Chapter 2
General Topology

2.1 Topological Spaces

If a set X and a collection of its subsets τ satisfy the following three properties (i) τ contains X and the empty set $\emptyset$; (ii) τ is closed under finite intersections; (iii) τ is closed under arbitrary unions; then, the pair (X, τ) is called a ***topological space***. The elements of τ are said to be *open* and their complements in X are said to be *closed*. We assume all our topological spaces (X, τ) to be ***Hausdorff***, that is, for any two points $p_1, p_2 \in \mathsf{X}$ there exist open neighbourhoods $U_1, U_2 \in \tau$ of p_1 and p_2, respectively, such that $U_1 \cap U_2 = \emptyset$. In what follows we frequently drop the explicit declaration of the topology τ. A point $p \in \mathsf{X}$ is called a ***limit*** of the sequence $\{p_k\}_{k\geq 0}$ if for any open neighbourhood U of p there is a $K \in \mathbb{N}$ such that $p_k \in U$ for all $k \geq K$. As X is a Hausdorff space, this limit is unique, which is important in the context of defining dynamical systems and their stability. Moreover, as we will appeal to Whitney's approximation theorems, we assume all our topological spaces (X, τ) to be ***second countable***, that is, there is a set $\mathscr{B} \subseteq \tau$ such that every element in τ can be written as a union of countably many elements in $\mathscr{B}$, i.e., τ admits a countable basis. Then, we call the topological space (X, τ) a n-dimensional ***topological manifold***, when for each $p \in \mathsf{X}$ there is an open neighbourhood $U \in \tau$ of p such that U is ***homeomorphic*** to $\mathbb{R}^n$ (or equivalently, some open set of $\mathbb{R}^n$), that is, there is a continuous bijection between U and $\mathbb{R}^n$ with the inverse of this map also being continuous (see below). When these homeomorphisms fail to exist, but *do* exist when elements of τ are also allowed to be homeomorphic to open subsets of $\mathbb{H}^n = \{p \in \mathbb{R}^n : p_n \geq 0\}$, X is said to be a manifold with ***boundary***, frequently denoted as $\partial\mathsf{X} \neq \emptyset$. Indeed, $\partial(\partial\mathsf{X}) = \emptyset$.

Example 2.1 (*The standard topology on* $\mathbb{R}^n$) Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and let $\mathbb{B}^n_r(p) = \{y \in \mathbb{R}^n : \|p - y\| < r\}$ be an open ball in $\mathbb{R}^n$. The collection of all these open balls gives rise to a topology on $\mathbb{R}^n$, called the norm topology, or the ***standard topology***, denoted τ_{std}. Now it can be shown that the set of all open balls $\mathbb{B}^n_r(p)$, with a rational radius r, centred at a point p with rational coordinates, is a countable basis for the standard topology [4, Chap. IV]. As any two points $p_1, p_2 \in \mathbb{R}^n$ admit

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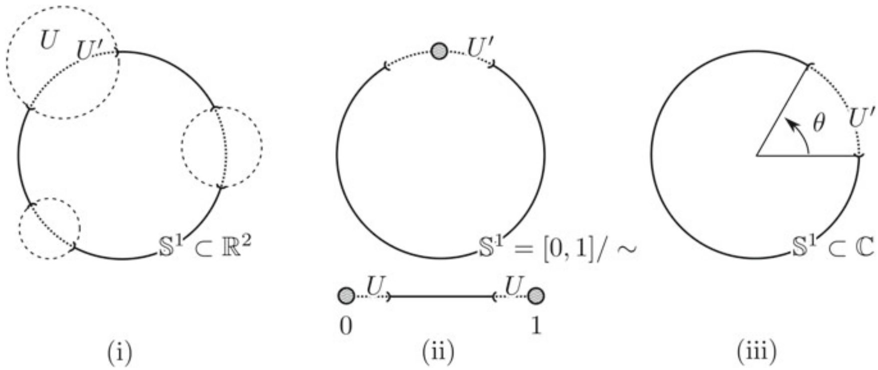

