Closed Surfaces

11

We define a *closed surface* as a surface $f\colon M \to \mathbb{R}^3$ whose boundary components have been matched in pairs in such a way that f as well as its unit normal N are continuous across the boundary. This allows us to prove an analog of the fact that the tangent winding number of a closed plane curve is an integer: The total Gaussian curvature $\int_M K\,\det$ of a closed surface $f\colon M \to \mathbb{R}^3$ is equal to $2\pi\chi(M)$ where $\chi(M)$ is the Euler characteristic.

11.1 History of Closed Surfaces

Our goal here is to define "closed surfaces" in such a way that we are able to prove an analog of Theorem 3.8, which says that the turning number of a plane curve is an integer. Furthermore, in Sect. 13.1 we want to discuss for closed surfaces the analog of the total squared curvature of a curve.

Our approach will be based on the very idea that was already at the heart of the 1845 paper by Möbius where closed surfaces were studied for the first time: By cutting them into horizontal slices, Möbius decomposed closed surfaces into pieces each of which can be parametrized by a compact domain with smooth boundary in $\mathbb{R}^2$. Figure 11.1 is adapted from the paper by Möbius. This very idea was already the motivation for us to allow for disconnected domains in the case of surfaces and will be formalized in Sect. 11.2.

More details on the early history of surface theory can be found in an article by Peter Dombrowski [11].

A more advanced way to define closed surfaces in $\mathbb{R}^n$ (that would not need to cut the surface into pieces that can be parametrized by planar domains) would be to define them in terms of smooth maps $f\colon M \to \mathbb{R}^n$ defined on 2-dimensional compact manifolds M. Such manifolds were first defined in 1910 by Hermann Weyl in a famous book with the title "Die Idee der Riemannschen Fläche" [44].

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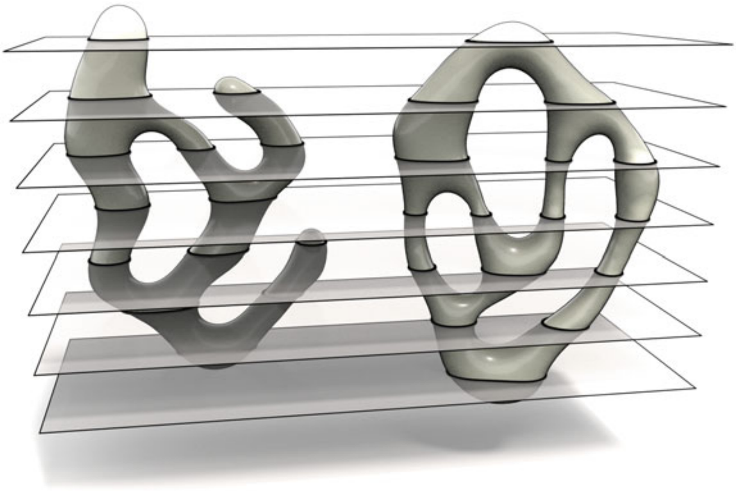

Fig. 11.1 Möbius decomposed closed surfaces into pieces that can be parametrized by compact domains in $\mathbb{R}^2$ with smooth boundary (modeled after Möbius' original sketch in [29])

On the other hand, the fully developed version of the Gauss-Bonnet theorem (which we will prove in the next chapter) is already contained in the 1903 thesis of Werner Boy [8], that he did under the supervision of David Hilbert.

Modern treatments of Differential Topology (like the books by Andrew Wallace [42] and Morris Hirsch [16]) often discuss surface topology in their last chapters. The main work there goes into proving (with the help of Morse theory) that indeed every compact 2-dimensional manifold can be decomposed into pieces each of which can be parametrized by a compact domain with smooth boundary in $\mathbb{R}^2$. Therefore, the work that will be done in the next two chapters would not become obsolete even if we had manifolds at our disposal.

