Chapter 2
2d Sigma-Models and DAHA

In this section, we study representation theory of DAHA, strictly speaking, the spherical subalgebra of DAHA of type A_1, in terms of *brane quantization* in the 2d A-model [85] on the moduli space of flat $\mathrm{SL}(2,\mathbb{C})$-connections on a once-punctured torus. The brane quantization lends itself well to a geometric approach to representation theory of spherical DAHA, which provides novel viewpoints. The main goal of this section is to explicitly show the correspondence between A-branes with compact Lagrangian submanifolds and finite-dimensional representations of spherical DAHA with respect to dimensions, shortening conditions and morphisms. This matching enables us to find new finite-dimensional representations. The geometric picture also allows us to identify $\mathrm{PSL}(2,\mathbb{Z})$ actions on some finite-dimensional modules. As another advantage, we generalize Cherednik's polynomial representation from a geometric viewpoint. These results play a crucial role in higher-dimensional physical theories and categorical structures in the subsequent sections.

DAHA associated to a root system R (or, equivalently, to a semisimple Lie algebra $\mathfrak{g}$) can be constructed by beginning with the quantum torus algebra $QT(\mathsf{P}\oplus\mathsf{P}^\vee,\omega)$ defined on the direct sum of the weight and coweight lattices of $\mathfrak{g}$ with the symplectic pairing ω between P and $\mathsf{P}^\vee$. More concretely, $QT(\mathsf{P}\oplus\mathsf{P}^\vee,\omega)$ can be understood as the group algebra of the Heisenberg group with the relation

$$X^\mu Y^\lambda = q^{(\mu,\lambda)} Y^\lambda X^\mu, \quad \text{for} \quad \mu\in\mathrm{P},\lambda\in\mathrm{P}^\vee\ ,$$

where (μ,λ) is the symplectic pairing. Note that this lattice is isomorphic to the standard pairing on $\mathbb{Z}^{2\dim\mathsf{P}}\cong\mathbb{Z}^{2n}$, so that the algebra has outer automorphism group $\mathrm{Out}(QT(\mathsf{P}\oplus\mathsf{P}^\vee,\omega))=\mathrm{Sp}(2n,\mathbb{Z})$.

However, we have the additional data of the action of the Weyl group W on P and $\mathsf{P}^\vee$. This gives a distinguished embedding of W into $\mathrm{Sp}(2n,\mathbb{Z})$, which therefore determines an extension

$$0\to QT(\mathsf{P}\oplus\mathsf{P}^\vee,\omega)\to \ddot{H}_{t=1}(W)\to\mathbb{C}[W]\to 0 \tag{2.1}$$

S. Gukov et al., *Branes and DAHA Representations*,
SpringerBriefs in Mathematical Physics,
https://doi.org/10.1007/978-3-031-28154-9_2

up to equivalence. The algebra $\ddot{H}_{t=1}(W)$ is known to be the group algebra of the *double affine Weyl group* $\ddot{W}$: $\ddot{H}_{t=1}(W) \cong \mathbb{C}[\ddot{W}]$. Since the representation of W is just on P (and contragredient on $\mathsf{P}^\vee$), this extension leaves the "diagonal" $\mathrm{Sp}(2, \mathbb{Z})$ subgroup unbroken as outer automorphisms of $\ddot{H}_{t=1}(W)$. For the Cartan type A_1, this construction is equivalent to the algebra $\ddot{H}_{t=1}$ in Appendix C.3. Moreover, the algebra $\ddot{H}_{t=1}(W)$ can be further deformed by other formal parameters t, transforming the group algebra $\mathbb{C}[W]$ to the Hecke algebra. The result is DAHA $\ddot{H}(W)$. We will give a concrete description of the deformation in the Cartan type A_1 in this section. DAHAs of general Cartan types are explained in Appendix B. Through this construction, the quantum torus algebra and DAHA are closely related, and we can take the same approach to representation theory of the quantum torus algebra. Although the representation theory of the quantum torus algebra is well-known, it can be a useful guide for DAHA. Therefore, the reader can refer to Appendix C for the brane quantization of the quantum torus algebra and symmetrized quantum torus.

The algebra $\ddot{H}(W)$ is not commutative, even in the $q = 1$ limit. Nonetheless, it contains the *spherical subalgebra* $S\ddot{H}(W)$, obtained by an idempotent projection, which is commutative as $q = 1$. In the limit $t = 1$, $S\ddot{H}_{t=1}(W)$ is isomorphic to the Weyl-invariant subalgebra of $QT(\mathsf{P} \oplus \mathsf{P}^\vee, \omega)$ (after a lift of the Weyl group action is chosen). In the further specialization $q = 1$, $S\ddot{H}$ becomes precisely the algebra of Weyl-invariant functions on

$$(\mathfrak{t}_\mathbb{C}/\mathsf{Q}^\vee) \times (\mathfrak{t}_\mathbb{C}^\vee/\mathsf{Q}) = T_\mathbb{C} \times T_\mathbb{C} \ .$$

Note that we take the coroot and root lattices $\mathsf{Q}^\vee \oplus \mathsf{Q} = \mathrm{Hom}(\mathsf{P}, \mathbb{Z}) \oplus \mathrm{Hom}(\mathsf{P}^\vee, \mathbb{Z})$ (namely the dual lattice) as the quotient lattice. This space with group action is nothing other than the moduli space of flat connections on a two-torus T^2, valued in the corresponding complex Lie group $G_\mathbb{C}$:

$$\begin{aligned} \mathcal{M}_{\mathsf{flat}}(T^2, G_\mathbb{C}) &= \mathrm{Hom}(\pi_1(T^2), G_\mathbb{C})/G_\mathbb{C} \\ &\cong \frac{T_\mathbb{C} \times T_\mathbb{C}}{W} . \end{aligned} \tag{2.2}$$

We would like to consider an additional deformation of this moduli space to study the representation theory of spherical DAHA geometrically. Happily, for type A, this can be achieved just by adding a "puncture" on a two-torus T^2. Despite this rather simple "addition", the story becomes incredibly deeper and more interesting. This section focuses on DAHA of rank one to illustrate and highlight all the delicate features and interesting phenomena. In rank one, we can perform concrete computations as explicitly as possible. For that reason, we will first review some necessary background on the moduli space of flat $\mathrm{SL}(2, \mathbb{C})$-connections on a once-punctured torus, which will play the role of the target space $\mathfrak{X}$ in the 2d sigma-model.

Then, we will carve out A-branes in $\mathfrak{X}$ for salient modules of the spherical DAHA. This will give solid evidence of the functor (1.3) from the categories of A-branes in $\mathfrak{X}$ to the representation category of the spherical DAHA.

2.1 Higgs Bundles and Flat Connections

Figuratively speaking, the target space of the 2d sigma-model is the stage where our main characters (branes) will make their appearance. Thus, let us begin by setting the stage.

The target space of our system will be the moduli space of $G = \mathrm{SU}(2)$ Higgs bundles on a genus-one curve C_p, ramified at one point p:

$$\mathfrak{X} := \mathcal{M}_H(C_p, G). \tag{2.3}$$

Although the geometry of this space, also called the Hitchin moduli space, is a fairly familiar character in mathematical physics literature, we review those aspects that will be especially important for applications to DAHA representations.

Recall [94, 138], that a ramified (or stable parabolic) Higgs bundle is a pair (E, φ) of a holomorphic SU(2)-bundle E over a curve C and a holomorphic section φ, called the Higgs field, of the bundle $K_C \otimes \mathrm{ad}(E) \otimes \mathcal{O}(p)$. Here, K_C denotes the canonical bundle of C, and $\mathcal{O}(p)$ is the line bundle whose holomorphic sections are functions holomorphic away from p with a first-order pole at p. The ramification at p—more precisely called *tame ramification* since we are considering first-order pole—is described by the following conditions on the connection A on E and the Higgs field

$$\begin{aligned} A &= \alpha_p\, d\vartheta + \cdots \\ \varphi &= \frac{1}{2}(\beta_p + i\gamma_p)\frac{dz}{z} + \cdots \end{aligned} \tag{2.4}$$

Here, $z = re^{i\vartheta}$ is a local coordinate on a small disk centered at p, and the ramification data is a triple of continuous parameters, $(\alpha_p, \beta_p, \gamma_p) \in T \times \mathfrak{t} \times \mathfrak{t}$ where we denote the Cartan subgroup $T \subset G$ and the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$. With this prescribed behavior at p, the Hitchin moduli space is the space of solutions to the equations

$$\begin{aligned} F - [\varphi, \overline{\varphi}] &= 0 \\ \overline{D}_A\, \varphi &= 0\ , \end{aligned} \tag{2.5}$$

modulo gauge transformations. We denote this moduli space $\mathcal{M}_H(C_p, G)$, where C_p is a Riemann surface C with the tame ramification (2.4) at $p \in C$. It is a hyper-Kähler space and the corresponding Kähler forms are

$$\begin{aligned}
\omega_I &= -\frac{i}{2\pi}\int_C |d^2z|\,\mathrm{Tr}\Big(\delta A_{\bar z}\wedge\delta A_z - \delta\bar\varphi\wedge\delta\varphi\Big)\,,\\
\omega_J &= \frac{1}{2\pi}\int_C |d^2z|\,\mathrm{Tr}\Big(\delta\bar\varphi\wedge\delta A_z + \delta\varphi\wedge\delta A_{\bar z}\Big)\,,\\
\omega_K &= \frac{i}{2\pi}\int_C |d^2z|\,\mathrm{Tr}\Big(\delta\bar\varphi\wedge\delta A_z - \delta\varphi\wedge\delta A_{\bar z}\Big)\,.
\end{aligned} \tag{2.6}$$

There is also a triplet of holomorphic symplectic forms $\Omega_I = \omega_J + i\omega_K$, $\Omega_J = \omega_K + i\omega_I$, and $\Omega_K = \omega_I + i\omega_J$, holomorphic in complex structures I, J, and K, respectively. In the absence of ramification, it is easy to check that ω_J and ω_K are cohomologically trivial [117, Sect. 4.1], whereas ω_I is non-trivial and, if properly normalized, can be taken as a generator of $H^2(\mathfrak{X},\mathbb{Z})$. On the other hand, in the presence of ramification (2.4), the cohomology classes of ω_J and ω_K are proportional to β_p and γ_p, respectively.

The description of $\mathcal{M}_H(C_p, G)$ as the moduli space of Higgs bundles given above is in complex structure I. Another useful description, in complex structure J, comes from identifying a complex combination $A_{\mathbb{C}} = A + i\phi$ with a $G_{\mathbb{C}}$-valued connection, where $\phi = \varphi + \bar\varphi$. The Hitchin equations then become the flatness condition $F_{\mathbb{C}} = dA_{\mathbb{C}} + A_{\mathbb{C}}\wedge A_{\mathbb{C}} = 0$ for this $G_{\mathbb{C}}$-valued connection $A_{\mathbb{C}}$. According to (2.4), it has a non-trivial monodromy around the point p:

$$U = \exp(2\pi(\gamma_p + i\alpha_p))\,. \tag{2.7}$$

which depends holomorphically on $\gamma_p + i\alpha_p$ and is independent of β_p. Indeed, in complex structure J, β_p is a Kähler parameter and $\gamma_p + i\alpha_p$ is a complex structure parameter. Another useful fact, also explained in [84], is that the cohomology class of the holomorphic symplectic form $\Omega_J = \omega_K + i\omega_I$ is proportional to $\gamma_p + i\alpha_p$ and independent of β_p.

Similarly, in complex structure I the Kähler modulus is α_p, while $\beta_p + i\gamma_p$ is a complex structure parameter. The cohomology class of the holomorphic symplectic form $\Omega_I = \omega_J + i\omega_K$ is $\beta_p + i\gamma_p$. There is a similar story for complex structure K and all these statements are summarized in Table 2.1.

In a supersymmetric sigma-model with target $\mathfrak{X}$, the Kähler modulus of the target space is always complexified. This fact plays an important role in mirror symmetry. In the present setup, too, the Kähler moduli are all complexified by the periods of the 2-form field B. For example, in complex structure I, the complexified Kähler

Table 2.1 Complex and Kähler moduli of the moduli space $\mathcal{M}_H$ with one ramification point

Complex structure	Complex modulus	Kähler modulus
I	$\beta_p + i\gamma_p$	α_p
J	$\gamma_p + i\alpha_p$	β_p
K	$\alpha_p + i\beta_p$	γ_p

modulus is $\alpha_p + i\eta_p$, where $\eta_p \in T^\vee = \mathrm{Hom}(\Lambda^\vee, \mathrm{U}(1))$ and $\Lambda^\vee$ is the cocharacter lattice of G. Therefore, taking into account the "quantum" parameter η_p, the ramification data consists of the quadruple of parameters $(\alpha_p, \beta_p, \gamma_p, \eta_p)$.

All of these structures can be made completely explicit in the case when C_p is a punctured torus. In complex structure J, where $\mathfrak{X} = \mathcal{M}_H(C_p, G)$ is the moduli space of complex flat connections on C_p, we can then use an explicit presentation of the fundamental group

$$\pi_1(C_p) = \langle \mathfrak{m}, \mathfrak{l}, \mathfrak{c} | \mathfrak{m}\mathfrak{l}\mathfrak{m}^{-1}\mathfrak{l}^{-1} = \mathfrak{c} \rangle \ . \tag{2.8}$$

to describe flat connections concretely, in terms of holonomies along the $(1, 0)$-cycle $\mathfrak{m}$, the $(0, 1)$-cycle $\mathfrak{l}$, and the loop $\mathfrak{c}$ around p:

$$x = \mathrm{Tr}(\rho(\mathfrak{m})), \ \ y = \mathrm{Tr}(\rho(\mathfrak{l})), \ \text{ and } z = \mathrm{Tr}(\rho(\mathfrak{m}\mathfrak{l}^{-1})) \ . \tag{2.9}$$

In terms of these holonomy variables, the space of $SL(2, \mathbb{C})$-representations $\rho : \pi_1(C_p) \to SL(2, \mathbb{C})$ is a cubic surface (see e.g. [72, 82]):

$$\mathcal{M}_{\text{flat}}(C_p, SL(2, \mathbb{C})) = \{(x, y, z) \in \mathbb{C}^3 | x^2 + y^2 + z^2 - xyz - 2 = \mathrm{Tr}(\rho(\mathfrak{c})) = \tilde{t}^2 + \tilde{t}^{-2}\} \ . \tag{2.10}$$

Here we used the fact that, according to (2.7), the holonomy of the complex flat connection around p is conjugate to

$$\rho(\mathfrak{c}) \sim \begin{pmatrix} \tilde{t}^{-2} & 0 \\ 0 & \tilde{t}^2 \end{pmatrix} \ . \tag{2.11}$$

This section will be devoted to studying the deformation quantization $\mathscr{O}^q(\mathfrak{X})$ of this coordinate ring holomorphic in complex structure J, which is generated by x, y, z, and its representations geometrically.

For a complex surface defined by the zero locus of a polynomial $f(x, y, z)$, the holomorphic symplectic form (a.k.a. Atiyah-Bott-Goldman symplectic form) can be written as

$$\Omega_J = \frac{1}{2\pi i} \frac{dx \wedge dy}{\partial f / \partial z} = \frac{1}{2\pi i} \frac{dx \wedge dy}{2z - xy} \ . \tag{2.12}$$

and the Kähler form is

$$\omega_J = \frac{i}{4\pi}(dx \wedge d\bar{x} + dy \wedge d\bar{y} + dz \wedge d\bar{z}) \ . \tag{2.13}$$

In the special case $\alpha_p = \beta_p = \gamma_p = 0$, the moduli space of flat $SL(2, \mathbb{C})$-connections on C_p is simply a quotient space

$$(\mathbb{C}^\times \times \mathbb{C}^\times)/\mathbb{Z}_2 \tag{2.14}$$

by the Weyl group $\mathbb{Z}_2$. It can be understood as a moduli space of flat $SL(2, \mathbb{C})$-connections on a torus (without ramification), such that holonomy eigenvalues along A- and B-cycles each parametrize a copy of $\mathbb{C}^\times$. The "real slice" $(S^1 \times S^1)/\mathbb{Z}_2$ is the moduli space of SU(2) flat connections on the (punctured) torus, and it is sometimes called the "pillowcase". According to the theorem of [128] (resp. [125]), it can be identified with the moduli space $\mathrm{Bun}(C_p, G)$ of stable (resp. parabolic) G-bundles on C_p. It is easy to see that $\mathrm{Bun}(C_p, G)$ is a holomorphic submanifold of $\mathcal{M}_H(C_p, G)$ in complex structure I. Furthermore, because $\delta\varphi = 0$ on $\mathrm{Bun}(C_p, G)$, it follows from (2.6) that $\mathrm{Bun}(C_p, G)$ is a holomorphic Lagrangian submanifold with respect to Ω_I (in particular, Lagrangian with respect to ω_J and ω_K). Following the notation in Sect. 2.4, we write it by $\mathbf{V}$ as a Lagrangian submanifold in the target $(\mathfrak{X}, \omega_{\mathfrak{X}})$.

In addition to $\mathbf{V}$, other special submanifolds of $\mathcal{M}_H(C_p, G)$ will play a role in what follows. For example, in complex structure I, the Hitchin moduli space is a completely integrable Hamiltonian system [94], i.e. a fibration

$$\pi : \mathcal{M}_H(C_p, G) \to \mathcal{B}_H \tag{2.15}$$

over an affine space, the "Hitchin base" $\mathcal{B}_H$, whose generic fibers are abelian varieties (sometimes called "Liouville tori"). For $G = SU(2)$, the map π takes a pair (E, φ) to $\mathrm{Tr}\varphi^2$, which is holomorphic in complex structure I. Specializing further to the case where C_p is a genus-one curve gives a particularly simple integrable system: its generic fiber $\mathbf{F}$ is a torus that, just like $\mathbf{V}$, is holomorphic in complex structure I and Lagrangian with respect to ω_J and ω_K. We also note that the only singular fiber of the Hitchin fibration $\pi : \mathcal{M}_H(C_p, G) \to \mathcal{B}_H$ is the pre-image $\mathbf{N} = \pi^{-1}(0)$ of $0 \in \mathcal{B}_H$ which, in the limit $\alpha_p = \beta_p = \gamma_p = 0$, is the "pillowcase" $\mathbf{V} \cong (S^1 \times S^1)/\mathbb{Z}_2$ with four orbifold points.

Now let us consider what happens when we go away from the limit $\alpha_p = \beta_p = \gamma_p = 0$ and consider generic values of the ramification parameters. From the viewpoint of the complex structure J, the equation (2.10) describes the *deformation* of the four A_1 singularities of the singular surface (2.14), where $\tilde{t}^2$ (or, equivalently, $\gamma_p + i\alpha_p$) plays the role of the complex structure deformation. On the other hand, turning on $\beta_p \neq 0$ leads to a *resolution* of the A_1-singularities. In other words, β_p is the Kähler structure parameter in complex structure J, *cf.* Table 2.1.

Recall that α_p is the Kähler structure parameter in complex structure I. If we turn on α_p while keeping $\beta_p = \gamma_p = 0$, then the four orbifold points are blown up in the Hitchin fibration. Consequently, the singular fiber in the Hitchin fibration, called the *global nilpotent cone* $\mathbf{N} := \pi^{-1}(0)$, now contains five compact irreducible components (all rational) [82, 90]:

$$\mathbf{N} = \mathbf{V} \cup \bigcup_{i=1}^{4} \mathbf{D}_i \ . \tag{2.16}$$

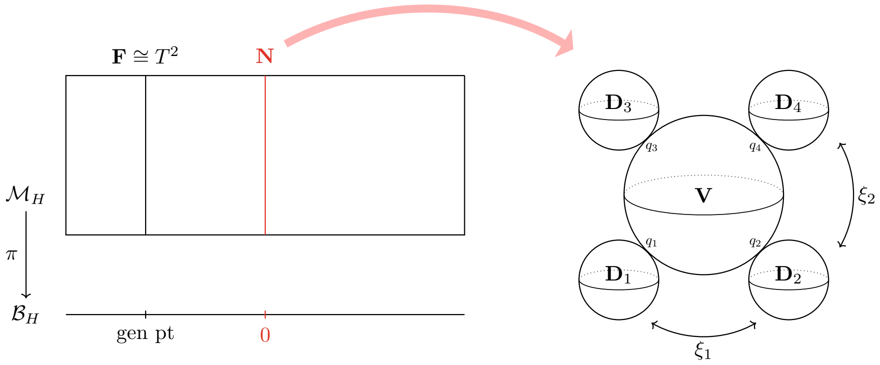

Fig. 2.1 Schematic illustration of the Hitchin fibration $\mathcal{M}_H(C_p, \mathrm{SU}(2)) \to \mathcal{B}_H$ and global nilpotent cone at $\beta_p = 0 = \gamma_p$ and a generic value of $\alpha\alpha_p$

In fact, it is a singular fiber of Kodaira type I_0^* [108, 109] in the elliptic fibration π. The irreducible components $\mathbf{V}$ and $\mathbf{D}_i$ of the global nilpotent cone are holomorphic Lagrangians with respect to Ω_I, sometimes called Lagrangians of type (B, A, A). The homology classes of $\mathbf{V}$ and $\mathbf{D}_i$ provide a basis for the second homology groups $H_2(\mathcal{M}_H(C_p, G), \mathbb{Z})$, and their intersection form is the affine Cartan matrix of type $\widehat{D}_4$, as illustrated in Fig. 2.1. The intersection form has only one null vector, which must be identified with the class of a generic fiber $\mathbf{F}$ of the Hitchin fibration, resulting in the relation

$$[\mathbf{F}] = 2[\mathbf{V}] + \sum_{i=1}^{4} [\mathbf{D}_i] \,. \tag{2.17}$$

Once we move away from $\beta_p = \gamma_p = 0$, we are deforming the complex structure modulus $\beta_p + i\gamma_p$ of complex structure I, and so the structure of the Hitchin fibration drastically changes. For generic values of (β_p, γ_p), the embeddings of the two-cycles $\mathbf{V}$ and $\mathbf{D}_i$ $(i = 1, \ldots, 4)$ into $\mathcal{M}_H(C_p, G)$ are no longer holomorphic with respect to complex structure I, and the singular fiber of type I_0^* splits into three singular fibers of type I_2 [61, Sect. 3.4]. If we write the base genus-one curve C_p of the Hitchin system by an algebraic equation $y^2 = (x - e_1)(x - e_2)(x - e_3)$ with $e_1 + e_2 + e_3 = 0$ where the ramification point p is located at infinity, then the singular fibers of type I_2 are preimages of points

$$\mathcal{B}_H \ni b_i := e_i \mathrm{Tr}\,(\beta_p + i\gamma_p)^2 \qquad (i = 1, 2, 3) \,, \tag{2.18}$$

under the Hitchin fibration as depicted in Fig. 2.2. In the singular fiber at $b_i \in \mathcal{B}_H$, two irreducible components $\mathbf{U}_{2i-1}$ and $\mathbf{U}_{2i}$, which are topologically $\mathbb{C}\mathbf{P}^1$, meet at two double points.

