# Curves in $\mathbb{R}^n$

# 1

Differential Geometry studies smoothly curved shapes, called *manifolds*. One-dimensional shapes are called *curves* and two-dimensional shapes are called *surfaces*. In this chapter we look at curves in $n$-dimensional Euclidean space. The basic properties of curves in $\mathbb{R}^n$ (length, tangent, bending energy) were explored right after the invention of calculus by Newton, Bernoulli and Euler.

## 1.1 What is a Curve in $\mathbb{R}^n$?

Since many interesting curves (for example a figure eight) have self-intersections, it is not a good idea to define a curve as a special kind of subset in $\mathbb{R}^n$. Intuitively, a curve is something that can be traced out ("parametrized") as the path of a moving point (cf. Fig. 1.1).

**Definition 1.1**

A **curve** in $\mathbb{R}^n$ is a smooth map $\gamma\colon [a,b] \to \mathbb{R}^n$ such that its velocity vector $\gamma'(x)$ never vanishes, i.e.

$$\gamma'(x) \neq 0$$

for all $x \in [a,b]$.

**Remark 1.2** If $M \subset \mathbb{R}^n$ is an arbitrary subset, then a map $f\colon M \to \mathbb{R}^k$ is called **smooth** (or $C^\infty$) if there is an open set $U \subset \mathbb{R}^n$ with $M \subset U$ and an infinitely often differentiable map $\tilde{f}\colon U \to \mathbb{R}^k$ such that $f = \tilde{f}|_M$ (cf. Appendix A.1). Instead of a closed interval $[a,b]$ one could also allow an open or semi-open interval (or even a finite union of intervals) as the domain of definition for a curve. The only problem that would arise is that then the integral of a smooth function would not always be defined. For all of our applications we can stick to closed intervals.



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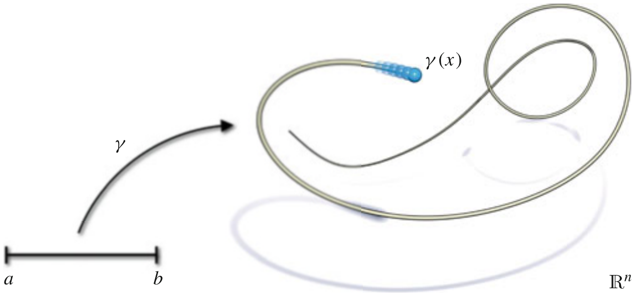



**Fig. 1.1** A curve can be described as the trajectory of a particle moving in space. The particles position at time $x$ is given by $\gamma(x)$

**Definition 1.3**

A curve $\gamma : [a, b] \to \mathbb{R}^n$ is called **closed** if $\gamma$ can be extended to a smooth map $\tilde{\gamma} : \mathbb{R} \to \mathbb{R}^n$ with *period* $b - a$, which means

$$\tilde{\gamma}(x + (b - a)) = \tilde{\gamma}(x)$$

for all $x \in \mathbb{R}$.

**Example 1.4**

(i) The quarter circle is a curve:

$$\gamma : \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right] \to \mathbb{R}^2, \ \gamma(x) = \begin{pmatrix} t \\ \sqrt{1 - x^2} \end{pmatrix}.$$

(ii) Another version of the quarter circle is also a curve:

$$\gamma : \left[\frac{\pi}{4}, \frac{3\pi}{4}\right] \to \mathbb{R}^2, \ \gamma(x) = \begin{pmatrix} \cos x \\ \sin x \end{pmatrix}.$$

(iii) The full circle

$$\gamma : [0, 2\pi] \to \mathbb{R}^2, \ \gamma(x) = \begin{pmatrix} \cos x \\ \sin x \end{pmatrix}$$

is a closed curve with period $2\pi$. It can be extended to

$$\tilde{\gamma} : \mathbb{R} \to \mathbb{R}^2, \ \gamma(x) = \begin{pmatrix} \cos x \\ \sin x \end{pmatrix}.$$