11.2 Defining Closed Surfaces

Suppose that for a surface $f : M \to \mathbb{R}^3$ the boundary components of M match up in pairs in such a way that, given suitable parametrizations of the boundary curves, corresponding points of ∂M are mapped to the same points in $\mathbb{R}^3$. If in addition also the unit normals of f fit together up to sign on ∂M, we consider f (together with a specification of the boundary matching) as a closed surface (Fig. 11.2):

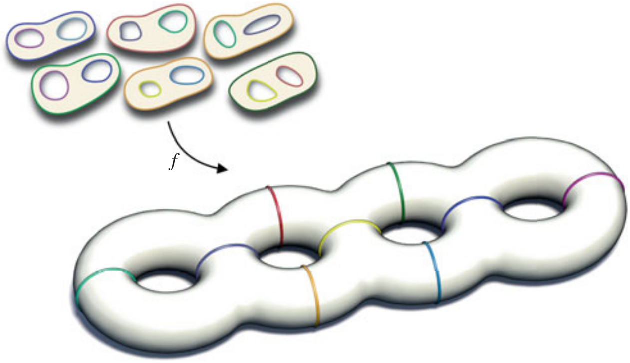

Fig. 11.2 A closed surface f

Definition 11.1

Let $M \subset \mathbb{R}^2$ be a compact domain with smooth boundary and $f: M \to \mathbb{R}^3$ a surface with unit normal N. We parametrize the boundary curves of M by closed curves

$$\gamma_1, \ldots, \gamma_n: [-\pi, \pi] \to \mathbb{R}^2$$

and define curves $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n: [-\pi, \pi] \to \mathbb{R}^3$ by

$$\tilde{\gamma}_j := f \circ \gamma_j.$$

As in Definition 10.1, we equip the closed space curves $\tilde{\gamma}_j$ with unit normal fields $\tilde{N}_j := N \circ \gamma_j$. Let

$$\rho: \{1, \ldots, n\} \to \{1, \ldots, n\}$$

a bijective map such that

$$(\rho \circ \rho)(j) = j$$

for all j. Then the pair (f, ρ) is called a **closed surface** if there are signs $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$ such that for all $j \in \{1, \ldots, n\}$ we have:

(i) If $\rho(j) \neq j$ then

$$\tilde{\gamma}_{\rho(j)}(x) = \tilde{\gamma}_j(\epsilon_j x)$$

$$\tilde{N}_{\rho(j)}(x) = -\epsilon_j \tilde{N}_j(\epsilon_j x).$$

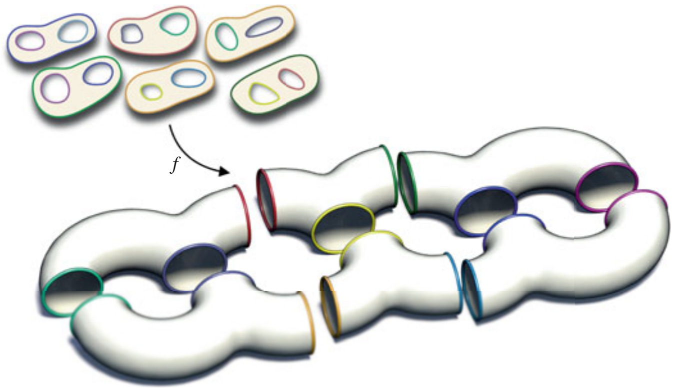

Fig. 11.3 The surface in Fig. 11.2 made into a non-closed surface by applying a small translation to each piece

(ii) If $\rho(j) = j$ then $\epsilon_j = 1$ and

$$\tilde{\gamma}_j(x) = \begin{cases} \tilde{\gamma}_j(x+\pi) & \text{for } x \in [-\pi, 0) \\ \tilde{\gamma}_j(x-\pi) & \text{for } x \in [0, \pi] \end{cases}$$

$$\tilde{N}_j(x) = \begin{cases} -\tilde{N}_j(x+\pi) & \text{for } x \in [-\pi, 0) \\ -\tilde{N}_j(x-\pi) & \text{for } x \in [0, \pi]. \end{cases}$$

It is easy to see that such $\epsilon_1, \ldots, \epsilon_n$ are uniquely determined by f and ρ. We say that a closed surface is *oriented* if $\epsilon_j = -1$ for all $j \in \{1, \ldots, n\}$.

Figure 11.3 shows the shape of the individual pieces that are being glued in Fig. 11.2. It has $k = 6$ components and $n = 18$ boundary curves.