Hence, the two-cycles $\mathbf{V}$ and $\mathbf{D}_i$ $(i = 1, \ldots, 4)$ are not projected to a point by the Hitchin fibration with a generic ramification, though they still give a basis of

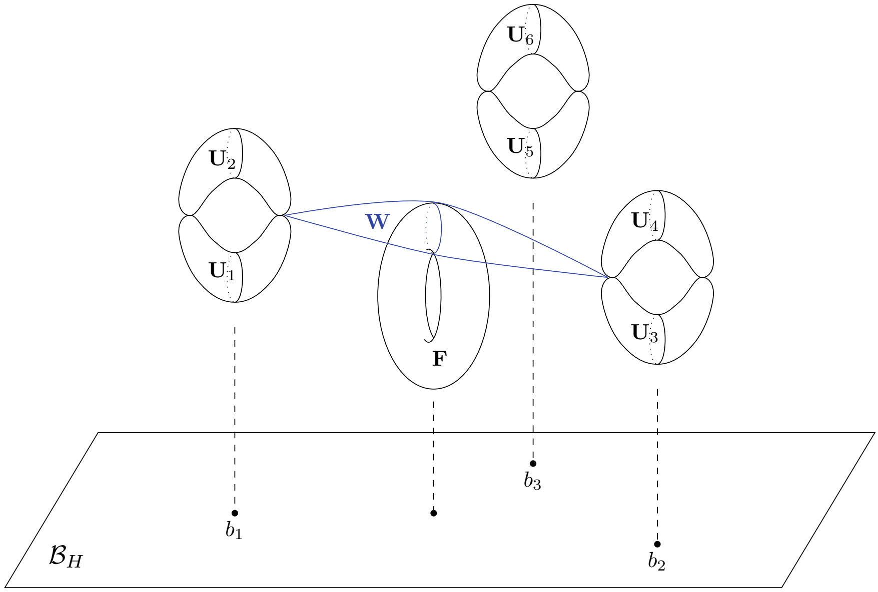

Fig. 2.2 The Hitchin fibration with a generic ramification contains three singular fibers of Kodaira type I_2 at the base points b_i $(i = 1, 2, 3)$

$H_2(\mathcal{M}_H(C_p, G), \mathbb{Z})$ and satisfy the relation (2.17). An analysis by the Mayer–Vietoris sequence tells us that the homology class of each irreducible component in a singular fiber I_2 can be expressed as

$$\begin{aligned}
&[\mathbf{U}_1] = [\mathbf{V}] + [\mathbf{D}_1] + [\mathbf{D}_2]\,, \quad [\mathbf{U}_3] = [\mathbf{V}] + [\mathbf{D}_1] + [\mathbf{D}_3]\,, \quad [\mathbf{U}_5] = [\mathbf{V}] + [\mathbf{D}_1] + [\mathbf{D}_4]\,, \\
&[\mathbf{U}_2] = [\mathbf{V}] + [\mathbf{D}_3] + [\mathbf{D}_4]\,, \quad [\mathbf{U}_4] = [\mathbf{V}] + [\mathbf{D}_2] + [\mathbf{D}_4]\,, \quad [\mathbf{U}_6] = [\mathbf{V}] + [\mathbf{D}_2] + [\mathbf{D}_3]\,,
\end{aligned} \tag{2.19}$$

and there is another two-cycle $\mathbf{W}$ as in Fig. 2.2 with homology class $[\mathbf{W}] = [\mathbf{D}_1]$. With respect to the new basis

$$[\mathbf{U}_1], [\mathbf{U}_2], [\mathbf{U}_3], [\mathbf{U}_5], [\mathbf{W}] \in H_2(\mathcal{M}_H(C_p, G), \mathbb{Z})\,, \tag{2.20}$$

the intersection form becomes

$$\begin{pmatrix} 2 & -2 & 0 & 0 & 1 \\ -2 & 2 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 \\ 1 & -1 & 1 & 1 & 2 \end{pmatrix}. \tag{2.21}$$

Note that the upper-left two-by-two matrix is the Cartan matrix of the affine type $\widehat{A}_1$ as the intersection form of a singular fiber of type I_2.

For our applications to branes and representations, we also need to know the type of the five compact two-cycles $\mathbf{V}$, $\mathbf{D}_i$ $(i = 1, \ldots, 4)$ and periods of the Kähler forms over them. The integrals of Ω_J over $\mathbf{V}$ and over $\mathbf{F}$ were computed e.g. in [82]. They can be expressed as the following set of relations:

$$\begin{aligned}
&\int_{\mathbf{V}} \frac{\omega_I}{2\pi} = \frac{1}{2} - |\alpha_p| \,, && \mathrm{diag}(\alpha_p, -\alpha_p) \sim \alpha_p \in T \,, \\
&\int_{\mathbf{V}} \frac{\omega_J}{2\pi} = -\mathrm{sign}(\alpha_p)\beta_p \,, && \mathrm{diag}(\beta_p, -\beta_p) \sim \beta_p \in \mathfrak{t} \,, \\
&\int_{\mathbf{V}} \frac{\omega_K}{2\pi} = -\mathrm{sign}(\alpha_p)\gamma_p \,, && \mathrm{diag}(\gamma_p, -\gamma_p) \sim \gamma_p \in \mathfrak{t}
\end{aligned} \tag{2.22}$$

and

$$\int_{\mathbf{F}} \frac{\omega_I}{2\pi} = 1 \,, \qquad \int_{\mathbf{F}} \frac{\omega_J}{2\pi} = 0 = \int_{\mathbf{F}} \frac{\omega_K}{2\pi} \,, \tag{2.23}$$

where in the latter we used the fact that the Hitchin fiber $\mathbf{F}$ is holomorphic in complex structure I and Lagrangian with respect to Ω_I for any $(\alpha_p, \beta_p, \gamma_p)$. We assume that α_p takes its value in the range $-\frac{1}{2} < \alpha_p \leq \frac{1}{2}$. Although we did not compute the periods of the 2-forms (2.12) and (2.13) over exceptional divisors $\mathbf{D}_i$ directly, we claim

$$\frac{|\alpha_p|}{2} = \int_{\mathbf{D}_i} \frac{\omega_I}{2\pi} \,, \qquad \mathrm{sign}(\alpha_p)\frac{\beta_p}{2} = \int_{\mathbf{D}_i} \frac{\omega_J}{2\pi} \,, \qquad \mathrm{sign}(\alpha_p)\frac{\gamma_p}{2} = \int_{\mathbf{D}_i} \frac{\omega_K}{2\pi} \,, \tag{2.24}$$

independently of $i = 1, 2, 3, 4$. One way to justify this claim is to compute the periods for small values of $\gamma_p + i\alpha_p \approx 0$, i.e. for $\tilde{t} \approx 1$. Another way is to use (2.17) together with the symmetries of $\mathcal{M}_H(C_p, G)$ that we discuss next. The formulae above are compatible with the fact that the Weyl group symmetry of the ramification parameters given by an overall sign change

$$(\alpha_p, \beta_p, \gamma_p) \to (-\alpha_p, -\beta_p, -\gamma_p) \tag{2.25}$$

leaves the moduli space completely invariant.

Furthermore, the "quantum" parameter that complexifies a Kähler parameter can be understood as the period of the B-field in a 2d sigma-model over $\mathbf{D}_i$

$$\mathrm{sign}(\alpha_p)\eta_p = \int_{\mathbf{D}_i} \frac{B}{2\pi} \,, \qquad \mathrm{diag}(\eta_p, -\eta_p) \sim \eta_p \in T^\vee \,. \tag{2.26}$$

In the following, we often use the parameters $(\alpha_p, \beta_p, \gamma_p, \eta_p) \in S^1 \times \mathbb{R} \times \mathbb{R} \times S^1$ and the quadruple $(\alpha_p, \beta_p, \gamma_p, \eta_p) \in T \times \mathfrak{t} \times \mathfrak{t} \times T^\vee$ of the tame ramification (2.4) at $p \in C$ in the same meaning.

Symmetries

The target space (2.3) of our sigma-model has the symmetry group[1]

$$\Xi \times \mathrm{MCG}(C_p) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathrm{SL}(2,\mathbb{Z}) \tag{2.27}$$

where $\Xi = \mathbb{Z}_2 \times \mathbb{Z}_2$ is the group of "sign changes" generated by twists of a Higgs bundle $E \to C_p$ by line bundles of order 2. Abusing notation, this group can be identified with $H^1(C,\mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ where $\mathbb{Z}_2$ is the center of SU(2). Obviously, $\mathrm{SL}(2,\mathbb{Z})$ is the mapping class of the (punctured) torus:

$$\mathrm{MCG}(C_p) \cong \mathrm{SL}(2,\mathbb{Z}) \ . \tag{2.28}$$

Both Ξ and $\mathrm{MCG}(C_p)$ are symmetries in all complex and symplectic structures. In particular, in what follows, we will need their explicit presentations as holomorphic symplectomorphisms with respect to Ω_J.

In complex structure J, the "sign changes" $\Xi = \mathbb{Z}_2 \times \mathbb{Z}_2$ are holomorphic involutions, and its generators ξ_1, ξ_2 and their combination $\xi_3 := \xi_1 \circ \xi_2$ act as

$$\begin{aligned} \xi_1 : &\quad (x,y,z) \mapsto (-x,y,-z) \ , \\ \xi_2 : &\quad (x,y,z) \mapsto (x,-y,-z) \ , \\ \xi_3 : &\quad (x,y,z) \mapsto (-x,-y,z) \ , \end{aligned} \tag{2.29}$$

respectively. The "sign changes" symmetry plays a very important role to understand mirror symmetry [82] and connections to 4d physics in Sect. 4.

The symmetry group Ξ leaves $\mathbf{V}$ invariant (as a set, not pointwise) and acts on the exceptional divisors $\mathbf{D}_i$ as follows:

$$\begin{aligned} \xi_1 : \mathbf{D}_1 \leftrightarrow \mathbf{D}_2 &\quad \text{and} \quad \mathbf{D}_3 \leftrightarrow \mathbf{D}_4 \ , \\ \xi_2 : \mathbf{D}_1 \leftrightarrow \mathbf{D}_3 &\quad \text{and} \quad \mathbf{D}_2 \leftrightarrow \mathbf{D}_4 \ , \\ \xi_3 : \mathbf{D}_1 \leftrightarrow \mathbf{D}_4 &\quad \text{and} \quad \mathbf{D}_2 \leftrightarrow \mathbf{D}_3 \ . \end{aligned} \tag{2.30}$$

This symmetry, illustrated in Fig. 2.1, provides supporting evidence to our assumption in (2.24).

In complex structure I, a point in the Hitchin base $\mathcal{B}_H$ is invariant under Ξ so that it acts on each fiber as translations of order two in the Hitchin fibration $\mathcal{M}_H \to \mathcal{B}_H$ [61, §3.5]. It acts freely on a generic fiber. On the other hand, ξ_i acts on each irreducible component of the singular fiber $\pi^{-1}(b_i)$, namely $\mathbf{U}_{2i-1}$ and $\mathbf{U}_{2i}$, respectively, where the fixed points are exactly the two double points. At the other singular fibers, it exchanges the two double points and swaps the two irreducible components

[1] The symmetry of the A-model can be larger or smaller than the group of geometric symmetries. It can be larger due to quantum symmetries not directly visible from geometry, and it can be smaller if some geometric symmetries are Q-exact from the A-model viewpoint.

$$\xi_i : \mathbf{U}_{2i+1} \leftrightarrow \mathbf{U}_{2i+2} \quad \text{and} \quad \mathbf{U}_{2i+3} \leftrightarrow \mathbf{U}_{2i+4} \ , \tag{2.31}$$

where the indices of $\mathbf{U}$ are counted modulo 6. This is consistent with the homology classes (2.19) and the actions (2.30).

The action of $SL(2, \mathbb{Z})$ on the eigenvalues of the holonomies $\rho(\mathfrak{m})$ and $\rho(\mathfrak{l})$ is indeed given in (C.30). In particular, the non-trivial central element -1 of $SL(2, \mathbb{Z})$ indeed exchanges the eigenvalues of the holonomies $\rho(\mathfrak{m})$ and $\rho(\mathfrak{l})$ as well as the one around the puncture (2.11) to their inverses. Therefore, it acts as the Weyl group symmetry of $SL(2, \mathbb{C})$. Subsequently, the trace coordinates x, y, z are invariant under the non-trivial central element -1 so that $SL(2, \mathbb{Z})$ acts projectively on the coordinate ring $\mathscr{O}(\mathfrak{X})$ holomorphic in complex structure J. However, the eigenvalues of the holonomy around the puncture are exchanged, which we denote

$$\iota : \tilde{t} \to \tilde{t}^{-1} \ . \tag{2.32}$$

A quotient of $MCG(C_p) \cong SL(2, \mathbb{Z})$ by the center is $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\pm 1$, which is the mapping class group of a 4-punctured sphere. In order to find an explicit presentation of $PSL(2, \mathbb{Z})$, it is convenient to note that $T^2 \to S^2$ is a double cover branched at 4 points, *cf.* (2.14)

$$PSL(2, \mathbb{Z}) \cong Br_3 / \mathcal{Z} \tag{2.33}$$

where the second equality is a well-known relation to the Artin braid group Br_3. In terms of standard generators τ_+ and τ_-^{-1}, which satisfy the braid relation $\tau_+\tau_-^{-1}\tau_+ = \tau_-^{-1}\tau_+\tau_-^{-1}$, the center $\mathcal{Z}$ of Br_3 is generated by $(\tau_+\tau_-^{-1})^3$. Under the surjective map onto $PSL(2, \mathbb{Z})$, we have

$$\tau_+ \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \tau_- \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{2.34}$$

and

$$\sigma := \tau_+\tau_-^{-1}\tau_+ = \tau_-^{-1}\tau_+\tau_-^{-1} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \tau_+\tau_-^{-1} \mapsto \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} . \tag{2.35}$$

In the quotient (2.33), the latter two elements have order 2 and 3, respectively.

Using (2.33), we can relate our present problem to the mapping class group action on the character variety of the 4-punctured sphere[2] which is also a cubic surface of the form (2.10) and on various branes (submanifolds) on this surface [81]:

[2] In the notations of [81] we need to take $(x_1, x_2, x_3) = (-x, -y, -z)$, $\theta_1 = \theta_2 = \theta_3 = 0$, and $\theta_4 = -2 - \tilde{t}^2 - \tilde{t}^{-2}$.

$$\begin{aligned} \tau_+ : \quad & (x, y, z) \mapsto (x, xy - z, y) \,, \\ \tau_- : \quad & (x, y, z) \mapsto (xy - z, y, x) \,, \\ \sigma : \quad & (x, y, z) \mapsto (y, x, xy - z) \,. \end{aligned} \tag{2.36}$$

It is easy to verify that these are indeed polynomial automorphisms of the cubic surface (2.10) and that they satisfy the braid relation.

Note, the action of $\mathrm{PSL}(2, \mathbb{Z})$ leaves $\mathbf{V}$ invariant (as a set, not pointwise) and acts on the exceptional divisors $\mathbf{D}_i$ as on the set of $\mathbb{Z}_2$ torsion points on an elliptic curve, In other words, $\mathbf{D}_1$ is fixed by the $\mathrm{PSL}(2, \mathbb{Z})$, also as a set, not pointwise, whereas $\mathbf{D}_2$, $\mathbf{D}_3$ and $\mathbf{D}_4$ transform as points $\frac{1}{2}$, $\frac{\tau}{2}$, and $\frac{1}{2} + \frac{\tau}{2}$, respectively. In terms of the generators of $\mathrm{PSL}(2, \mathbb{Z})$, we have explicit transformation rules

$$\begin{aligned} \tau_+ : \mathbf{D}_2 \leftrightarrow \mathbf{D}_4 \quad & \text{and} \quad \mathbf{D}_1,\ \mathbf{D}_3 \quad \text{are fixed as a set} \,, \\ \tau_- : \mathbf{D}_3 \leftrightarrow \mathbf{D}_4 \quad & \text{and} \quad \mathbf{D}_1,\ \mathbf{D}_2 \quad \text{are fixed as a set} \,, \\ \sigma : \mathbf{D}_2 \leftrightarrow \mathbf{D}_3 \quad & \text{and} \quad \mathbf{D}_1,\ \mathbf{D}_4 \quad \text{are fixed as a set} \,. \end{aligned} \tag{2.37}$$

In addition, these generators permute the singular fibers of type I_2 in the Hitchin fibration as $\mathfrak{S}_3$:

$$\begin{tikzcd} & \pi^{-1}(b_2) \arrow[loop above, "\tau_+"] \arrow[dl, <->, "\sigma"'] \arrow[dr, <->, "\tau_-"] & \\ \pi^{-1}(b_1) \arrow[loop left, "\tau_-"] \arrow[rr, <->, "\tau_+"'] & & \pi^{-1}(b_3) \arrow[loop right, "\sigma"] \end{tikzcd} \tag{2.38}$$

In the above, we pointed out that $\mathbf{V}$ is invariant under both symmetries Ξ and $\mathrm{PSL}(2, \mathbb{Z})$ only as a set, not pointwise. Also, the same is true about $\mathrm{PSL}(2, \mathbb{Z})$ action on $\mathbf{D}_1$. While in the case of $\mathbf{V}$ the reason for both claims is fairly clear (e.g. it is manifest in the $\tilde{t} \to 1$ limit (2.14)), the fact that $\mathrm{PSL}(2, \mathbb{Z})$ fixes $\mathbf{D}_1$ only as a set and not pointwise is less obvious. In order to explain it, let us consider the limit $\tilde{t} = 1 + \epsilon$, with $\epsilon \ll 1$, and take $(x, y, z) = (2 + a, 2 + b, 2 + c)$. Then, for small values of (a, b, c), the surface (2.10) looks like a quadric

$$a^2 + b^2 + c^2 - 2(ab + bc + ca) = 4\epsilon^2 \,,$$

on which the generators $\tau_\pm$ act as linear reparametrizations:

$$\begin{aligned} \tau_+ : (a, b, c) &\mapsto (a, 2a + 2b - c, b) \,, \\ \tau_- : (a, b, c) &\mapsto (2a + 2b - c, b, a) \,. \end{aligned}$$

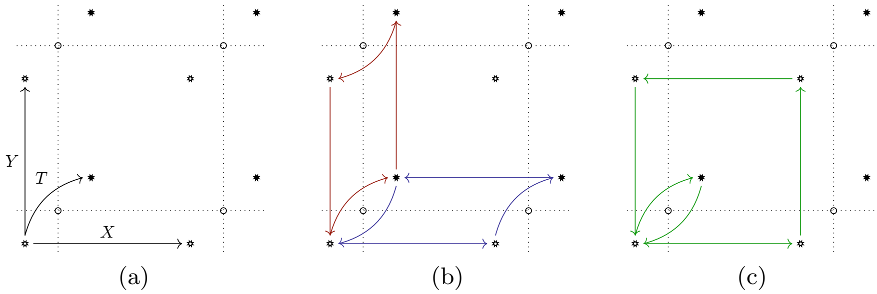

Fig. 2.3 Generators and relations in the orbifold fundamental group of the once-punctured torus. On the left, generators and relations are drawn on the double cover. The relations depicted are $TXT = X^{-1}$, $TY^{-1}T = Y$, and $Y^{-1}X^{-1}YXT^2 = 1$

2.2 DAHA of Rank One and Its Spherical Algebra

Now let us review a few necessary details of DAHA of rank one here. Much like the Hecke algebra sits, loosely speaking, between the Weyl group and the braid group—in the sense that the latter two can be obtained by either specialization or by omitting some of the relations—DAHA sits in between the double affine Weyl group and the double affine braid group. This perspective, reviewed in e.g. [83], will be useful to us in what follows. In Cartan type A_1, the *double affine braid group* (a.k.a. the *elliptic braid group*), denoted $\ddot{\mathrm{Br}}_{q=1}(\mathbb{Z}_2)$, is simply the orbifold fundamental group of the quotient space $(T^2\backslash p)/\mathbb{Z}_2$, the quotient of a once-punctured torus by $\mathbb{Z}_2$. It is generated by three generators X, Y, and T, illustrated in Fig. 2.3:

$$\pi_1^{\mathrm{orb}}\Big((T^2\backslash p)/\mathbb{Z}_2\Big) = \Big(T, X, Y \,|\, TXT = X^{-1},\ \ TY^{-1}T = Y,\ \ Y^{-1}X^{-1}YXT^2 = 1\Big)\,. \qquad (2.39)$$

Its central extension, denoted $\ddot{\mathrm{Br}}(\mathbb{Z}_2)$, is obtained by deforming the last relation to $Y^{-1}X^{-1}YXT^2 = q^{-1}$.

