A Critical Pair Criterion for Level-Commutation of Conditional Term Rewriting Systems

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Abstract. The rewrite relation of a conditional term rewriting system (CTRS) can be divided into a hierarchy of rewrite relations of term rewriting systems (TRSs) by the depth of the recursive use of rewrite relation in conditions; a CTRS is said to be level-confluent if each of these TRSs are confluent, and level-confluence implies confluence. We introduce level-commutation of CTRSs that extends the notion of level-confluence, in a way similar to extending confluence to commutation, and give a critical pair criterion for level-commutation of oriented CTRSs with extra variables (3-CTRSs). Our result generalizes a criterion for commutation of TRSs of (Toyama, 1987), and properly extends a criterion for level-confluence of orthogonal oriented 3-CTRSs (Suzuki et al., 1995). We also present criteria for level-confluence and commutation of join and semi-equational 3-CTRSs that may have overlaps.

Keywords: Level-commutation · Level-confluence · Commutation · Confluence · Critical pair · Conditional term rewriting systems

1 Introduction

Confluence, which guarantees unique results of computations, is an important property of term rewriting systems (TRSs). Commutativity between two TRSs is a natural generalization of confluence in the sense that self-commutativity coincides with confluence. It also allows to infer confluence of TRSs in a modular way—the union of two confluent TRSs is confluent if they commute.

Conditional term rewriting systems (CTRSs) are extensions of TRSs in which each rewrite rule can be equipped with conditions, where these conditions are supposed to be evaluated recursively using the underlying CTRS itself. Some type of CTRSs is known as a model of functional (and logic) programs. The underlying logic of TRSs is the equational logic, whereas the one of CTRSs is

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called the quasi-equational logic, constituting also an important class of systems for reasoning on a wider class of algebras.

From the computational point of view, the rewrite relation of a CTRS can be divided into a hierarchy of rewrite relations of TRSs by the depth of the recursive use of rewrite relation in conditions; a CTRS is said to be level-confluent if each of these TRSs are confluent. Suzuki et al. showed a criterion for orthogonal (i.e. left-linear non-overlapping) oriented CTRSs to be level-confluent [14]. Level-confluence implies confluence, and their result can be thought as a generalization of confluence of orthogonal TRSs. More crucially, since much fewer criterion have been obtained for CTRSs comparing to TRSs, level-confluence can be seen as an important approach to obtain confluence proofs of CTRSs. In contrast to TRSs, where many extensions of the orthogonality criterion for left-linear (possibly overlapping) TRSs to have confluence have been explored (e.g., [4,8,11,16]), similar extensions for CTRSs are not known. Similarly, several criteria for ensuring commutation for left-linear TRSs are known (e.g., [16,19]). Again, similar criteria for left-linear CTRSs are not known. In this paper, we give a criterion for a class of (possibly overlapping) left-linear oriented CTRSs, under which we prove level-commutation of such CTRSs. Our result is a generalization of the one given for TRSs in [16] and properly extends the result of [14] mentioned above. We also present criteria for level-confluence and commutation of left-linear join and semi-equational CTRSs that may have overlaps.

The rest of the paper is organized as follows. In the next section, we fix some notions and notations used in this paper, and explain two results that give starting points of our work. In Sect. 3, we present our main theorem on level-commutation of *oriented* CTRSs and its proof in detail, and explain relations to the previous results. We then give some results on *join* CTRSs and *semi-equational* CTRSs in Sect. 4. Section 5 concludes.

2 Preliminaries

We basically follow standard notions and notations (e.g., [3,10]). Below, we explain some key notions and fix notations that will be used in this paper, while omitting most of definitions of standard notions and notations.

We consider a set $\mathcal{F}$ of function symbols. The set of variables is denoted by $\mathcal{V}$ and the set of terms over $\mathcal{F}$ and $\mathcal{V}$ is by $\mathrm{T}(\mathcal{F},\mathcal{V})$. We sometimes specify a set $\mathcal{C} \subseteq \mathcal{F}$ of *constructors* to give the set of constructor terms $\mathrm{T}(\mathcal{C},\mathcal{V})$, i.e. terms over $\mathcal{C}$ and $\mathcal{V}$. The set of variables in a term t is denoted by $\mathcal{V}(t)$. A term t is *linear* if each variable occurs in t at most once; t is *ground* if no variable occurs in t. The size of a term t is denoted by $|t|$. The set of positions in a term t is denoted by $\mathrm{Pos}(t)$; the *root* position is written as ϵ. The symbol at a position $p \in \mathrm{Pos}(t)$ in a term t is written as $t(p)$. We put $\mathrm{Pos}_{\mathcal{F}}(t) = \{p \in \mathrm{Pos}(t) \mid t(p) \in \mathcal{F}\}$.

If $t = C[u]_p$ for a context C, we say u is a *subterm* of t (at a position $p \in \mathrm{Pos}(t)$). The subterm of t at a position $p \in \mathrm{Pos}(t)$ is written as $t|_p$. For terms $t = C[u]_p$ and s, the term $C[s]_p$ is denoted by $t[s]_p$. We speak of subterm *occurrences* when we consider subterms with their respective positions; see e.g.

[15] for a precise formalization of subterm occurrences. We will use capital letters $A, B, \ldots$ for subterm occurrences. For simplicity, a subterm occurrence A in a term is also treated as a term A (for example, we might write $\mathcal{V}(A)$). Suppose A, B are subterm occurrences in a term t. If $t = C[A]_p$ and $t = C'[B]_q$ with $p \leq q$ ($p < q$) we say that B is a (proper) subterm occurrence in a subterm occurrence A and write $B \subseteq A$ ($B \subset A$, respectively). Overlaps on subterm occurrences will be used to give a notion of weight on which our induction proof works.

A *term rewriting system* (*TRS*, for short) $\mathcal{R}$ is a set of *rewrite rules*, where each *rewrite rule* $l \to r$ satisfies the conditions $l \notin \mathcal{V}$ and $\mathcal{V}(r) \subseteq \mathcal{V}(l)$. Rewrite rules are identified modulo renaming. A TRS $\mathcal{R}$ is *left-linear* if l is linear for each $l \to r \in \mathcal{R}$. We write $s \to_{\mathcal{R}}^{p} t$ if $s|_p$ is the redex of this rewrite step; we also write $s \xrightarrow{A}_{\mathcal{R}} t$ to indicate the redex occurrence A of this rewrite step. The relation $\to_{\mathcal{R}}$ over terms is called the *rewrite relation* of $\mathcal{R}$, and its reflexive transitive closure is denoted by $\xrightarrow{*}_{\mathcal{R}}$. A *reduction* is a successive sequence of rewrite steps $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} \cdots \to_{\mathcal{R}} t_n$, where n is the *length* of this reduction. When no confusion arises, a reduction $s \to_{\mathcal{R}} \cdots \to_{\mathcal{R}} t$ is written as $s \xrightarrow{*}_{\mathcal{R}} t$ for brevity, whose length is denoted by $|s \xrightarrow{*}_{\mathcal{R}} t|$. We have a *parallel rewrite step* $s \mathrel{+\!\!\!\!\!\to\!\!\!\!\!\to}_{\mathcal{R}} t$ if $s = C[A_1, \ldots, A_n]$, $t = C[B_1, \ldots, B_n]$ ($n \geq 0$) for some context C and subterm occurrences A_i, B_i such that $A_i \to_{\mathcal{R}}^{\epsilon} B_i$ for all $i = 1, \ldots, n$; this rewrite step is written as $s \overset{A_1,\ldots,A_n}{\mathrel{+\!\!\!\!\!\to\!\!\!\!\!\to}}_{\mathcal{R}} t$ to indicate the redex occurrences $A_1, \ldots, A_n$.

A relation $\to$ is *confluent* if $\xleftarrow{*} \circ \xrightarrow{*} \subseteq \xrightarrow{*} \circ \xleftarrow{*}$; A TRS $\mathcal{R}$ is confluent if so is its rewrite relation $\to_{\mathcal{R}}$. Relations $\to$ and $\rightsquigarrow$ *commute* (or, are *commutative*) if $\xleftarrow{*} \circ \overset{*}{\rightsquigarrow} \subseteq \overset{*}{\rightsquigarrow} \circ \xleftarrow{*}$; TRSs $\mathcal{R}$ and $\mathcal{S}$ commute if so do their rewrite relations $\to_{\mathcal{R}}$ and $\to_{\mathcal{S}}$. Clearly, self-commutativity equals confluence, and from a sufficient criterion for commutativity the one for confluence naturally arises.

Let $l_1 \to r_1$ and $l_2 \to r_2$ be rewrite rules so that their sets of variables are renamed to be disjoint. If a non-variable subterm $l_2|_p$ of l_2 satisfies $l_2|_p\sigma = l_1\sigma$ for some substitution σ, we say that $l_1 \to r_1$ *overlaps on* $l_2 \to r_2$ (at p), provided that $p \neq \epsilon$ for the case $l_1 \to r_1$ and $l_2 \to r_2$ are identical. Suppose $l_1 \to r_1$ overlaps on $l_2 \to r_2$ at p and σ is an mgu of $l_2|_p$ and l_1. Then the pair $\langle l_2[r_1]_p\sigma, r_2\sigma\rangle$ is called a *critical pair* (obtained from that overlap); the pair is called *outer* if $p = \epsilon$ and is called *inner* if $p > \epsilon$. The set of critical pairs from overlaps of rules of $\mathcal{R}$ is denoted by $CP(\mathcal{R})$; the set of outer (inner) critical pairs are denoted by $CP_{out}(\mathcal{R})$ (resp. $CP_{in}(\mathcal{R})$). Let $\mathcal{R}, \mathcal{S}$ be TRSs. The set of critical pairs obtained from overlaps of $l_1 \to r_1 \in \mathcal{R}$ on $l_2 \to r_2 \in \mathcal{S}$ is denoted by $CP(\mathcal{R}, \mathcal{S})$. The sets $CP_{out}(\mathcal{R}, \mathcal{S})$ and $CP_{in}(\mathcal{R}, \mathcal{S})$ are defined similarly. We are now ready to state a sufficient criterion for commutativity of TRSs.

Proposition 1 ([16])**.** *Let $\mathcal{R}$ and $\mathcal{S}$ be left-linear TRSs. If both of the following conditions are satisfied, then $\mathcal{R}$ and $\mathcal{S}$ commute:*

1. *for any $\langle p, q\rangle \in CP(\mathcal{R}, \mathcal{S})$, $p \mathrel{+\!\!\!\!\!\to\!\!\!\!\!\to}_{\mathcal{S}} \circ \xleftarrow{*}_{\mathcal{R}} q$, and*
2. *for any $\langle q, p\rangle \in CP_{in}(\mathcal{S}, \mathcal{R})$, $q \mathrel{+\!\!\!\!\!\to\!\!\!\!\!\to}_{\mathcal{R}} p$ holds.*

The above criterion for commutativity arises a criterion for confluence: a left-linear TRS $\mathcal{R}$ is confluent if (1) for any $\langle p, q\rangle \in CP_{out}(\mathcal{R})$, $p \mathrel{+\!\!\!\!\!\to\!\!\!\!\!\to}_{\mathcal{R}} \circ \xleftarrow{*}_{\mathcal{R}} q$, and (2) for any $\langle q, p\rangle \in CP_{in}(\mathcal{R})$, $q \twoheadrightarrow\!\!\!\rightarrow_{\mathcal{R}} p$ holds. Note here in the condition (1), considering $\langle p, q\rangle \in CP_{out}(\mathcal{R})$ is sufficient, instead of considering $\langle p, q\rangle \in CP(\mathcal{R})$, because of the presence of condition (2).