Here is another example: M now consists of a disk with boundary γ_1 and an annulus with boundary curves γ_2 and γ_3. First, we tentatively define f on the disk bounded by γ_1 and obtain the cap on the upper right of Fig. 11.4. Postponing for the moment the task (indicated by the double-arrow on the right) of gluing γ_1 to γ_2, we first glue γ_3 to itself and obtain a Möbius band (on the bottom of the lower right of Fig. 11.4):

By growing the Möbius band (see Fig. 11.5) we finally obtain the closed surface we wanted to construct:

This surface (fully closed in Fig. 11.6) was found by Werner Boy in 1903 and is called the **Boy surface**.

Figure 11.7 shows two surfaces which are obtained by gluing the boundary curve of an annulus to itself appropriately. Even though both compact domains have $k = 1$ components and $n = 2$ boundary loops, the distinct maps f, $\tilde{f}$ lead to distinct closed surfaces. In particular, although the map ρ is the same, they have opposite sign ϵ.

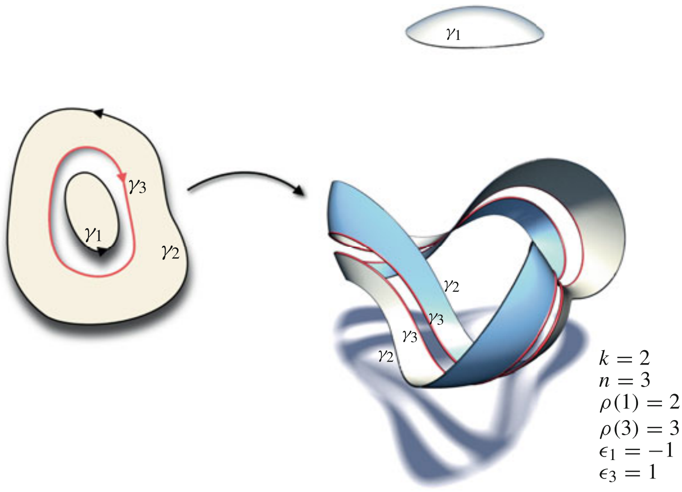

Fig. 11.4 The annulus part of the domain on the left has its red boundary component glued to itself. After growing the resulting Möbius strip, the other boundary component can be glued to the image of the disk part of the domain. The result is the so-called **Boy surface**

11.3 Boy's Theorem

Definition 11.2

We say that a surface $f : M \to \mathbb{R}^3$ **closes up** if there is ρ such that (f, ρ) is a closed surface in the sense of Definition 11.1.

Recall that for every closed plane curve $\gamma : [a, b] \to \mathbb{R}^2$ there was an integer $n \in \mathbb{Z}$ such that

$$\int_a^b \kappa \, ds = 2\pi n.$$

Surprisingly, the analog of this fact in the context of surfaces (cf. Theorem 11.3) does not involve any information about the specific way in which f closes up, but only depends on properties of the domain M. The theorem is a variant of the Gauss-Bonnet Theorem 10.6. Usually, it would be called by the same name. However, historically this is not quite correct. This theorem was in fact the main result of the