Then, rank-one DAHA $\ddot{H}(\mathbb{Z}_2)$ is obtained by imposing one more quadratic ("Hecke") relation:

$$\ddot{H}(\mathbb{Z}_2) = \mathbb{C}_{q,t}[T^{\pm 1}, X^{\pm 1}, Y^{\pm 1}] \Big/ \left\{ \begin{array}{l} TXT = X^{-1}\,,\ \ Y^{-1}X^{-1}YXT^2 = q^{-1}\,, \\ TY^{-1}T = Y\,,\ \ (T-t)(T+t^{-1}) = 0 \end{array} \right\}\,. \qquad (2.40)$$

This involves the second deformation parameter t. Here $\mathbb{C}_{q,t}$ is a ring of coefficients defined as follows. Let $\mathbb{C}[q^{\pm\frac{1}{2}}, t^{\pm}]$ be the ring of Laurent polynomials in the formal parameters $q^{1/2}$ and t, and consider a multiplicative system M in $\mathbb{C}[q^{\pm\frac{1}{2}}, t^{\pm}]$ generated by elements of the form $(q^{\ell}t - q^{-\ell}t^{-1})$ for any non-negative integer $\ell \in \mathbb{Z}_{\geq 0}$.

We define the coefficient ring $\mathbb{C}_{q,t}$ to be the localization (or formal "fraction")[3] of the ring $\mathbb{C}[q^{\pm\frac{1}{2}}, t^{\pm}]$ at M:

$$\mathbb{C}_{q,t} = M^{-1}\mathbb{C}[q^{\pm\frac{1}{2}}, t^{\pm}] \,. \tag{2.41}$$

This coefficient ring contains the two central generators of the algebra $\ddot{H}(\mathbb{Z}_2)$, q and t, which can be thought of as continuous deformation parameters and start life (in any irreducible representation) as arbitrary complex numbers. Many remarkable things happen when these two parameters assume special values, as will be further discussed in the sequel. In a way, the behavior of the algebra and its representations under such specializations—and the match of this behavior to the A-brane category—is one of the most interesting aspects of the geometric/physical approach.

Another standard notation for the second deformation parameter (which is convenient for some of the specializations) is

$$t = q^c \,. \tag{2.42}$$

where c is often called the "central charge". In what follows, we will use the shorthand notation $\ddot{H} = \ddot{H}(\mathbb{Z}_2)$ unless we wish to make a statement about DAHA of Cartan type other than A_1.

For further details and properties of DAHA, we refer the reader to the fundamental book [35]. The representation theory of DAHA there will be introduced throughout this section, as they emerge from physics and geometry. Also, some basics of DAHA are assembled in Appendix B.

The construction of $\ddot{H}$ based on the punctured torus allows us to see the action of the symmetry group (2.27), and the symmetry plays a pivotal role in the geometric understanding of the representation theory of (spherical) DAHA in what follows. Under Ξ, the generators are transformed as

$$\begin{aligned} \xi_1 &: T \mapsto T,\ X \mapsto -X,\ \ Y \mapsto Y,\ \ q \mapsto q,\ t \mapsto t, \\ \xi_2 &: T \mapsto T,\ \ X \mapsto X,\ \ Y \mapsto -Y,\ q \mapsto q,\ t \mapsto t \,. \end{aligned} \tag{2.43}$$

The mapping class group $SL(2, \mathbb{Z})$ acts on the generators of $\ddot{H}$ as follows[4]:

[3] In other words, $\mathbb{C}_{q,t}$ is the ring of rational functions in the formal parameters $q^{\frac{1}{2}}$ and t where denominators are always elements in the multiplicative system M such as

$$\frac{f(X)}{(t-t^{-1})^{k_0}(qt-q^{-1}t^{-1})^{k_1}\cdots(q^{\ell}t-q^{-\ell}t^{-1})^{k_\ell}}\,, \qquad f(X) \in \mathbb{C}[q^{\pm\frac{1}{2}}, t^{\pm}, X^{\pm}] \,.$$

.

[4] Although we follow the notation of [35] for the transformations $\tau_\pm$ on the generators of DAHA here and in (B.8), we change matrix assignments to $\tau_\pm$ as in (2.34) and (B.10) from [35] since it is consistent with the projective action (2.37) of $SL(2, \mathbb{Z})$ on the exceptional divisors geometrically.

$$\begin{aligned}\tau_+ :& \quad (X, Y, T) \mapsto (X, q^{-\frac{1}{2}} XY, T)\\ \tau_- :& \quad (X, Y, T) \mapsto (q^{\frac{1}{2}} YX, Y, T)\\ \sigma :& \quad (X, Y, T) \mapsto (Y^{-1}, XT^2, T)\end{aligned} \tag{2.44}$$

Since σ essentially exchanges the canonically conjugate variables X and Y, it is sometimes called the *Fourier transform* of $\ddot{H}$. Also, $\ddot{H}$ enjoys the following (non-inner) involution,

$$\tilde{\iota} : T \mapsto -T, \quad X \mapsto X, \quad Y \mapsto Y, \quad q \mapsto q, \quad t \mapsto t^{-1} . \tag{2.45}$$

It is easy to check from the Hecke relation that $\mathbf{e} = (T + t^{-1})/(t + t^{-1})$ is an idempotent element ($\mathbf{e}^2 = \mathbf{e}$) of $\ddot{H}$. Then, the spherical subalgebra $S\ddot{H}$ is defined by the idempotent projection

$$S\ddot{H} := \mathbf{e}\ddot{H}\mathbf{e} . \tag{2.46}$$

The generators of $S\ddot{H}$ can be identified with

$$x = (1 + t^2)\mathbf{e}X\mathbf{e} = (X + X^{-1})\mathbf{e} \tag{2.47}$$

$$y = (1 + t^{-2})\mathbf{e}Y\mathbf{e} = (Y + Y^{-1})\mathbf{e} \tag{2.48}$$

$$z = (q^{-\frac{1}{2}} Y^{-1} X + q^{\frac{1}{2}} X^{-1} Y)\mathbf{e} = \frac{[x, y]_q}{(q^{-1} - q)} , \tag{2.49}$$

and they satisfy relations

$$\begin{aligned}[x, y]_q &= (q^{-1} - q)z\\ [y, z]_q &= (q^{-1} - q)x\\ [z, x]_q &= (q^{-1} - q)y\\ q^{-1}x^2 + qy^2 + q^{-1}z^2 - q^{-\frac{1}{2}}xyz &= (q^{-\frac{1}{2}}t - q^{\frac{1}{2}}t^{-1})^2 + (q^{\frac{1}{2}} + q^{-\frac{1}{2}})^2 ,\end{aligned} \tag{2.50}$$

where $q = e^{2\pi i\hbar}$ and the q-commutator is defined by

$$[a, b]_q := q^{-\frac{1}{2}} ab - q^{\frac{1}{2}} ba .$$

See e.g. [142] for further details. The key point is that the spherical DAHA $S\ddot{H}$ is commutative at the "classical" limit $q = 1$ while the DAHA $\ddot{H}$ is not commutative even in the $q = 1$ limit. Indeed, it is easy to see that in the "classical" limit $q \to 1$, the Casimir relation (the last one) in (2.50) becomes the equation for the cubic surface (2.10):

$$S\ddot{H} \xrightarrow[q\to 1]{} \mathscr{O}(\mathcal{M}_{\text{flat}}(C_p, \mathrm{SL}(2, \mathbb{C}))) . \tag{2.51}$$

Thus, $S\ddot{H}$ is the deformation quantization $\mathscr{O}^q(\mathfrak{X})$ of the coordinate ring (2.10) of the moduli space of flat SL(2, $\mathbb{C}$)-connections $\mathfrak{X} = \mathcal{M}_{\text{flat}}(C_p, \text{SL}(2,\mathbb{C}))$ with respect to the Poisson bracket defined by Ω_J [130, 131].

Here, it is worth commenting on an important issue in the context of the deformation quantization of the coordinate ring on the affine cubic hypersurface of the form (2.10). It is clear that this equation is Weyl-group invariant, so that the monodromy parameter $\tilde{t}$ appears only through the symmetric combination $\tilde{t} + \tilde{t}^{-1}$, and that the same symmetry applies to the Poisson structure. Moreover, if we work with a specific value of $\tilde{t}$, we will obtain the deformation quantization at a specific value of the parameters, i.e. for a specific choice of the central character (at least for the formal parameter t).

Since the inputs to deformation quantization depend on $\tilde{t}$ only in a $\mathbb{Z}_2$-invariant fashion, the output $\mathscr{O}^q(\mathfrak{X}_{\tilde{t}})$ will also have the corresponding symmetry. However, this clarifies that $\tilde{t} \neq t$, since the relations (2.50) do *not* depend symmetrically on t. The proper identification is

$$\tilde{t} = tq^{-\frac{1}{2}} , \tag{2.52}$$

as will be made clear by the discussion of the formal outer automorphism ι below. There is no contradiction with the statement that $S\ddot{H}$ is the deformation quantization of $\mathscr{O}(\mathfrak{X})$, since the classical limit of $S\ddot{H}$ still recovers the same commutative Poisson algebra.

It is simple to check that the two involutions (2.43) straightforwardly reduce to the symmetry of $S\ddot{H}$, which is the same as (2.29). As in the classical case, the nontrivial central element $-1 \in \text{SL}(2,\mathbb{Z})$ acts trivially on the generators of $S\ddot{H}$, and the action of PSL(2, $\mathbb{Z}$) is quantized from (2.36)

$$\begin{aligned}
\tau_+ : \quad & (x, y, z) \mapsto \left(x, \frac{xy + yx}{q^{1/2} + q^{-1/2}} - z, y\right), \\
\tau_- : \quad & (x, y, z) \mapsto \left(\frac{xy + yx}{q^{1/2} + q^{-1/2}} - z, y, x\right), \\
\sigma : \quad & (x, y, z) \mapsto \left(y, x, \frac{xy + yx}{q^{1/2} + q^{-1/2}} - z\right).
\end{aligned} \tag{2.53}$$

Thus, the symmetries $\Xi \times \text{PSL}(2,\mathbb{Z})$ can be seen in outer automorphisms of $S\ddot{H}$. The other outer automorphism $\tilde{\iota}$ in (2.45) is somewhat more complicated; it does not preserve the idempotent, but it rather brings it into the other idempotent element

$$\tilde{\iota} : \quad \mathbf{e} = \frac{T + t^{-1}}{t + t^{-1}} \mapsto \tilde{\mathbf{e}} = \frac{-T + t}{t + t^{-1}} . \tag{2.54}$$

Thus, $\tilde{\iota}$ maps $S\ddot{H}$ to the other spherical subalgebra $\tilde{\mathbf{e}}\ddot{H}\tilde{\mathbf{e}}$ where the Casimir relations are different by $t \leftrightarrow t^{-1}$. However, the involution $\tilde{\iota}$ on $\ddot{H}$ does correspond in a sense to an outer automorphism of $S\ddot{H}$, which acts simply by

$$\iota : t \mapsto qt^{-1}. \tag{2.55}$$

Indeed, it is easy to check that this map preserves the Casimir relation in (2.50). (Note that this automorphism only acts nontrivially when q and t are regarded as formal elements; it does not preserve the central character.)

In general, we are free to think of any commutative algebra as the coordinate ring of a certain affine space. In addition to the example above, we consider the case of $\mathfrak{X} = \mathbb{C}^\times \times \mathbb{C}^\times$ for the quantum torus algebra in Appendix C, and $\mathfrak{X}$ as 3d $\mathcal{N} = 4$ Coulomb branches in Appendix D in this paper. What is common between all of these examples are certain key properties of $\mathfrak{X}$: First of all, it will always be a non-compact manifold, so that it has a large and interesting algebra $\mathcal{O}(\mathfrak{X})$ of holomorphic functions with polynomial growth at infinity. (In fact, in this paper, $\mathfrak{X}$ will always be an affine variety over $\mathbb{C}$.) It will also be a hyper-Kähler manifold, and an algebra is obtained by the deformation quantization of the coordinate ring of $\mathfrak{X}$ with respect to a certain holomorphic symplectic form. These conditions fit into the context of brane quantization [85] in a 2d sigma-model. It is the central idea of this paper, and this will pave the way towards a geometric angle on the representation theory of $S\ddot{H}$.

2.3 Canonical Coisotropic Branes in A-models

Here, we will obtain the deformation quantization of the coordinate ring of $\mathfrak{X}$ with respect to Ω_J by using the 2d A-model on a symplectic manifold $(\mathfrak{X}, \omega_{\mathfrak{X}})$. The main character in our story is the *canonical coisotropic brane*, denoted $\mathfrak{B}_{\text{cc}}$. Eventually, we will investigate the representation theory of $S\ddot{H}$ by the 2d A-model, but we begin by constructing the (presumably less familiar) canonical coisotropic brane $\mathfrak{B}_{\text{cc}}$ here. Subsequently, we will discuss standard Lagrangian branes and some methods for computing spaces of morphisms in what follows. Our review is necessarily cursory; for more details, we refer to the literature [82, 85].

In general, as was pointed out in [107], the A-model admits branes with support on coisotropic submanifolds which are equipped with a transverse holomorphic structure. The canonical coisotropic brane is supported on the target space $\mathfrak{X}$ itself, which is a coisotropic submanifold of the target space $\mathfrak{X}$ in a rather trivial way. More precisely, there is a family of such branes, labeled by a complex parameter

$$\hbar = |\hbar| e^{i\theta} , \tag{2.56}$$

and we will identify it with the parameter of deformation quantizations by $q = e^{2\pi i \hbar}$. The fact that the support involves no additional choice is (at least part of) the reason for the term "canonical." On a $2n$-dimensional target space, coisotropic branes can therefore be supported in dimension $n + 2j$ for integer j; when n is even, there can be branes supported throughout the entire target. In our example, $n = 2$, so that no other coisotropic branes can occur just for dimension reasons.

In complex structure $\mathcal{I} = I\cos\theta - K\sin\theta$, the data defining the brane $\mathfrak{B}_{\mathrm{cc}}$ is simply a holomorphic line bundle $\mathcal{L} \to \mathfrak{X}$, equipped with a connection whose curvature F is of course equal to the first Chern class:

$$\mathfrak{B}_{\mathrm{cc}} : \quad \begin{array}{c} \mathcal{L} \\ \downarrow \\ \mathfrak{X} \end{array} \qquad c_1(\mathcal{L}) = [F/2\pi] \in H^2(\mathfrak{X}, \mathbb{Z}) \; . \tag{2.57}$$

As usual, open strings ending on $\mathfrak{B}_{\mathrm{cc}}$ source the gauge-invariant combination $F + B$, where

$$B \in H^2(\mathfrak{X}, \mathrm{U}(1)) \tag{2.58}$$

is the 2-form B-field. For our family of the canonical coisotropic branes $\mathfrak{B}_{\mathrm{cc}}$ parametrized by $\hbar$ on a symplectic manifold $(\mathfrak{X}, \omega_{\mathfrak{X}})$, the values of $[B/2\pi] \in H^2(\mathfrak{X}, \mathrm{U}(1))$ and the integral class $[F/2\pi] \in H^2(\mathfrak{X}, \mathbb{Z})$ are determined by the equation

$$\Omega := F + B + i\omega_{\mathfrak{X}} = \frac{\Omega_J}{i\hbar} \; , \tag{2.59}$$

so that at a generic value of $\hbar$ in (2.56) we can write

$$\begin{aligned} F + B = \mathrm{Re}\;\Omega &= \frac{1}{|\hbar|}(\omega_I \cos\theta - \omega_K \sin\theta) \; , \\ \omega_{\mathfrak{X}} = \mathrm{Im}\;\Omega &= -\frac{1}{|\hbar|}(\omega_I \sin\theta + \omega_K \cos\theta) \; . \end{aligned} \tag{2.60}$$

Since the hyper-Kähler conditions ensure that $J = \omega_{\mathfrak{X}}^{-1}(F + B)$, we have the condition for $\mathfrak{B}_{\mathrm{cc}}$ to be a coisotropic A-brane [107]

$$\left(\omega_{\mathfrak{X}}^{-1}(B + F)\right)^2 = J^2 = -1 \; . \tag{2.61}$$

In particular, when $\hbar$ is real, $\omega_{\mathfrak{X}} = \omega_K$ and $\mathfrak{B}_{\mathrm{cc}}$ is a brane of type (B, A, A), whereas for $\hbar$ purely imaginary, $\omega_{\mathfrak{X}} = \omega_I$ and $\mathfrak{B}_{\mathrm{cc}}$ is an (A, A, B)-brane. $\mathfrak{B}_{\mathrm{cc}}$ is also called "canonical" because its extra data corresponds in this fashion to the holomorphic symplectic structure.

Now comes the key point. Under this circumstance, the space $\mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathrm{cc}})$ of open $(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathrm{cc}})$ strings with both ends on the canonical coisotropic brane $\mathfrak{B}_{\mathrm{cc}}$ is a non-commutative deformation of the Dolbeault cohomology $H^{0,*}_{\bar\partial}(\mathfrak{X})$ when $\mathfrak{X}$ is regarded as a complex manifold with J, and we are interested in its zeroth degree, namely the space of holomorphic functions. Moreover, for $\mathfrak{X}$ an affine variety, a suitable condition at infinity for a "good A-model" is to allow only functions of polynomial growth. In the presence of non-trivial background $F + B \neq 0$,

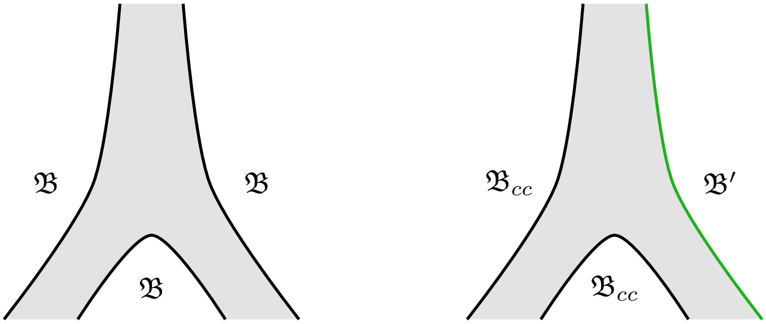

Fig. 2.4 (Left) Open strings that start and end on the same brane $\mathfrak{B}$ form an algebra. (Right) Joining a $(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathrm{cc}})$-string with a $(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}')$-string leads to another $(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}')$-string

$\mathrm{Hom}^0(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathrm{cc}})$ is therefore the deformation quantization of the coordinate ring on $\mathfrak{X}$, holomorphic in complex structure J [8, 85].[5]

In general, for any brane $\mathfrak{B}$, in either the A-model or the B-model, the space of open strings states $\mathrm{End}(\mathfrak{B})$ forms an algebra. This can be easily understood by considering the process of joining open strings, illustrated in Fig. 2.4 (left). However, generically, this algebra of $(\mathfrak{B}, \mathfrak{B})$ strings is rather simple and not very interesting. Even if the brane $\mathfrak{B}$ is "big enough," the algebra $\mathrm{End}(\mathfrak{B})$ can be interesting, but may be hard to identify or relate to more familiar algebras. For example, various (B, B, B) branes represented by hyper-holomorphic sheaves in [82] lead to interesting endomorphism algebras, but apart from some special cases it is hard to recognize them in the world of more familiar algebras. What makes the canonical coisotropic brane special is that the algebra $\mathrm{End}(\mathfrak{B}_{\mathrm{cc}})$ can be identified with the deformation quantization $\mathscr{O}^q(\mathfrak{X})$ of the target manifold $\mathfrak{X}$ [117].

2.3.1 Spherical DAHA as the Algebra of $(\mathfrak{B}_{cc}, \mathfrak{B}_{cc})$-Strings

In our example, the target space $\mathfrak{X} = \mathcal{M}_{\mathrm{flat}}(C_p, \mathrm{SL}(2, \mathbb{C}))$ is the moduli space of flat $\mathrm{SL}(2, \mathbb{C})$-connections over a punctured torus, which is a hyper-Kähler manifold. Then, by construction, the algebra of $(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathrm{cc}})$ strings is the deformation quantization $\mathscr{O}^q(\mathfrak{X})$ of the coordinate ring on $\mathfrak{X}$ with respect to Ω_J, which is the spherical DAHA $S\ddot{H}$.

The parameter q of $S\ddot{H}$ is identified with $\hbar$ in the data (2.59) of $\mathfrak{B}_{\mathrm{cc}}$ via $q = \exp(2\pi i \hbar)$. It is worth emphasizing that for a generic value of $q \in \mathbb{C}^\times$, the B-field needs to be turned on in the sigma-model. In fact, the target admits the Hitchin fibration (2.15) where a generic fiber is a two-torus T^2. Since a generic fiber $\mathbf{F}$ is

[5] Since we are mainly interested in the zeroth degree of morphism spaces, we will usually omit the superscript 0, meaning $\mathrm{Hom} = \mathrm{Hom}^0$ unless it is specified.

Lagrangian with respect to ω_J and ω_K and it sees only ω_I, the evaluation of Ω in (2.59) over $\mathbf{F}$ yields

$$\int_{\mathbf{F}} \frac{\Omega}{2\pi} = \frac{1}{\hbar} ,$$

where $F + B$ is responsible for its real part. Because $[F/2\pi] \in H^2(\mathfrak{X}, \mathbb{Z})$ is an element of the second integral cohomology class, the B-field needs to be switched on unless the real value of $1/\hbar$ is an integer. Thus, a 2d A-model has to incorporate the B-field for a generic value of $\hbar$, and we will moreover witness that the B-field plays a more important role in the subsequent sections.