A (directed) equation is an ordered pair $\langle u, v\rangle$ of terms, written as e.g. $u \approx v$. A *conditional rewrite rule* has the form $l \to r \Leftarrow u_1 \approx v_1, \ldots, u_k \approx v_k$ where $l \notin \mathcal{V}$; here, $u_1 \approx v_1, \ldots, u_k \approx v_k$ is a sequence of (directed) equations, called the *conditional part* of the rule. Often, we will use a meta-variable, say c, to denote the conditional part of the rule. Let $c = u_1 \approx v_1, \ldots, u_k \approx v_k$. Then, for any given substitution σ, we put $c\sigma = u_1\sigma \approx v_1\sigma, \ldots, u_k\sigma \approx v_k\sigma$. Also, we write e.g. $\mathcal{V}(l, c)$ to denote the set of variables occurring in l and c. We often also treat c as a set $\{u_1 \approx v_1, \ldots, u_k \approx v_k\}$ so as to write $u \approx v \in c$, $c\sigma \subseteq \rightsquigarrow$, etc., whose meaning should be apparent. The empty sequence is also written as $\emptyset$, and $l \to r \Leftarrow \emptyset$ is abbreviated as $l \to r$.

Conditional term rewriting system (CTRS, for short) is a set of conditional rewrite rules. In the literature, CTRSs are categorized into several types of CTRSs according the way of interpreting the conditions of the rules used in the definition of their rewrite steps. A *rewrite step* of *oriented* CTRS $\mathcal{R}$ is defined via the following TRSs $\mathcal{R}_n$ ($n \in \mathbb{N}$), which are inductively given as follows: $\mathcal{R}_0 = \emptyset$, $\mathcal{R}_{n+1} = \{l\sigma \to r\sigma \mid l \to r \Leftarrow c \in \mathcal{R}, c\sigma \subseteq \xrightarrow{*}_{\mathcal{R}_n}\}$. A rewrite step $s \to_{\mathcal{R}} t$ of CTRS $\mathcal{R}$ is given as $s \to_{\mathcal{R}} t$ iff $s \to_{\mathcal{R}_n} t$ for some n. Note that $m \le n$ implies $\to_{\mathcal{R}_m} \subseteq \to_{\mathcal{R}_n}$. The smallest n such that $s \to_{\mathcal{R}_n} t$ is called the *level* of the rewrite step $s \to_{\mathcal{R}} t$. We also use the notation $\to_{\mathcal{R}_{<n}} = \bigcup_{i<n} \to_{\mathcal{R}_i}$. We will also write $\mathcal{R}_n \vdash c\sigma$ to denote $c\sigma \subseteq \xrightarrow{*}_{\mathcal{R}_n}$. Except Sect. 4, we will only consider oriented CTRSs in this paper, and thus let us postpone to mention about join or semi-equational CTRSs until Sect. 4. A CTRS $\mathcal{R}$ is *level-confluent* if TRSs $\mathcal{R}_n$ are confluent for all $n \ge 0$. One can naturally extend the notion of level-confluence, in the similar way extending confluence to commutation.

Definition 1 (Level-commutation). *CTRSs $\mathcal{R}$ and $\mathcal{S}$ are* level-commutative *if for any $m, n \ge 0$, $\xleftarrow{*}_{\mathcal{R}_m} \circ \xrightarrow{*}_{\mathcal{S}_n} \subseteq \xrightarrow{*}_{\mathcal{S}_n} \circ \xleftarrow{*}_{\mathcal{R}_m}$.*

Clearly, level-commutativity (level-confluence) implies commutativity (resp. confluence), and self-level-commutativity implies level-confluence.

A conditional rewrite rule $l \to r \Leftarrow c$ has *type 1* if $\mathcal{V}(r, c) \subseteq \mathcal{V}(l)$, *type 2* if $\mathcal{V}(r) \subseteq \mathcal{V}(l)$, *type 3* if $\mathcal{V}(r) \subseteq \mathcal{V}(l, c)$, and *type 4* if "true". A CTRS $\mathcal{R}$ has type n if all rules have type n; CTRSs of type n are also referred to as *n-CTRSs*. We will mainly deal with 3-CTRSs below. Variables occurring in r, c which is not contained in $\mathcal{V}(l)$ are often called *extra* variables.

We now explain some notions necessary to give a sufficient criterion for level-confluence [14]. A CTRS $\mathcal{R}$ is *properly oriented* if $\mathcal{V}(r) \not\subseteq \mathcal{V}(l)$ implies $\mathcal{V}(u_i) \subseteq \mathcal{V}(l) \cup \bigcup_{j=1}^{i-1} \mathcal{V}(v_j)$ for all $1 \le i \le k$, for any $l \to r \Leftarrow u_1 \approx v_1, \ldots, u_k \approx v_k \in \mathcal{R}$. A CTRS $\mathcal{R}$ is *right-stable* if, for all $l \to r \Leftarrow u_1 \approx v_1, \ldots, u_k \approx v_k \in \mathcal{R}$, (1) $(\mathcal{V}(l) \cup (\bigcup_{j=1}^{i-1} \mathcal{V}(u_j, v_j)) \cup \mathcal{V}(u_i)) \cap \mathcal{V}(v_i) = \emptyset$ for all $1 \le i \le k$ and (2) for any $1 \le i \le k$, v_i is either a linear constructor term or a ground $\mathcal{R}_u$-normal form, where the constructors are given by $\mathcal{C} = \mathcal{F} \setminus \{l(\epsilon) \mid l \to r \Leftarrow c \in \mathcal{R}\}$ and the (extended) TRS $\mathcal{R}_u$ is given by $\mathcal{R}_u = \{l \to r \mid l \to r \Leftarrow c \in \mathcal{R}\}$. A CTRS $\mathcal{R}$ is left-linear if l is linear for all $l \to r \Leftarrow c \in \mathcal{R}$. Let $l_1 \to r_1 \Leftarrow c_1$ and $l_2 \to r_2 \Leftarrow c_2$ be conditional rewrite rules so that their sets of variables are renamed to be disjoint. We say $l_1 \to r_1 \Leftarrow c_1$ overlaps on $l_2 \to r_2 \Leftarrow c_2$ (at p) if a non-variable subterm $l_2|_p$ of l_2 satisfies $l_2|_p\sigma = l_1\sigma$ for some substitution σ, provided that $p \neq \epsilon$ for the case $l_1 \to r_1 \Leftarrow c_1$ and $l_2 \to r_2 \Leftarrow c_2$ are identical. A CTRS $\mathcal{R}$ is *non-overlapping* if there is no overlap between rules of $\mathcal{R}$; A CTRS $\mathcal{R}$ is *orthogonal* if it is left-linear and non-overlapping.

Proposition 2 ([14]). *Let $\mathcal{R}$ be an orthogonal, properly oriented, right-stable 3-CTRS. Then, $\overset{*}{\leftarrow}_{\mathcal{R}_m} \circ \overset{*}{\rightarrow}_{\mathcal{R}_n} \subseteq \overset{*}{\rightarrow}_{\mathcal{R}_n} \circ \overset{*}{\leftarrow}_{\mathcal{R}_m}$ for any $m, n \geq 0$. In particular, $\mathcal{R}$ is level-confluent.*

3 Level-Commutation of Oriented CTRSs

Proposition 1 only deals with TRSs but its scope is not limited to orthogonal ones. On the other hand, Proposition 2 can deal with CTRSs (not only TRSs) but limited to only orthogonal case. Also Proposition 2 only claims on (level-)confluence, whereas Proposition 1 claims on commutativity. A natural question is whether we can unify these two propositions and how—we will focus on this question in the this section.

Our basic idea is to unify proofs of [16, Theorem 3.1] and [14, Theorem 4.6]. The basic scenario of the former proof is showing that $\mathrel{\leftarrow\!\shortparallel\!-}_{\mathcal{R}} \circ \mathrel{-\!\shortparallel\!\rightarrow}_{\mathcal{S}} \subseteq \mathrel{-\!\shortparallel\!\rightarrow}_{\mathcal{S}} \circ \overset{*}{\mathrel{\leftarrow\!\shortparallel\!-}}_{\mathcal{R}}$. In the latter, an extended parallel rewriting $\mathrel{\lhook\!\shortparallel\!\rightarrow}_{\mathcal{R}_n}$ of $\mathrel{-\!\shortparallel\!\rightarrow}_{\mathcal{R}}$ was introduced and they showed $\mathrel{\leftarrow\!\shortparallel\!\rhook}_{\mathcal{R}_m} \circ \mathrel{\lhook\!\shortparallel\!\rightarrow}_{\mathcal{R}_n} \subseteq \mathrel{\lhook\!\shortparallel\!\rightarrow}_{\mathcal{R}_n} \circ \mathrel{\leftarrow\!\shortparallel\!\rhook}_{\mathcal{R}_m}$. Naturally, our first attempt was to prove $\mathrel{\leftarrow\!\shortparallel\!\rhook}_{\mathcal{R}_m} \circ \mathrel{\lhook\!\shortparallel\!\rightarrow}_{\mathcal{S}_n} \subseteq \mathrel{\lhook\!\shortparallel\!\rightarrow}_{\mathcal{S}_n} \circ \overset{*}{\mathrel{\leftarrow\!\shortparallel\!\rhook}}_{\mathcal{R}_m}$. Examining the details, however, it turned out that this scenario does not work (induction does not work). Thus, our first key ingredient is to modify our proof scenario as showing:

$$\mathrel{\leftarrow\!\shortparallel\!\rhook}_{\mathcal{R}_m} \circ \mathrel{\lhook\!\shortparallel\!\rightarrow}_{\mathcal{S}_n} \subseteq \mathrel{\lhook\!\shortparallel\!\rightarrow}_{\mathcal{S}_n} \circ \overset{*}{\mathrel{\lhook\!\shortparallel\!\rightarrow}}_{\mathcal{S}_{<n}} \circ \overset{*}{\mathrel{\leftarrow\!\shortparallel\!\rhook}}_{\mathcal{R}_m} \qquad (*)$$

We now reason why this scenario is sound using an abstract setting.

Let $(\to_n)_{n \in \mathbb{N}}$ be an $\mathbb{N}$-indexed family of relations on a set X. We put $\to_{<n} = \bigcup_{i<n} \to_i$. We say $(\to_n)_{n \in \mathbb{N}}$ is *up-simulated* if $\overset{*}{\to}_{<n} \subseteq \to_n$ for any $n \in \mathbb{N}$.

Lemma 1. *Let $(\to_n)_{n \in \mathbb{N}}, (\rightsquigarrow_n)_{n \in \mathbb{N}}$ be up-simulated families of relations on a set X. Suppose that[1], for any $m, n \in \mathbb{N}$, $\leftarrow_m \circ \rightsquigarrow_n \subseteq \rightsquigarrow_n \circ \overset{*}{\rightsquigarrow}_{<n} \circ \overset{*}{\leftarrow}_m$. Then, for any $m, n \in \mathbb{N}$, we have (1) $\overset{*}{\leftarrow}_m \circ \rightsquigarrow_n \subseteq \rightsquigarrow_n \circ \overset{*}{\rightsquigarrow}_{<n} \circ \overset{*}{\leftarrow}_m$, (2) $\overset{*}{\leftarrow}_m \circ \rightsquigarrow_n \subseteq \overset{*}{\rightsquigarrow}_n \circ \overset{*}{\leftarrow}_m$ and (3) $\overset{*}{\leftarrow}_m \circ \overset{*}{\rightsquigarrow}_n \subseteq \overset{*}{\rightsquigarrow}_n \circ \overset{*}{\leftarrow}_m$.*

Proof. Use induction. Use (1) to show (2), and then (2) to (3). □

[1] The criterion has some similarity with the *decreasing diagrams*; however, because multiple $\to_m$-steps are allowed, it is not at all apparent (currently, to the authors) whether the criterion can be obtained via the decreasing diagrams.