Fig. 2.1 Example 2.2, for the subspace- (i), quotient- (ii) and the standard topology (iii) on $\mathbb{S}^1$, we show a typical open set U'. When applicable, U denotes the corresponding open set in the topological space the topology on $\mathbb{S}^1$ is inherited from

open non-intersecting neighbourhoods $\mathbb{B}^n_r(p_1), \mathbb{B}^n_r(p_2) \in \tau_{\text{std}}$ for $r = \frac{1}{2}\|p_1 - p_2\|$, it readily follows that $(\mathbb{R}^n, \tau_{\text{std}})$ is Hausdorff and second-countable. As any open ball is homeomorphic to $\mathbb{R}^n$, e.g., consider without loss of generality $\mathbb{B}^n_1(0)$ and see that the homeomorphism $\varphi : \mathbb{B}^n_1(0) \to \mathbb{R}^n$ is given by $\varphi : p \mapsto p/(1 - \|p\|)$ with the inverse map $\varphi^{-1} : y \mapsto y/(1 + \|y\|)$, it follows that $(\mathbb{R}^n, \tau_{\text{std}})$ is in fact a topological manifold.

Example 2.2 (*Topologies on the circle* $\mathbb{S}^1$) When looking at the circle as a subset of the plane, i.e., $\mathbb{S}^1 = \{x \in \mathbb{R}^2 : \|x\|_2 = 1\}$, one can define a topology on $\mathbb{S}^1$ via a topology on $\mathbb{R}^2$. Generally, let (X, τ) be a topological space and let $\mathsf{A} \subseteq \mathsf{X}$, then $\tau_{\mathsf{A}} = \{\mathsf{A} \cap U : U \in \tau\}$ is the ***subspace topology*** on A. The circle can also be described as $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ or $\mathbb{S}^1 = [0, 1]/\sim$ for $0 \sim 1$, that is, one identifies all integers. Now again, the topology on $\mathbb{R}$ can be used to generate a topology on $\mathbb{R}/\mathbb{Z}$. Generally, let $\sim$ be an equivalence relation on the topological space (X, τ) and define the surjective map $q : \mathsf{X} \to \mathsf{X}/\sim$, then, the ***quotient topology*** on $\mathsf{X}/\sim$ is defined as $\tau_{/\sim} = \{U \subseteq \mathsf{X}/\sim : q^{-1}(U) \in \tau\}$. A third option would be to directly employ open sets of the form $\{e^{i\theta} : \theta \in (a, b) \subseteq [0, 2\pi]\} \subset \mathbb{C}$ and proceed as in Example 2.1. See Fig. 2.1 for a visualization of these topologies.

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *continuous* at $x \in \mathbb{R}$ when for each $\varepsilon > 0$ there is $\delta > 0$ such that for all $y \in \mathbb{R}$ satisfying $|x - y| < \delta$ one has $|f(x) - f(y)| < \varepsilon$. Some refer to this construction as the $\varepsilon - \delta$ definition of continuity. Imposing a topology on spaces X and Y allows for generalizing the notion of continuity beyond Euclidean spaces. Let (X, τ) and (Y, τ') be topological spaces, then $f : \mathsf{X} \to \mathsf{Y}$ is said to be ***continuous*** when for each $V \in \tau'$ the preimage under f is a contained in τ, i.e., $f^{-1}(V) = \{p \in \mathsf{X} : f(p) \in V\} \in \tau$. Indeed, under the standard topology on $\mathbb{R}$, one recovers the $\varepsilon - \delta$ definition. Another concept of importance is that of *compactness*. An *open cover* of a topological space (X, τ) is a collection of open sets $\mathscr{U} = \{U_j\}_{j \in \mathcal{J}}$ with $U_j \subseteq \mathsf{X}$ for all $j \in \mathcal{J}$, such that $\mathsf{X} = \cup_{j \in \mathcal{J}} U_j$. Then, if a subset of $\mathscr{U}$ still covers X, this subset is said to be a *subcover*. Now a topological space X is ***compact*** when every open cover of X has a *finite* subcover. The notion of compactness is fundamental in topology since for any continuous map $f : \mathsf{X} \to \mathsf{Y}$ between topological spaces X and Y, when X is compact, so is $f(\mathsf{X})$ [9, Theorem 4.32]. A useful result is the Heine-Borel theorem, stating that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded [9, Theorem 4.40]. One should observe that continuity and compactness can be in conflict, i.e., a *fine* topology is desirable from a continuity point of view, yet a *coarse* topology is easier to work with when it comes to compactness.