The parameter t of $S\ddot{H}$ is related to the ramification parameters of the target space. In fact, the monodromy parameter (2.11) around the puncture can be expressed by the ramification parameters (2.7) as

$$\tilde{t} = \exp(-\pi(\gamma_p + i\alpha_p)) .$$

Furthermore, (2.52) compares the monodromy parameter $\tilde{t}$ with the central character t of $S\ddot{H}$. Then, it is easy to see from (2.24) that the evaluation of (2.59) on an exceptional divisor yields

$$\frac{1}{2\pi}\int_{\mathbf{D}_i} F + B + i\omega_{\mathfrak{X}} = \int_{\mathbf{D}_i} \frac{\Omega_J}{2\pi i\hbar} = \frac{\gamma_p + i\alpha_p}{2i\hbar} = -c + \frac{1}{2} . \tag{2.62}$$

where c is the central charge in (2.42).

The canonical coisotropic brane enjoys the symmetries $\Xi \times \mathrm{PSL}(2, \mathbb{Z})$ of the target space $\mathfrak{X}$ analyzed in Sect. 2.1, which become the outer automorphisms of $S\ddot{H}$ given by (2.29) and (2.53). The symmetry (2.55) of $S\ddot{H}$ is indeed the Weyl group symmetry $\tilde{t} \leftrightarrow \tilde{t}^{-1}$ of the monodromy matrix (2.11). In fact, the Weyl group symmetry (2.25) of the ramification parameters preserves the target space. Since the canonical coisotropic brane is sensitive only to (α_p, γ_p) or $\tilde{t}$, the symmetry (2.55) of $S\ddot{H}$ is equivalent to the fact that the canonical coisotropic branes supported on $\mathfrak{X}_{\tilde{t}}$ and $\mathfrak{X}_{\tilde{t}^{-1}}$ related by the Weyl group symmetry give rise to the isomorphic algebra

$$\mathrm{End}(\mathfrak{B}_{\mathrm{cc}}) \cong S\ddot{H} \cong \mathrm{End}(\iota(\mathfrak{B}_{\mathrm{cc}})) . \tag{2.63}$$

2.4 Lagrangian A-Branes and Modules of $\mathscr{O}^q(\mathfrak{X})$

Now we lay out the approach to the representation theory of $\mathscr{O}^q(\mathfrak{X})$ by the 2d A-model on $(\mathfrak{X}, \omega_{\mathfrak{X}})$. This subsection also serves as a lightning review about the category of A-branes.

The approach to the representation theory of $\mathscr{O}^q$ from the 2d A-model arises from a simple idea: given an open string boundary condition (or an A-brane) $\mathfrak{B}'$, the space of $(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}')$ open strings forms a vector space. As in the right of Fig. 2.4, a joining of $(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathrm{cc}})$ and $(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}')$ string leads to another $(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}')$ string. This implies that the space of $(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}')$ strings receives an action of the algebra of $(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathrm{cc}})$ strings [85]. Namely, other A-branes $\mathfrak{B}'$ on $\mathfrak{X}$ give rise to modules for $\mathscr{O}^q(\mathfrak{X})$:

$$\begin{array}{ccc} \mathscr{O}^q(\mathfrak{X}) & = & \mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathrm{cc}}) \\ \circlearrowleft & & \circlearrowleft \\ \mathscr{B}' & = & \mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}') \end{array} \tag{2.64}$$

In our example, supports of other branes $\mathfrak{B}'$ are always Lagrangian submanifolds so that we will review Lagrangian A-branes $\mathfrak{B}_{\mathbf{L}}$ in the next subsection. If the support of $\mathfrak{B}'$ is a Lagrangian submanifold contained in the fixed point set of an antiholomorphic involution $\zeta : \mathfrak{X} \to \mathfrak{X}$ with $\zeta^* J = -J$, then the corresponding representation admits unitarity.

We now briefly recall a few standard facts about Lagrangian A-branes [56, 57] and their mathematical incarnation, the Fukaya category $\mathsf{Fuk}(\mathfrak{X}, \omega_{\mathfrak{X}})$. For more detail, the reader is referred to the literature, which is substantial; [7] is a good starting point, or [110] for the fundamentals of homological mirror symmetry.

The *Lagrangian Grassmannian*, denoted LGr, of a symplectic vector space parameterizes the collection of its Lagrangian subspaces. We can obtain a description of this space by thinking of the standard symplectic vector space $(\mathbb{R}^{2n}, \omega)$, which can be equipped with a metric via a contractible choice. By the two-of-three property, the group preserving both the symplectic and orthogonal structures is $\mathrm{U}(n)$, which therefore acts on $\mathrm{LGr}(2n)$; the subgroup stabilizing a fixed Lagrangian subspace is $\mathrm{O}(n)$, so that

$$\mathrm{LGr}(2n) = \mathrm{U}(n)/\,\mathrm{O}(n)\ . \tag{2.65}$$

There is furthermore an obvious map

$$\det{}^2 : \mathrm{LGr}(2n) \to \mathrm{U}(1) \tag{2.66}$$

which can be shown to induce an isomorphism of fundamental groups. The *Maslov index* [3] of a loop in $\mathrm{LGr}(2n)$ is its image under this induced map in $\pi_1(\mathrm{U}(1)) \cong \mathbb{Z}$; it is responsible for both obstructions and gradings in the Fukaya category. The universal cover $\widetilde{\mathrm{LGr}}(2n)$ of $\mathrm{LGr}(2n)$ thus has deck group $\mathbb{Z}$, and the Maslov index of a loop is simply the element of $\mathbb{Z}$ that connects the endpoints of a lift to $\widetilde{\mathrm{LGr}}(2n)$.

Let $(\mathfrak{X}, \omega_{\mathfrak{X}})$ be a symplectic manifold with zero first Chern class (as is obviously the case in our hyper-Kähler examples). There is a bundle

$$\mathrm{LGr}(\mathfrak{X}) \to \mathfrak{X} \tag{2.67}$$

whose fiber over $x \in \mathfrak{X}$ is $\mathrm{LGr}(T_x\mathfrak{X})$. (We hope the reader will forgive the moderately abusive notation.) We can furthermore define a bundle $\widetilde{\mathrm{LGr}}(\mathfrak{X})$, which is a covering space of the total space $\mathrm{LGr}(\mathfrak{X})$, such that the covering map is a bundle map and restricts over each fiber to the universal covering map.

A Lagrangian subspace $\mathbf{L} \subset \mathfrak{X}$ comes with an obvious lift

$$\begin{tikzcd} & \mathrm{LGr}(\mathfrak{X}) \arrow[d] \\ \mathbf{L} \arrow[ur, hook] \arrow[r, "\subset"] & \mathfrak{X} \end{tikzcd} \tag{2.68}$$

defined by the Lagrangian subbundle $T\mathbf{L} \subset T\mathfrak{X}|_{\mathbf{L}}$. Lifting this canonical map to $\widetilde{\mathrm{LGr}}(\mathfrak{X})$ is obstructed by the image of $\pi_1(\mathbf{L})$ under the Maslov map, which is an element of $H^1(\mathbf{L}, \mathbb{Z})$ called the *Maslov class*. Lagrangians with zero Maslov class admit so-called *graded lifts*, which are maps

$$\begin{tikzcd} \widetilde{\mathrm{LGr}}(\mathfrak{X}) \arrow[r, "\cdot/\mathbb{Z}"] & \mathrm{LGr}(\mathfrak{X}) \arrow[d] \\ \mathbf{L} \arrow[u, dashed, "g"] \arrow[ur, hook] \arrow[r, "\subset"] & \mathfrak{X} \end{tikzcd} \tag{2.69}$$

making the square commute. The set of such maps is naturally a $\mathbb{Z}$-torsor under the action of deck transformations, but no canonical choice of graded lift exists. Given a Lagrangian object of A-$\mathsf{Brane}(\mathfrak{X}, \omega_{\mathfrak{X}})$, the set of graded lifts plays the role of its shifts.

A (rank-one) Lagrangian object of A-$\mathsf{Brane}(\mathfrak{X}, \omega_{\mathfrak{X}})$ is supported on a Lagrangian submanifold $\mathbf{L} \subset \mathfrak{X}$ of zero Maslov class, which is considered up to Hamiltonian isotopy. The additional data required to define a Lagrangian A-brane consists of a "Chan-Paton" bundle with unitary connection; a flat Spin^c structure on $\mathbf{L}$; and a grade lift. A Chan-Paton bundle for a Lagrangian A-brane is generally endowed with a flat Spin^c structure [60, 85, 114, 154]. A Spin^c structure arises if $\mathcal{L}'$ does not exist as a line bundle, but is obstructed by the same cocycle that obstructs the existence of the square root $K_{\mathbf{L}}^{-1/2}$ of the canonical bundle over the Lagrangian $\mathbf{L}$. Namely, putative transition functions g_{ij} and w_{ij} of $\mathcal{L}'$ and $K_{\mathbf{L}}^{-1/2}$, respectively, obey $g_{ij}g_{jk}g_{ki} = \phi_{ijk} = w_{ij}w_{jk}w_{ki}$ where $\phi_{ijk} = \pm 1$. In this case, the cocycle cancels out in the transition functions $g_{ij}w_{ij}$ of an honest vector bundle $\mathcal{L}' \otimes K_{\mathbf{L}}^{-1/2} \to \mathbf{L}$, called a Spin^c structure. The $K_{\mathbf{L}}^{-1/2}$ part in a Spin^c structure arises from boundary fermions of the open worldsheet [96, Sect. 5] [92, Sect. 3.2], which gives rise to a Spin^c structure of the normal bundle to the support of a brane. (This proposal is explicitly checked by Hemisphere partition functions in [104].) Thus, the canonical coisotropic brane $\mathfrak{B}_{\mathrm{cc}}$ is endowed with an ordinary line bundle whereas a Lagrangian A-brane is equipped with a Spin^c structure. Since most of the Lagrangian submanifolds in this paper are of real two dimensions, there always exists a spin bundle of $\mathbf{L}$, which is a square-root of the canonical bundle $K_{\mathbf{L}}^{\pm 1/2}$ of $\mathbf{L}$, though it is not necessarily unique. Hence, both $\mathcal{L}'$ and $K_{\mathbf{L}}^{-1/2}$ exist as genuine line bundles in most of the examples in this paper and we treat their tensor product $\mathcal{L}' \otimes K_{\mathbf{L}}^{-1/2}$ as a Spin^c structure. However, a subtlety arises when an A-brane degenerates into a different spin structure, which will be considered in Sect. 2.7. Moreover, a Lagrangian A-brane is endowed with a flat Spin^c structure: if $\mathcal{L}'$ exists as a line bundle, the curvature $F'_{\mathbf{L}}$ of $\mathcal{L}'$ must obey a gauge-invariant version of the flatness condition

$$F'_{\mathbf{L}} + B|_{\mathbf{L}} = 0 \,, \tag{2.70}$$

in the presence of a B-field. Even if $\mathcal{L}'$ does not exist as a line bundle, its square $(\mathcal{L}')^2$ does so that a half of the curvature of $(\mathcal{L}')^2$ is subject to (2.70). In sum, for a Lagrangian A-brane, we have a Chan-Paton bundle

$$\mathfrak{B}_{\mathbf{L}} : \quad \begin{array}{c} \mathcal{L}' \otimes K_{\mathbf{L}}^{-1/2} \\ \downarrow \\ \mathbf{L} \end{array} \tag{2.71}$$

with a flat Spin^c structure (2.70). We will sometimes denote a Chan-Paton bundle by $\mathfrak{B}_{\mathbf{L}} \to \mathbf{L}$, abusing notation. Morphisms between Lagrangian objects are defined in the usual way using the Floer–Fukaya complex generated by intersection points between the Lagrangians; see [7] for details.

Defining the space of morphisms between Lagrangian and coisotropic objects is a bit more subtle, and is discussed in detail for flat targets in [8]. The essential idea is that the morphism space should be thought of as related to the space of holomorphic functions on the intersection, with respect to the transverse holomorphic structure on coisotropic objects. For Lagrangian objects, this complex structure obviously plays no role, but instanton corrections can appear, in the guise of the contributions of holomorphic disks to the differential in the Floer–Fukaya complex. On the other hand, for $\mathfrak{B}_{\text{cc}}$, the transverse holomorphic structure is just a standard complex structure and plays an essential role, but instanton corrections are forbidden. In the case of general coisotropic branes, both phenomena can be expected to be relevant. (For some further discussion of this fact from the worldsheet perspective, as well as generalizations to branes of higher rank, see [91].)

In a hyper-Kähler manifold, we can make use of a B-model analysis as in [82, 85] to compute the dimension of open strings. The dimension of the representation space $\mathscr{L} := \text{Hom}(\mathfrak{B}_{\text{cc}}, \mathfrak{B}_{\mathbf{L}})$ associated to a compact Lagrangian brane $\mathfrak{B}_{\mathbf{L}}$ can be computed with the help of the Grothendieck–Riemann–Roch formula:

$$\begin{aligned} \dim \mathscr{L} &= \dim H^0(\mathbf{L}, \mathfrak{B}_{\text{cc}} \otimes \mathfrak{B}_{\mathbf{L}}^{-1}) \\ &= \int_{\mathbf{L}} \text{ch}(\mathfrak{B}_{\text{cc}}) \wedge \text{ch}(\mathfrak{B}_{\mathbf{L}}^{-1}) \wedge \text{Td}(T\mathbf{L}) \,, \end{aligned} \tag{2.72}$$

Here we denote, by $\mathfrak{B}$, a bundle for the corresponding brane including an effect of the B-field, abusing notation.

If a Lagrangian $\mathbf{L}$ is of real two dimensions, then the Todd class $\text{Td}(T\mathbf{L}) = \text{ch}(K_{\mathbf{L}}^{-1/2})\widehat{A}(T\mathbf{L})$ is equivalent to $\text{ch}(K_{\mathbf{L}}^{-1/2})$. Consequently, the formula becomes a very simple form

$$\dim \mathscr{L} = \int_{\mathbf{L}} \mathrm{ch}(\mathfrak{B}_{cc}) = \int_{\mathbf{L}} \frac{F+B}{2\pi} , \tag{2.73}$$

for a real two-dimensional Lagrangian $\mathbf{L}$.

As explained in [85], for a Lagrangian brane $\mathfrak{B}_{\mathbf{L}}$, the space of open strings $\mathrm{Hom}(\mathfrak{B}_{cc}, \mathfrak{B}_{\mathbf{L}})$ can be understood as a geometric quantization of $\mathbf{L}$ with a curvature on a "prequantum line bundle" $\mathfrak{B}_{cc} \otimes \mathfrak{B}_{\mathbf{L}}^{-1}$. If $\mathfrak{X}$ is a complexification of $\mathbf{L}$ in the sense of [85], then the action of $\mathrm{End}(\mathfrak{B}_{cc})$ on the quantization $\mathrm{Hom}(\mathfrak{B}_{cc}, \mathfrak{B}_{\mathbf{L}})$ plays the role of the quantized algebra of operators.

Finally, let us mention a brief word about coefficients. In general, the Fukaya category is defined with coefficients in the Novikov ring; this is necessary because the sums over instanton contributions that define the differential are formal and not necessarily guaranteed to converge. Similarly, deformation quantization of a Poisson manifold [53, 77, 111] is not guaranteed to produce convergent series, but only a formal deformation in general. We will restrict ourselves to target spaces $\mathfrak{X}$ for which a "good A-model" is expected to exist, meaning that all the series involved should in fact converge. The existence of a complete hyper-Kähler metric on $\mathfrak{X}$ should be sufficient to ensure this; see [85] for further discussion of this issue.

We will proceed to compare the two categories A-$\mathsf{Brane}(\mathfrak{X}, \omega_{\mathfrak{X}})$ and $\mathsf{Rep}(S\ddot{H})$ via the brane quantization.[6] For the comparison, the symmetries play a crucial role. In fact, the symmetries of the target space $\mathfrak{X}$ become the group of auto-equivalences of the categories. More concretely, we will investigate the action of $\Xi \times \mathrm{PSL}(2, \mathbb{Z})$ ((2.29) and (2.53)) and the Weyl group $\mathbb{Z}_2$ generated by ι (2.55) on both categories.

Now we set up the framework so that we will start our expedition to "see" and "touch" representations of $S\ddot{H}$ as if they were geometric objects in the target $\mathfrak{X}$.

2.5 (A, B, A)-Branes for Polynomial Representations

DAHA was introduced by Cherednik in the study of Macdonald polynomials from the viewpoint of representation theory [33] in which the distinguished infinite-dimensional representation on the ring $\mathscr{P} := \mathbb{C}_{q,t}[X^{\pm}]^{\mathbb{Z}_2}$ of symmetric Laurent polynomials, called *polynomial representation*, plays an important role. (See also [38] for finite-dimensional modules.) Here, Laurent polynomials in a single variable X over $\mathbb{C}_{q,t}$ are symmetrized under the inversion $\mathbb{Z}_2 : X \mapsto X^{-1}$ so that the ring can also be expressed as $\mathscr{P} = \mathbb{C}_{q,t}[X + X^{-1}]$. This polynomial representation of $S\ddot{H}$ is defined by the following formulas:

---

[6] Note, that spherical DAHA is Morita-equivalent to DAHA (2.40), i.e. the category of representations of DAHA is equivalent to the category of representations of its spherical subalgebra [131]:

$$\mathsf{Rep}(\ddot{H}) \cong \mathsf{Rep}(S\ddot{H}) . \tag{2.74}$$

See also (4.26) for the explanation from the 2d sigma-model.

$$\mathrm{pol} : S\ddot{H} \to \mathrm{End}(\mathscr{P}), \quad \begin{aligned} x &\mapsto X + X^{-1}, \\ y &\mapsto \frac{tX - t^{-1}X^{-1}}{X - X^{-1}}\varpi + \frac{t^{-1}X - tX^{-1}}{X - X^{-1}}\varpi^{-1}, \\ z &\mapsto q^{\frac{1}{2}}X\frac{tX - t^{-1}X^{-1}}{X - X^{-1}}\varpi + q^{\frac{1}{2}}X^{-1}\frac{t^{-1}X - tX^{-1}}{X - X^{-1}}\varpi^{-1}, \end{aligned} \tag{2.75}$$

where $\varpi^{\pm}(X) = q^{\pm}X$ is the exponentiated degree operator, often called the q-shift operator, that appeared in (C.15) for the quantum torus algebra. In particular, $\mathrm{pol}(y)$ is the so-called *Macdonald difference operator*, whose eigenfunctions are *symmetric Macdonald polynomials* [35, 121]. The Macdonald functions of type A_1 are labeled by spin-$\frac{j}{2}$ representations, and can be expressed in terms of the basic hypergeometric series

$$P_j(X; q, t) := X^j\, {}_2\phi_1(q^{-2j}, t^2; q^{-2j+2}t^{-2}; q^2; q^2t^{-2}X^{-2})\ . \tag{2.76}$$

They are acted on diagonally by the Macdonald difference operator, with eigenvalues

$$\mathrm{pol}(y) \cdot P_j(X; q, t) = (q^jt + q^{-j}t^{-1})P_j(X; q, t)\ . \tag{2.77}$$

Under this basis, the actions of the other generators are

$$\begin{aligned} \mathrm{pol}(x) \cdot P_j(X; q, t) =& P_{j+1}(X; q, t) + \frac{\left(1 - q^{2j}\right)\left(1 - q^{2j-2}t^4\right)}{\left(1 - q^{2j-2}t^2\right)\left(1 - q^{2j}t^2\right)} P_{j-1}(X; q, t)\ , \\ \mathrm{pol}(z) \cdot P_j(X; q, t) =& tq^{j+\frac{1}{2}}P_{j+1}(X; q, t) + t^{-1}q^{-j+\frac{1}{2}}\frac{\left(1 - q^{2j}\right)\left(1 - q^{2j-2}t^4\right)}{\left(1 - q^{2j-2}t^2\right)\left(1 - q^{2j}t^2\right)} P_{j-1}(X; q, t)\ . \end{aligned} \tag{2.78}$$

In fact, the Macdonald polynomials P_j form a basis for the ring $\mathscr{P}$ over $\mathbb{C}_{q,t}$, so that the polynomial representation can be studied with the help of raising and lowering operators [105]:

$$\begin{aligned} \mathsf{R}_j &:= x - q^{j-\frac{1}{2}}tz = X(q^jt^{-1}Y - q^{2j}t^2) + X^{-1}(q^jtY^{-1} - q^{2j}t^2)\ , \\ \mathsf{L}_j &:= x - q^{-j-\frac{1}{2}}t^{-1}z = X(q^{-j}t^{-3}Y - q^{-2j}t^{-2}) + X^{-1}(q^{-j}t^{-1}Y^{-1} - q^{-2j}t^{-2})\ . \end{aligned} \tag{2.79}$$

These operators relate adjacent Macdonald polynomials, respectively increasing or decreasing the value of j:

$$\mathrm{pol}(\mathsf{R}_j) \cdot P_j(X; q, t) = (1 - q^{2j}t^2)P_{j+1}(X; q, t)\ , \tag{2.80}$$

$$\mathrm{pol}(\mathsf{L}_j) \cdot P_j(X; q, t) = \frac{(1 - q^{2j})(1 - q^{2(j-1)}t^4)}{q^{2j}t^2(q^{2(j-1)}t^2 - 1)} P_{j-1}(X; q, t)\ . \tag{2.81}$$

See Fig. 2.5 for a schematic diagram of this action. At $t = 1$, this representation reduces to the pullback of the lift of $\mathscr{P}^{y_1=1}$ in Proposition C.6 so that

$$
\begin{tikzcd}
1 \arrow[r, bend left, "\mathsf{R}_0"] & P_1 \arrow[l, bend left, "\mathsf{L}_1"] \arrow[r, bend left, "\mathsf{R}_1"] & \cdots \arrow[l, bend left, "\mathsf{L}_2"] & \cdots \arrow[r, bend left, "\mathsf{R}_{j-2}"] & P_{j-1} \arrow[l, bend left, "\mathsf{L}_{j-1}"] \arrow[r, bend left, "\mathsf{R}_{j-1}"] & P_j \arrow[l, bend left, "\mathsf{L}_j"] \arrow[r, bend left, "\mathsf{R}_j"] & \cdots \arrow[l, bend left, "\mathsf{L}_{j+1}"]
\end{tikzcd}
$$

Fig. 2.5 The action of raising and lowering operators on Macdonald polynomials

Cherednik's polynomial representation can be understood as its deformation from the symmetrized quantum torus to DAHA. Since the classical limit ($q = 1$) of the Macdonald eigenvalues (2.77) is always $t + t^{-1}$, the support of the corresponding A-brane $\mathfrak{B}_{\mathbf{P}}$ is given by

$$
\mathbf{P} = \{y = \tilde{t} + \tilde{t}^{-1}\ ,\ z = \tilde{t}^{-1}x\}\ . \tag{2.82}
$$

While the parameter t in $S\ddot{H}$ coincides with the monodromy parameter $\tilde{t}$ at the classical limit ($q = 1$) (see (2.52)), we use $\tilde{t}$ to specify the position of the brane because it is the geometric parameter of $\mathfrak{X}$. Since it is of type (A, B, A), it is happily a Lagrangian submanifold with respect to $\omega_{\mathfrak{X}}$ for any value of $\hbar$ or q.