Now let us adopt our abstract framework to CTRSs. Let $\mathcal{R}$ be a CTRS. The notion of extended parallel rewriting [14] is given as follows: we write $s \mathrel{\hookrightarrow\!\!\!\!\!\shortparallel\;}_{\mathcal{R}_n} t$ if $s = C[A_1, \ldots, A_p]$, $t = C[B_1, \ldots, B_p]$ $(p \geq 0)$ for some context C and subterm occurrences A_i, B_i such that either $A_i \to^{\epsilon}_{\mathcal{R}_n} B_i$ or $A_i \xrightarrow{*}_{\mathcal{R}_{<n}} B_i$ for all $i = 1, \ldots, p$. We put $\mathrel{\hookrightarrow\!\!\!\!\!\shortparallel\;}_{\mathcal{R}} = \bigcup_{n \geq 0} \mathrel{\hookrightarrow\!\!\!\!\!\shortparallel\;}_{\mathcal{R}_n}$, which is called the *extended parallel rewrite step* of $\mathcal{R}$. We will also write $s \overset{A_1,\ldots,A_p}{\mathrel{\hookrightarrow\!\!\!\!\!\shortparallel\;}}_{\mathcal{R}} t$ to indicate subterm occurrences $A_1, \ldots, A_p$.

Then, from the Lemma 1, it easily follows:

Lemma 2. *Let $\mathcal{R}, \mathcal{S}$ be CTRSs. Suppose $\mathrel{\hookleftarrow\!\!\!\!\!\shortparallel\;}_{\mathcal{R}_m} \circ \mathrel{\hookrightarrow\!\!\!\!\!\shortparallel\;}_{\mathcal{S}_n} \subseteq \mathrel{\hookrightarrow\!\!\!\!\!\shortparallel\;}_{\mathcal{S}_n} \circ \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\shortparallel\;}}_{\mathcal{S}_{<n}} \circ \xleftarrow{*}_{\mathcal{R}_m}$ for any $m, n \geq 0$. Then, for any m, n, we have $\overset{*}{\mathrel{\hookleftarrow\!\!\!\!\!\shortparallel\;}}_{\mathcal{R}_m} \circ \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\shortparallel\;}}_{\mathcal{S}_n} \subseteq \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\shortparallel\;}}_{\mathcal{S}_n} \circ \overset{*}{\mathrel{\hookleftarrow\!\!\!\!\!\shortparallel\;}}_{\mathcal{R}_m}$. Hence, for any m, n, we have $\xleftarrow{*}_{\mathcal{R}_m} \circ \xrightarrow{*}_{\mathcal{S}_n} \subseteq \xrightarrow{*}_{\mathcal{S}_n} \circ \xleftarrow{*}_{\mathcal{R}_m}$.*

Proof. Suppose $t_1 \xleftarrow{*}_{\mathcal{R}_m} t \xrightarrow{*}_{\mathcal{S}_n} t_2$. As $\to_{\mathcal{R}_k} \subseteq \mathrel{\hookrightarrow\!\!\!\!\!\shortparallel\;}_{\mathcal{R}_k}$ for each k we have $t_1 \overset{*}{\mathrel{\hookleftarrow\!\!\!\!\!\shortparallel\;}}_{\mathcal{R}_m} t \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\shortparallel\;}}_{\mathcal{R}_n} t_2$ (and similarly for $\mathcal{S}$). From the fact $\to_{\mathcal{R}_m} \subseteq \to_{\mathcal{R}_n}$ for $m < n$, it immediately follows that $(\mathrel{\hookrightarrow\!\!\!\!\!\shortparallel\;}_n)_{n \in \mathbb{N}}$ is up-simulated (again, similarly for $\mathcal{S}$). Thus, it follows $t_1 \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\shortparallel\;}}_{\mathcal{S}_n} t' \overset{*}{\mathrel{\hookleftarrow\!\!\!\!\!\shortparallel\;}}_{\mathcal{R}_m} t_2$ by using Lemma 1 and our hypothesis. Because $\overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\shortparallel\;}}_{\mathcal{R}_k} \subseteq \xrightarrow{*}_{\mathcal{R}_k}$ for each k (and similarly for $\mathcal{S}$), we obtain $t_1 \xrightarrow{*}_{\mathcal{S}_n} t' \xleftarrow{*}_{\mathcal{R}_m} t_2$. $\square$

It is now concluded from this lemma that our proof scenario $(*)$ works to obtain the level-confluence.

For our proof below, we need to use the induction hypothesis to claim a more general statement as in the above. The following lemma is presented for this purpose.

Lemma 3. *Let $\mathcal{R}, \mathcal{S}$ be CTRSs and $k \in \mathbb{N}$. Suppose $\mathrel{\hookleftarrow\!\!\!\!\!\shortparallel\;}_{\mathcal{R}_m} \circ \mathrel{\hookrightarrow\!\!\!\!\!\shortparallel\;}_{\mathcal{S}_n} \subseteq \mathrel{\hookrightarrow\!\!\!\!\!\shortparallel\;}_{\mathcal{S}_n} \circ \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\shortparallel\;}}_{\mathcal{S}_{<n}} \circ \xleftarrow{*}_{\mathcal{R}_m}$ for any m, n such that $m + n < k$. Then, for any m, n such that $m + n < k$, we have (1) $\overset{*}{\mathrel{\hookleftarrow\!\!\!\!\!\shortparallel\;}}_{\mathcal{R}_m} \circ \mathrel{\hookrightarrow\!\!\!\!\!\shortparallel\;}_{\mathcal{S}_n} \subseteq \mathrel{\hookrightarrow\!\!\!\!\!\shortparallel\;}_{\mathcal{S}_n} \circ \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\shortparallel\;}}_{\mathcal{S}_{<n}} \circ \overset{*}{\mathrel{\hookleftarrow\!\!\!\!\!\shortparallel\;}}_{\mathcal{R}_m}$, (2) $\overset{*}{\mathrel{\hookleftarrow\!\!\!\!\!\shortparallel\;}}_{\mathcal{R}_m} \circ \mathrel{\hookrightarrow\!\!\!\!\!\shortparallel\;}_{\mathcal{S}_n} \subseteq \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\shortparallel\;}}_{\mathcal{S}_n} \circ \overset{*}{\mathrel{\hookleftarrow\!\!\!\!\!\shortparallel\;}}_{\mathcal{R}_m}$ and (3) $\overset{*}{\mathrel{\hookleftarrow\!\!\!\!\!\shortparallel\;}}_{\mathcal{R}_m} \circ \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\shortparallel\;}}_{\mathcal{S}_n} \subseteq \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\shortparallel\;}}_{\mathcal{S}_n} \circ \overset{*}{\mathrel{\hookleftarrow\!\!\!\!\!\shortparallel\;}}_{\mathcal{R}_m}$.*

Proof. Use an abstract version of the lemma, which can be proved in the way similar to Lemma 1. $\square$

Our second key ingredient is the following alternative definition of conditional critical pairs.

Definition 2 (Condition-separated CCP). *Suppose $l_1 \to r_1 \Leftarrow c_1$ overlaps on $l_2 \to r_2 \Leftarrow c_2$ at p and σ is an mgu of $l_2|_p$ and l_1. Then the quadruple $\langle l_2[r_1]_p\sigma, r_2\sigma \rangle \Leftarrow \langle c_1\sigma, c_2\sigma \rangle$ is called a* (condition-separated) conditional critical pair (CCP, for short) *(obtained from that overlap); when $p = \epsilon$, the pair is called* outer *and $p > \epsilon$, the pair is called* inner*. The set of (outer, inner) critical pairs obtained from overlaps of $l_1 \to r_1 \Leftarrow c_1 \in \mathcal{R}$ on $l_2 \to r_2 \Leftarrow c_2 \in \mathcal{S}$ is denoted by $CCP(\mathcal{R}, \mathcal{S})$ (resp. $CCP_{out}(\mathcal{R}, \mathcal{S})$, $CCP_{in}(\mathcal{R}, \mathcal{S})$). The set of (outer, inner) critical pairs from overlaps of rules of $\mathcal{R}$ is denoted by $CCP(\mathcal{R})$ (resp. $CCP_{out}(\mathcal{R})$, $CCP_{in}(\mathcal{R})$).*

In most literature, we see that instead of distinguishing two sequences $c_1\sigma$ and $c_2\sigma$, the combined sequence of $c_1\sigma$ and $c_2\sigma$ is employed in the definition of CCPs. But, in our case where CTRSs $\mathcal{R}$ and $\mathcal{S}$ may be different, this distinction is important to state a precise condition of our theorem.

We now present one more preparation: the following lemma is used several times as a part of the proof of our main theorem—when the lemma is used in the proof of our main theorem, the assumption (†) of the lemma can be inferred from the induction hypothesis (of the proof of the main theorem), using Lemma 3.

Lemma 4. *Let $\mathcal{R}$ and $\mathcal{S}$ be 3-CTRSs and suppose that $\mathcal{R}$ is left-linear and right-stable. Suppose that $M = l\sigma$, $N = r\sigma$, $\mathcal{R}_{m-1} \vdash c\sigma$ with $l \to r \Leftarrow c \in \mathcal{R}$. Assume moreover that $M \overset{P_1,\ldots,P_p}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\|\;\,}}_{\mathcal{S}_n} P$ and $P_1, \ldots, P_p$ occurs in the substitution σ. Assume that (†) $\overset{*}{\mathrel{\hookleftarrow\!\!\!\!\!\!\!\|\;\,}}_{\mathcal{R}_i} \circ \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\|\;\,}}_{\mathcal{S}_j} \subseteq \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\|\;\,}}_{\mathcal{S}_j} \circ \overset{*}{\mathrel{\hookleftarrow\!\!\!\!\!\!\!\|\;\,}}_{\mathcal{R}_i}$ for any i, j such that $i + j < m + n$. Then, there exists Q such that $N \mathrel{\hookrightarrow\!\!\!\!\!\!\!\|\;\,}_{\mathcal{S}_n} Q$ and $P \to_{\mathcal{R}_m} Q$.*

Now we present our critical pair criterion for commutativity.

Theorem 1. *Let $\mathcal{R}$ and $\mathcal{S}$ be left-linear, properly oriented, right-stable 3-CTRSs. If the following conditions are satisfied, then $\mathcal{R}$ and $\mathcal{S}$ are level-commutative:*

1. *for any $\langle u, v\rangle \Leftarrow \langle c, c'\rangle \in CCP(\mathcal{R}, \mathcal{S})$, $m, n \geq 1$ and substitution ρ, if $c\rho \subseteq \overset{*}{\to}_{\mathcal{R}_{m-1}}$ and $c'\rho \subseteq \overset{*}{\to}_{\mathcal{S}_{n-1}}$ then $u\rho \mathrel{-\!\!\!\|\!\!\!\rightarrow}_{\mathcal{S}_n} \circ \overset{*}{\to}_{\mathcal{S}_{<n}} \circ \overset{*}{\leftarrow}_{\mathcal{R}_m} v\rho$, and*
2. *for any $\langle v, u\rangle \Leftarrow \langle c', c\rangle \in CCP_{in}(\mathcal{S}, \mathcal{R})$, $m, n \geq 1$ and substitution ρ, if $c\rho \subseteq \overset{*}{\to}_{\mathcal{R}_{m-1}}$ and $c'\rho \subseteq \overset{*}{\to}_{\mathcal{S}_{n-1}}$ then $v\rho \mathrel{-\!\!\!\|\!\!\!\rightarrow}_{\mathcal{R}_m} \circ \overset{*}{\to}_{\mathcal{R}_{<m}} u\rho$.*

Proof. Let $M \overset{A_1,\ldots,A_{\bar{m}}}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\|\;\,}}_{\mathcal{R}_m} N$ and $M \overset{B_1,\ldots,B_{\bar{n}}}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\|\;\,}}_{\mathcal{S}_n} P$. We show $N \mathrel{\hookrightarrow\!\!\!\!\!\!\!\|\;\,}_{\mathcal{S}_n} \circ \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\|\;\,}}_{\mathcal{S}_{<n}} Q$ and $P \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\|\;\,}}_{\mathcal{R}_m} Q$ for some Q. For the rewrite steps used in the critical pairs conditions above, note that $\mathrel{-\!\!\!\|\!\!\!\rightarrow}_{\ell} \circ \overset{*}{\to}_{<\ell} = \mathrel{\hookrightarrow\!\!\!\!\!\!\!\|\;\,}_{\ell} \circ \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\|\;\,}}_{<\ell}$ as well as $\overset{*}{\to}_{\ell} = \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\|\;\,}}_{\ell}$ for any ℓ. Let Γ and Δ be sets of subterm occurrences in the term M given as follows:

$$\begin{aligned} \Gamma &= \{A_i \mid \exists B_j.\ A_i \subset B_j\} \cup \{B_i \mid \exists A_j.\ B_i \subseteq A_j\} \\ \Delta &= \{A_i \mid \forall B_j.\ A_i \not\subset B_j\} \cup \{B_i \mid \forall A_j.\ B_i \not\subseteq A_j\} \end{aligned}$$

Thus, Γ consists of subterm occurrences A_i's that is a proper subterm occurrence of some B_j and subterm occurrences B_j's that is a subterm occurrence of some A_i; Δ consists of subterm occurrences A_i's and B_j's not contained in Γ. Clearly, for any $1 \leq i \leq \bar{m}$, either one of $A_i \in \Gamma$ or $A_i \in \Delta$ holds, and for any $1 \leq j \leq \bar{n}$, either one of $B_j \in \Gamma$ or $B_j \in \Delta$ holds. In the case $A_{i'}$ and $B_{j'}$ are the same subterm occurrence, we put $A_{i'}$ to Δ and $B_{j'}$ to Γ.