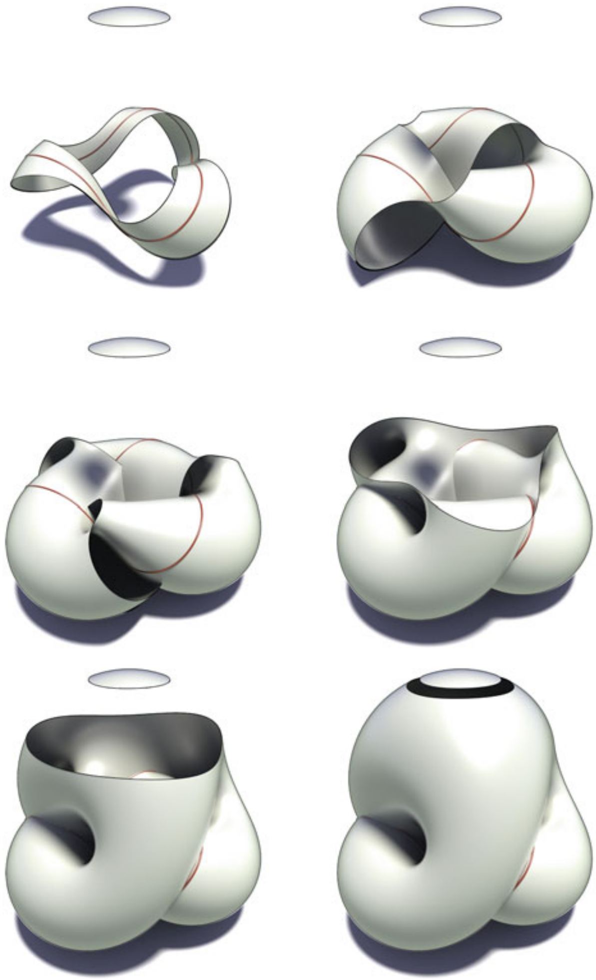

Fig. 11.5 A growing Möbius strip can be capped off to form a Boy surface

thesis of Werner Boy [8], written in 1903 under the supervision of David Hilbert. For this reason, we name it after Boy:

Theorem 11.3 (Boy's Theorem)
Let $f : M \to \mathbb{R}^3$ be a surface that closes up. Then the Gaussian curvature K of f satisfies

$$\int_M K \ \det = 2\pi\ \chi(M).$$

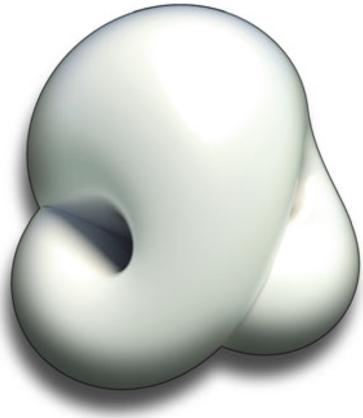

Fig. 11.6 The Boy surface is a closed, non-oriented surface

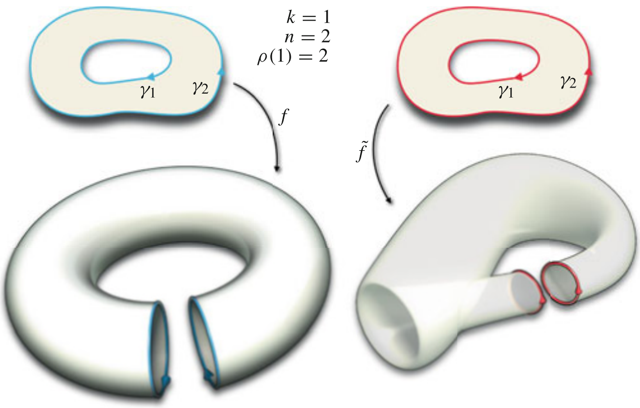

Fig. 11.7 After pushing the two boundary curves together, we obtain a closed surface which is oriented—a torus *(left)*, or a closed surface that is not oriented—a so-called **Klein bottle** *(right)*

Before we give the proof, we introduce the notion of an **orientation cover** of a closed surface. Given a closed surface (f, ρ) with $f: M \to \mathbb{R}^3$, we can define an oriented closed surface $(\tilde{f}, \tilde{\rho})$ in the following way:

Let us use M_{-1} as another name for M and, using an orientation-reversing isometry $g: \mathbb{R}^2 \to \mathbb{R}^2$, we place a second copy $M_1 = g(M)$ into $\mathbb{R}^2$ in such a way that M_{-1} and M_1 are disjoint. Then we define

$$\tilde{M} := M_{-1} \cup M_1$$

and

$$\tilde{f}\colon \tilde{M} \to \mathbb{R}^3,\ \ \tilde{f}(p) = \begin{cases} f(p) & \text{if} \quad p \in M_{-1} \\ (f \circ g^{-1})(p) & \text{if} \quad p \in M_1. \end{cases}$$

We can label the boundary curves of $\tilde{M}$ by the elements of $\{-1, 1\} \times \{1, \dots, n\}$ and parametrize them by maps

$$\gamma_{(i,j)} : \mathbb{R} \to \partial\tilde{M},\ \ \gamma_{(i,j)} = \begin{cases} \gamma_j & \text{if} \quad i = -1 \\ x \mapsto g \circ \gamma_j(-x) & \text{if} \quad i = 1. \end{cases}$$

Finally, we define

$$\tilde{\rho}\colon \{-1, 1\} \times \{1, \dots, n\} \to \{-1, 1\} \times \{1, \dots, n\},\ \ \tilde{\rho}(i, j) = (-\epsilon_j\, i, \rho(j)).$$

We now leave it to the reader to check that $(\tilde{f}, \tilde{\rho})$ is an oriented closed surface, i.e. we obtain a closed surface by setting $\tilde{\epsilon}_{(i,j)} = -1$ for all $(i, j) \in \{-1, 1\}\times\{1, \dots, n\}$.