Regarding notation, we will drop the explicit dependency on τ as the upcoming material is invariant under the particular choice of the topology, as long as the topology satisfies the properties as highlighted above. Besides, the dimension (of the component(s) under consideration) is frequently added by means of a superscript, i.e., X^n denotes a n-dimensional topological manifold. Unless stated otherwise, n will be finite. As mentioned above, maps of interest are *homeomorphisms*, i.e., continuous bijections with a continuous inverse. When two objects are homeomorphic, we speak of ***topological equivalence***, denoted $\simeq_t$. Here, mapping the interval $[0, 1)$ to the circle $\mathbb{S}^1$ is the prototypical example of a map that is continuous and one-to-one, yet not a homeomorphism as the inverse cannot be chosen to be continuous.

2.2 Homotopy and Retractions

It turns out that many topological invariants (under homeomorphims) are invariant under a weaker notion; that of *homotopy*.[1] Let X and Y be topological spaces with g_1 and g_2 continuous maps from X to Y. A continuous map $H : [0, 1] \times \mathsf{X} \to \mathsf{Y}$ is said to be a ***homotopy*** from g_1 to g_2 when for all $p \in \mathsf{X}$ we have $H(0, p) = g_1(p)$ and $H(1, p) = g_2(p)$. If such a map exists, g_1 and g_2 are homotopic, which is an *equivalence relation*, denoted $g_1 \simeq_h g_2$. Moreover, if H is stationary with respect to some set $A \subseteq \mathsf{X}$, that is, $H(t, p) = g_1(p) = g_2(p)$ for all $p \in A$ and $t \in [0, 1]$, then, H is a ***homotopy relative to*** A. We note that not only homotopies give rise to an equivalence class, but also homotopies relative to some subset [14, p. 24]. Two topological spaces X and Y are called ***homotopy equivalent***, or simply *homotopic*, when there are continuous maps $g_1 : \mathsf{X} \to \mathsf{Y}, g_2 : \mathsf{Y} \to \mathsf{X}$ such that $g_1 \circ g_2 \simeq_h \mathrm{id}_{\mathsf{Y}}$ and $g_1 \circ g_2 \simeq_h \mathrm{id}_{\mathsf{X}}$, e.g., generalizing the concept of a homeomorphism to maps that are not necessarily invertible. It is imperative to remark that when colloquially referring to "*the topology of a space* X" one commonly refers to the homotopy type of X.

Definition 2.1 (*Retractions*) Given a topological space X, a subset $A \subseteq \mathsf{X}$ is a ***retract*** of X if there is a continuous map $r : \mathsf{X} \to A$, called a retraction, such that $r \circ \iota_A = \mathrm{id}_A$, for $\iota_A : A \hookrightarrow \mathsf{X}$ the inclusion map. The retraction is said to be a ***deformation retract*** when $\iota_A \circ r \simeq_h \mathrm{id}_{\mathsf{X}}$. We speak of a ***strong deformation retract*** when the homotopy is relative to A. On the other hand, A is ***weak deformation retract*** of X if every open neighbourhood $U \subseteq \mathsf{X}$ of A contains a strong deformation retract V of X such that $A \subseteq V$.

[1] One can argue that homotopy theory is a field of its own and not merely a branch of topology [1].