Δ denotes the set of maximal redexes occurrences in the following sense. Let $\Delta = \{M_1, \ldots, M_{\bar{p}}\}$. Then we have $M = C[M_1, \ldots, M_{\bar{p}}]$ for some context C. Furthermore, we have $N = C[N_1, \ldots, N_{\bar{p}}]$ and $P = C[P_1, \ldots, P_{\bar{p}}]$ for some $N_1, \ldots, N_{\bar{p}}, P_1, \ldots, P_{\bar{p}}$ such that $M_i \mathrel{\hookrightarrow\!\!\!\!\!\!\!\|\;\,}_{\mathcal{R}_m} N_i$, $M_i \mathrel{\hookrightarrow\!\!\!\!\!\!\!\|\;\,}_{\mathcal{S}_n} P_i$ $(i = 1, \ldots, \bar{p})$. Thus, it suffices to show for each M_i, there exists Q_i such that $N_i \mathrel{\hookrightarrow\!\!\!\!\!\!\!\|\;\,}_{\mathcal{S}_n} \circ \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\|\;\,}}_{\mathcal{S}_{<n}} Q_i$ and $P_i \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\|\;\,}}_{\mathcal{R}_m} Q_i$. On the other hand, Γ is used to count the size of overlaps and

is used to give the induction weight. Let $|\Gamma| = \sum_{D\in\Gamma} |D|$. Our proof proceeds on induction on lexicographic combination of $\langle m+n, |\Gamma|\rangle$.

The cases for $m = 0$ or $n = 0$ are easy, thus we consider the cases for $m > 0, n > 0$. We distinguish two cases:

1. Case $M_i \notin \{B_1, \ldots, B_{\bar{n}}\}$. Note that $M_i \in \{A_1, \ldots, A_{\bar{m}}\}$ and $M_i \subseteq B_j$ for no B_j. Let $\{B'_1, \ldots, B'_{\bar{q}}\} = \{B_j \mid 1 \le j \le \bar{n}, B_j \subset M_i\}$. Then we have $M_i = C_i[B'_1, \ldots, B'_{\bar{q}}]$ and $P_i = C_i[\tilde{B}'_1, \ldots, \tilde{B}'_{\bar{q}}]$ so that $M_i \overset{M_i}{\hookrightarrow\!\!\!\!\!\shortparallel\!\!\rightarrow}_{\mathcal{R}_m} N_i$ and $M_i \overset{B'_1,\ldots,B'_{\bar{q}}}{\hookrightarrow\!\!\!\!\!\shortparallel\!\!\rightarrow}_{\mathcal{S}_n} P_i$. We distinguish the cases.
   (a) Case $M_i \overset{*}{\rightarrow}_{\mathcal{R}_{m-1}} N_i$. Since $\overset{*}{\rightarrow}_{\mathcal{R}_{m-1}} \subseteq \overset{*}{\hookrightarrow\!\!\!\!\!\shortparallel\!\!\rightarrow}_{\mathcal{R}_{m-1}}$, we have $M_i \overset{*}{\hookrightarrow\!\!\!\!\!\shortparallel\!\!\rightarrow}_{\mathcal{R}_{m-1}} N_i$. Thus, the desired Q_i is obtained by induction hypothesis and Lemma 3.
   (b) Case $M_i \overset{M_i}{\rightarrow}_{\mathcal{R}_m} N_i$. Then $M_i = l\theta$, $N_i = r\theta$ and $\mathcal{R}_{m-1} \vdash c\theta$ for some $l \rightarrow r \Leftarrow c \in \mathcal{R}$ and θ. If all redex occurrences B'_j in M_i are contained in the substitution θ, then the desired Q_i exists by Lemmas 3, 4 and induction hypothesis. Suppose otherwise, i.e. there exists B'_j which is not contained in θ. Let $X = \{B'_j \mid 1 \le j \le \bar{q},\ B'_j \text{ is not contained in } \theta\}$ and $Y = \{B'_j \mid 1 \le j \le \bar{q},\ B'_j \text{ is contained in } \theta\}$. For each $B'_j \in X$, either $B'_j \overset{B'_j}{\rightarrow}_{\mathcal{S}_n} \tilde{B}'_j$ or $B'_j \overset{*}{\rightarrow}_{\mathcal{S}_{<n}} \tilde{B}'_j$. We distinguish two cases.
      i Case that there exists $B'_j \in X$ such that $B'_j \overset{B'_j}{\rightarrow}_{\mathcal{S}_n} \tilde{B}'_j$. W.l.o.g. suppose $j = 1$, i.e. $B'_1 \in X$ and $B'_1 \overset{B'_1}{\rightarrow}_{\mathcal{S}_n} \tilde{B}'_1$. Let $M_i \overset{B'_1}{\rightarrow}_{\mathcal{S}_n} \tilde{M}_i$. Note also here $\tilde{M}_i \overset{B'_2,\ldots,B'_{\bar{q}}}{\hookrightarrow\!\!\!\!\!\shortparallel\!\!\rightarrow}_{\mathcal{S}_n} P_i$. The proof of this case is illustrated in Fig. 1. Let $l' \rightarrow r' \Leftarrow c' \in \mathcal{S}$, $B'_1 = l'\theta'$ and $\mathcal{S}_{n-1} \vdash c'\theta'$. Then, since B'_1 is not contained in θ, $l \rightarrow r \Leftarrow c \in \mathcal{R}$ and $l' \rightarrow r' \Leftarrow c' \in \mathcal{S}$ overlap. Furthermore, as $B'_1 \subset M_i$, we have $\langle v, u\rangle \Leftarrow \langle c', c\rangle \in CCP_{in}(\mathcal{S}, \mathcal{R})$ and there exists a substitution θ'' such that $\tilde{M}_i = v\theta''$ and $N_i = u\theta''$. By our critical pair condition (2), we obtain $\tilde{M}_i \hookrightarrow\!\!\!\!\!\shortparallel\!\!\rightarrow_{\mathcal{R}_m} \tilde{Q}_i \overset{*}{\hookleftarrow\!\!\!\!\!\shortparallel\!\!\rightarrow}_{\mathcal{R}_{<m}} N_i$; let $\tilde{M}_i \overset{C_1,\ldots,C_{\bar{r}}}{\hookrightarrow\!\!\!\!\!\shortparallel\!\!\rightarrow}_{\mathcal{R}_m} \tilde{Q}_i$. Let $\Gamma' = \{C_i \mid \exists B'_j (j \neq 1).\ C_i \subset B'_j\} \cup \{B'_i \mid i \neq 1, \exists C_j.\ B'_i \subseteq C_j\}$. Occurrences in Γ' are distinct, and for any $\tilde{B} \in \Gamma'$, there exists B'_j $(2 \le j \le \bar{q})$ such that $\tilde{B} \subseteq B'_j$. Thus, $|\Gamma'| \le \sum_{j=2}^{\bar{q}} |B'_j|$ holds. Hence, we obtain $|\Gamma'| \le \sum_{j=2}^{\bar{q}} |B'_j| < \sum_{j=1}^{\bar{q}} |B'_j| \le |\Gamma|$. Thus, one can apply induction hypothesis to $\tilde{Q}_i \overset{C_1,\ldots,C_{\bar{r}}}{\hookleftarrow\!\!\!\!\!\shortparallel\!\!\leftarrow}_{\mathcal{R}_m} \tilde{M}_i \overset{B'_2,\ldots,B'_{\bar{q}}}{\hookrightarrow\!\!\!\!\!\shortparallel\!\!\rightarrow}_{\mathcal{S}_n} P_i$ so as to obtain $\tilde{Q}'_i, \tilde{P}_i$ such that $\tilde{Q}_i \hookrightarrow\!\!\!\!\!\shortparallel\!\!\rightarrow_{\mathcal{S}_n} \tilde{Q}'_i \overset{*}{\hookrightarrow\!\!\!\!\!\shortparallel\!\!\rightarrow}_{\mathcal{S}_{<n}} \tilde{P}_i$ and $P_i \overset{*}{\hookrightarrow\!\!\!\!\!\shortparallel\!\!\rightarrow}_{\mathcal{R}_m} \tilde{P}_i$. Since we have $N_i \overset{*}{\hookleftarrow\!\!\!\!\!\shortparallel\!\!\rightarrow}_{\mathcal{R}_{<m}} \tilde{Q}_i \hookrightarrow\!\!\!\!\!\shortparallel\!\!\rightarrow_{\mathcal{S}_n} \tilde{Q}'_i$, by applying induction hypothesis and Lemma 3, it follows that there exists $\tilde{N}_i$ such that $N_i \hookrightarrow\!\!\!\!\!\shortparallel\!\!\rightarrow_{\mathcal{S}_n} \circ \overset{*}{\hookrightarrow\!\!\!\!\!\shortparallel\!\!\rightarrow}_{\mathcal{S}_{<n}} \tilde{N}_i$ and $\tilde{Q}'_i \overset{*}{\hookrightarrow\!\!\!\!\!\shortparallel\!\!\rightarrow}_{\mathcal{R}_{<m}} \tilde{N}_i$. Then, by induction hypothesis and Lemma 3, it follows that there exists Q_i such that $\tilde{N}_i \overset{*}{\hookrightarrow\!\!\!\!\!\shortparallel\!\!\rightarrow}_{<\mathcal{S}_n} Q_i$ and $\tilde{P}_i \overset{*}{\hookrightarrow\!\!\!\!\!\shortparallel\!\!\rightarrow}_{\mathcal{R}_{<m}} Q_i$.
      ii Case that $B'_j \overset{*}{\rightarrow}_{\mathcal{S}_{n-1}} \tilde{B}'_j$ holds for any $B'_j \in X$. As $M_i \overset{B'_1,\ldots,B'_{\bar{q}}}{\hookrightarrow\!\!\!\!\!\shortparallel\!\!\rightarrow}_{\mathcal{S}_n} P_i$ and $B'_1, \ldots, B'_{\bar{q}}$ are parallel, we can first rewrite all $B'_j \in Y$ $(1 \le$

