simulation of reductions also holds (i) for the reductions involving $(!p)$ as well as (ii) for reductions occurring ***above*** a promotion rule (aka. in a box) since the encoding of $[!p]$ uses an infinite, circular derivation. In the promotion commutation case for instance, we have:

$$\dfrac{\dfrac{\vdash F,?^\bullet G,?^\bullet\Gamma}{\vdash !^\bullet F,?^\bullet G,?^\bullet\Gamma}\,(!\mathsf{p}^\bullet) \qquad \dfrac{\vdash G^\perp,?^\bullet\Delta}{\vdash !^\bullet G^\perp,?^\bullet\Delta}\,(!\mathsf{p}^\bullet)}{\vdash !^\bullet F,?^\bullet\Gamma,?^\bullet\Delta}\,(\mathsf{Cut}) \quad \longrightarrow_\omega \quad \dfrac{\dfrac{\vdash F,?^\bullet G,?^\bullet\Gamma \qquad \dfrac{\vdash G^\perp,?^\bullet\Delta}{\vdash !^\bullet G^\perp,?^\bullet\Delta}\,(!\mathsf{p}^\bullet)}{\vdash F,?^\bullet\Gamma,?^\bullet\Delta}\,(\mathsf{Cut})}{\vdash !^\bullet F,?^\bullet\Gamma,?^\bullet\Delta}\,(!\mathsf{p}^\bullet)$$

Proposition 34. *Each μLL$^\infty$ mcut-reduction r can be simulated in μMALL$^\infty$ by a (possibly infinite) sequence of mcut-reductions, denoted $r^\bullet$.*

Remark 35. Conversely, one can wonder whether a possible reduction in $\pi^\bullet$ necessarily comes from the simulation of a reduction step in π. It is *almost* the case except when the reduction in $\pi^\bullet$ comes from exponential cuts requiring a (!p)-ready context (*ie.* (!p) commutation as well as (?w)/(!p) and (?c)/(!p) key cases, see above): in those cases indeed, if the context is "partially ready" – meaning that some, but not all, the required premises are promoted – a prefix of the sequence simulating the reduction step can indeed be performed, before being stuck. As consequence – and we shall exploit it in the next section when proving μLL$^\infty$ cut-elimination – the simulation of a fair reduction sequence is not necessarily fair, *but only as long as the above cases are involved*:

Proposition 36. *There exists a fair reduction ρ from some $\mu\mathsf{LL}^\infty$ (pre-)proof π such that $\rho^\bullet$ is an ω-indexed unfair $\mu\mathsf{MALL}^\infty$ cut-reduction sequence.*

3.4 Proof of $\mu\mathsf{LL}^\infty$ Cut-Elimination Theorem

$\mu\mathsf{LL}^\infty$ cut-elimination theorem follows from the following two lemmas:

Lemma 37. *Let π be a $\mu\mathsf{LL}^\infty$-proof of $\vdash \Gamma$ and $\sigma = (\pi_i, r_i, p_i)_{i\in\omega}$ a fair $\mu\mathsf{LL}^\infty$ cut-reduction sequence from π. σ converges to a cut-free $\mu\mathsf{LL}^\infty$-pre-proof of $\vdash \Gamma$.*

Lemma 38. *Let π be a $\mu\mathsf{LL}^\infty$ pre-proof of $\vdash \Gamma$ and let us consider a cut-reduction sequence $\sigma = (\pi_i, r_i, p_i)_{i\in\omega}$ in $\mu\mathsf{LL}^\infty$ from π that converges to a cut-free $\mu\mathsf{LL}^\infty$ pre-proof π'. $\sigma^\bullet$ is a strongly converging (possibly transfinite) sequence.*

Proof (Sketch for Thm. 27). Let π be a $\mu\mathsf{LL}^\infty$-proof of $\vdash \Gamma$ and $\sigma = (\pi_i, r_i, p_i)_{i\in\omega}$ be a fair $\mu\mathsf{LL}^\infty$ mcut-reduction sequence from π. Consider the associated (transfinite) $\mu\mathsf{MALL}^\infty$ mcut-reduction sequence $\sigma^\bullet$ from $\pi^\bullet$ obtained by simulation. By Lemma 37, σ converges (*strongly*) to a cut-free $\mu\mathsf{LL}^\infty$ pre-proof π'.