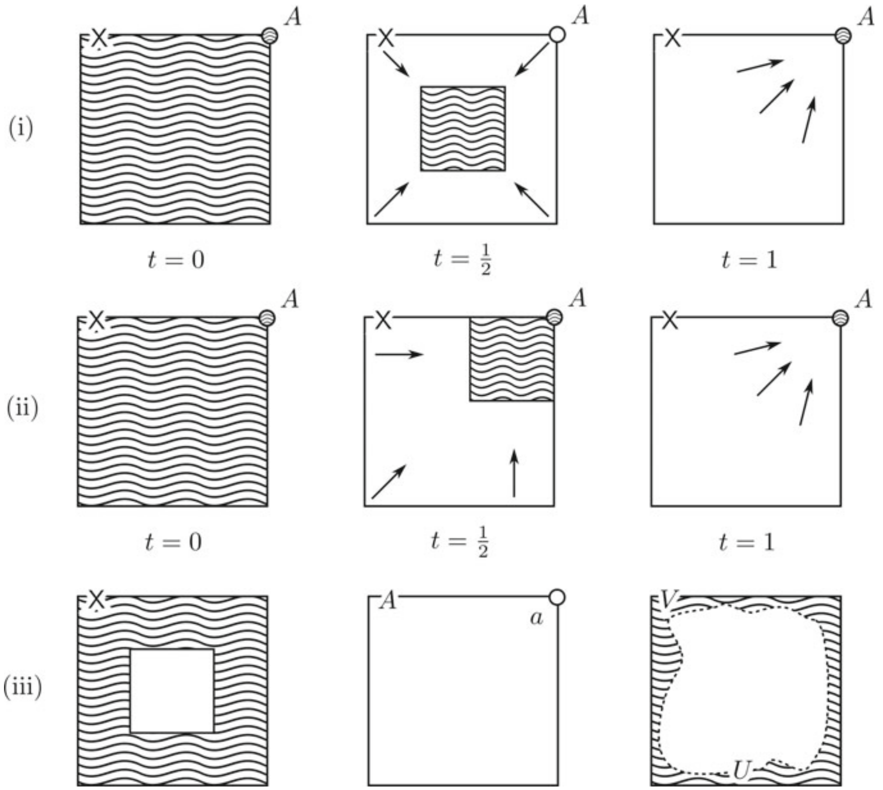

Fig. 2.2 Definition 2.1, (i) a deformation retract of X onto the point A; (ii) a strong deformation retract of X onto the point A; (iii) a weak deformation retract of X onto the set A, with A being the boundary of the cube without the point a, U an open neighbourhood of A and $V \subset U$ a strong deformation retract of X

A deformation retraction maps all of X, continuously, to A, but with A free to move throughout the process. On the other hand, a *strong* deformation retract keeps A stationary, see also Fig. 2.2. A mere retraction to a point is not particularly interesting as one can retract to any point via the constant map. As will be clarified below, deformation retracts, however, relate to stability notions indeed. For more on retraction theory, see [2, 7], it is imperative to remark that the literature does not agree on the terminology used in Definition 2.1 cf. [5].

Lemma 2.1 (Subset deformation retract) *Let both A and B be deformation retracts of C. Then, if $A \subseteq B$, A is a deformation retract of B.*

Proof As C deformation retracts on $A \subseteq C$ there is a map $r_A : C \to A$ such that $r_A \circ \iota_{AC} = \mathrm{id}_A$, $\iota_{AC} \circ r_A \simeq_h \mathrm{id}_C$ for $\iota_{AC} : A \hookrightarrow C$. Similarly for $B \subseteq C$, there is a map $r_B : C \to B$ such that $r_B \circ \iota_{BC} = \mathrm{id}_B$, $\iota_{BC} \circ r_B \simeq_h \mathrm{id}_C$. Now construct the map $r : B \to A$ via the inclusion map $\iota_{BC} : B \hookrightarrow C$, that is, $r = r_A \circ \iota_{BC}$. As $A \subseteq B$, we have that $r \circ \iota_{AB} = r_A \circ \iota_{BC} \circ \iota_{AB} = r_A \circ \iota_{AC} = \mathrm{id}_A$. Moreover, as $\iota_{BC} \circ \iota_{AB} \circ r_A \simeq_h \mathrm{id}_C$ and $r_B \circ \iota_{BC} = \mathrm{id}_B$ we have that $\iota_{AB} \circ r \simeq_h \mathrm{id}_B$, as desired.

For more on the relation between homotopies and deformation retractions, see, [7], [14, Chap. 1], [5, Chap. 0] or [10, Chap. 7].

When for a closed subset $A \subseteq \mathsf{X}$ there is an open neighbourhood $U \subseteq \mathsf{X}$ of A such that A is any retraction type from Definition 2.1 of U, then A is said to be a ***neighbourhood retract***, of that particular type, e.g., a neighbourhood deformation retract, reconsider Fig. 2.2(iii).