To understand the brane $\mathfrak{B}_{\mathbf{P}}$ for the polynomial representation $\mathscr{P}$ of $S\ddot{H}$ better, it is illuminating to consider its relation to the skein module. The skein module of type A_1 [135, 143] of an oriented 3-manifold M_3 is defined as

$$
\mathrm{Sk}(M_3, \mathrm{SU}(2)) := \mathrm{Sk}(M_3) = \frac{\mathbb{C}[q^{\pm\frac{1}{2}}](\text{isotopy classes of framed links in } M_3)}{\left(\times = q^{-1/2} \right)\left(+ q^{1/2} \asymp\ ,\ \bigcirc = -q - q^{-1}\right)} \tag{2.83}
$$

The skein algebra $\mathrm{Sk}(C)$ associated to an oriented closed surface C is defined as

$$
\mathrm{Sk}(C) := \mathrm{Sk}(C \times [0, 1], \mathrm{SU}(2))\ \ , \tag{2.84}
$$

where the multiplication $\mathrm{Sk}(C) \times \mathrm{Sk}(C) \to \mathrm{Sk}(C)$ is given by stacking. As a result, $\mathrm{Sk}(C)$ is a $\mathbb{C}[q^{\pm\frac{1}{2}}]$-associative algebra [144].

At the $q = 1$ specialization, the skein module $\mathrm{Sk}(M_3)$ becomes a commutative algebra. Moreover, it was shown in [26, 136] that by assigning a loop $\gamma : S^1 \to M_3$ to $\mathrm{Tr}(\rho(\gamma))$ where $\rho : \pi_1(M_3) \to \mathrm{SL}(2, \mathbb{C})$ is the holonomy homomorphism, the classical limit $q = 1$ of $\mathrm{Sk}(M_3)$ is isomorphic to the coordinate ring of the character variety $\mathcal{M}_{\text{flat}}(M_3, \mathrm{SL}(2, \mathbb{C}))$. Hence, the skein module $\mathrm{Sk}(M_3)$ can be understood as a BV quantization [66]

$$
\mathrm{Sk}(M_3) \cong \mathrm{BV}^q(\mathcal{M}_{\text{flat}}(M_3, \mathrm{SL}(2, \mathbb{C})))\ .
$$

The skein module of a closed 3-manifold will be studied in Sect. 3.2.

If a 3-manifold has a boundary $\partial M_3 = C$, then we have a module $\mathrm{Sk}(C) \circlearrowright \mathrm{Sk}(M_3)$ by pushing a framed links in a thickened boundary $C \times [0, 1]$ into the bulk M_3. In fact, $\mathcal{M}_{\mathrm{flat}}(M_3, \mathrm{SL}(2, \mathbb{C}))$ is a holomorphic Lagrangian submanifold of $\mathcal{M}_{\mathrm{flat}}(C, \mathrm{SL}(2, \mathbb{C}))$ with respect to the holomorphic symplectic form Ω_J. Therefore, it can be understood as an (A, B, A)-brane $\mathfrak{B}_H$ on $\mathcal{M}_{\mathrm{flat}}(C, \mathrm{SL}(2, \mathbb{C}))$, called a *Heegaard brane*. From the viewpoint of brane quantization, the action of the skein algebra can be understood as

$$\begin{array}{ccc} \mathrm{Sk}(C) & \cong & \mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathrm{cc}}) \\ \circlearrowleft & & \circlearrowleft \\ \mathrm{Sk}(M_3) & \cong & \mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_H) \end{array} . \tag{2.85}$$

Of our particular interest is the skein algebra $\mathrm{Sk}(T^2)$ of a torus, which is the $t = q$ specialization of $S\ddot{H}$ [24]. Also, the skein module $\mathrm{Sk}(S^1 \times D^2)$ of the solid torus is the Grothendieck ring of the category of finite-dimensional representations of $U_q(\mathfrak{sl}(2))$

$$\mathrm{Sk}(S^1 \times D^2) \cong K^0(\mathsf{Rep}\, U_q(\mathfrak{sl}(2))) \otimes \mathbb{C}[q^{\pm\frac{1}{2}}] \ , \tag{2.86}$$

which is spanned by Chebyshev polynomials $S_j(z)$ of the second kind [54]. They are recursively defined by

$$z S_j(z) = S_{j+1}(z) + S_{j-1}(z) \tag{2.87}$$

with the initial conditions $S_0(z) = 1$, $S_1(z) = z$, and they are actually the $t = q$ specialization of the Macdonald polynomials

$$S_j(X + X^{-1}) = P_j(X; q, t = q) = \frac{X^{j+1} - X^{-j-1}}{X - X^{-1}} \ . \tag{2.88}$$

Consequently, the polynomial representation $\mathscr{P}$ of $S\ddot{H}_{t=q}$ is indeed the skein module $\mathrm{Sk}(T^2) \circlearrowright \mathrm{Sk}(S^1 \times D^2)$. In fact, the support of the Heegaard brane for the solid torus is given by $y = 2$, which is the A-polynomial of the unknot complement in S^3. Indeed the eigenvalue of the y operator on $S_j(X + X^{-1})$ under the polynomial representation $\mathscr{P}$ at $t = q$ is $q^{j+1} + q^{-j-1}$ and its classical limit is $y = 2$. Thus, the polynomial representation $\mathscr{P}$ of $S\ddot{H}$ can be understood as the t-deformation of the skein module $\mathrm{Sk}(T^2) \circlearrowright \mathrm{Sk}(S^1 \times D^2)$ [93].

Let us consider how the symmetries of $S\ddot{H}$ act on the polynomial representation $\mathscr{P}$. For instance, an action of $\mathrm{PSL}(2, \mathbb{Z})$ on $\mathscr{P}$ can be seen by using the maps (2.53) combined with (2.75). It is easy to see from (2.53) that the generators of $\mathrm{PSL}(2, \mathbb{Z})$ maps $\mathfrak{B}_{\mathbf{P}}$ to another (A, B, A) brane

$$\begin{aligned}\tau_+(\mathbf{P}) &= \{z = \tilde{t} + \tilde{t}^{-1}\ ,\ x = \tilde{t}^{-1}y\}\ ,\\ \tau_-(\mathbf{P}) &= \{y = \tilde{t} + \tilde{t}^{-1}\ ,\ z = \tilde{t}^{-1}x\}\ ,\\ \sigma(\mathbf{P}) &= \{x = \tilde{t} + \tilde{t}^{-1}\ ,\ y = \tilde{t}^{-1}z\}\ .\end{aligned} \tag{2.89}$$

Under the modular T-transformation τ_-, the support does not change and the polynomial representation $\mathscr{P}$ is invariant, $\tau_-(\mathscr{P}) \cong \mathscr{P}$ since the Macdonald polynomials are transformed under the modular T-transformation τ_- as

$$\tau_-(P_j) = q^{-\frac{j^2}{2}} t^{-j} P_j \quad \rightsquigarrow \quad T_{jj'} = q^{-\frac{j^2}{2}} t^{-j} \delta_{jj'}\ . \tag{2.90}$$

The proof is given at the end (B.36) of Appendix B.2.1. The image $\sigma(\mathscr{P})$ of the polynomial representation of $S\ddot{H}$ under the S-transformation σ is called the *functional representation*, which is explained in Appendix B.2.2. As for the group Ξ of the sign changes, the image $\xi_1(\mathscr{P})$ is isomorphic to itself $\mathscr{P} \cong \xi_1(\mathscr{P})$. On the other hand, the image under the involution ξ_2 can be obtained by multiplying the minus sign to y and z as in (2.29) and the support of the corresponding brane is

$$\xi_2(\mathbf{P}) = \{y = -\tilde{t} - \tilde{t}^{-1}\ ,\ z = -\tilde{t}^{-1}x\}\ . \tag{2.91}$$

Finally, the outer automorphism (2.55) changes the Chan-Paton bundle of $\mathfrak{B}_{\mathbf{P}}$ as explained in Sect. 2.3.1 and the support becomes

$$\iota(\mathbf{P}) = \{y = \tilde{t} + \tilde{t}^{-1}\ ,\ z = \tilde{t}x\}\ . \tag{2.92}$$

Note that the ι image $\iota(\mathscr{P})$ of the polynomial representation can be obtained by changing $t \to q/t$ in (2.75).

The perspective from the brane quantization also sheds new light on infinite-dimensional representations. We have seen that Cherednik's polynomial representation (2.75) corresponds to the A-brane $\mathfrak{B}_{\mathbf{P}}$ (2.82) at the particular value of y. It is natural to expect that it can be deformed in such a way that the corresponding brane is supported on a generic point of y.

This consideration leads us to the following. Let us consider the multiplicative system $\widetilde{M} \subset \mathbb{C}_{q,t}[X^{\pm}]$ generated by all elements of the form $(q^{\ell}X - q^{-\ell}X^{-1})$ for all integers $\ell \in \mathbb{Z}$. Then there is a family of representations of $S\ddot{H}$ on the localization[7] of the ring of Laurent polynomials by $\widetilde{M}$

$$\mathscr{P}^{y_1} = \widetilde{M}^{-1}\mathbb{C}_{q,t}[X^{\pm}]\ , \tag{2.93}$$

[7] In other words, $\mathscr{P}^{y_1}$ is the ring of rational functions with coefficients in $\mathbb{C}_{q,t}$ where denominators are always elements in the multiplicative system $\widetilde{M}$ such as

$$\frac{f(X)}{(q^{-m}X - q^{m}X^{-1})^{k_{-m}} \cdots (X - X^{-1})^{k_0} \cdots (q^{\ell}X - q^{-\ell}X^{-1})^{k_\ell}}\ , \qquad f(X) \in \mathbb{C}_{q,t}[X^{\pm}]\ .$$

.

labeled by a parameter $y_1 \in \mathbb{C}^\times$ where the representations are defined by

$$\mathrm{pol}_{y_1} : S\ddot{H} \to \mathrm{End}(\mathscr{P}^{y_1}), \quad \begin{aligned} &x \mapsto X + X^{-1}, \\ &y \mapsto y_1 \frac{tX - t^{-1}X^{-1}}{X - X^{-1}} \varpi + y_1^{-1} \frac{t^{-1}X - tX^{-1}}{X - X^{-1}} \varpi^{-1}, \\ &z \mapsto q^{\frac{1}{2}} y_1 X \frac{tX - t^{-1}X^{-1}}{X - X^{-1}} \varpi + q^{\frac{1}{2}} (y_1 X)^{-1} \frac{t^{-1}X - tX^{-1}}{X - X^{-1}} \varpi^{-1} . \end{aligned} \tag{2.94}$$

Concretely, one is free to deform Cherednik's polynomial representation (2.75) to this larger representation parametrized by y_1, as long as we allow denominators to be elements of the multiplicative system $\widetilde{M}$. Only at $y_1 = 1$, it decomposes into two irreducible representations where one is Cherednik's polynomial representation, and the other irreducible representation is

$$\widetilde{M}^{-1}\mathbb{C}_{q,t}[X^{\pm}]^{\mathbb{Z}_2} .$$

When $t = 1$, the story reduces to the polynomial representations of the symmetrized quantum torus discussed in Appendix C.3.2. Thus, the support of the corresponding brane $\mathfrak{B}_{\mathbf{P}}^{y_1}$ is expected to be

$$\mathrm{supp}\, \mathfrak{B}_{\mathbf{P}}^{y_1} = \{y = y_1^{-1}\tilde{t} + y_1\tilde{t}^{-1}\} .$$

In fact, the eigenfunction of y under pol_{y_1} that generalizes the Macdonald polynomials is constructed in [22, 112, 118][8]

$$\mathcal{Z}(X, y_1, q, t) = {}_2\phi_1 \left(y_1^2, t^2; q^2t^{-2}y_1^2; q^2; q^2t^{-2}X^{-2}\right) , \tag{2.95}$$

where the eigenvalue is

$$\mathrm{pol}_{y_1}(y) \cdot \mathcal{Z} = (y_1^{-1}t + y_1t^{-1})\mathcal{Z} . \tag{2.96}$$

Thus, for a generic value of y_1, the eigenfunction is an infinite hypergeometric series (2.95). However, as easily seen from (2.76), the series truncates to the symmetric Macdonald polynomial

$$\mathcal{Z}(X, y_1 = q^{-j}, q, t) = X^{-j} P_j(X; q, t) . \tag{2.97}$$

at the specialization $y_1 = q^{-j}$ $(j \in \mathbb{Z}_{\geq 0})$.

A geometric interpretation of the multiplicative system $\widetilde{M}$ can be given by thinking about the $t = 1$ limit, where we are interested in the quotient map $\mathbb{C}^\times \times \mathbb{C}^\times \to (\mathbb{C}^\times \times \mathbb{C}^\times)/\mathbb{Z}_2$. After deforming the target of this covering map, no natural ramified twofold cover by $\mathbb{C}^\times \times \mathbb{C}^\times$ exists. However, such a cover can be constructed once we extract the $\mathbb{Z}_2$-invariant points $X = \pm 1$. In fact, $\mathscr{O}(\mathbb{C}^\times \backslash \{X = \pm 1\})$ admits

[8] $\mathcal{Z}$ is the so-called uncapped vertex function in the quantum K-theory of $T^*\mathbb{C}\mathbf{P}^1$.

the generator $\frac{1}{X-X^{-1}}$. A related story exists in the rational limit, where the relevant geometry is the deformation of the A_1 singularity $(\mathbb{C} \times \mathbb{C})/\mathbb{Z}_2$ to the total space of $T^*\mathbb{C}P^1$; we discuss this in detail in Appendix B.2.4.

2.6 Branes with Compact Supports and Finite-Dimensional Representations: Object Matching

Cherednik's polynomial representation is of particular significance due to the theorems of Cherednik [35, Sect. 2.8–9], which classify finite-dimensional representations of $S\ddot{H}$ obtained as quotients of the polynomial representation paired with the action of outer automorphisms. Similar to the theory of Verma modules, the polynomial representation is generically irreducible. A raising operator (2.80) never be null since the Macdonald polynomial P_{2j} always has a factor $(1 - q^{2j}t^2)$ in the denominator. However, it can occur that a lowering operator L_j annihilates one of the Macdonald polynomials P_j when certain conditions on the central character are satisfied. If this occurs, P_j generates a subrepresentation, and a finite-dimensional representation of the spherical DAHA appears as the quotient $\mathscr{P}/(P_j)$. We can therefore study finite-dimensional representations by asking that the condition $\mathrm{pol}(\mathsf{L}_j) \cdot P_j = 0$ be satisfied for some j, i.e. that the factor

$$\frac{(1-q^{2j})(1-q^{(j-1)}t^2)(1+q^{(j-1)}t^2)}{q^{2j}t^2(q^{2(j-1)}t^2-1)} \tag{2.98}$$

on the right hand side of (2.81) vanishes.

This amounts to the following three cases:

$$q^{2n} = 1 \ , \tag{2.99a}$$

$$t^2 = -q^{-k} \ , \tag{2.99b}$$

$$t^2 = q^{-(2\ell-1)} \ . \tag{2.99c}$$

Here, the exponent in the right hand side of (2.99c) must be an odd integer in order for the denominators of Macdonald polynomials as well as (2.98) to be non-zero; even exponents are excluded by the definition of the coefficient ring $\mathbb{C}_{q,t}$ in (2.41). We write this odd integer as $2\ell - 1$. Each of these separate shortening conditions will naturally appear as an existence condition of an A-brane with compact support in what follows; we will examine each of the resulting finite-dimensional representations and the corresponding compact Lagrangian branes in turn.

2.6.1 Generic Fibers of the Hitchin Fibration

First we consider analogous A-branes in this setting; the ones supported on generic fibers in the Hitchin fibration. As explained in Sect. 2.1, the Hitchin fibration (2.15) is completely integrable, and a generic Hitchin fiber $\mathbf{F}$ is holomorphic in complex structure I while it is a complex Lagrangian submanifold from the viewpoint of the holomorphic two-form Ω_I for a generic ramification data (2.4). Namely, it is a Lagrangian submanifold of type (B, A, A) for any values of $(\alpha_p, \beta_p, \gamma_p)$-triple. Therefore, a generic fiber $\mathbf{F}$ can be Lagrangian in a symplectic manifold $(\mathfrak{X}, \omega_{\mathfrak{X}})$ only when the canonical coisotropic brane $\mathfrak{B}_{\text{cc}}$ obeys the condition $\theta = 0$ in (2.60) so that

$$\omega_{\mathfrak{X}} = -\frac{\omega_K}{\hbar}\ , \quad \text{and} \quad F + B = \frac{\omega_I}{\hbar}\ . \tag{2.100}$$

With $\theta \neq 0$, there is no A-brane supported on $\mathbf{F}$ in the symplectic manifold $(\mathfrak{X}, \omega_{\mathfrak{X}})$. Accordingly, $\hbar = |\hbar|$ is real (i.e. $|q| = 1$), and the canonical coisotropic brane $\mathfrak{B}_{\text{cc}}$ is an A-brane of type (B, A, A).

An analogous brane appears in the brane quantization of $\mathbb{C}^\times \times \mathbb{C}^\times$ for the quantum torus algebra. As in Appendix C.2.1, a brane is supported on a fiber T^2 of the elliptic fibration $T^*T^2 \cong \mathbb{C}^\times \times \mathbb{C}^\times$, which gives rise to a finite-dimensional representation, called the cyclic representation. Therefore, we can study a brane supported on a generic fiber $\mathbf{F}$ of the Hitchin fibration, comparing with the case of the quantum torus algebra.

The branes are indexed by a position of the Hitchin base $\mathcal{B}_H$ (see also Appendix C.2.1). Also, the flatness condition (2.70) of the line bundle $\mathcal{L}'$ an A-brane supported $\mathfrak{B}_{\mathbf{F}}$ is

$$F'_{\mathbf{F}} + B\big|_{\mathbf{F}} = 0\ .$$

Since $\mathbf{F}$ is topologically a two-torus, the flat Spin^c structure $\mathcal{L}' \otimes K_{\mathbf{L}}^{-1/2}$ of $\mathfrak{B}_{\mathbf{F}}$ can have non-trivial $\mathrm{U}(1)^2$ holonomy with a choice of spin structure [85]. The branes $\mathfrak{B}_{\mathbf{F}}^{\lambda}$ are parametrized by $\lambda = (x_m, y_m) \in \mathbb{C}^\times \times \mathbb{C}^\times$ where the absolute values $(|x_m|, |y_m|)$ describe its position and the angular phases illustrate the $\mathrm{U}(1)^2$ holonomy with a choice of spin structures. Namely, the angular phase $\mathrm{U}(1)$ encodes the holonomy $\mathrm{U}(1)$ and a choice of spin structures $\mathbb{Z}_2$ along a one-cycle of a Riemann surface via

$$1 \to \mathbb{Z}_2 \to \mathrm{U}(1) \to \mathrm{U}(1) \to 1\ .$$

We assign the plus sign $+$ for $1 \in \mathbb{Z}_2$ to the Ramond spin structure, and the minus sign $-$ for $-1 \in \mathbb{Z}_2$ to the Neveu-Schwarz spin structure. The choice of spin structures appears in the representation of the symmetrized quantum torus discussed in Appendix C.3.2.

Consequently, the computation of the dimension (2.73) of the space $\mathrm{Hom}(\mathfrak{B}_{\text{cc}}, \mathfrak{B}_{\mathbf{F}}^{\lambda})$ is reduced to the period integral (2.23)

$$\dim \mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathbf{F}}^{\lambda}) = \int_{\mathbf{F}} \frac{F+B}{2\pi} = \int_{\mathbf{F}} \frac{\omega_I}{2\pi\hbar} = \frac{1}{\hbar} \tag{2.101}$$

for arbitrary λ. Hence, this leads to the Bohr-Sommerfeld quantization condition $\hbar = 1/m$, or equivalently that $q = e^{2\pi i/m}$ is a primitive mth root of unity for $m \in \mathbb{Z}_{>0}$. In fact, since $[F/2\pi]$ is an integral cohomology class in $H^2(\mathfrak{X}, \mathbb{Z})$, the fiber class relation (2.17) requires $\int_{\mathbf{F}} F/2\pi$ to be an even integer. Thus, if m is an odd positive integer, then we need non-trivial B-flux with

$$\int_{\mathbf{F}} \frac{B}{2\pi} = -\int_{\mathbf{F}} \frac{F'_{\mathbf{F}}}{2\pi} = 1\ , \tag{2.102}$$

up to an even integer shift. For instance, this can be achieved if the B-field flux over $\mathbf{V}$ is $1/2$ and those over the exceptional divisors $\mathbf{D}_i$ ($i = 1, \ldots, 4$) are zero.