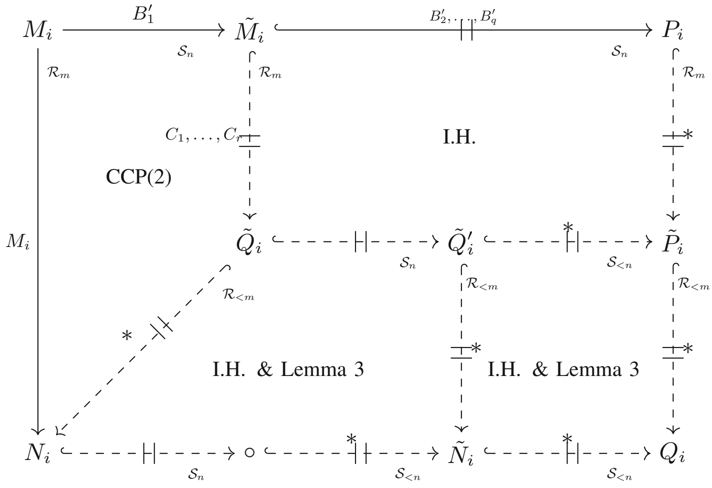

Fig. 1. Case 1.(b).i

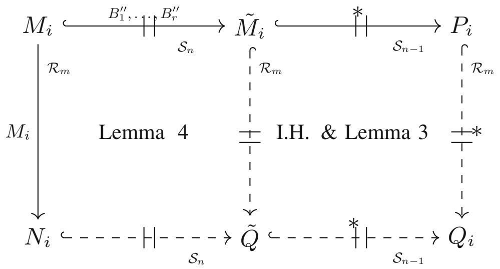

Fig. 2. Case 1.(b).ii

$j \le \bar{q}$). Namely, let $Y = \{B''_1, \ldots, B''_{\bar{r}}\}$, and we have $M_i \overset{B''_1,\ldots,B''_{\bar{r}}}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\shortparallel\;}}_{\mathcal{S}_n} \tilde{M}_i \overset{*}{\rightarrow}_{\mathcal{S}_{n-1}} P_i$. The proof of this case is illustrated in Fig. 2. Here, since each B''_j in contained in the substitution θ, one can use Lemma 4 to obtain $\tilde{Q}$ such that $N_i \mathrel{\hookrightarrow\!\!\!\!\!\!\!\shortparallel\;}_{\mathcal{S}_n} \tilde{Q}$ and $\tilde{M}_i \rightarrow_{\mathcal{R}_m} \tilde{Q}$. Now, since $\rightarrow_{\mathcal{R}_m} \subseteq \mathrel{\hookrightarrow\!\!\!\!\!\!\!\shortparallel\;}_{\mathcal{R}_m}$ and $\overset{*}{\rightarrow}_{\mathcal{S}_{n-1}} \subseteq \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\shortparallel\;}}_{\mathcal{S}_{n-1}}$, we have $\tilde{Q} \mathrel{\hookleftarrow\!\!\!\shortparallel\!\!\!\hookrightarrow}_{\mathcal{R}_m} \tilde{M}_i \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\shortparallel\;}}_{\mathcal{S}_{n-1}} P_i$. Then, using induction hypothesis and Lemma 3, we can obtain Q_i such that $\tilde{Q} \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\shortparallel\;}}_{\mathcal{S}_{n-1}} Q_i$, $P_i \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\shortparallel\;}}_{\mathcal{R}_m} Q_i$. As a side remark, we mention that our first key ingredient becomes necessary to solve this case.

2. Case $M_i \in \{B_1, \ldots, B_{\bar{n}}\}$. Let $\{A'_1, \ldots, A'_{\bar{q}}\} = \{A_j \mid 1 \le j \le \bar{n}, A'_j \subseteq M_i\}$. Then one can put $M_i = C_i[A'_1, \ldots, A'_{\bar{q}}]$, $N_i = C_i[\tilde{A}'_1, \ldots, \tilde{A}'_{\bar{q}}]$, $M_i \overset{A'_1,\ldots,A'_{\bar{q}}}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\shortparallel\;}}_{\mathcal{R}_m} N_i$ and $M_i \overset{M_i}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\shortparallel\;}}_{\mathcal{S}_n} P_i$. By definition, $M_i \overset{M_i}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\shortparallel\;}}_{\mathcal{S}_n} P_i$ is either of the form $M_i \overset{*}{\rightarrow}_{\mathcal{S}_{n-1}} P_i$ or $M_i \overset{M_i}{\rightarrow}_{\mathcal{S}_n} P_i$.

Suppose $M_i \overset{*}{\to}_{\mathcal{S}_{n-1}} P_i$. Then, we have $M_i \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\Vert}}_{\mathcal{S}_{n-1}} P_i$ and thus the desired Q_i exists by induction hypothesis and Lemma 3.
Thus, it remains to consider the case $M_i \overset{M_i}{\to}_{\mathcal{S}_n} P_i$. Then there exists $l' \to r' \Leftarrow c' \in \mathcal{S}$ and θ' such that $M_i = l'\theta'$, $P_i = r'\theta'$ and $c'\theta' \subseteq \overset{*}{\to}_{\mathcal{S}_{n-1}}$. We distinguish whether all redex occurrences A'_j in M_i are contained in θ' or not. If all redex occurrences A'_j in M_i are contained in θ', then using $\to_{\mathcal{S}_n} \subseteq \mathrel{\hookrightarrow\!\!\!\!\!\!\!\Vert}_{\mathcal{S}_n} \circ \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\Vert}}_{\mathcal{S}_{<n}}$ and $\mathrel{\hookrightarrow\!\!\!\!\!\!\!\Vert}_{\mathcal{R}_m} \subseteq \overset{*}{\to}_{\mathcal{R}_m}$, one obtains desired Q_i by Lemma 4.
So, let us consider there exists A'_j which is not contained in θ'. Let $X' = \{A'_j \mid 1 \leq j \leq \bar{q}, A'_j$ is not contained in $\theta'\}$ and $Y' = \{A'_j \mid 1 \leq j \leq \bar{q}, A'_j$ is contained in $\theta'\}$. Then for each $A'_j \in X'$, we have either $A'_j \overset{A'_j}{\to}_{\mathcal{R}_m} \tilde{A}'_j$, or $A'_j \overset{*}{\to}_{\mathcal{R}_{<m}} \tilde{A}'_j$. We distinguish two cases.

(a) Case that $A'_j \overset{A'_j}{\to}_{\mathcal{R}_m} \tilde{A}'_j$ for some $A'_j \in X'$. W.l.o.g. assume $j = 1$, i.e. $A'_1 \in X'$ and $A'_1 \overset{A'_1}{\to}_{\mathcal{R}_m} \tilde{A}'_1$. Then there exists $l \to r \Leftarrow c \in \mathcal{R}$ such that $A'_1 = l\theta$ and $c\theta \subseteq \overset{*}{\to}_{\mathcal{R}_{m-1}}$. We further distinguish two cases: (α) the case $A'_1 = M_i$ and $l \to r \Leftarrow c \in \mathcal{R}$ are $l' \to r' \Leftarrow c' \in \mathcal{S}$ are identical, and (β) the case $A'_1 \neq M_i$ or $l \to r \Leftarrow c \in \mathcal{R}$ and $l' \to r' \Leftarrow c' \in \mathcal{S}$ are distinct. We remark that a construction similar to the one in [14] will be used in case of (α) and that our assumption that $\mathcal{R}$ and $\mathcal{S}$ are properly oriented and right-stable will be used here.

i Case (α). Then we have $M_i = A'_1 \overset{A'_1}{\to}_{\mathcal{R}_m} \tilde{A}'_1 = N_i$ and $M_i \overset{M_i}{\to}_{\mathcal{S}_n} P_i$. By $l\theta = M_i = l\theta'$, $x\theta = x\theta'$ for any $x \in \mathcal{V}(l)$. We also have $\mathcal{R}_{m-1} \vdash c\theta$ and $\mathcal{S}_{n-1} \vdash c\theta'$. Thus, if $\mathcal{V}(r) \subseteq \mathcal{V}(l)$, then $r\theta = r\theta'$, and it suffices to take $r\theta$ as Q_i. Suppose otherwise, i.e. $\mathcal{V}(r) \nsubseteq \mathcal{V}(l)$. Below, let $c = s_1 \approx t_1, \ldots, s_j \approx t_j$ and $c_k = s_1 \approx t_1, \ldots, s_k \approx t_k$ $(1 \leq k \leq j)$. We now show there are substitution ρ_k $(k \in \{0, \ldots, j\})$ satisfying the following properties (a)–(c) by induction.

(a) $\rho_k = \theta = \theta'$ $[\mathcal{V}(l)]$.
(b) $dom(\rho_k) \subseteq \mathcal{V}(l) \cup \mathcal{V}(c_k)$.
(c) for any $x \in \mathcal{V}(l) \cup \mathcal{V}(c_k)$, we have $x\theta' \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\Vert}}_{\mathcal{R}_{m-1}} x\rho_k$ and $x\theta \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\Vert}}_{\mathcal{S}_{n-1}} x\rho_k$.

If $k = 0$ then take $\rho_0 = \theta|_{\mathcal{V}(l)}$, and (a)–(c) follow. Suppose $k > 0$. Since r contains an extra variable and $\mathcal{R}$ (or $\mathcal{S}$) is properly oriented, we have $\mathcal{V}(s_k) \subseteq \mathcal{V}(l) \cup \mathcal{V}(c_{k-1})$. Thus, by induction hypothesis on (c), we have $s_k\theta \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\Vert}}_{\mathcal{S}_{n-1}} s_k\rho_{k-1}$ and $s_k\theta' \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\Vert}}_{\mathcal{R}_{m-1}} s_k\rho_{k-1}$. Furthermore, we have $s_k\theta \overset{*}{\to}_{\mathcal{R}_{m-1}} t_k\theta$ and $s_k\theta' \overset{*}{\to}_{\mathcal{S}_{n-1}} t_k\theta'$ by $\mathcal{R}_{m-1} \vdash c\theta$ and $\mathcal{S}_{n-1} \vdash c\theta'$, respectively. Hence, $s_k\rho_{k-1} \overset{*}{\mathrel{\Vert\!\!\!\!\!\!\!\hookleftarrow}}_{\mathcal{S}_{n-1}} s_k\theta \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\Vert}}_{\mathcal{R}_{m-1}} t_k\theta$ and $t_k\theta' \overset{*}{\mathrel{\Vert\!\!\!\!\!\!\!\hookleftarrow}}_{\mathcal{S}_{n-1}} s_k\theta' \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\Vert}}_{\mathcal{R}_{m-1}} s_k\rho_{k-1}$. Then, by applying induction hypothesis and Lemma 3, we obtain q', r' such that $s_k\rho_{k-1} \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\Vert}}_{\mathcal{R}_{m-1}} q' \overset{*}{\mathrel{\Vert\!\!\!\!\!\!\!\hookleftarrow}}_{\mathcal{S}_{n-1}} t_k\theta$ and $t_k\theta' \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\Vert}}_{\mathcal{R}_{m-1}} r' \overset{*}{\mathrel{\Vert\!\!\!\!\!\!\!\hookleftarrow}}_{\mathcal{S}_{n-1}} s_k\rho_{k-1}$. Thus, one obtains $r' \overset{*}{\mathrel{\Vert\!\!\!\!\!\!\!\hookleftarrow}}_{\mathcal{S}_{n-1}} s_k\rho_{k-1} \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\Vert}}_{\mathcal{R}_{m-1}} q'$. Again, by applying induction hypoth-