Let us prove that π' is valid. By Lemma 38, $\sigma^\bullet$ is a *transfinite* mcut-reduction sequence from $\pi^\bullet$ *strongly converging* to $\pi'^\bullet$. By Prop. 26, $\sigma^\bullet$ can be compressed into $\rho = (\pi'_i, r'_i, p'_i)_{i\in\omega}$ an *ω-indexed* depth-increasing $\mu\mathsf{MALL}^\infty$ mcut-reduction sequence which converges to $\pi'^\bullet$ and contains the same reductions as $\sigma^\bullet$. By Proposition 36, ρ may not be fair: this prevents us from concluding directly by Proposition 33 but we can still conclude. Let us consider ρ_f a fair reduction sequence obtained from ρ by reducing those redexes which cause the lack of fairness of ρ and let us consider the limit of ρ_f, π_f. To any infinite branch β of $\pi'^\bullet$, one can associate a branch β_f of π_f: it coincides with β except when the next inference of β_f is on a $(!\,F)^\bullet$ (in a sequent, say, $\vdash (!\,F)^\bullet, ?^\bullet\Delta^\bullet$ which is not principal along β). In that case, we expand β_f by following the unique premise of the (ν) rule, the second premise of the first $(\&)$ rule and the first premise of the second $(\&)$ rule, reaching $\vdash 1, ?^\bullet\Delta^\bullet$, in which case we know that the 1 is not principal (and never will be) and we follow back β. β_f has exactly the same threads as β: finite threads may only be extended *finitely* on occurrences of $(!\,F)^\bullet$. Since ρ_f is fair, β_f is valid and so is β.

We can then conclude that $\pi'^\bullet$ is cut-free and valid and, using preservation of validity (Proposition 33), that π' is a valid cut-free $\mu\mathsf{LL}^\infty$-proof. □

Infinitary cut-elimination for $\mu\mathsf{LL}^\infty$ two-sided sequent calculus is an easy corollary of Theorem 27. Indeed, fair cut-reduction sequences in two-sided $\mu\mathsf{LL}^\infty$ are mapped to fair reduction sequences in one-sided $\mu\mathsf{LL}^\infty$ from which follows:

Corollary 39. *Fair 2-sided $\mu\mathsf{LL}^\infty$ valid mcut-reduction sequences eliminate cuts.*

4 Cut-Elimination Theorem for $\mu\mathsf{LK}^\infty$ and $\mu\mathsf{LJ}^\infty$

Cut-elimination theorems for both $\mu\mathsf{LK}^\infty$ and $\mu\mathsf{LJ}^\infty$ can be established as corollaries of Theorem 27. For lack of space, we directly go to our results and postpone to future work a detailed study of the generalizations to non-wellfounded sequent calculi of the linear embeddings of LK and LJ into LL developed since Girard seminal paper. We shall comment on those translations in the conclusion.

4.1 $\mu\mathsf{LK}^\infty$ Cut-Elimination: Skeletons and Decorations

To any $\mu\mathsf{LL}^\infty$ formulas and $\mu\mathsf{LL}^\infty$ proofs, one can associate their skeletons, that is corresponding $\mu\mathsf{LK}^\infty$ formulas and proofs, after erasing of the linear information:

Definition 40 (Skeleton). $\mathsf{Sk}(A)$ *is defined by induction on* $A \in \mu\mathsf{LL}^\infty$*:*

$\mathsf{Sk}(A \otimes B) = \mathsf{Sk}(A) \wedge \mathsf{Sk}(B)$	$\mathsf{Sk}(A \parr B) = \mathsf{Sk}(A) \vee \mathsf{Sk}(B)$	$\mathsf{Sk}(!\,A) = \mathsf{Sk}(A)$
$\mathsf{Sk}(A \& B) = \mathsf{Sk}(A) \wedge \mathsf{Sk}(B)$	$\mathsf{Sk}(A \oplus B) = \mathsf{Sk}(A) \vee \mathsf{Sk}(B)$	$\mathsf{Sk}(?\,A) = \mathsf{Sk}(A)$
$\mathsf{Sk}(1) = \mathsf{Sk}(\top) = \top$	$\mathsf{Sk}(\bot) = \mathsf{Sk}(0) = \mathsf{F}$	$\mathsf{Sk}(a) = a$
$\mathsf{Sk}(A \multimap B) = \mathsf{Sk}(A) \Rightarrow \mathsf{Sk}(B)$	$\mathsf{Sk}(\sigma X.A) = \sigma X.\mathsf{Sk}(A)$	$\mathsf{Sk}(X) = X$

with $\sigma \in \{\mu, \nu\}$*.*

Given a 2-sided $\mu\mathsf{LL}^\infty$ *pre-proof* π *of* $\Gamma \vdash \Delta$ *with last rule* r *and premises* $(\pi_i)_{1 \le i \le n}$*,* $\mathsf{Sk}(\pi)$ *is the* $\mu\mathsf{LK}^\infty$ *pre-proof of* $\mathsf{Sk}(\Gamma) \vdash \mathsf{Sk}(\Delta)$ *defined corecursively, by case on* r*: (i) if* $r \in \{(!\mathsf{p}), (?\mathsf{d})\}$*,* $\mathsf{Sk}(\pi) = \mathsf{Sk}(\pi_1)$*; (ii) otherwise, apply the* $\mu\mathsf{LK}^\infty$ *rule corresponding to* r *with premises* $(\mathsf{Sk}(\pi_i))_{1 \le i \le n}$*.*