In order for the $(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathbf{F}}^{\lambda})$-strings exist, q has to be a primitive mth root of unity whereas t can be generic. Under this condition, the action of $S\ddot{H}$ under the generalized polynomial representation pol_{y_1} in (2.94) commute with $X^m - x_m$ for $x_m \in \mathbb{C}^\times$ because the shift operator ϖ acts trivially on it. Consequently, the ideal $(X^m - x_m)$ is invariant under pol_{y_1} so that the quotient space

$$\mathscr{F}_m^{\lambda} = \mathscr{P}^{y_1}/(X^m - x_m)\ ,$$

is also a representation of $S\ddot{H}$. Since the Taylor expansion of a denominator in the multiplicative system $\widetilde{M}$ always truncates under the condition $X^m = x_m$, this is indeed an m-dimensional representation parametrized by $\lambda = (x_m, y_m)$ where y_1 is any mth root of y_m. Hence, we can identify $\mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathbf{F}}^{\lambda})$ with $\mathscr{F}_m^{\lambda}$ when q is a primitive mth root of unity where the parameter $\lambda \in \mathbb{C}^\times \times \mathbb{C}^\times$ exactly matches.

For generic values of $\lambda = (x_m, y_m)$, the support of a brane $\mathfrak{B}_{\mathbf{F}}^{\lambda}$ is mapped to another Hitchin fiber up to Hamiltonian isotopy under the $\mathrm{PSL}(2, \mathbb{Z})$ action, and the holonomy of the Chan-Paton bundle, which is a point in the dual torus $\mathrm{Jac}(\mathbf{F})$, is also transformed appropriately. Namely, $\mathrm{PSL}(2, \mathbb{Z})$ acts on λ. On the other hand, a generic fiber is invariant as a set under the group Ξ of the sign changes as we have seen in Sect. 2.1. Correspondingly, the representation $\mathscr{F}_m^{\lambda}$ is invariant under Ξ at a generic value of λ.

Setting $y_1 = 1$, we can symmetrize the story [35, Theorem 2.8.5 (iv)]. Namely, since the ideal $(X^m + X^{-m} - x_m - x_m^{-1})$ is invariant under Cherednik's polynomial representation $\mathscr{P}$ due to the same reason, we have an m-dimensional representation

$$\mathscr{F}_m^{(x_m,+)} = \mathscr{P}/(X^m + X^{-m} - x_m - x_m^{-1})\ . \tag{2.103}$$

In this case, the corresponding brane $\mathfrak{B}_{\mathbf{F}}^{(x_m,+)}$ supported on a Hitchin fiber intersects with the support $\mathbf{P}$ (2.82) of the polynomial representation. Also, the Chan-Paton bundle has the trivial holonomy and the Ramond spin structure around one generator, say the (0, 1)-cycle, of $\pi_1(\mathbf{F}) \cong \mathbb{Z} \oplus \mathbb{Z}$. The parameter x_m encodes its position in the x coordinate and the holonomy around the other generator of $\pi_1(\mathbf{F})$.

Therefore, the representations of this family are analogous to the finite-dimensional representations of both symmetrized and ordinary quantum torus in terms of A-branes on fibers of the elliptic fibration of the target in the 2d A-models as illustrated in Appendix C. As in the case of the symmetrized quantum torus Appendix C.3, if a brane $\mathfrak{B}_{\mathbf{F}}$ with trivial holonomies moves to a special position, we will see below that a special phenomenon occurs.

2.6.2 Irreducible Components in Singular Fibers of Type I_2

As in Fig. 2.2, the Hitchin fibration has three singular fibers of Kodaira type I_2 for generic ramification parameters of $(\alpha_p, \beta_p, \gamma_p)$. Since they are still fibers in the Hitchin fibration, the irreducible components $\mathbf{U}_i$ $(i = 1, \ldots, 6)$ in a singular fiber are also Lagrangian submanifolds of type (B, A, A). Therefore, $\mathfrak{B}_{\text{cc}}$ needs to satisfy (2.100) in order for $\mathfrak{B}_{\mathbf{U}_i}$ to be A-branes as in the previous subsection.

For instance, let us investigate a module that the brane $\mathfrak{B}_{\mathbf{U}_1}$ gives rise to. The curvature of the line bundle $\mathcal{L}'$ should obey the flatness condition (2.70)

$$F'_{\mathbf{U}_1} + B|_{\mathbf{U}_1} = 0\,. \tag{2.104}$$

Since $\mathbf{U}_1$ is topologically $\mathbb{C}\mathbf{P}^1$ and a position is fixed, there is no deformation parameter associated to the brane $\mathfrak{B}_{\mathbf{U}_1}$. Subsequently, one can evaluate the dimension formula (2.73)

$$\dim \mathrm{Hom}(\mathfrak{B}_{\text{cc}}, \mathfrak{B}_{\mathbf{U}_1}) = \int_{\mathbf{U}_1} \frac{F+B}{2\pi} = \int_{\mathbf{U}_1} \frac{\omega_I}{2\pi\hbar} = \frac{1}{2\hbar} \tag{2.105}$$

Consequently, the brane $\mathfrak{B}_{\mathbf{U}_1}$ can exist only at $1/(2\hbar) = n \in \mathbb{Z}_{>0}$, or equivalently when q is a primitive $2n$th root of unity.

This is exactly one (2.99a) of the shortening conditions, and under this condition a lowering operator (2.81) annihilates the Macdonald polynomial

$$\mathrm{pol}(\mathsf{L}_n) \cdot P_n(X; q, t) = 0 \qquad \text{where} \qquad P_n(X; q, t) = X^n + X^{-n}\ . \tag{2.106}$$

Therefore, the quotient space

$$\mathscr{U}_n^{(1)} := \mathscr{P}/(P_n) \tag{2.107}$$

by an ideal (P_n) is an n-dimensional irreducible representation of spherical DAHA [35, Theorem 2.8.5 (ii)] so that one can identify

$$\mathscr{U}_n^{(1)} = \mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathbf{U}_1}) \ .$$

As seen in Sect. 2.1, the irreducible component $\mathbf{U}_1$ is invariant under the sign change ξ_1 whereas it is mapped to $\mathbf{U}_2$ under ξ_2 (2.31). In fact, it follows from the form (2.106) of $P_n(X)$ that the finite-dimensional module $\mathscr{U}_n^{(1)}$ is invariant under the sign flip ξ_1. On the other hand, the sign change ξ_2 leads to another non-isomorphic finite-dimensional module. Thus, the brane $\xi_2(\mathfrak{B}_{\mathbf{U}_1})$ corresponds to a brane supported on the other irreducible component $\mathbf{U}_2$ in the same singular fiber from which the module comes from

$$\mathscr{U}_n^{(2)} := \xi_2(\mathscr{U}_n^{(1)}) = \mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathbf{U}_2}) \ .$$

In a similar fashion, a brane $\mathfrak{B}_{\mathbf{U}_i}$ supported on another irreducible component in a singular fiber gives rise to an image of $\mathscr{U}_n^{(1)}$ under $\mathrm{PSL}(2, \mathbb{Z})$ and the sign changes $\xi_{1,2}$. The transformation rule can be read off from (2.38) so that the branes $\mathfrak{B}_{\mathbf{U}_{1,2}}$ are invariant under τ_- whereas they are mapped as

$$\begin{aligned} \sigma(\mathfrak{B}_{\mathbf{U}_1}) &= \mathfrak{B}_{\mathbf{U}_3} \ , \qquad & \sigma(\mathfrak{B}_{\mathbf{U}_2}) &= \mathfrak{B}_{\mathbf{U}_4} \ , \\ \tau_+(\mathfrak{B}_{\mathbf{U}_1}) &= \mathfrak{B}_{\mathbf{U}_5} \ , \qquad & \tau_+(\mathfrak{B}_{\mathbf{U}_2}) &= \mathfrak{B}_{\mathbf{U}_6} \ . \end{aligned} \tag{2.108}$$

The corresponding modules $\mathscr{U}_n^{(i)}$ are obtained from $\mathscr{U}_n^{(1)}$ in the same way.

2.6.3 Moduli Space of G-Bundles

Next, we consider a brane $\mathfrak{B}_{\mathbf{V}}$ supported on the moduli space $\mathbf{V}$ of G-bundles. For the sake of brevity, let us first see the case of $\beta_p = 0$. If $\hbar$ is real, only α_p can be turned on while γ_p must vanish in order for $\mathbf{V}$ to be Lagrangian with respect to ω_K. As $\hbar = |\hbar|e^{i\theta}$ is rotated $\theta \neq 0$ in the complex plane, the symplectic form we are interested in is also rotated from ω_K to $\omega_{\mathfrak{X}}$ according to (2.60). However, this rotation can be actually compensated by switching on γ_p so that $\mathbf{V}$ can stay Lagrangian with respect to $\omega_{\mathfrak{X}}$. According to (2.22) and (2.60), the set $\mathbf{V}$ is Lagrangian with respect to $\omega_{\mathfrak{X}}$ when the following condition holds:

$$\mathrm{Im}\ \frac{\left(\frac{1}{2} - \alpha_p\right) + i\gamma_p}{\hbar} \ = \ 0 \tag{2.109}$$

As a simple check, one can easily see from (2.22) and (2.62) that the integral of the symplectic form is zero

$$\int_{\mathbf{V}} \frac{\mathrm{Im}\ \Omega}{2\pi} = \int_{\mathbf{V}} \frac{\omega_{\mathfrak{X}}}{2\pi} = 0 \ , \tag{2.110}$$

In addition, if $\beta_p = 0$, the submanifold $\mathbf{V}$ is also Lagrangian with respect to ω_J. Namely, it is a complex Lagrangian submanifold with respect to a holomorphic two-form $\omega_{\mathfrak{X}} + i\omega_J$. When β_p is varied, $\mathbf{V}$ stays as a Lagrangian submanifold with respect to $\omega_{\mathfrak{X}}$ while they are no longer Lagrangian with respect to ω_J. In fact, the variation of β_p does not change the holomorphic symplectic form $\Omega_J = \omega_K + i\omega_I$, and therefore keeps $\omega_{\mathfrak{X}}$ fixed. In conclusion, $\mathbf{V}$ can be Lagrangian with respect to $\omega_{\mathfrak{X}}$ only when (2.109) holds. Since our concern is the A-model in the symplectic manifold $(\mathfrak{X}, \omega_{\mathfrak{X}})$, the value of β_p can be arbitrary. For generic (β_p, γ_p), $\mathbf{V}$ is no longer a Lagrangian of type (B, A, A), and it is therefore not contained in a fiber of the Hitchin fibration. Nonetheless, unlike a Hitchin fiber, we can consider the A-model in a generic symplectic form $\omega_{\mathfrak{X}}$ in (2.60) where $\hbar$ can take any complex value.

Under the condition (2.109) with a generic value of $\hbar$, $\mathbf{V}$ is a unique compact Lagrangian submanifold, which is topologically $\mathbb{C}\mathbf{P}^1$. Hence, there is no deformation parameter for $\mathfrak{B}_{\mathbf{V}}$. Consequently, we obtain the dimension of the space of $(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathbf{V}})$-strings from (2.62)

$$\dim \mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathbf{V}}) = \int_{\mathbf{V}} \frac{F+B}{2\pi} = \frac{1}{2\hbar} - \frac{\gamma_p + i\alpha_p}{i\hbar} = \frac{1}{2\hbar} + 2c - 1 \; . \quad (2.111)$$

The Bohr-Sommerfeld quantization condition imposes its dimension as a positive integer $1/2\hbar + 2c - 1 = k + 1 \in \mathbb{Z}_{>0}$, or equivalently $t^2 = -q^{k+2}$.

One can observe that this quantization condition is equivalent to the image of the shortening condition (2.99b) under the involution ι. In fact, under the shortening condition $t^2 = -q^{k+2}$, the lowering operator in the ι-image of the polynomial representation becomes an annihilation operator

$$\mathrm{pol}(\mathsf{L}_{k+1}) \cdot P_{k+1}(X; q, t)\Big|_{t \to \frac{q}{t}} = 0 \; .$$

Consequently, the quotient space by an ideal (P_{k+1})

$$\iota(\mathscr{V}_{k+1}) := \iota(\mathscr{P})/(P_{k+1}(X; q, \tfrac{q}{t})) \quad (2.112)$$

is a $(k+1)$-dimensional irreducible representation of $\ddot{SH}$ [132]. This representation is called the *additional series* in [35, §2.8.2], and we identify

$$\iota(\mathscr{V}_{k+1}) = \mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathbf{V}}) \; .$$

In fact, the support (2.92) of the brane $\iota(\mathfrak{B}_{\mathbf{P}})$ intersects with $\mathbf{V}$ at $t^2 = -q^{k+2}$ so that $\mathrm{Hom}(\iota(\mathfrak{B}_{\mathbf{P}}), \mathfrak{B}_{\mathbf{V}}) \cong \mathbb{C}$ becomes non-trivial. Hence, $\iota(\mathscr{V}_{k+1})$ can be obtained as the quotient of $\iota(\mathscr{P})$ as in (2.112).

As we have seen at the end of Sect. 2.1, the submanifold $\mathbf{V}$ is geometrically invariant under the sign changes $\xi_{1,2}$ so that we expect that the corresponding module $\mathscr{V}_{k+1}$ is also endowed with the same property. When $t^2 = -q^{k+2}$, the Macdonald polynomials obey

$$P_{k+1}(-X; q, \tfrac{q}{t}) = (-1)^k P_{k+1}(X; q, \tfrac{q}{t}) ,$$

which implies that $\iota(\mathscr{V}_{k+1})$ is indeed invariant under ξ_1. In addition, it is easy to check that the full set of y-eigenvalues (the ι-image of (2.77)) of $\iota(\mathscr{V}_{k+1})$ is also invariant under ξ_2.

What makes the space of $(\mathfrak{B}_{cc}, \mathfrak{B}_{\mathbf{V}})$-strings even more interesting is that it also carries a $\mathrm{PSL}(2, \mathbb{Z})$ action. Indeed, as also explained in Sect. 2.1, the submanifold $\mathbf{V}$ is invariant under $\mathrm{PSL}(2, \mathbb{Z})$ symmetry and, as a result, the module $\iota(\mathscr{V}_{k+1})$ is a $\mathrm{PSL}(2, \mathbb{Z})$ representation.

Of course, it is then natural to ask *which* representation it is, and in particular, what the corresponding S and T matrices are. To this end, it is more convenient to consider the space of $(\mathfrak{B}_{cc}, \mathfrak{B}_{\mathbf{V}})$-strings in the target $\mathfrak{X}_{\tilde{t}^{-1}}$ under (2.32) or (2.25). Then, the corresponding representation is given by

$$\mathscr{V}_{k+1} := \mathscr{P}/(P_{k+1}) \tag{2.113}$$

under the shortening condition (2.99b). Since the basis fo $\mathscr{V}_{k+1}$ is spanned by the Macdonald polynomials $P_j(X)$ $(j = 0, \ldots, k)$, the modular T-transformation τ_- acts diagonally in this basis due to (2.90). Under the modular S transformation, this basis is transformed to $P_j(Y)$ and the submanifold $\mathbf{V}$ intersects with both the support (2.82) of the branes $\mathfrak{B}_{\mathbf{P}}$ and that (2.89) of $\sigma(\mathfrak{B}_{\mathbf{P}})$. Hence, the modular S-matrix can be written as

$$S_{jj'} = \mathrm{pol}(P_j(Y^{-1})) \cdot P_{j'}(X)\big|_{X=t^{-1}} = P_j(tq^{j'}; q, t)\, P_{j'}(t^{-1}; q, t) \,. \tag{2.114}$$

This is first introduced by Cherednik [34] as a symmetric bilinear pairing of Macdonald polynomials, which we also denote by $[P_j, P_{j'}]$ as in (B.21). Moreover, it becomes of rank $(k+1)$ when $t^2 = -q^{-k}$, and it acts on $\mathscr{V}_{k+1}$. Therefore, we find explicit forms of the S and T matrices as follows, and we will also find a 3d interpretation of this $\mathrm{PSL}(2, \mathbb{Z})$ representation in §3.1.1.

Conjecture 2.1 The space $\mathscr{V}_{k+1}$ is a $(k+1)$-dimensional $\mathrm{PSL}(2, \mathbb{Z})$ representation, with modular S and T matrices given by

$$\begin{aligned} T_{jj'}\big|_{\mathscr{V}_{k+1}} &= e^{\frac{\pi i k}{12}} q^{-\frac{k(k-1)}{12}}\, i^{-j} q^{\frac{j(k-j)}{2}} \delta_{jj'} \qquad 0 \le j, j' \le k \\ S_{jj'}\big|_{\mathscr{V}_{k+1}} &= a_k^{-1} g_j(q, t = iq^{-k/2})^{-1} P_j(iq^{j'-k/2}; q, t = iq^{-k/2})\, P_{j'}(iq^{k/2}; q, t = iq^{-k/2}) \,. \end{aligned} \tag{2.115}$$

These matrices provide the $\mathrm{PSL}(2, \mathbb{Z})$ representation for "refined Chern-Simons theory".

Here we normalize the modular S-transformation (2.114) by the Macdonald norm of type A_1 (See (B.15) for the definition)

$$g_j(q,t) := \frac{(q^{2j};q^{-2})_j(t^4;q^2)_j}{(q^{2j-2};q^{-2})_j(t^2q^2;q^2)_j} \tag{2.116}$$

and

$$a_k = \begin{cases} \sqrt{2}\prod_{i=0}^{\frac{k-3}{2}}(q^{\frac{1}{4}+\frac{i}{2}}+q^{-\frac{1}{4}-\frac{i}{2}}) & k:\ \text{odd} \\ 2\prod_{i=0}^{\frac{k-4}{2}}(q^{\frac{1}{2}+\frac{i}{2}}+q^{-\frac{1}{2}-\frac{i}{2}}) & k:\ \text{even} \end{cases}$$

so that $S^2=1$. We also normalize the T-transformation (2.90) by $e^{\pi ik/12}q^{-k(k-1)/12}$ so that $(ST)^3=1$. For example, the first non-trivial case occurs at $k=1$

$$T\big|_{\mathscr{V}_2} = e^{\pi i/12}\begin{pmatrix}1 & 0\\ 0 & -i\end{pmatrix}, \qquad S\big|_{\mathscr{V}_2} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i(q^{\frac{1}{2}}-q^{-\frac{1}{2}})\\ i(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^{-1} & -1\end{pmatrix}.$$

Next, we turn to less familiar and more interesting modular representation that arises from another Lagrangian A-brane in a similar fashion.

2.6.4 Exceptional Divisors

Now let us consider an interesting A-brane $\mathfrak{B}_{\mathbf{D}_i}$ supported on an exceptional divisor $\mathbf{D}_i$, $i=1,\ldots,4$. As we reviewed in the earlier part of this section, the ramification parameters $(\alpha_p,\beta_p,\gamma_p)$ play the role of resolution/deformation parameters for $\mathbf{D}_i$. In particular, when $\beta_p=0$ and $\hbar$ is real, only α_p can be turned on while γ_p must vanish in order for $\mathbf{D}_i$ to be Lagrangian with respect to ω_K. As $\hbar=|\hbar|e^{i\theta}$ is rotated $\theta\neq 0$ in the complex plane, the exceptional divisors $\mathbf{D}_i$ stay Lagrangian with respect to $\omega_{\mathfrak{X}}$ if the deformation parameter $\gamma_p+i\alpha_p\in\mathbb{C}$ in complex structure J is proportional to $i\hbar$, namely,

$$\mathrm{Im}\,\frac{\gamma_p+i\alpha_p}{2i\hbar}=0\,. \tag{2.117}$$

Here the value of β_p can be arbitrary as in the previous case. It is easy to verify from (2.24) and (2.62) that

$$\int_{\mathbf{D}_i}\frac{\mathrm{Im}\ \Omega}{2\pi}=\int_{\mathbf{D}_i}\frac{\omega_{\mathfrak{X}}}{2\pi}=0\,.$$

The story goes as before. The flatness condition (2.70) of the Chan-Paton bundle for the brane $\mathfrak{B}_{\mathbf{D}_i}$ is

$$F'_{\mathbf{D}_i}+B\big|_{\mathbf{D}_i}=0\,,$$

Since it is topologically $\mathbb{C}\mathbf{P}^1$, there is no holonomy and no deformation parameter for $\mathfrak{B}_{\mathbf{D}_i}$. Subsequently, the dimension can be computed as

$$\dim \mathrm{Hom}(\mathfrak{B}_{cc}, \mathfrak{B}_{\mathbf{D}_i}) = \int_{\mathbf{D}_i} \frac{F+B}{2\pi} = -c + \frac{1}{2} \,. \tag{2.118}$$

The Bohr-Sommerfeld quantization condition imposes its dimension as a positive integer $-c + 1/2 = \ell \in \mathbb{Z}_{>0}$, or equivalently $t^2 = q^{-(2\ell-1)}$, which is (2.99c).