Definition 11.4

The closed surface $(\tilde{f}, \tilde{\rho})$ constructed above is called an **orientation cover** of f.

Proof of Theorem 11.3—Boy's Theorem If ρ has no fixed points (no boundary component is glued to itself), one just has to note that the existence of ρ (making (f, ρ) into a closed surface) implies that in Theorem 10.6 the total geodesic curvatures of the individual boundary curves cancel in pairs. If ρ has fixed points, we note that the $\tilde{\rho}$ of the orientation cover has no fixed points and therefore our theorem holds for $\tilde{f}$. Dividing both sides of the resulting equation by two, we see that our theorem also holds for f. □

11.4 The Genus of a Closed Surface

The Euler characteristic of a closed surface was solely a property of its domain M, the specific way the various boundary curves are glued is irrelevant for the Euler characteristic. There is another number associated with a closed surface (f, ρ), the so-called **genus**, that depends on the gluing correspondence ρ:

Suppose $M \subset \mathbb{R}^2$ is a domain with k components and n boundary curves. Consider the map that assigns to each $j \in \{1, \dots, n\}$ the index $c(j) \in \{1, \dots, k\}$ of the component of M to which the jth boundary component belongs. Let us consider the graph G whose vertex set is $\{1, \dots, k\}$ and in which two vertices $\ell, \tilde{\ell}$ with $\ell \neq \tilde{\ell}$ are connected by an edge if and only if there is an index $j \in \{1, \dots, n\}$ for which $c(j) = \ell$ and $c(\rho(j)) = \tilde{\ell}$, which means that the components of M with indices j and $\tilde{j}$ are glued via one (or more) of their respective boundary curves. We say that two vertices ℓ and $\tilde{\ell}$ of G are **connectable** in G if it is possible to travel from ℓ to $\tilde{\ell}$ by following edges. Connectability is an equivalence relation and the corresponding equivalence classes are called the connected components of G.

Definition 11.5

If $\{\ell_1, \dots, \ell_{\tilde{k}}\}$ is a component of the graph G, then

$$\tilde{f} = f|_{M_{\ell_1} \cup \ldots \cup M_{\ell_{\tilde{k}}}}$$

closes up with boundary gluing $\tilde{\rho}$ read off from (f, ρ). We call the resulting closed surface $(\tilde{f}, \tilde{\rho})$ a **component** of (f, ρ). We call (f, ρ) **connected** if it has only one component.

So the components of a closed surface are in one-to-one correspondence with the components of its associated graph G.

Definition 11.6

Let M be a compact domain with k components and n boundary curves. Let (f, ρ) be a closed surface with $f : M \to \mathbb{R}^3$. If (f, ρ) has m connected components, we define the **genus** of (f, ρ) as

$$g := \frac{n}{2} - k + m.$$

In terms of the genus, the Gauss-Bonnet formula takes the form

$$\int_M K \ \det = 4\pi(m - g).$$

The first surface featured in Sect. 11.2 has genus $g = 4$, the Klein bottle has genus $g = 1$ und the Boy surface has genus $g = \frac{1}{2}$. The two surfaces in Fig. 11.8 have genus $g = \frac{5}{2}$ and genus $g = 2$ respectively.

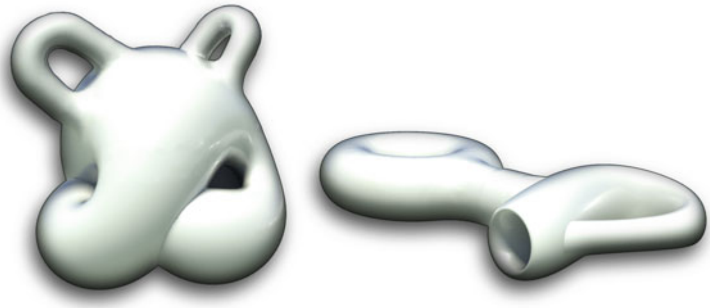

Fig. 11.8 Non-oriented surfaces of genus $g = \frac{5}{2}$ *(left)* and $g = 2$ *(right)*. They are obtained by smoothly gluing handles onto a Boy surface or respectively a Klein bottle