esis and Lemma 3, we obtain s' such that $r' \overset{*}{\hookrightarrow\!\!\!\!\!\shortparallel}_{\mathcal{R}_{m-1}} s' \overset{*}{\shortparallel\!\!\!\!\!\hookleftarrow}_{\mathcal{S}_{n-1}} q'$. Thus, we have $t_k\theta \overset{*}{\hookrightarrow\!\!\!\!\!\shortparallel}_{\mathcal{S}_{n-1}} s'$ and $t_k\theta' \overset{*}{\hookrightarrow\!\!\!\!\!\shortparallel}_{\mathcal{R}_{m-1}} s'$.
We know that t_k is either a ground $\mathcal{R}_u$-normal form or a linear constructor term (w.r.t. $\mathcal{R}$) by the right-stability of $\mathcal{R}$, and that t_k is either a ground $\mathcal{S}_u$-normal form or a linear constructor term (w.r.t. $\mathcal{S}$) by the right-stability of $\mathcal{R}$. Suppose t_k is a ground $\mathcal{R}_u$-normal form or t_k is a ground $\mathcal{S}_u$-normal form. Then, $t_k\theta' = t_k\theta = t_k$ by $\mathcal{V}(t_k) = \emptyset$, and thus, $t_k = s'$ by $t_k\theta' \overset{*}{\hookrightarrow\!\!\!\!\!\shortparallel}_{\mathcal{R}_{m-1}} s'$. Furthermore, as we are assuming $\mathcal{V}(r) \nsubseteq \mathcal{V}(l)$, we know $\mathcal{V}(s_i) \subseteq \mathcal{V}(l) \cup \mathcal{V}(c_{i-1})$ from the proper-orientedness of $\mathcal{R}$ (or $\mathcal{S}$). Thus, $\mathcal{V}(l) \cup \mathcal{V}(c_k) = \mathcal{V}(l) \cup \mathcal{V}(c_{k-1})$. Hence, $\rho_k := \rho_{k-1}$ satisfies (a)–(c). Suppose otherwise. Then t_k is linear and is a constructor term w.r.t. both $\mathcal{R}$ and $\mathcal{S}$. Then, by $t_k\theta \overset{*}{\hookrightarrow\!\!\!\!\!\shortparallel}_{\mathcal{S}_{n-1}} s'$, there exists a substitution ρ such that $s' = t_k\rho$ and $dom(\rho) \subseteq \mathcal{V}(t_k)$ such that for any $x \in \mathcal{V}(t_k)$, $x\theta \overset{*}{\hookrightarrow\!\!\!\!\!\shortparallel}_{\mathcal{S}_{n-1}} x\rho$. Furthermore, by $t_k\theta' \overset{*}{\hookrightarrow\!\!\!\!\!\shortparallel}_{\mathcal{R}_{m-1}} s'$, there exists a substitution ρ' such that $s' = t_k\rho'$ and $dom(\rho') \subseteq \mathcal{V}(t_k)$ such that for any $x \in \mathcal{V}(t_k)$, $x\theta' \overset{*}{\hookrightarrow\!\!\!\!\!\shortparallel}_{\mathcal{R}_{m-1}} x\rho'$. Now, because $t_k\rho = s' = t_k\rho'$, we know $x\rho = x\rho'$ for any $x \in \mathcal{V}(t_k)$, and thus $\rho = \rho'$ from $dom(\rho), dom(\rho') \subseteq \mathcal{V}(t_k)$. We also have $\mathcal{V}(t_k) \cap (\mathcal{V}(l) \cup \mathcal{V}(c_{k-1})) = \emptyset$ by the right-stability of $\mathcal{R}$ (or $\mathcal{S}$), and thus, $dom(\rho) \cap dom(\rho_{k-1}) = \emptyset$. Hence, $\rho_k := \rho_{k-1} \cup \rho$ is a substitution, and ρ_k satisfies (a)–(c). This completes the induction proof for existence of substitutions ρ_k satisfying (a)–(c) $(1 \leq k \leq j)$.
Now consider the substitution ρ_j. Since $\mathcal{R}$ (and $\mathcal{S}$) is a 3-CTRS, we have $\mathcal{V}(r) \subseteq \mathcal{V}(l) \cup \mathcal{V}(c_j)$. Thus, by the condition (c), $N_i = r\theta \overset{*}{\hookrightarrow\!\!\!\!\!\shortparallel}_{\mathcal{S}_{n-1}} r\rho_j$ and $P_i = r\theta' \overset{*}{\hookrightarrow\!\!\!\!\!\shortparallel}_{\mathcal{R}_{m-1}} r\rho_j$ hold. Thus, taking $Q_i := r\rho_j$, and we have $N_i \overset{*}{\hookrightarrow\!\!\!\!\!\shortparallel}_{\mathcal{S}_{n-1}} Q_i$ and $P_i \overset{*}{\hookrightarrow\!\!\!\!\!\shortparallel}_{\mathcal{R}_{m-1}} Q_i$.

ii Case (β). Let $M_i \overset{A_1'}{\to}_{\mathcal{R}_m} \tilde{M}_i \overset{A_2',\ldots,A_{\bar{q}}'}{\hookrightarrow\!\!\!\!\!\shortparallel}_{\mathcal{S}_n} N_i$. The proof of this case is illustrated in Fig. 3 (left). Because there exists an overlap between $l \to r \Leftarrow c \in \mathcal{R}$ and $l' \to r' \Leftarrow c' \in \mathcal{S}$, there is substitution θ'' and a position $p \in \mathrm{Pos}_{\mathcal{F}}(l')$ such that $M_i = l'\theta'' = l'\theta''[l\theta'']_p = l\theta''[A_1']_p$. Then, $\tilde{M}_i = l'[r]_p\theta''$, $P_i = r'\theta''$, $\mathcal{R}_{m-1} \vdash c\theta''$ and $\mathcal{S}_{n-1} \vdash c'\theta''$. Then, there exists an CCP $\langle u, v\rangle \Leftarrow \langle d, d'\rangle \in CCP(\mathcal{R}, \mathcal{S})$, where $u = l'[r]_p\sigma$, $v = r'\sigma$, $d = c\sigma$ and $d' = c'\sigma$ for the mgu σ of $l'|_p$ and l. Then, as $(l'\theta'')_p = l\theta''$, we have $\theta'' = \rho \circ \sigma$ for some ρ. Thus, $P_i = r'\theta'' = (r'\sigma)\rho = v\rho$, $\tilde{M}_i = l'[r]_p\theta'' = (l'[r]_p\sigma)\rho = u\rho$, $\mathcal{R}_{m-1} \vdash d\rho$, and $\mathcal{S}_{n-1} \vdash d'\rho$. Hence, by our critical pair condition (2), $u\rho \hookrightarrow\!\!\!\!\!\shortparallel_{\mathcal{S}_n} \circ \overset{*}{\hookrightarrow\!\!\!\!\!\shortparallel}_{\mathcal{S}_{<n}} s$ and $v\rho \overset{*}{\hookrightarrow\!\!\!\!\!\shortparallel}_{\mathcal{R}_m} s$ for some s, and thus, by taking $\tilde{P}_i := s\rho$, we have $\tilde{M}_i \hookrightarrow\!\!\!\!\!\shortparallel_{\mathcal{S}_n} \tilde{P}_i' \overset{*}{\hookrightarrow\!\!\!\!\!\shortparallel}_{\mathcal{S}_{<n}} \tilde{P}_i$ and $P_i \overset{*}{\hookrightarrow\!\!\!\!\!\shortparallel}_{\mathcal{R}_m} \tilde{P}_i$ for some $\tilde{P}_i'$.
Suppose $\tilde{M}_i \overset{C_1,\ldots,C_{\bar{r}}}{\hookrightarrow\!\!\!\!\!\shortparallel}_{\mathcal{S}_n} \tilde{P}_i'$. Let $\Gamma' = \{A_i' \mid \exists C_j.A_i' \subset C_j\} \cup \{C_i \mid \exists A_j'.C_i \subseteq A_j'\}$. Occurrences in Γ' are distinct, and for any $\tilde{C} \in \Gamma'$, there exists A_j' $(2 \leq j \leq \bar{q})$ such that $\tilde{C} \subseteq A_j'$. Hence, $|\Gamma'| \leq$

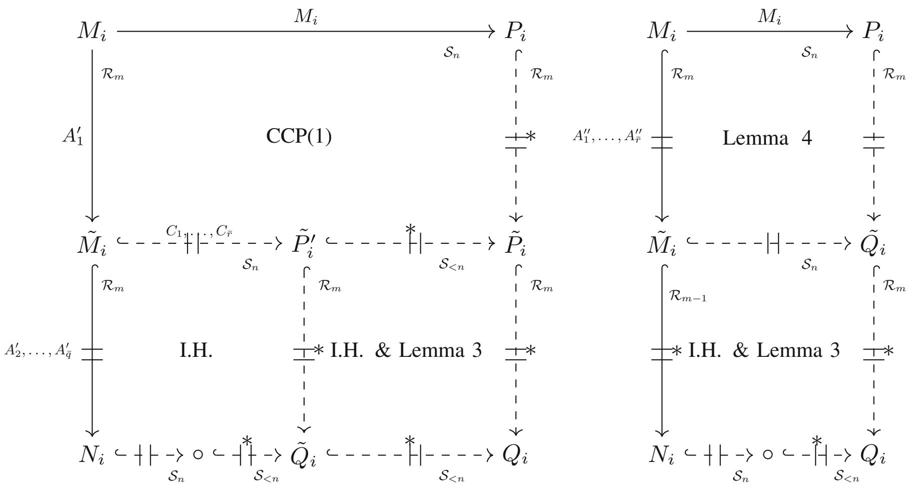

Fig. 3. Case 2.(a).ii (left) and Case 2.(b) (right)

$\sum_{j=2}^{\bar{q}} |A'_j| < \sum_{j=1}^{\bar{q}} |A'_j| \leq |\Gamma|$. Thus, one can apply induction hypothesis to obtain $\tilde{Q}_i$ such that $N_i \mathrel{\hookrightarrow\!\!\!\!\!\!\!\shortparallel\;\;}_{\mathcal{S}_n} \circ \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\shortparallel\;\;}}_{\mathcal{S}_{<n}} \tilde{Q}_i$ and $\tilde{P}'_i \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\shortparallel\;\;}}_{\mathcal{R}_m} \tilde{Q}_i$. By applying induction hypothesis and Lemma 3 once again, we know that there exists Q_i such that $\tilde{Q} \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\shortparallel\;\;}}_{\mathcal{S}_{<n}} Q_i \overset{*}{\mathrel{\leftarrow\!\!\!\!\!\shortparallel\!\!\!\!\!\hookrightarrow}}_{\mathcal{R}_m} \tilde{P}_i$.

(b) Case that $A'_j \xrightarrow{*}_{\mathcal{R}_{m-1}} \tilde{A}'_j$ for any $A'_j \in X'$. Since $M_i \overset{A'_1,\ldots,A'_{\bar{q}}}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\shortparallel\;\;}}_{\mathcal{R}_m} N_i$ and $A'_1,\ldots,A'_{\bar{q}}$ are parallel, one can rewrite $A'_j \in Y'$ first. That is, $M_i \overset{A''_1,\ldots,A''_{\bar{r}}}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\shortparallel\;\;}}_{\mathcal{R}_m} \tilde{M}_i \xrightarrow{*}_{\mathcal{R}_{m-1}} N_i$ where $Y' = \{A''_1,\ldots,A''_{\bar{r}}\}$. The proof of this case is illustrated in Fig. 3 (right). Then, as each A''_j is contained in θ', by Lemma 4, there exists $\tilde{Q}$ such that $\tilde{M}_i \rightarrow_{\mathcal{S}_n} \tilde{Q}$ and $P_i \mathrel{\hookrightarrow\!\!\!\!\!\!\!\shortparallel\;\;}_{\mathcal{R}_m} \tilde{Q}$. Furthermore, as $\rightarrow_{\mathcal{S}_n} \subseteq \mathrel{\hookrightarrow\!\!\!\!\!\!\!\shortparallel\;\;}_{\mathcal{S}_n}$ and $\xrightarrow{*}_{\mathcal{R}_{m-1}} \subseteq \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\shortparallel\;\;}}_{\mathcal{R}_{m-1}}$, one can apply induction hypothesis and Lemma 3 to $N_i \xleftarrow{*}_{\mathcal{R}_{m-1}} \tilde{M}_i \rightarrow_{\mathcal{S}_n} \tilde{Q}$ to obtain Q_i such that $N_i \mathrel{\hookrightarrow\!\!\!\!\!\!\!\shortparallel\;\;}_{\mathcal{S}_n} \circ \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\shortparallel\;\;}}_{\mathcal{S}_{<n}} Q_i$ and $\tilde{Q} \overset{*}{\mathrel{\hookrightarrow\!\!\!\!\!\!\!\shortparallel\;\;}}_{\mathcal{R}_m} Q_i$.

Finally, from Lemma 2 we conclude that $\mathcal{R}$ and $\mathcal{S}$ are level-commutative. □

A level-confluence criterion is obtained by taking $\mathcal{R} = \mathcal{S}$. Note that one can use CCP_{out} instead of CCP in the first condition, contrast to the commutativity criterion, as the second condition implies the part for $CCP_{in}(\mathcal{R})$ of it.