When $t = q^{-(2\ell-1)/2}$, the lowering operator annihilates the Macdonald polynomial

$$\mathrm{pol}(\mathsf{L}_{2\ell}) \cdot P_{2\ell}(X; q, t) = 0 \,. \tag{2.119}$$

Therefore, the quotient space

$$\mathscr{D}_{2\ell} := \mathscr{P}/(P_{2\ell}) \tag{2.120}$$

by an ideal $(P_{2\ell})$ is a 2ℓ-dimensional representation of $S\ddot{H}$. In fact, it is not irreducible, and decomposes into two irreducible representations

$$\mathscr{D}_{2\ell} = \mathscr{D}_\ell^{(1)} \oplus \mathscr{D}_\ell^{(2)} \,. \tag{2.121}$$

Because P_j and $P_{2\ell-j-1}$ have the same eigenvalue of the Macdonald difference operator (2.77) when $t = q^{-(2\ell-1)/2}$, their combinations indeed form bases of $\mathscr{D}_\ell^{(1,2)}$

$$\mathscr{D}_\ell^{(1)} = \bigoplus_{j=0}^{\ell-1} \mathbb{C}_{q,t}\left[\frac{P_j(X)}{P_j(t^{-1})} + \frac{P_{2\ell-j-1}(X)}{P_{2\ell-j-1}(t^{-1})}\right], \quad \mathscr{D}_\ell^{(2)} = \bigoplus_{j=0}^{\ell-1} \mathbb{C}_{q,t}\left[\frac{P_j(X)}{P_j(t^{-1})} - \frac{P_{2\ell-j-1}(X)}{P_{2\ell-j-1}(t^{-1})}\right]. \tag{2.122}$$

Consequently, they are related by the sign change $\mathscr{D}_\ell^{(2)} = \xi_1(\mathscr{D}_\ell^{(1)})$. In fact, the support (2.82) of the brane $\mathfrak{B}_{\mathbf{P}}$ intersects with $\mathbf{D}_{1,2}$ at $t = q^{-(2\ell-1)/2}$ so that $\mathscr{D}_\ell^{(1)} \oplus \mathscr{D}_\ell^{(2)}$ can be obtained as the quotient of $\mathscr{P}$ as in (2.120).

Even when $t = -q^{-(2\ell-1)/2}$, the shortening condition (2.119) holds, but the eigenvalues (2.77) of the y-operator have the opposite sign as in (2.91). Therefore, the corresponding irreducible representations can be obtained by the sign change ξ_2 in (2.29) from $\mathscr{D}_\ell^{(1,2)}$.

As a result, for $t^2 = q^{-(2\ell-1)}$, there are four irreducible finite-dimensional representations [35, Theorem 2.8.1] that are obtained from $\mathscr{D}_\ell^{(1)}$ by the sign changes $\xi_{1,2}$. This is analogous to the relationship among the exceptional divisors under the sign changes (2.30). Therefore, we identify these modules to the spaces of open $(\mathfrak{B}_{cc}, \mathfrak{B}_{\mathbf{D}_i})$-strings as

$$\begin{aligned} \mathscr{D}_\ell^{(1)} &= \mathrm{Hom}(\mathfrak{B}_{cc}, \mathfrak{B}_{\mathbf{D}_1}) \,, & \mathscr{D}_\ell^{(2)} &= \xi_1(\mathscr{D}_\ell^{(1)}) = \mathrm{Hom}(\mathfrak{B}_{cc}, \mathfrak{B}_{\mathbf{D}_2}) \,, \\ \mathscr{D}_\ell^{(3)} := \xi_2(\mathscr{D}_\ell^{(1)}) &= \mathrm{Hom}(\mathfrak{B}_{cc}, \mathfrak{B}_{\mathbf{D}_3}) \,, & \mathscr{D}_\ell^{(4)} &:= \xi_2(\mathscr{D}_\ell^{(2)}) = \xi_3(\mathscr{D}_\ell^{(1)}) = \mathrm{Hom}(\mathfrak{B}_{cc}, \mathfrak{B}_{\mathbf{D}_4}) \,. \end{aligned} \tag{2.123}$$

The modules $\mathscr{D}_\ell^{(1,2)}$ can be obtained as the quotient of the polynomial representation because the support (2.82) of $\mathfrak{B}_{\mathbf{P}}$ intersects with $\mathbf{D}_1$ and $\mathbf{D}_2$. On the other hand, its ξ_2-image (2.92) intersects with $\mathbf{D}_3$ and $\mathbf{D}_4$. (See also Fig. 2.7.)

Under the PSL$(2,\mathbb{Z})$ action, the four irreducible representations are transformed as in (2.37). Namely, the modular T-transformation τ_- exchanges $\mathscr{D}_\ell^{(3)}$ and $\mathscr{D}_\ell^{(4)}$ whereas $\mathscr{D}_\ell^{(1)}$ and $\mathscr{D}_\ell^{(2)}$ are invariant. Also, the modular S-transformation σ exchanges $\mathscr{D}_\ell^{(2)}$ and $\mathscr{D}_\ell^{(3)}$ whereas the modules $\mathscr{D}_\ell^{(1)}$ and $\mathscr{D}_\ell^{(4)}$ are invariant.

$$\begin{aligned} \tau_+ : \mathscr{D}_\ell^{(2)} &\leftrightarrow \mathscr{D}_\ell^{(4)} \quad \text{and} \quad \mathscr{D}_\ell^{(1)},\ \mathscr{D}_\ell^{(3)} \quad \text{are invariant}\ , \\ \tau_- : \mathscr{D}_\ell^{(3)} &\leftrightarrow \mathscr{D}_\ell^{(4)} \quad \text{and} \quad \mathscr{D}_\ell^{(1)},\ \mathscr{D}_\ell^{(2)} \quad \text{are invariant}\ , \\ \sigma : \mathscr{D}_\ell^{(2)} &\leftrightarrow \mathscr{D}_\ell^{(3)} \quad \text{and} \quad \mathscr{D}_\ell^{(1)},\ \mathscr{D}_\ell^{(4)} \quad \text{are invariant}\ . \end{aligned} \tag{2.124}$$

Thus, only the module $\mathscr{D}_\ell^{(1)} = \mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathbf{D}_1})$ among the four modules becomes a PSL$(2,\mathbb{Z})$ representation.

Let us find the modular S and T matrices for this PSL$(2,\mathbb{Z})$ representation. As we have seen, the polynomial representation $\mathscr{P}$ captures both $\mathscr{D}_\ell^{(1)}$ and $\mathscr{D}_\ell^{(2)}$ so that the S-matrix (2.114) truncates a matrix of size $2\ell \times 2\ell$ under the shortening condition (2.99c). However, the matrix has rank ℓ and it acts non-trivially only on $\mathscr{D}_\ell^{(1)}$ under the change (2.122) of basis

$$\widetilde{S}_{jj'} := G^{-1} S_{jj'} G(q, t = q^{-(2\ell-1)/2})\big|_{\mathscr{D}_\ell^{(1)}}\ , \qquad 0 \le j, j' \le \ell - 1 \tag{2.125}$$

where G is a matrix of size $2\ell \times 2\ell$ that changes the basis according to (2.122). This gives the geometric interpretation of the basis change in [116, §4.1]. As a result, we find the following explicit forms of the S and T matrices, and a 3d interpretation of our A-model setup in §3.1.1 will identify an intrinsic physical meaning of the PSL$(2,\mathbb{Z})$ representation:

Conjecture 2.2 The space $\mathscr{D}_\ell^{(1)}$ is an ℓ-dimensional PSL$(2,\mathbb{Z})$ representation, with modular S and T matrices given by

$$\begin{aligned} T_{jj'}\big|_{\mathscr{D}_\ell^{(1)}} &= e^{\frac{(\ell-1)\pi i}{6}} q^{-\frac{(2\ell-1)(\ell-1)}{6}} q^{\frac{j(k-j)}{2}} \delta_{jj'} \qquad 0 \le j, j' \le \ell - 1 \\ S_{jj'}\big|_{\mathscr{D}_\ell^{(1)}} &= b_\ell^{-1} g_j(q, t = q^{-(2\ell-1)/2})^{-1} \widetilde{S}_{jj'}\ . \end{aligned} \tag{2.126}$$

The PSL$(2,\mathbb{Z})$ representation comes from a modular tensor category associated to the Argyres-Douglas theory of type $(A_1, A_{2(\ell-1)})$. These matrices coincide with those of the $(2, 2\ell+1)$ Virasoro minimal model at $q = e^{-2\pi i/(2\ell+1)}$.

Here we normalize (2.125) by the Macdonald norm (2.116) and

$$b_\ell = 2 \prod_{i=0}^{\ell-2} (q^{1/2+i} - q^{-1/2-i})$$

so that $S^2 = 1$. We also normalize (2.90) by $e^{(\ell-1)\pi i/6} q^{-(2\ell-1)(\ell-1)/6}$ so that $(ST)^3 = 1$.

Table 2.2 A summary of finite-dimensional representations of $S\ddot{H}$ with corresponding shortening and A-brane conditions

finite-dim rep	shortening condition	A-brane condition
$\mathscr{F}_m^{(x_m,y_m)}$	$q^m = 1$	$m = \frac{1}{\hbar}$
$\mathscr{U}_n$	$q^{2n} = 1$	$n = \frac{1}{2\hbar}$
$\mathscr{V}_{k+1}$	$t^2 = -q^{-k}$	$k = \frac{1}{2\hbar} + \frac{\gamma_p + i\alpha_p}{i\hbar}$
$\mathscr{D}_\ell$	$t^2 = q^{-\ell+1/2}$	$\ell = \frac{\gamma_p + i\alpha_p}{2i\hbar}$

For instance, when $\ell = 2$, these matrices become

$$T\big|_{\mathscr{D}^{(1)}_{\ell=2}} = e^{\frac{\pi i}{6}} \begin{pmatrix} q^{-\frac{1}{2}} & 0 \\ 0 & q^{\frac{1}{2}} \end{pmatrix}, \qquad S\big|_{\mathscr{D}^{(1)}_{\ell=2}} = \frac{i}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \begin{pmatrix} 1 & -(q - 1 + q^{-1}) \\ 1 & -1 \end{pmatrix} .$$

When $q = e^{-2\pi i/5}$, they coincide with the modular matrices of the $(2,5)$ Virasoro minimal model although an appropriate change of basis is required to bring the S-matrix into the standard form (Table 2.2).

2.7 Bound States of Branes and Short Exact Sequences: Morphism Matching

We have hitherto studied generic conditions when an individual A-brane supported on a compact irreducible Lagrangian can exist. Next, we will figure out the situation in which two distinct A-branes are present at a singular fiber of the Hitchin fibration. When two distinct A-branes intersect at a singular fiber, they will form a bound state. In this section, we will study a bound state of compact A-branes and identify the corresponding $S\ddot{H}$-module. This provide evidence of the equivalent morphism structure under the functor (1.3), restricting to the subcategory of compact Lagrangian A-branes with that of finite-dimensional $S\ddot{H}$-modules.

2.7.1 At Singular Fiber of Type I_2

As seen in Sects. 2.6.1 and 2.6.2, the compact branes $\mathfrak{B}_{\mathbf{F}}$ and $\mathfrak{B}_{\mathbf{U}_i}$ can exist when q is a root of unity and t is generic. As Fig. 2.6 shows, the irreducible components $\mathbf{U}_1$ and $\mathbf{U}_2$ at the singular fiber $\pi^{-1}(b_1)$ of type I_2 intersect at two points p_1 and p_2. Therefore, the Floer complex [56, 57] (or morphisms) of the two A-branes $\mathfrak{B}_{\mathbf{U}_1}$ and $\mathfrak{B}_{\mathbf{U}_2}$ is

$$\mathrm{Hom}^*(\mathfrak{B}_{\mathbf{U}_1}, \mathfrak{B}_{\mathbf{U}_2}) := CF^*(\mathfrak{B}_{\mathbf{U}_1}, \mathfrak{B}_{\mathbf{U}_2}) \cong \mathbb{C}\langle p_1 \rangle \oplus \mathbb{C}\langle p_2 \rangle . \tag{2.127}$$

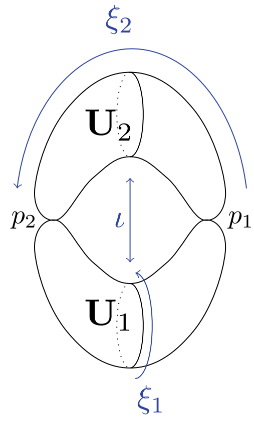

Fig. 2.6 At the singular fiber $\pi^{-1}(b_1)$, ξ_2 exchanges the irreducible components, $\mathbf{U}_1$ and $\mathbf{U}_2$, by the 180° rotation along the $(0,1)$-circle (longitude). Therefore, ξ_2 exchanges p_1 and p_2. On the other hand, ι exchanges $\mathbf{U}_1$ and $\mathbf{U}_2$ by fixing p_1 and p_2. Besides, ξ_1 maps each irreducible component to itself by the 180° rotation along the $(1,0)$-circle (meridian)

Note that the Floer complexes $CF^*(\mathfrak{B}_{\mathbf{U}_1}, \mathfrak{B}_{\mathbf{U}_2})$ and $CF^*(\mathfrak{B}_{\mathbf{U}_2}, \mathfrak{B}_{\mathbf{U}_1})$ and the differentials on them are Poincaré-dual to each other. Namely, each intersection point p_i defines generators of both complexes, whose degrees sum to 2 (the complex dimension of the target).

This implies that there are two bound states of $\mathfrak{B}_{\mathbf{U}_1}$ and $\mathfrak{B}_{\mathbf{U}_2}$ as A-branes. Let us consider one natural candidate for them: a brane $\mathfrak{B}_{\mathbf{F}}^{\lambda}$ degenerating into the singular fiber $\pi^{-1}(b_1)$ of type I_2. First of all, the dimension m of $\mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathbf{F}}^{\lambda})$ needs to be even $m = 2n$ in order for the brane to be supported on a singular fiber because the evaluation of the integral cohomology class $[F'_{\mathbf{F}}/(2\pi)]$ over a singular fiber cannot be odd like (2.102). There is also a topological constraint to be a bound state of the branes $\mathfrak{B}_{\mathbf{U}_1}$ and $\mathfrak{B}_{\mathbf{U}_2}$. As illustrated in Fig. 2.6, a one-cycle, say the $(1,0)$-cycle, of a torus is pinched to a double point at two locations so that the singular fiber $\pi^{-1}(b_1)$ topologically consists of two copies of $\mathbb{C}\mathbf{P}^1$. Therefore, it has the unique bounding spin structure along the $(1,0)$-cycle, which is Neveu-Schwarz. Consequently, only a brane $\mathfrak{B}_{\mathbf{F}}^{(-,+)}$ with trivial holonomy and the Neveu-Schwarz spin structure along the $(1,0)$-cycle of $\mathbf{F}$ can degenerate to a bound state of the branes $\mathfrak{B}_{\mathbf{U}_1}$ and $\mathfrak{B}_{\mathbf{U}_2}$ at the singular fiber $\pi^{-1}(b_1)$.

There is indeed a corresponding representation of $S\ddot{H}$. We see that the support $\mathbf{U}_1 \cup \mathbf{U}_2$ is invariant under τ_- (as a set). Thus, a brane $\mathfrak{B}_{\mathbf{F}}^{(x_{2n},+)}$ can enter the singular fiber when the corresponding module $\mathscr{F}_{2n}^{(-,+)}$ is τ_--invariant, namely when the two ideals

$$(X^{2n} + X^{-2n} - x_{2n} - x_{2n}^{-1}), \qquad (\tau_-(X^{2n} + X^{-2n} - x_{2n} - x_{2n}^{-1}))$$

coincide. Under the condition (2.99a), the $2n$th Macdonald polynomial takes the form $P_{2n} = X^{2n} + X^{-2n} + 2 = (X^n + X^{-n})^2$, and (2.90) yields $\tau_-(1) = 1$ and $\tau_-(P_{2n}) = t^{-2n} P_{2n}$. For a generic value of t, only when $x_{2n} = -1$, we therefore have the τ_--invariant module $\mathscr{F}_{2n}^{(-,+)} \cong \mathscr{P}/(P_{2n})$. Moreover, since $P_{2n} = (P_n)^2$

under (2.99a), the quotient of the polynomial representation $\mathscr{P}$ yields a short exact sequence

$$0 \to \mathscr{U}_n^{(2)} \to \mathscr{F}_{2n}^{(-,+)} \to \mathscr{U}_n^{(1)} \to 0\,. \tag{2.128}$$

The representation $\mathscr{F}_{2n}^{(-,+)}$ corresponds to the bound state $\mathfrak{B}_{\mathbf{F}}^{(-,+)}$. As explained in Sect. 2.6, the raising operator (2.80) of $\mathscr{P}$ does not become null because the prefactor $(1-q^{2j}t^2)$ cancels with the denominator of P_{j+1}. Consequently, this short exact sequence (2.128) does not split as a direct sum, but rather is a nontrivial extension of $\mathscr{U}_n^{(1)}$ by $\mathscr{U}_n^{(2)}$. This is analogous to the fact that $\mathbb{C}[X]/(X^{2n}) \to \mathbb{C}[X]/(X^n)$ cannot split as a $\mathbb{C}[X]$-module. As such, when the gradings are chosen such that $\mathscr{U}_n^{(1,2)}$ are in degree zero, the degree of the corresponding morphism between the A-branes is one, and corresponds to the class in $\mathrm{Hom}^1(\mathfrak{B}_{\mathbf{U}_1}, \mathfrak{B}_{\mathbf{U}_2})$ represented by $\mathfrak{B}_{\mathbb{F}}^{(-,+)}$.[9] Although this paper does not determine the degree of the morphism in the A-brane category, the representation category of $S\ddot{H}$ predicts one. Even in what follows, non-trivial extensions in the representation category give a description of degree-one morphisms (extensions or bound states) of various distinct compact A-branes. Determining the degree of the morphisms directly in the A-brane category is left for future work.

Since $\mathrm{Hom}^*(\mathfrak{B}_{\mathbf{U}_1}, \mathfrak{B}_{\mathbf{U}_2})$ is two-dimensional, there must be another generator. To identify it, we consider the symmetries. As Fig. 2.6 illustrates, ξ_2 and ι exchange the irreducible components $\mathbf{U}_1$ and $\mathbf{U}_2$ at the singular fiber. More precisely, ξ_2 acts on the singular fiber as the 180° rotation along the (0, 1)-circle (longitude) so that the intersection points $p_{1,2}$ are exchanged by ξ_2. On the other hand, ι exchanges $\mathbf{U}_1$ and $\mathbf{U}_2$ by fixing $p_{1,2}$. Consequently, the images of the brane $\mathfrak{B}_{\mathbf{F}}^{(-,+)}$ under the symmetries ξ_2 and ι are non-isomorphic objects in the A-brane category. They indeed span the morphism space

$$\mathrm{Hom}^1(\mathfrak{B}_{\mathbf{U}_2}, \mathfrak{B}_{\mathbf{U}_1}) \cong \mathbb{C}\langle \xi_2(\mathfrak{B}_{\mathbf{F}}^{(-,+)})\rangle \oplus \mathbb{C}\langle \iota(\mathfrak{B}_{\mathbf{F}}^{(-,+)})\rangle\,. \tag{2.129}$$

As a result, two irreducible branes can form bound states in more than one way. Similarly, the images of the brane $\mathscr{F}_{2n}^{(-,+)}$ under the symmetries ξ_2 and ι are non-isomorphic in the representation category of $S\ddot{H}$ for $n > 1$. The image of the short exact sequence (2.128) under ξ_2 becomes

$$0 \to \mathscr{U}_n^{(1)} \to \xi_2(\mathscr{F}_{2n}^{(-,+)}) \to \mathscr{U}_n^{(2)} \to 0\,. \tag{2.130}$$

Likewise, The image of the short exact sequence (2.128) under ι becomes

$$0 \to \mathscr{U}_n^{(1)} \to \iota(\mathscr{F}_{2n}^{(-,+)}) \to \mathscr{U}_n^{(2)} \to 0\,. \tag{2.131}$$

By using the polynomial representation (2.75), one can read off the action of the generators x and y on these representations as

[9] Often literature in mathematics uses the notation Ext^1 instead of Hom^1. Here they have the same meaning.

$$
x\Big|_{\xi_2(\mathscr{F}_{2n}^{(-,+)})} = \begin{array}{c} \begin{array}{cc} \overbrace{\hphantom{0 \ * \ \ldots \ 0 \ 0}}^{\mathscr{U}_n^{(2)}} & \overbrace{\hphantom{0 \ * \ \ldots \ 0 \ 0}}^{\mathscr{U}_n^{(1)}} \end{array} \\ \left(\begin{array}{ccccc|ccccc}
0 & * & \ldots & 0 & 0 & & & & & \\
* & 0 & * & 0 & 0 & & & & & \\
0 & * & \ddots & * & 0 & & & 0 & & \\
0 & 0 & & 0 & * & & & & & \\
0 & 0 & \ldots & * & 0 & & & & & \\
\hline
 & & & & {\color{red}*} & 0 & * & \ldots & 0 & 0 \\
 & & & & & * & 0 & * & 0 & 0 \\
 & & & & & 0 & * & \ddots & * & 0 \\
 & & & & & 0 & 0 & & 0 & * \\
 & & & & & 0 & 0 & \ldots & * & 0
\end{array}\right) \end{array} , \quad
x\Big|_{\iota(\mathscr{F}_{2n}^{(-,+)})} = \begin{array}{c} \begin{array}{cc} \overbrace{\hphantom{0 \ * \ \ldots \ 0 \ 0}}^{\mathscr{U}_n^{(2)}} & \overbrace{\hphantom{0 \ * \ \ldots \ 0 \ 0}}^{\mathscr{U}_n^{(1)}} \end{array} \\ \left(\begin{array}{ccccc|ccccc}
0 & * & \ldots & 0 & 0 & & & & & \\
* & 0 & * & 0 & 0 & & & & & \\
0 & * & \ddots & * & 0 & & & 0 & & \\
0 & 0 & & 0 & * & & & & & \\
0 & 0 & \ldots & * & 0 & & & & & \\
\hline
 & & & & 0 & 0 & * & \ldots & 0 & 0 \\
 & & & & & * & 0 & * & 0 & 0 \\
 & & & & & 0 & * & \ddots & * & 0 \\
 & & & & & 0 & 0 & & 0 & * \\
{\color{red}*} & & & & & 0 & 0 & \ldots & * & 0
\end{array}\right) \end{array} ,
\tag{2.132}
$$

on the basis where y acts diagonally as $\mathrm{diag}(t+t^{-1}, qt+q^{-1}t^{-1}, \ldots, q^{2n-1}t+q^{1-2n}t^{-1})$. Note that the upper-left block and lower-right matrices of the x actions are the same whereas the lower-left matrices are different. These matrices explicitly show that $\xi_2(\mathscr{F}_{2n}^{(-,+)})$ and $\iota(\mathscr{F}_{2n}^{(-,+)})$ are not isomorphic.