Proposition 41. $\mathsf{Sk}(\cdot)$ *transports valid* $\mu\mathsf{LL}^\infty$*-proofs to valid* $\mu\mathsf{LK}^\infty$ *proofs.*

$\mu\mathsf{LK}^\infty$ cut-elimination follows from the existence of $\mu\mathsf{LK}^\infty$ linear decorations.

Proposition 42. *For any* $\mu\mathsf{LK}^\infty$ *sequent* s *and any* $\mu\mathsf{LK}^\infty$ *proof* π *of* s*, there is a linear decoration of* π*, that is a* $\mu\mathsf{LL}^\infty$ *proof* π^d *such that* $\mathsf{Sk}(\pi^d) = \pi$*.*

Definition 43 ($\mu\mathsf{LK}^\infty$cut-reduction). $\mu\mathsf{LK}^\infty$ *mcut-reduction relation is defined as follows:* $\longrightarrow_{\mu\mathsf{LK}^\infty} = \{(\mathsf{Sk}(\pi), \mathsf{Sk}(\pi')) \mid \pi \longrightarrow_{mcut} \pi' \;\&\; \pi \neq \pi'\}$*.*

Theorem 44. $\mu\mathsf{LK}^\infty$ *enjoys cut-elimination.*

4.2 $\mu\mathsf{LJ}^\infty$ Cut-Elimination

The linear decoration for $\mu\mathsf{LJ}^\infty$ is simply Girard's call-by-value translation [21] extended to fixed-points on formulas and proofs as follows:

$$[X]^j = !\,X; \qquad [\mu X.F]^j = !\,\mu X.[F]^j; \qquad [\nu X.F]^j = !\,\nu X.[F]^j.$$

$$\left[\dfrac{\dfrac{\pi}{\Gamma \vdash F[\sigma X.F/X]}}{\Gamma \vdash \sigma X.F}\,(\sigma_r)\right]^j = \dfrac{\dfrac{\dfrac{[\pi]^j}{[\Gamma]^j \vdash [F]^j[\sigma X.[F]^j/X]}}{[\Gamma]^j \vdash \sigma X.[F]^j}\,(\sigma_r)}{[\Gamma]^j \vdash [\sigma X.F]^j}\,(!\mathsf{p}_r) \quad \text{and}$$

$$\left[\dfrac{\dfrac{\pi}{\Gamma, F[\sigma X.F/X] \vdash G}}{\Gamma, \sigma X.F \vdash G}\,(\sigma_l)\right]^j = \dfrac{\dfrac{\dfrac{[\pi]^j}{[\Gamma]^j, [F]^j[\sigma X.[F]^j/X], \vdash [G]^j}}{[\Gamma]^j, \sigma X.[F]^j \vdash [G]^j}\,(\sigma_l)}{[\Gamma]^j, [\sigma X.F]^j \vdash [G]^j}\,(!\mathsf{d}_l)\;.$$

The translation is consistent with $\mu\mathsf{LJ}^\infty$- and $\mu\mathsf{LL}^\infty$-positivity conditions.

Definition 45 (μILL$^\infty$). *μILL formulas are defined inductively as:*
$I, J ::= a \mid !X \mid I \multimap J \mid I \& J \mid I \oplus J \mid \top \mid 0 \mid \mu X.I \mid \nu X.I \mid !I.$

A μILL sequent is a sequent of μILL formulas with exactly one formula in the succedent. A μILL$^\infty$ proof is a μLL$^\infty$ proof containing only μILL sequents.

The translation preserves validity, following from $[X]^j = !X$, by induction.