Corollary 1. *Let $\mathcal{R}$ be a left-linear, properly oriented, right-stable 3-CTRS. If the following conditions are satisfied, then $\mathcal{R}$ is level-confluent:*

1. *for any $\langle u, v\rangle \Leftarrow \langle c, c'\rangle \in CCP_{out}(\mathcal{R})$, $m, n \geq 1$ and substitution ρ, if $c\rho \subseteq \xrightarrow{*}_{\mathcal{R}_{m-1}}$ and $c'\rho \subseteq \xrightarrow{*}_{\mathcal{R}_{n-1}}$ then $u\rho \mathrel{\rightarrow\!\!\!\!\!\!\!\shortparallel\;\;}_{\mathcal{R}_n} \circ \xrightarrow{*}_{\mathcal{R}_{<n}} \circ \xleftarrow{*}_{\mathcal{R}_m} v\rho$, and*
2. *for any $\langle v, u\rangle \Leftarrow \langle c', c\rangle \in CCP_{in}(\mathcal{R})$, $m, n \geq 1$ and substitution ρ, if $c\rho \subseteq \xrightarrow{*}_{\mathcal{R}_{m-1}}$ and $c'\rho \subseteq \xrightarrow{*}_{\mathcal{R}_{n-1}}$ then $v\rho \mathrel{\rightarrow\!\!\!\!\!\!\!\shortparallel\;\;}_{\mathcal{R}_m} \circ \xrightarrow{*}_{\mathcal{R}_{<m}} u\rho$.*

Example 1. Let $\mathcal{R}$ and $\mathcal{S}$ be the following CTRSs:

$$\mathcal{R} = \left\{ \begin{array}{ll} \mathsf{p}(x) \to \mathsf{q}(x) & \\ \mathsf{r}(x) \to \mathsf{s}(\mathsf{p}(x)) & \\ \mathsf{s}(x) \to \mathsf{f}(y) & \Leftarrow \mathsf{p}(x) \approx y \end{array} \right\} \quad \mathcal{S} = \left\{ \begin{array}{ll} \mathsf{p}(x) \to \mathsf{r}(x) & \\ \mathsf{q}(x) \to \mathsf{s}(\mathsf{p}(x)) & \\ \mathsf{s}(x) \to \mathsf{f}(y) & \Leftarrow \mathsf{p}(x) \approx y \end{array} \right\}$$

We have $CCP(\mathcal{R}, \mathcal{S}) = \{\langle \mathsf{q}(x), \mathsf{r}(x)\rangle \Leftarrow \langle \emptyset, \emptyset\rangle\}$ and $CCP_{in}(\mathcal{S}, \mathcal{R}) = \emptyset$. Note that the overlap of $\mathsf{s}(x) \to \mathsf{f}(y) \Leftarrow p(x) \approx y \in \mathcal{R}$ and $\mathsf{s}(x) \to \mathsf{f}(y) \Leftarrow \mathsf{p}(x) \approx y \in \mathcal{S}$ is not considered, as these rules are identical; the case 2.(a).i of the proof above treats this case. Now, because we have $\mathsf{q}(x) \to_{\mathcal{S}_n} \mathsf{s}(\mathsf{p}(x))$ and $\mathsf{r}(x) \to_{\mathcal{R}_m} \mathsf{s}(\mathsf{p}(x))$ $(n, m \geq 1)$ the condition (1) of the Theorem 1 is satisfied. Other conditions of the theorem are also satisfied. Thus, $\mathcal{R}$ and $\mathcal{S}$ are level-commutative. Similarly, one can show $\mathcal{R} \cup \mathcal{S}$ is level-confluent.

Example 2. Take CTRSs $\mathcal{R} = \mathcal{R}' \cup \mathcal{R}_f$ and $\mathcal{S} = \mathcal{S}' \cup \mathcal{R}_f$ such that

$$\mathcal{R}' = \left\{ \begin{array}{l} \mathsf{p}(x, y) \to \mathsf{r}(x, y) \Leftarrow x \approx \mathsf{a} \\ \mathsf{q}(x, y) \to \mathsf{p}(x, y) \Leftarrow x \approx \mathsf{a} \end{array} \right\} \quad \mathcal{S}' = \left\{ \begin{array}{l} \mathsf{p}(x, y) \to \mathsf{q}(x, y) \Leftarrow y \approx \mathsf{b} \\ \mathsf{r}(x, y) \to \mathsf{p}(x, y) \Leftarrow y \approx \mathsf{b} \end{array} \right\}$$

and $\mathcal{R}_f = \{\mathsf{f}(0) \to \mathsf{a},\ \mathsf{f}(\mathsf{s}(x)) \to \mathsf{b} \Leftarrow \mathsf{f}(x) \approx \mathsf{a},\ \mathsf{f}(\mathsf{s}(x)) \to \mathsf{a} \Leftarrow \mathsf{f}(x) \approx \mathsf{b}\}$. We have $CCP(\mathcal{R}, \mathcal{S}) = \{\ (a) : \langle \mathsf{r}(x, y), \mathsf{q}(x, y)\rangle \Leftarrow \langle \{x \approx \mathsf{a}\}, \{y \approx \mathsf{b}\}\rangle,\ (b) : \langle \mathsf{a}, \mathsf{b}\rangle \Leftarrow \langle \{\mathsf{f}(x) \approx \mathsf{b}\}, \{\mathsf{f}(x) \approx \mathsf{a}\}\rangle,\ (c) : \langle \mathsf{b}, \mathsf{a}\rangle \Leftarrow \langle \{\mathsf{f}(x) \approx \mathsf{a}\}, \{\mathsf{f}(x) \approx \mathsf{b}\}\rangle\}$, and $CCP_{in}(\mathcal{S}, \mathcal{R}) = \emptyset$. For the CCP (a), let $m, n \geq 1$ and ρ be any substitution, and suppose that $\rho(x) \to_{\mathcal{R}_{m-1}} \mathsf{a}$ and $\rho(y) \to_{\mathcal{S}_{n-1}} \mathsf{b}$. Then, we have $\mathsf{r}(\rho(x), \rho(y)) \to_{\mathcal{S}_n} \mathsf{p}(\rho(x), \rho(y))$ and $\mathsf{q}(\rho(x), \rho(y)) \to_{\mathcal{R}_m} \mathsf{p}(\rho(x), \rho(y))$. Also, note that there is no term t such that $t \xrightarrow{*}_{\mathcal{R}} \mathsf{b}$ and $t \xrightarrow{*}_{\mathcal{S}} \mathsf{a}$ (or $t \xrightarrow{*}_{\mathcal{R}} \mathsf{a}$ and $t \xrightarrow{*}_{\mathcal{S}} \mathsf{b}$). Thus, the condition (1) of the Theorem 1 holds for CCPs (a)–(c). Other conditions of the theorem are also satisfied. Thus, $\mathcal{R}$ and $\mathcal{S}$ are level-commutative. Similarly, one can show $\mathcal{R} \cup \mathcal{S}$ is level-confluent.

Since TRSs can be regarded as CTRSs with no conditions and they are trivially properly-oriented, right-stable, and of type 3, this theorem covers Proposition 1. However, this does not mean our theorem broaden the scope of TRSs that can be guaranteed to commute—because rewrite steps of TRSs are level 1 rewrite steps in CTRSs, our condition reduces to the one of Proposition 1 in TRSs. Thus, when restricting to TRSs, Theorem 1 coincides Proposition 1.

On the other hand, Corollary 1 properly extends Proposition 2, as witnessed by $\mathcal{R} \cup \mathcal{S}$ in Examples 1, 2.

4 Critical Pair Criteria for Join and Semi-Equational CTRSs

In this section, we explore critical pair criteria for join and semi-equational CTRSs, following our approach in the previous section.

First, let us fix additional notions and notations that will be used in this section. A rewrite step of *join* CTRS $\mathcal{R}$ is defined via the following TRS $\mathcal{R}_n$

($n \in \mathbb{N}$), which are inductively given as follows: $\mathcal{R}_0 = \emptyset$, $\mathcal{R}_{n+1} = \{l\sigma \to r\sigma \mid l \to r \Leftarrow c \in \mathcal{R}, c\sigma \subseteq \xrightarrow{*}_{\mathcal{R}_n} \circ \xleftarrow{*}_{\mathcal{R}_n}\}$. For *semi-equational* CTRS $\mathcal{R}$, we modify the second clause as: $\mathcal{R}_{n+1} = \{l\sigma \to r\sigma \mid l \to r \Leftarrow c \in \mathcal{R}, c\sigma \subseteq \xleftrightarrow{*}_{\mathcal{R}_n}\}$. Similarly to the oriented case, a rewrite step $s \to_{\mathcal{R}} t$ of $\mathcal{R}$ is given as $s \to_{\mathcal{R}} t$ iff $s \to_{\mathcal{R}_n} t$ for some n, and the smallest n such that $s \to_{\mathcal{R}_n} t$ is called the *level* of the rewrite step $s \to_{\mathcal{R}} t$. We write $\downarrow_{\mathcal{R}_n}$ ($\downarrow_{\mathcal{R}}$) for the relation $\xrightarrow{*}_{\mathcal{R}_n} \circ \xleftarrow{*}_{\mathcal{R}_n}$ (resp. $\xrightarrow{*}_{\mathcal{R}} \circ \xleftarrow{*}_{\mathcal{R}}$).

In this section (except Subsect. 4.2), in order to distinguish three types of CTRSs, we write $\mathcal{R}^{\mathsf{o}}$ for an oriented CTRS, $\mathcal{R}^{\mathsf{j}}$ for a join CTRS, and $\mathcal{R}^{\mathsf{s}}$ for a semi-equational CTRS. Similarly, notations $\mathcal{R}^{\mathsf{o}}_n, \mathcal{R}^{\mathsf{j}}_n, \ldots$ are employed. Notations $\mathcal{R}^{\mathsf{o}}_n \vdash c\sigma$ ($\mathcal{R}^{\mathsf{j}}_n \vdash c\sigma$, $\mathcal{R}^{\mathsf{s}}_n \vdash c\sigma$) stands for $c\sigma \subseteq \xrightarrow{*}_{\mathcal{R}^{\mathsf{o}}_n}$ (resp. $c\sigma \subseteq \downarrow_{\mathcal{R}^{\mathsf{j}}_n}$, $c\sigma \subseteq \xleftrightarrow{*}_{\mathcal{R}^{\mathsf{s}}_n}$).

The following basic relations between rewrite relation on three types of CTRSs on each level are essentially proved in [18, Lemmas 1 and 2].

Lemma 5. *Let $\mathcal{R}$ be a CTRS. Then $\to_{\mathcal{R}^{\mathsf{o}}_n} \subseteq \to_{\mathcal{R}^{\mathsf{j}}_n} \subseteq \to_{\mathcal{R}^{\mathsf{s}}_n}$ for each n.*

Notions of orthogonality, proper-orientedness and right-stability are syntax-oriented, and their definitions remain same for other types of CTRSs. Note that even under the conditions of proper-orientedness and right-stability, $\to_{\mathcal{R}^{\mathsf{o}}_n} = \to_{\mathcal{R}^{\mathsf{j}}_n}$ does not hold in general.

4.1 Level-Confluence of Join and Semi-Equational 3-CTRSs

In [14, Corollary 5.3], Proposition 2 is applied to show the corresponding class of join CTRSs are level-confluent:

Proposition 3 ([14]). *Let $\mathcal{R}$ be an orthogonal, properly oriented, right-stable 3-CTRS. Then $\mathcal{R}^{\mathsf{j}}$ is level-confluent.*

Given our Theorem 1, a natural question is whether a similar extension is possible for our theorem. In this subsection, we give a partially positive answer to this question—we generalize the result above to the level-confluence part (Corollary 1) of our theorem, even though a similar extension does not work for level-commutation. Indeed, we show that above proposition can be extended to a more general setting of CTRSs where the orthogonality requirement is replaced with level-confluence of $\mathcal{R}^{\mathsf{o}}$. Furthermore, the generalization is obtained not only for join CTRSs but also for semi-equational CTRSs.