Lemma 2.2 (Neighbourhood retracts [12, Theorem 4]) *Let $A \subseteq \mathsf{X}$ be a weak deformation retract of $B \subseteq \mathsf{X}$, then the following hold:*

(i) if A is a neighbourhood retract of X, then A is a retract of B;
(ii) if A is a neighbourhood deformation retract of X, then A is a deformation retract of B;
(iii) if A is a strong neighbourhood deformation retract of X, then A is a strong deformation retract of B.

The intuition behind Lemma 2.2 is that B strongly deformation retracts onto a *neighbourhood* of A, which can be subsequently retracted to A itself.

The prototypical retraction example is that of the sphere $\mathbb{S}^{n-1}$ being a strong deformation retract of the punctured Euclidean space $\mathbb{R}^n \setminus \{0\}$. To see this, consider $r(p) = p/\|p\|_2$ and let the homotopy, relative to $\mathbb{S}^{n-1}$, be the convex combination of r and $\mathrm{id}_{\mathbb{R}^n}$, that is, $H(t, p) = tr(p) + (1-t)p$. See Example 3.1 for a retraction in the context of vector bundles, Example 5.3 for a homotopy in the context of Lyapunov functions and Example 6.10 for strong deformation retracts of Lie groups.

A set S is ***contractible*** when id_{S} is homotopic to a constant map. Equivalently, S is homotopy equivalent to a point or a point $p \in \mathsf{S}$ is a deformation retract of S. For example, X in Fig. 2.2(i) is contractible, while X in Fig. 2.2(iii) is not. Note, contractability does not imply that the deformation is *strong* [5, Exercise 0.6].

Remark 2.1 (*On contractible sets*) One might expect that all n-dimensional contractible sets are homeomorphic to $\mathbb{R}^n$. In 1935, Whitehead provided the first counterexample. Namely, there is an open, 3-dimensional manifold which is contractible but not homeomorphic to $\mathbb{R}^3$, see [15]. Although we focus on the finite-dimensional setting, more counter-intuitive phenomena appear in the infinite-dimensional setting. For example, $\mathbb{S}^\infty$ is contractible [5, Example 1B.3].

2.3 Comments on Triangulation

Motivated by Morse [3, p. 913], triangulations were formally introduced by Cairns, with further initial work by Brouwer, Freudenthal and Whitehead [8, Chap. 15]. A topological space X is called ***triangulable*** when the space is homeomorphic to some

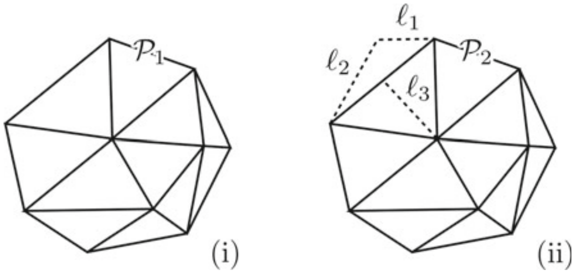

Fig. 2.3 Adding the lines ℓ_1, ℓ_2 and ℓ_3 preserves the Euler characteristic

polyhedron $\mathcal{P}$. Then, the *Euler characteristic* for surfaces of polyhedra is given by $\chi(\mathcal{P}) = \mathcal{V} - \mathcal{E} + \mathcal{F}$, for $\mathcal{V}$ the number of vertices (0-dimensional), $\mathcal{E}$ the number of edges (1-dimensional) and $\mathcal{F}$ the number of faces[2] (2-dimensional) of the polyhedron $\mathcal{P}$ at hand. It turns out that this number $\chi(\mathcal{P})$ equals $2-2g$, for g the number of holes in $\mathcal{P}$ and is independent of how one selects the triangulation, as such, χ is a *topological invariant* of X, see Fig. 2.3. This invariance is why in what follows one will keep seeing alternating sums akin to $\chi(\mathcal{P})$. Studying a topological space X via a naïve triangulation, however, requires attention above dimension 3, those topological spaces do not have a canonical triangulation e.g., see [13], [9, Chap. 5] for more on the so-called *Hauptvermuntung*.

For further references on general topology, see [5, 7, 9, 14] and see [11] for how homotopies appeared in the context of robust control, albeit not explicitly.

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[2] See [6] for comments on prevailing misunderstandings of $\chi(\mathcal{P})$.