Lemma 46. *The following hold:*

- *For any μLJ formulas A, B, $\sigma \in \{\mu, \nu\}$, $[A[\sigma X.B/X]]^j = [A]^j[\sigma X.[B]^j/X]$.*
- *For any μLJ formula A, $[A]^j$ is a μILL formula.*
- *If π is a μLJ$^\infty$ proof of $\Gamma \vdash F$, then $[\pi]^j$ is a μILL$^\infty$ proof of $[\Gamma]^j \vdash [F]^j$.*

On μILL$^\infty$ proofs, the skeletons of the previous section can be reused: $\mathsf{Sk}(\cdot)$ transports valid μILL$^\infty$ proof to valid μLJ$^\infty$ proofs. Moreover μILL$^\infty$ proofs are closed by μLL$^\infty$ cut-reductions from which we deduce, as for μLK$^\infty$, that:

Theorem 47. *μLJ$^\infty$ enjoys cut-elimination.*

5 Conclusion

In the present paper, we established several cut-elimination results for non-wellfounded proof systems for logics with least and greatest fixed-points expanding on previous works [4,20]: (i) for μMALL$^\infty$ with sequents as lists in contrast sequents as sets of locative occurrences [4], (ii) for the 1-sided and 2-sided sequent calculi of μLL$^\infty$, (iii) for μLK$^\infty$ and (iv) for μLJ$^\infty$. We also established additional results from a compression lemma for μMALL$^\infty$ strongly converging cut-reduction sequences to linear embeddings of μLK$^\infty$ and μLJ$^\infty$ into μLL$^\infty$.

On the Meaning and Expressiveness of Tree-Exponential Modalities. The proof of our main result proceeds by encoding LL exponentials in μMALL$^\infty$ following an encoding first considered by Baelde and Miller, and studying μLL$^\infty$ cut-reduction sequences through their simulation in μMALL$^\infty$, which was first conjectured in Doumane's thesis [18]. We think that the present paper does not only demonstrate the usefulness of the encoding but that it also suggests new questions. Indeed, this encoding has interesting features:

- this "rigid" tree-like exponential does not exhibit the Seely isomorphism but, even though those isomorphisms are common in axiomatizations of categorical models of linear logic, it is not necessary to have them as isomorphisms to build a denotational model of linear logic (that is, which quotients proofs up to cut-equivalence): the present work is actually an example of this fact. (They are crucial, though, to encode the λ-calculus in linear logic, as additional equations are needed, which are realized by Seely isos.)

– These exponentials allow for a realization of a somehow non-uniform promotion: indeed, while a proof of $\vdash\, !^\bullet F, ?^\bullet \Gamma$ has to provide a proof of $\vdash F, ?^\bullet \Gamma$, the circular definition of the promotion is not the only possible definition: one can consider as well promotions that would provide a distinct value each time a box is opened (*e.g.* a proof of $\vdash\, !^\bullet \mu X.1 \oplus X$ may provide distinct integers depending on how structural rules managed the resource). See [30] for a detailed discussion.

This tree-like exponential is being investigated with Ehrhard and Jafarrahmani.

Benefiting from Advances in Infinitary Rewriting. Our cut-elimination proof by encoding $\mu\mathsf{LL}^\infty$ into $\mu\mathsf{MALL}^\infty$ relies on a simulation of reductions sequences which makes use of transfinite reductions sequences and compression results. Those techniques are inspired and adapted from the literature on infinitary rewriting. We plan to make clearer the connection between non-wellfounded proof theory and infinitary rewriting in the future, even though in the present state it was not possible to readily apply results from infinitary rewriting such as the compression lemma which we has to reprove in our setting [31]. Moreover, we did not make use of coinductive formulations of infinitary rewriting [19]. That is another direction for future work: currently, we do not know how to use those formulations of infinitary rewriting because the sequences we consider by simulation are not given as (strongly) converging sequences. We plan to reconsider this and benefit from the coinductive approach to infinite reduction sequences.

On Linear Translations for Fixed-Point Logics and Non-Wellfounded Proofs. We obtained a cut-elimination theorem for $\mu\mathsf{LK}^\infty$ and $\mu\mathsf{LJ}^\infty$ thanks to linear translations which deserve some comments. While the linear translation used for $\mu\mathsf{LJ}^\infty$ is standard (it is a call-by-value translation dating back to Girard's seminal paper), the treatment of classical logic was more complex. Indeed, usual linear translation for classical logic introduce, at places, cuts. Due to the sensitivity of the straight-thread validity condition with respect to the presence of cuts in cycles, we could not use those translations. However, we plan to investigate whether a more standard translation can be used in the specific case of bouncing validity [3].

A Treatment of Cut-Elimination Which Is Agnostic to Validity Conditions. Last but not least, a major advantage of our approach is that $\mu\mathsf{MALL}^\infty$ cut-elimination proof and, to some extent, the validity conditions, are regarded as black boxes, simplifying the presentation of the proof and making it reusable wrt. other validity condition or $\mu\mathsf{MALL}^\infty$ proof techniques. The proof seems to be reusable easily with bouncing validity for instance (even though setting up an adequate definition of bouncing validity for $\mu\mathsf{LL}^\infty$ is quite tricky). A fragment which seems promising and that we wish to investigate in the near future, is $\mu\mathsf{MELL}^\infty$ equipped with bouncing validity [3].

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