The next two lemmas are abstractions of the ones [14, Lemmas 5.1 and 5.2], where the proofs remain almost the same.

Lemma 6. *Let $\mathcal{R}$ be a properly oriented, right-stable 3-CTRS such that $\mathcal{R}^{\mathsf{o}}$ is level-confluent. Let $l \to r \Leftarrow s_1 \approx t_1, \ldots, s_j \approx t_j \in \mathcal{R}$. If $s_i\sigma \downarrow_{\mathcal{R}^{\mathsf{o}}_{n-1}} t_i\sigma$ for any $1 \le i \le j$ then $l\sigma \downarrow_{\mathcal{R}^{\mathsf{o}}_n} r\sigma$.*

Lemma 7. *Let $\mathcal{R}$ be a properly oriented, right-stable 3-CTRS such that $\mathcal{R}^{\mathsf{o}}$ is level-confluent. If $s \to_{\mathcal{R}^{\mathsf{s}}_n} t$ then $s \downarrow_{\mathcal{R}^{\mathsf{o}}_n} t$.*

Now we present the claimed result:

Theorem 2. *Let $\mathcal{R}$ be a properly oriented, right-stable 3-CTRS. If $\mathcal{R}^{\mathrm{o}}$ is level-confluent then $\mathcal{R}^{\mathrm{j}}$ and $\mathcal{R}^{\mathrm{s}}$ are level-confluent.*

Proof. Let $\mathcal{R}$ be a properly oriented, right-stable 3-CTRS such that $\mathcal{R}^{\mathrm{o}}$ is level-confluent. Suppose $t_1 \xleftarrow{*}_{\mathcal{R}^{\mathrm{j}}_n} s \xrightarrow{*}_{\mathcal{R}^{\mathrm{j}}_n} t_2$ ($t_1 \xleftarrow{*}_{\mathcal{R}^{\mathrm{s}}_n} s \xrightarrow{*}_{\mathcal{R}^{\mathrm{s}}_n} t_2$). Then $t_1 \xleftarrow{*}_{\mathcal{R}^{\mathrm{s}}_n} s \xrightarrow{*}_{\mathcal{R}^{\mathrm{s}}_n} t_2$ by Lemma 5. Thus, by Lemma 7, $t_1 \xleftrightarrow{*}_{\mathcal{R}^{\mathrm{o}}_n} t_2$. Hence, $t_1 \downarrow_{\mathcal{R}^{\mathrm{o}}_n} t_2$ follows by the level-confluence of $\mathcal{R}^{\mathrm{o}}$. Using again Lemma 5, this implies $t_1 \downarrow_{\mathcal{R}^{\mathrm{j}}_n} t_2$ (resp. $t_1 \downarrow_{\mathcal{R}^{\mathrm{s}}_n} t_2$). □

Thus, Corollary 1 can be applied to show the level-confluence of join and semi-equational CTRSs. Note here that the conditions of Corollary 1 is stated in terms of $\to^{\mathrm{o}}_{\mathcal{R}}$ not in that of $\to^{\mathrm{j}}_{\mathcal{R}}$ or $\to^{\mathrm{s}}_{\mathcal{R}}$.

4.2 Commutation of Semi-Equational 3-CTRSs

A most fundamental ingredient of the proof presented (inherited from [14]) is to use induction on the level of rewrite relation. It seems, however, applying this approach for join and semi-equational CTRSs contains fundamental difficulty. Without the induction on the level, what can we do within the parallel-closed approach? In this subsection, we will exhibit one alternative approach for semi-equational CTRSs.

In [1], it is reported that left-linear parallel-closed semi-equational 1-CTRSs are confluent. By examining its proof detail, we can extend it to commutativity of 3-CTRSs as follows. Below, notation $\mathcal{R} \vdash c\sigma$ (etc.) stands for $c\sigma \subseteq \xleftrightarrow{*}_{\mathcal{R}}$.

Theorem 3. *Let $\mathcal{R}, \mathcal{S}$ be semi-equational left-linear 3-CTRSs. Suppose the following conditions are satisfied:*

1. *for any $\langle u, v\rangle \Leftarrow \langle c, c'\rangle \in CCP(\mathcal{R}, \mathcal{S})$ and any substitution ρ, if $\mathcal{R} \vdash c\sigma$ and $\mathcal{S} \vdash c'\sigma$, then $u\rho \mathrel{+\!\!\!\!\!\twoheadrightarrow}_{\mathcal{S}} \circ \xleftarrow{*}_{\mathcal{R}} v\rho$, and*
2. *for any $\langle v, u\rangle \Leftarrow \langle c', c\rangle \in CCP_{in}(\mathcal{S}, \mathcal{R})$ and any substitution ρ, if $\mathcal{R} \vdash c\rho$ and $\mathcal{S} \vdash c'\rho$, then $v\rho \mathrel{+\!\!\!\!\!\twoheadrightarrow}_{\mathcal{R}} u\rho$.*

Furthermore, assume $\mathrel{+\!\!\!\!\!\twoheadrightarrow}_{\mathcal{S}} \subseteq \xleftrightarrow{*}_{\mathcal{R}}$, $\mathrel{+\!\!\!\!\!\twoheadrightarrow}_{\mathcal{R}} \subseteq \xleftrightarrow{*}_{\mathcal{S}}$ and $\mathcal{R} \cap \mathcal{S}$ is a 2-CTRS. Then, $\mathcal{R}$ and $\mathcal{S}$ commute.

We remark that conditions $\mathrel{+\!\!\!\!\!\twoheadrightarrow}_{\mathcal{S}} \subseteq \xleftrightarrow{*}_{\mathcal{R}}$ and $\mathrel{+\!\!\!\!\!\twoheadrightarrow}_{\mathcal{R}} \subseteq \xleftrightarrow{*}_{\mathcal{S}}$ are used to close nested peaks, and that the condition that $\mathcal{R} \cap \mathcal{S}$ is a 2-CTRS is required to resolve for peaks obtained by the same rule.

Example 3. Let $\mathcal{R}$ and $\mathcal{S}$ be the following left-linear semi-equational 3-CTRSs:

$$\mathcal{R} = \{\mathsf{q}(x,y) \to \mathsf{p}(y,x),\ \mathsf{p}(x,y) \to \mathsf{q}(x',y') \Leftarrow x \approx x', y \approx y'\}$$
$$\mathcal{S} = \{\mathsf{p}(x,y) \to \mathsf{q}(y,x),\ \mathsf{q}(x,y) \to \mathsf{p}(x',y') \Leftarrow x \approx x', y \approx y'\}$$

By induction on the level n, one can show $\to_{\mathcal{S}_n} \subseteq \xrightarrow{*}_{\mathcal{R}_n}$ and $\to_{\mathcal{R}_n} \subseteq \xrightarrow{*}_{\mathcal{S}_n}$. Thus, conditions $\mathrel{+\!\!\!\!\!\twoheadrightarrow}_{\mathcal{S}} \subseteq \xleftrightarrow{*}_{\mathcal{R}}$ are $\mathrel{+\!\!\!\!\!\twoheadrightarrow}_{\mathcal{R}} \subseteq \xleftrightarrow{*}_{\mathcal{S}}$ are satisfied. Clearly, $\mathcal{R} \cap \mathcal{S} = \emptyset$ is a 2-CTRS. We have $CCP(\mathcal{R}, \mathcal{S}) = \{\langle \mathsf{q}(x',y'), \mathsf{q}(y,x)\rangle \Leftarrow \langle\{x \approx x', y \approx$

$y'\}, \emptyset\rangle, \{\langle \mathsf{p}(y,x), \mathsf{p}(x',y')\rangle \Leftarrow \langle \emptyset, \{x \approx x', y \approx y'\}\rangle\}$ and $CCP_{in}(\mathcal{S},\mathcal{R}) = \emptyset$. Clearly, $\rho(x) \xleftrightarrow{*}_{\mathcal{R}} \rho(x')$ and $\rho(y) \xleftrightarrow{*}_{\mathcal{R}} \rho(y')$ imply $\mathsf{p}(\rho(x'),\rho(y')) \to_{\mathcal{R}} \mathsf{q}(\rho(y),\rho(x))$, and $\rho(x) \xleftrightarrow{*}_{\mathcal{S}} \rho(x')$ and $\rho(y) \xleftrightarrow{*}_{\mathcal{S}} \rho(y')$ imply $\mathsf{q}(\rho(x),\rho(y)) \leftarrow_{\mathcal{S}} \mathsf{p}(\rho(y),\rho(x))$. Thus, all conditions of the Theorem 3 are satisfied. Thus, $\mathcal{R}$ and $\mathcal{S}$ commute.

Note the conditions $\twoheadrightarrow\!\!\!\!\!\!+_{\mathcal{S}} \subseteq \xleftrightarrow{*}_{\mathcal{R}}$ and $\twoheadrightarrow\!\!\!\!\!\!+_{\mathcal{R}} \subseteq \xleftrightarrow{*}_{\mathcal{S}}$ of Theorem 3 imply $\xleftrightarrow{*}_{\mathcal{R}} = \xleftrightarrow{*}_{\mathcal{S}}$, i.e. $\mathcal{R}$ and $\mathcal{S}$ have the same underlying logic.

5 Conclusion

We have given a critical pair criterion for ensuring level-commutativity of left-linear properly-oriented right-stable oriented 3-CTRSs. Our result generalizes a sufficient criterion for commutativity of left-linear TRSs of Toyama [16]. It also properly extends level-confluence of orthogonal properly-oriented right-stable oriented 3-CTRSs of Suzuki et al. [14]. We then have showed this result can be applied to obtain a criterion for level-confluence of left-linear properly-oriented right-stable join and semi-equational 3-CTRSs, generalizing a result of [14]. We have also explored a similar but different approach of Aoto and Toyama [1] to obtain a criterion for the commutation of semi-equational 3-CTRSs.

Wirth [17] also gave a criterion of level-confluence for possibly non-orthogonal CTRSs that generalizes a sufficient criterion for confluence of left-linear TRSs of [16]. He adapted the approach of [16] for a framework of join CTRSs. It also incorporates some ideas of [14] so as to give the notions of (weak-)quasi-normal CTRSs, etc. A critical key difference with the usual conditional rewriting such as employed in our paper, however, is that the validity of conditions needs to be satisfied under a kind of constructor discipline. This restriction considerably simplifies proof arguments dealing with conditional parts, paying the penalty of going apart from the standard framework. On the other hand, despite these sharp differences on the underlying frameworks of ours and [17], interestingly, the critical pair criterion of Theorem 3 and Wirth's critical pair criterion ([17, Definition 28]) resemble very much.

Over various formalisms of rewriting, considerable efforts have been spent on automating confluence checks in recent years. Yearly competition[2] of confluence tools started in 2012; the category of CTRS has been also introduced in 2014. In recent competitions, confluence of *oriented 3-CTRSs*, which our main theorem deal with, has been focused in the category of CTRS. Known confluence tools for CTRSs include CONFident [6], ConCon [13], CO3 [9] and ACP [2]. We note here that all these tools fail to show confluence of $\mathcal{R} \cup \mathcal{S}$ of Example 2[3]. Among these tools (at least) ConCon and ACP incorporate checking of confluence criterion of [14]. We have been working on the automation of our results, but it is yet under

[2] http://project-coco.uibk.ac.at/.

[3] Experimented for CoCo 2022 participants ACP, CO3, CONFident and a CoCo 2020 participant ConCon, via CoCoWeb [7].

development. Recent advances in confluence tools for CTRSs include automation of infeasibility checking [5]—we believe some approaches for automation of infeasibility checking can be adapted for automation of our criterion.

Formalization by interactive theorem provers such as Isabelle/HOL, Coq, PVS4, etc. have been of great interest in recent years. Formalization is also indispensable for certification of results obtained by confluence tools. Regarding for results of [14], a formalization in Isabelle/HOL has been reported by Sternagel and Sternagel [12]. On the other hand, formalization of our results remains completely as a future work.

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