In fact, the composition $\xi_2 \circ \iota$ leaves $\mathfrak{B}_{\mathbf{U}_1}$ and $\mathfrak{B}_{\mathbf{U}_2}$ as they are, respectively. However, it maps $\mathfrak{B}_{\mathbf{F}}^{(-,+)}$ to a different object. Correspondingly, we have a short exact sequence

$$0 \to \mathscr{U}_n^{(2)} \to \xi_2 \circ \iota(\mathscr{F}_{2n}^{(-,+)}) \to \mathscr{U}_n^{(1)} \to 0 \,, \tag{2.133}$$

which is not isomorphic to (2.128). Therefore, they span the morphism space of two dimensions

$$\mathrm{Hom}^1(\mathfrak{B}_{\mathbf{U}_1}, \mathfrak{B}_{\mathbf{U}_2}) \cong \mathbb{C}\langle \mathfrak{B}_{\mathbf{F}}^{(-,+)} \rangle \oplus \mathbb{C}\langle \xi_2 \circ \iota(\mathfrak{B}_{\mathbf{F}}^{(-,+)}) \rangle \,, \tag{2.134}$$

which is Poincaré-dual to (2.129). In conclusion, when two compound branes intersect two points, they can form non-isomorphic bound states with the same support in the A-brane category, and these bound states are related to subtleties defining A-branes supported on singular submanifolds.

At the other singular fibers $\pi^{-1}(b_{2,3})$, there are similar bound states. As in (2.108), $\sigma \in \mathrm{PSL}(2,\mathbb{Z})$ maps (2.129) to $\mathrm{Hom}^1(\mathfrak{B}_{\mathbf{U}_3}, \mathfrak{B}_{\mathbf{U}_4})$. Also, $\tau_+ \in \mathrm{PSL}(2,\mathbb{Z})$ maps (2.129) to $\mathrm{Hom}^1(\mathfrak{B}_{\mathbf{U}_5}, \mathfrak{B}_{\mathbf{U}_6})$.

2.7.2 At Global Nilpotent Cone of Type I_0^*

Next, let us consider the case in which both the A-branes $\mathfrak{B}_{\mathbf{V}}$ and $\mathfrak{B}_{\mathbf{D}_i}$ exist. In order for both $\mathfrak{B}_{\mathbf{V}}$ and $\mathfrak{B}_{\mathbf{D}_i}$ to be Lagrangian, (2.109) and (2.117) need to be satisfied, which implies $\gamma_p = 0$ and $\hbar$ is real whereas α_p and β_p can be arbitrary. Therefore, the symplectic form must be $\omega_{\mathfrak{X}} = \omega_K/\hbar$. In this situation, $\mathbf{F}$ and $\mathbf{U}_i$ are also Lagrangian with respect to the symplectic form. Moreover, the quantization conditions, (2.118) and (2.111), for both $\mathfrak{B}_{\mathbf{D}_i}$ and $\mathfrak{B}_{\mathbf{V}}$ are

$$-c+\frac{1}{2}=\ell\ , \qquad \frac{1}{2\hbar}+2c-1=k+1\ , \tag{2.135}$$

which implies that $1/2\hbar = 2\ell + k + 1$. In other words, the two shortening conditions lead to the other one

$$(2.99c)\ \text{ and }\ \iota(2.99\text{b}) \quad \longrightarrow \quad (2.99a)\ \text{ where }\ n=2\ell+k+1\ .$$

Under this condition, there are therefore finite-dimensional representations of three kinds, $\mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathbf{U}_i})$, $\mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathbf{V}})$ and $\mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathbf{D}_i})$. On the representation theory side, the quotient of the polynomial representation yields a short exact sequence

$$0 \longrightarrow \iota(\mathscr{V}_{k+1}) \longrightarrow \mathscr{U}_n^{(1)} \xrightarrow{f} \mathscr{D}_\ell^{(1)} \oplus \mathscr{D}_\ell^{(2)} \longrightarrow 0\ . \tag{2.136}$$

We also note that there exist similar short exact sequences for the images of $\mathscr{U}_n^{(1)}$ under the symmetry $\Xi \times \mathrm{PSL}(2,\mathbb{Z})$ in Sect. 2.6.2 under the same shortening condition.

In a similar fashion, if the branes $\mathfrak{B}_{\mathbf{D}_i}$ and $\mathfrak{B}_{\mathbf{U}_i}$ exist simultaneously, their quantization conditions guarantee the existence of $\mathfrak{B}_{\mathbf{V}}$. Also, if we assume the presence of the branes $\mathfrak{B}_{\mathbf{V}}$ and $\mathfrak{B}_{\mathbf{U}_i}$, then the quantization condition for $\mathfrak{B}_{\mathbf{D}_i}$ follows. In fact, it is straightforward to check that, under the relation $n = k + 1 + 2\ell$, we have

$$\begin{aligned} (2.99c)\ \text{ and }\ (2.99a) \quad &\longrightarrow \quad \iota(2.99b)\ , \\ (2.99a)\ \text{ and }\ \iota(2.99b) \quad &\longrightarrow \quad (2.99c)\ . \end{aligned} \tag{2.137}$$

Subsequently, we have the short exact sequence (2.136).

If $\beta_p \neq 0$, the Hitchin fibration has the three singular fibers of type I_2 (Fig. 2.2), and the Lagrangians $\mathbf{V}$ and $\mathfrak{B}_{\mathbf{D}_i}$ are not contained in a Hitchin fiber. Thus, the short exact sequence (2.136) implies that a Hamiltonian isotopy can deform the brane $\mathfrak{B}_{\mathrm{U}_1}$ in such a way that it contains $\mathfrak{B}_{\mathbf{V}}$ as subbranes. The situation becomes much more lucid when $\beta_p = 0$. As $\beta_p \to 0$, the three singular fibers meet simultaneously and transform into the singular fiber of type I_0^*, which is the global nilpotent cone. In this process, the A-brane $\mathfrak{B}_{\mathbf{U}_1}$ becomes a bound state of $\mathfrak{B}_{\mathbf{D}_1}$, $\mathfrak{B}_{\mathbf{D}_2}$ and $\mathfrak{B}_{\mathbf{V}}$ because of (2.19). The short exact sequence (2.136) indeed corresponds to the bound state as illustrated in Fig. 2.7. A similar story holds for the other branes $\mathfrak{B}_{\mathbf{U}_i}$ and they become bound states of irreducible branes according to the relation (2.19) of the second homology group.

As explained above, the short exact sequence (2.136) does not split into the direct sum because the raising operator (2.80) of $\mathscr{P}$ never becomes null. Geometrically, the choice of the direction of the arrows in (2.136) comes from how the support (2.82) of $\mathfrak{B}_{\mathbf{P}}$ intersects with the global nilpotent cone. As explained in Sects. 2.6.3

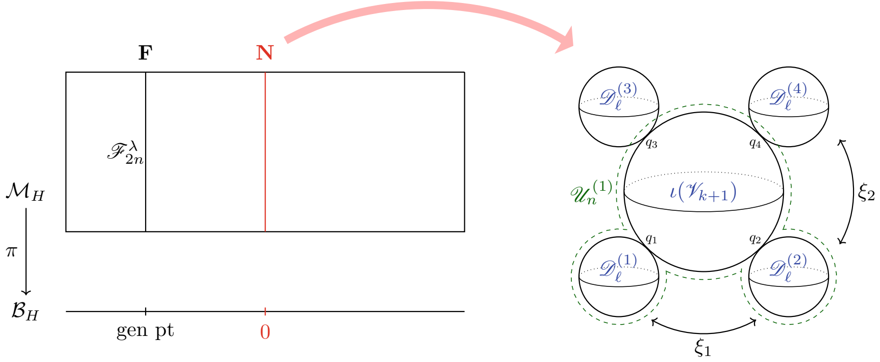

Fig. 2.7 This figure depicts the correspondence between compact supports of (B, A, A)-branes and finite-dimensional modules of the spherical DAHA when $\hbar = 1/2n$, $\alpha_p/2\hbar = \ell$ and $\beta_p = 0 = \gamma_p$. Note that $n = 2\ell + k + 1$

and 2.6.4, the support (2.82) of $\mathfrak{B}_{\mathbf{P}}$ cuts through real one-dimensional slices of the exceptional divisors $\mathbf{D}_{1,2}$, but it does not intersect with $\mathbf{V}$. As a result, the brane $\mathfrak{B}_{\mathbf{V}}$ becomes a subbrane of $\mathfrak{B}_{\mathrm{U}_1}$ whereas $\mathfrak{B}_{\mathbf{D}_1} \oplus \mathfrak{B}_{\mathbf{D}_2}$ becomes its quotient.

On the other hand, the ι-image $\iota(\mathfrak{B}_{\mathbf{P}})$ intersects with $\mathbf{V}$ whereas it does not with exceptional divisors $\mathbf{D}_i$. Consequently, there is a short exact sequence

$$0 \longrightarrow \mathscr{D}^{(3)} \oplus \mathscr{D}^{(4)} \longrightarrow \iota(\mathscr{U}_n^{(1)}) \longrightarrow \iota(\mathscr{V}_{n-2\ell}) \longrightarrow 0\,. \tag{2.138}$$

Once we take ξ_2-image of this short exact sequence, we have

$$0 \longrightarrow \mathscr{D}_\ell^{(1)} \oplus \mathscr{D}_\ell^{(2)} \xrightarrow{g} \xi_2 \circ \iota(\mathscr{U}_n^{(1)}) \longrightarrow \iota(\mathscr{V}_{n-2\ell}) \longrightarrow 0\,. \tag{2.139}$$

because $\iota(\mathscr{V}_{n-2\ell})$ is ξ_2-invariant.

Now we are ready to compare the morphism structures of the two categories under the shortening condition $\hbar = 1/2n$ and $\alpha_p/\hbar = \ell$. As Fig. 2.7 illustrates, the supports of branes $\mathfrak{B}_{\mathbf{V}}$ and $\mathfrak{B}_{\mathbf{D}_1}$ intersect at one point q_1 so that the morphism space between them is one-dimensional:

$$\mathrm{Hom}^1(\mathfrak{B}_{\mathbf{D}_1}, \mathfrak{B}_{\mathbf{V}}) \cong \mathbb{C}\langle q_1 \rangle\,. \tag{2.140}$$

This means that there is one bound state of $\mathfrak{B}_{\mathbf{V}}$ and $\mathfrak{B}_{\mathbf{D}_1}$. Indeed, we find the corresponding representation from (2.136):

$$0 \longrightarrow \iota(\mathscr{V}_{k+1}) \longrightarrow f^{-1}(\mathscr{D}_\ell^{(1)}) \longrightarrow \mathscr{D}_\ell^{(1)} \longrightarrow 0\,. \tag{2.141}$$

Its Poincare dual in the representation category can be obtained from (2.139)

$$0 \longrightarrow \mathscr{D}_\ell^{(1)} \longrightarrow \xi_2 \circ \iota(\mathscr{U}_n^{(1)})/g(\mathscr{D}_\ell^{(2)}) \longrightarrow \iota(\mathscr{V}_{n-2\ell}) \longrightarrow 0\,. \tag{2.142}$$

By using the sign change group Ξ, we obtain short exact sequences analogous to (2.141), which changes from $\mathscr{D}_\ell^{(1)}$ to $\mathscr{D}_\ell^{(i)}$ ($i = 2, 3, 4$). We can further pursue the comparison of the morphism structure. In the A-brane category, the morphism space between $\mathfrak{B}_{\mathbf{V}}$ and $\mathfrak{B}_{\mathbf{D}_1} \oplus \mathfrak{B}_{\mathbf{D}_2}$ is two-dimensional:

$$\mathrm{Hom}^1(\mathfrak{B}_{\mathbf{D}_1} \oplus \mathfrak{B}_{\mathbf{D}_2}, \mathfrak{B}_{\mathbf{V}}) \cong \mathbb{C}\langle q_1 \rangle \oplus \mathbb{C}\langle q_2 \rangle \ . \tag{2.143}$$

It is easy to find the corresponding representations

$$\begin{aligned} 0 \longrightarrow \iota(\mathscr{V}_{k+1}) \longrightarrow f^{-1}(\mathscr{D}_\ell^{(1)}) \oplus \mathscr{D}_\ell^{(2)} \longrightarrow \mathscr{D}_\ell^{(1)} \oplus \mathscr{D}_\ell^{(2)} \longrightarrow 0 \ , \\ 0 \longrightarrow \iota(\mathscr{V}_{k+1}) \longrightarrow f^{-1}(\mathscr{D}_\ell^{(2)}) \oplus \mathscr{D}_\ell^{(1)} \longrightarrow \mathscr{D}_\ell^{(1)} \oplus \mathscr{D}_\ell^{(2)} \longrightarrow 0 \ . \end{aligned} \tag{2.144}$$

In fact, the short exact sequence (2.136) can be understood as the diagonal element corresponding to $q_1 + q_2 \in \mathrm{Hom}^1(\mathfrak{B}_{\mathbf{D}_1} \oplus \mathfrak{B}_{\mathbf{D}_2}, \mathfrak{B}_{\mathbf{V}})$. More generally, we have

$$\mathrm{Hom}^1(\oplus_{i\in I}\mathfrak{B}_{\mathbf{D}_i}, \mathfrak{B}_{\mathbf{V}}) \cong \oplus_{i\in I}\mathbb{C}\langle q_i \rangle \ , \tag{2.145}$$

where I is a subset of $\{1, 2, 3, 4\}$. The diagonal element in the representation category is

$$0 \longrightarrow \iota(\mathscr{V}_{k+1}) \longrightarrow \mathscr{N}^I_{|I|\ell+k+1} \longrightarrow \oplus_{i\in I}\mathscr{D}_\ell^{(i)} \longrightarrow 0 \ , \tag{2.146}$$

where $|I|$ is the cardinality of the set I. We write the corresponding A-brane

$$\mathfrak{B}_{\mathbf{N}^I} \in \mathrm{Hom}^1(\oplus_{i\in I}\mathfrak{B}_{\mathbf{D}_i}, \mathfrak{B}_{\mathbf{V}}) \ , \tag{2.147}$$

which is supported on $\mathbf{N}^I := \cup_{i\in I}\mathbf{D}_i \cup \mathbf{V}$.

If the cardinality $|I|$ is three, the corresponding brane is supported on $\mathbf{V}$ plus three exceptional divisors, and the representation $\mathscr{N}^I$ is not obtained by a quotient of the polynomial representation. Therefore, these are *new finite-dimensional representations*, which do not appear in the theorems of Cherednik [35, Sects. 2.8–9].

When $I = \{1, 2, 3, 4\}$, the support of the corresponding brane is the entire global nilpotent cone $\mathbf{N}$ (2.16) so that we simply write it as $\mathfrak{B}_{\mathbf{N}}$. It turns out that this brane gives rise to another interesting bound state in the A-brane, which we will see below.

The global nilpotent cone. In fact, when $n - k - 1$ is odd (or equivalently $c \in \mathbb{Z}_{\leq 0}$), there is a short exact sequence

$$0 \longrightarrow \iota(\mathscr{V}_{k+1}) \longrightarrow \mathscr{F}^{(+,+)}_{2n} \longrightarrow \mathscr{N}_{2n-k-1} \longrightarrow 0 \ , \tag{2.148}$$

In fact, when both (2.99a) and ι(2.99b) are satisfied, we have $\mathrm{pol}(\mathsf{L}_{2n-k-1}) \cdot P_{2n-k-1} = 0$. Furthermore, when $n - k - 1$ is odd, then $\mathscr{N}_{2n-k-1} := \mathscr{P}/(P_{2n-k-1})$ becomes an irreducible module of dimension $2n - k - 1$. The short exact sequence (2.148) illustrates that the module $\mathscr{N}_{2n-k-1}$ can also be obtained by the quotient $\mathscr{F}^{(+,+)}_{2n}/\iota(\mathscr{V}_{k+1})$.

When $\beta_p = 0 = \gamma_p$, the Hitchin fibration has one singular fiber of type I_0^*, and the entire global nilpotent cone $\mathbf{N}$ is Lagrangian with respect to $\omega_{\mathfrak{X}}$. The short exact sequence (2.148) indeed depicts the situation where the brane $\mathfrak{B}_{\mathbf{F}}^{\lambda=(+,+)}$ with the Ramond spin structures enters the global nilpotent cone. Since it has a different spin structure, the brane is *not* decomposed into each irreducible component. As a result, the brane $\mathfrak{B}_{\mathbf{F}}^{\lambda=(+,+)}$ becomes the bound state of two branes; $\mathfrak{B}_{\mathbf{V}}$ and $\mathfrak{B}_{\mathbf{N}}$. Actually, using the fiber class relation (2.17) with (2.101) and (2.111), one can evaluate the dimension formula for an A-brane $\mathfrak{B}_{\mathbf{N}}$

$$\dim \operatorname{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathbf{N}}) = \int_{\mathbf{N}} \frac{F+B}{2\pi} = \int_{\mathbf{N}} \frac{\omega_I}{2\pi\hbar} = \frac{1}{2\hbar} + 2c - 1\ . \tag{2.149}$$

From (2.135), this is equal to $2n-k-1$, and the space of $(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathbf{N}})$-strings therefore corresponds to the module $\mathscr{N}_{2n-k-1}$ in (2.148).

One delicate point arises for constructing the Chan-Paton bundle for $\mathfrak{B}_{\mathbf{N}}$ because $\mathbf{N}$ is *not* a manifold. Since $\mathbf{V}$ is linked with the exceptional divisors $\mathbf{D}_i$ in $\mathfrak{B}_{\mathbf{N}}$, the Chan-Paton bundle for $\mathfrak{B}_{\mathbf{N}}$ is no longer well-defined at the four joining points of $\mathbf{V}$ and $\mathbf{D}_i$. The Chan-Paton bundle becomes a putative "line bundle" $\mathcal{L}'$ over each exceptional divisor $\mathbf{D}_i$ and the curvature $F'_{\mathbf{N}}$ of its connection has a half-integral flux over it [60]

$$\int_{\mathbf{D}_i} F'_{\mathbf{N}} = -\frac{1}{2}\ ,$$

while it cancels with the B-field due to (2.70)

$$F'_{\mathbf{N}} + B\big|_{\mathbf{N}} = 0\ .$$

In other words, $\mathcal{L}'$ restricted to an exceptional divisor $\mathbf{D}_i$ is a "square root" of the $\mathcal{O}(-1) \to \mathbb{C}\mathbf{P}^1$ bundle and the B-field flux over it is $1/2$. As a result, we have

$$\int_{\mathbf{D}_i} \frac{F+B}{2\pi} = \int_{\mathbf{D}_i} \frac{\omega_I}{2\pi\hbar} = \frac{n-k-1}{2} \in \frac{1}{2} + \mathbb{Z}\ ,$$

which gives the condition that $n-k-1$ is odd.

Under this circumstance, the line bundle $\mathcal{L} \to \mathfrak{X}$ (2.57) for $\mathfrak{B}_{\mathrm{cc}}$ is actually the $2n$th tensor product of the determinant line bundle [95, Sect. 8] of the Hitchin moduli space. As a result, the geometric quantization of $\mathbf{V}$ provides the quantum Hilbert space $\mathscr{V}_{k+1}$ on a once-punctured torus in Chern-Simons theory [82]. The additional series $\mathscr{V}_{k+1}$ at a primitive $2n$th root of unity $q = e^{\pi i/n}$ is called *perfect representation* [35, Sect. 2.9.3]. Moreover, when $n = k+2$, the additional series $\mathscr{V}_{k+1}$ of dimension $k+1$ is isomorphic to the well-known Verlinde formula of $\widehat{\mathfrak{sl}}(2)_k$ with level k for a torus (without puncture) [147].

Let us end this section by commenting briefly on future directions. There are an enormous number of non-compact Lagrangian submanifolds in the moduli space of Higgs bundles that have been studied in their own right: for example, the image of the Hitchin section, the brane of opers (see [25, 86, 124, 129] in a similar context), or the A-polynomial of any knot [80]. Each of these geometric objects should naturally be associated with an $S\ddot{H}$-module whose behavior precisely matches the geometric properties of the object, just as we demonstrate occurs for compact Lagrangians and for the (generalized) polynomial representation. It would be of great interest to further pursue this correspondence for infinite-dimensional representations, even just in the rank-one case.

It would also be interesting to connect explicitly with other mathematical contexts in which algebraic approaches to the Fukaya category or equivalences between Fukaya categories and module categories appear. To give just one example, in [51], Etgü and Lekili study the Chekanov-Eliashberg dg-algebra associated with a Legendrian link in a Weinstein four-manifold for a given graph. They show that this algebra is A_∞-quasi-isomorphic to, roughly speaking, the endomorphism algebra of a collection of generating objects of the wrapped Fukaya category of the surface, and go on to recover the multiplicative preprojective algebra studied in [31] in the context of the Deligne–Simpson problem from the Legendrian link. When the graph in question is the affine D_4 Dynkin diagram, it is expected that the corresponding preprojective algebra is related to DAHA. (We thank A. Oblomkov for private communication related to this point.) The computations of the (wrapped) Fukaya category of the above four-manifolds in [51] thus may provide an interesting perspective on our Claim 1.1 as well as its generalization to other algebras.