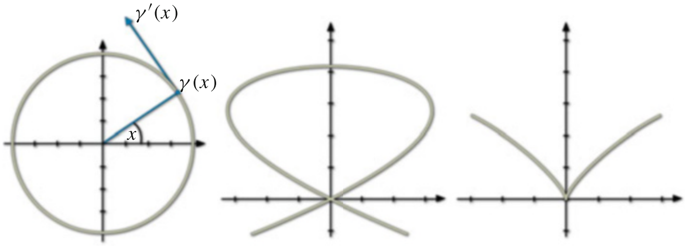



**Fig. 1.2** A circle *(left)*, the Cartesian leaf *(middle)* and Neil's parabola *(right)*

(iv) The **Helix** is a curve:

$$\gamma\colon [a,b] \to \mathbb{R}^3,\ \gamma(x) = \begin{pmatrix} \cos x \\ \sin x \\ x \end{pmatrix}.$$

(v) The **Cartesian leaf** (see Fig. 1.2) is a curve:

$$\gamma\colon [a,b] \to \mathbb{R}^2,\ \gamma(t) = \begin{pmatrix} x^3 - 4x \\ x^2 - 4 \end{pmatrix}$$

so that

$$\gamma'(t) = \begin{pmatrix} 3x^2 - 4 \\ 2x \end{pmatrix}.$$

(vi) **Neil's parabola** (see Fig. 1.2) is given by

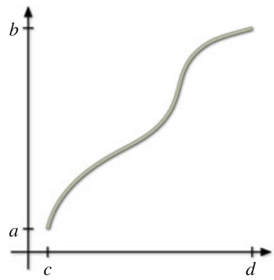
$$\gamma\colon [a,b] \to \mathbb{R}^2,\ \gamma(t) = \begin{pmatrix} x^3 \\ x^2 \end{pmatrix}.$$

It is not a curve if $0 \in [a,b]$, because at $t = 0$

$$\gamma'(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For the purposes of geometry, the speed with which we run through a curve does not really matter, nor does the particular time interval $[a,b]$ that we use for the parametrization. However, we will always assume that our curves are *oriented*, so we want to keep track of the direction in which we run through the curve. This means that we are only interested in properties of a curve that do not change under orientation-preserving reparametrization (see Fig. 1.3):

**Fig. 1.3** A reparametrization of a curve is given by a strictly increasing function with nowhere vanishing derivative which maps $[c, d]$ onto $[a, b]$




**Definition 1.5**

Let $\gamma\colon [a, b] \to \mathbb{R}^n$ and $\tilde{\gamma}\colon [c, d] \to \mathbb{R}^n$ be two curves. Then $\tilde{\gamma}$ is called an **orientation-preserving reparametrization** of $\gamma$ if there is a bijective smooth map $\varphi\colon [c, d] \to [a, b]$ such that $\varphi'(x) > 0$ for all $x \in [c, d]$ and $\tilde{\gamma} = \gamma \circ \varphi$.

**Example 1.6**
For the two curves $\gamma$ from Example 1.4 (i) and $\tilde{\gamma}$ from Example 1.4 (ii) we have $\tilde{\gamma} = \gamma \circ \varphi$ with

$$\varphi\colon \left[\tfrac{\pi}{4}, \tfrac{3\pi}{4}\right] \to \left[-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right], \quad \varphi(x) = \cos x.$$

**Remark 1.7** Orientation-preserving reparametrization is an equivalence relation on the set of curves in $\mathbb{R}^n$. Although we are ultimately only interested in properties shared by all curves in the same equivalence class, we will always work with a particular representative curve $\gamma$.

## 1.2 Length and Arclength

The most simple numerical quantity that can be assigned to a curve as a whole is its length.

**Definition 1.8**

Let $\gamma\colon [a, b] \to \mathbb{R}^n$ be a curve. Then the function

$$v\colon [a, b] \to \mathbb{R}, \ t \mapsto |\gamma'(t)|$$

is called the **speed** of $\gamma$ and

$$\mathcal{L}(\gamma) := \int_a^b v$$

is called the **length** of $\gamma$.

The length of a curve does not change under reparametrization:

**Theorem 1.9**
*Suppose $\gamma\colon [a,b] \to \mathbb{R}^n$ and $\tilde{\gamma}\colon [c,d] \to \mathbb{R}^n$ are two curves such that $\tilde{\gamma} = \gamma \circ \varphi$ for some diffeomorphism $\varphi\colon [c,d] \to [a,b]$. Then $\gamma$ and $\tilde{\gamma}$ have the same length.*

*Proof.* By the substitution rule, we have

$$\mathcal{L}(\tilde{\gamma}) = \int_c^d |(\gamma \circ \varphi)'| = \int_c^d |\gamma' \circ \varphi|\varphi' = \int_a^b |\gamma'| = \mathcal{L}(\gamma).$$

□

**Example 1.10**

(i) For the half circle $\gamma\colon [0,\pi] \to \mathbb{R}^2$,

$$\gamma(x) = \begin{pmatrix} \cos x \\ \sin x \end{pmatrix}$$

we have $|\gamma'| = 1$ and therefore $\mathcal{L}(\gamma) = \pi$.

(ii) The line segment $\gamma\colon [a,b] \to \mathbb{R}^2$,

$$\gamma(x) = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

has length $\mathcal{L}(\gamma) = b - a$.

**Definition 1.11**

A **rigid motion** of $\mathbb{R}^n$ is a map $g\colon \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$g(\mathbf{y}) = A\mathbf{y} + \mathbf{b}$$

where $A \in O(n)$ is an orthogonal matrix and $\mathbf{b} \in \mathbb{R}^n$ is a vector.

Rigid motions are those transformations of the ambient space $\mathbb{R}^n$ which preserve distances between points. Two shapes that differ only by a rigid motion are said to be **congruent**. Matching the physical intuition for curves as trajectories of a particle moving in space, the length of a curve is invariant under rigid motions:

**Theorem 1.12**
*Let $\gamma : [a, b] \to \mathbb{R}^n$ be a curve and $g : \mathbb{R}^n \to \mathbb{R}^n$ a rigid motion. Then*

$$\mathcal{L}(g \circ \gamma) = \mathcal{L}(\gamma).$$

*Proof.* For $\tilde{\gamma} = g \circ \gamma$ we have $\tilde{\gamma} = A\gamma + \mathbf{b}$ and $\tilde{\gamma}' = A\gamma'$. Therefore,

$$\mathcal{L}(\tilde{\gamma}) = \int_a^b |A\gamma'| = \int_a^b |\gamma'| = \mathcal{L}(\gamma).$$

□

**Definition 1.13**

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a curve. Then the function

$$s : [a, b] \to \mathbb{R}, \ \ s(t) := \mathcal{L}\left(\gamma|_{[a,t]}\right) = \int_a^t |\gamma'|$$

is called the **arclength** function (or **arclength coordinate**) of $\gamma$.

In most situations however, the arclength function $s$ itself is less useful than its derivative, the speed $s' = v = |\gamma'|$. Using only $v$, not $s$, we can define the derivative with respect to arclength:

**Definition 1.14**

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a curve and $v = |\gamma'|$ its speed. Let $g : [a, b] \to \mathbb{R}^k$ be a smooth function. Then we define the **derivative with respect to arclength** of $g$ as the function

$$\frac{dg}{ds} := \frac{g'}{v}$$

and the **integral over arclength** of $g$ as

$$\int_a^b g \, ds := \int_a^b g \, v.$$

▶ **Remark 1.15** Once we have learned about 1-forms in Sect. 7.2, we will be able to interpret $ds$ as a 1-form on $[a, b]$ and $\frac{dg}{ds}$ as quotient of 1-forms, just as it had been the dream of Leibniz. For now, they are just $\mathbb{R}^k$-valued functions on $[a, b]$.

**Theorem 1.16**
*The arclength fucnction* $s\colon [a,b] \to \mathbb{R}$ *of a curve* $\gamma\colon [a,b] \to \mathbb{R}^n$ *is an orientation-preserving diffeomorphism of the interval* $[a,b]$ *onto the interval* $[0,L]$ *where* $L = \mathcal{L}(\gamma)$*. The reparametrization*

$$\tilde{\gamma}\colon [0,L] \to \mathbb{R}^n,\ \tilde{\gamma} = \gamma \circ s^{-1}$$

*has unit speed, i.e.* $|\tilde{\gamma}'| = 1$.

▶ **Remark 1.17** It is common in the literature on curves to routinely assume that the curves under consideration have unit speed, usually expressed by saying that they are "**parametrized by arclength**". We will not do this here, for the following reasons:

(i) Making use of the derivative with respect to arclength defined in 1.14 gives us the same elegant formulas as they arise in the context of unit speed curves, without actually changing the parametrization.
(ii) When dealing with one-parameter families $t \mapsto \gamma_t$ of curves of varying length, one cannot assume that all curves $\gamma_t$ are parametrized by unit speed. Therefore, in this situation one has to resort anyway to formulas that remain valid for arbitrary curves.
(iii) In the context of surfaces, there is no obvious analog for the unit speed parametrization of a curve. Therefore, habitual reliance on unit speed parametrizations makes the theory of surfaces look more different from the theory of curves than it actually is.

## 1.3 Unit Tangent and Bending Energy

**Definition 1.18**

For a curve $\gamma\colon [a,b] \to \mathbb{R}^n$, the normalized velocity vector field

$$T\colon [a,b] \to S^{n-1},\ T = \frac{d\gamma}{ds} = \frac{\gamma'}{|\gamma'|}$$

is called the **unit tangent field** of $\gamma$.

Next to the length, the most important numerical quantity that can be assigned to a curve as a whole is its bending energy:

**Definition 1.19**

Let $T$ be the unit tangent field of a curve $\gamma\colon [a,b] \to \mathbb{R}^n$. Then

$$\mathcal{B}(\gamma) = \frac{1}{2}\int_a^b \left\langle \frac{dT}{ds}, \frac{dT}{ds} \right\rangle ds$$

is called the **bending energy** of $\gamma$.

The bending energy is invariant under orientation-preserving reparametrization. The name comes from the following physical picture:

Consider a rod manufactured out of some elastic material in the shape of a thin cylinder of length $L$ and radius $\epsilon$. Then we bend the cylinder into the shape of a curve $\gamma$ of length $L$. While doing this, we make sure that we do not force any twisting on the cylinder, for example we place the cylinder in a hollow tube with shape $\gamma$, leaving it free to untwist itself within the tube (see Fig. 1.4). Then, in the limit of $\epsilon \to 0$, the energy needed to bring the initially straight rod into its new shape will be proportional to $\mathcal{B}(\gamma)$.

In later sections we will find out what curves we obtain if we hold a curve fixed near its end points but otherwise let it minimize bending energy (cf. Fig. 2.3). We also will find a way to deal with twisting.

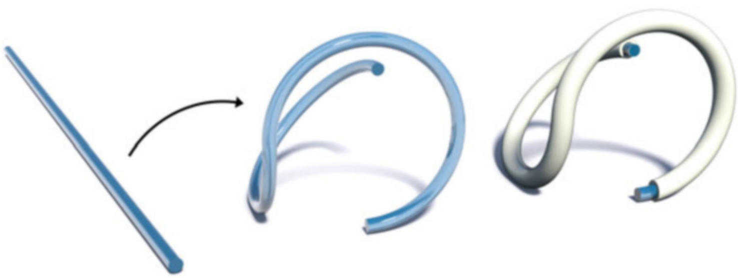


**Fig. 1.4** A rod is bent into the shape of a curve. Then it is fixed in its position by a porcelain case within which it can untwist while staying in shape