Chapter 7
Geometric Construction of the Covering Manifold

In this chapter, we provide a geometric construction of a manifold extending a given Galois cover to a wreath product, using composita and fiber products. For this to be possible, a certain assumption on the homology, previously called $(*)$, needs to be strengthened to a new condition $(**)$ (equivalent in most cases). To motivate and use this new condition, we first recall the connection between homology of a quotient and coinvariants. Apart from geometric tools, the construction is also based on the vanishing of certain group cohomology, which is used to prove the existence of certain isometries of manifolds. In the final section, we give a universal property of the wreath product in relation to coverings of manifolds, just like there is such a universal property in the theory of Galois extensions of fields.

7.1 From Quotient to Submodule

If ℓ is a prime number coprime to $|G|$, by Maschke's theorem, any short exact sequence of $\mathbf{F}_\ell[G]$-modules splits, and condition $(*)$ from Theorem 6.4.1 is equivalent to

$$(**)\ (\mathrm{Ind}_{H_1}^{G}\,\mathbf{1}) \otimes_{\mathbf{Z}} \mathbf{F}_\ell \textit{ is an } \mathbf{F}_\ell[G]\textit{-submodule of } \mathrm{H}_1(M, \mathbf{F}_\ell).$$

7.2 Homology of a Quotient as Coinvariants

We recall the following tool from invariant theory, see, e.g. [21, II §2]. If R is a commutative ring (for us, R is $\mathbf{Z}, \mathbf{Q}$ or $\mathbf{F}_\ell$), H a finite group, and $\mathscr{M}$ is a (left) $R[H]$-module, its *coinvariants* are defined as the R-module $\mathscr{M}_H := \mathscr{M}/I\mathscr{M}$ where

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I is the kernel of the augmentation map $R[H] \to R\colon \sum k_h h \mapsto \sum k_h$. An explicit description is given by

$$I\mathscr{M} = \langle h(x) - x\colon h \in H, x \in \mathscr{M}\rangle$$

(by linearly, it suffices to let x run over a set of generators of $\mathscr{M}$). Denote the projection map by

$$\underline{\mathrm{t}}_R\colon \mathscr{M} \to \mathscr{M}_H = \mathscr{M}/I\mathscr{M}. \tag{7.1}$$

When R is clear from the context, we will leave it out of the notation and simply write $\underline{\mathrm{t}}$ for this map.

This map is particularly easy if $\mathscr{M} = \bigoplus R[H]x_i$ is *free* as an $R[H]$-module with generators x_i; then $\mathscr{M}_H = \bigoplus Rx_i$ with the obvious map, i.e.,

$$\underline{\mathrm{t}}_R\colon \bigoplus R[H]x_i \to \bigoplus Rx_i\colon \sum_i \sum_h k_h h x_i \mapsto \sum_i \left(\sum_h k_h\right) x_i, \tag{7.2}$$

cf. [21, (2.3)].

One may use “transfer” to prove the following (the case of a free action is also in [21, II.(2.4)]).

Lemma 7.2.1 ([16, III.2.4]) *If H is a finite group of isometries of a closed smooth manifold M with quotient map*

$$q\colon M \to H\backslash M,$$

and the order of H is coprime to the characteristic of the field K, then the first K-homology of the quotient, $\mathrm{H}_1(H\backslash M, K)$*, is isomorphic to the coinvariants* $\mathrm{H}_1(M, K)_H$ *of the first K-homology of M, and under this identification, the map q_* that q induces on the first homology groups is the map $\underline{\mathrm{t}}_K$ from (7.1), i.e., we have a diagram*

$$\begin{tikzcd} & \mathrm{H}_1(H\backslash M, K) \arrow[d, "\cong"] \\ \mathrm{H}_1(M, K) \arrow[ur, "q_*"] \arrow[r, "\underline{\mathrm{t}}_K"'] & \mathrm{H}_1(M, K)_H \end{tikzcd}$$

7.3 Geometric Construction

We refer back to the situation of diagram (6.10), and keep our assumption that $\mathbf{F}_\ell$ is a field of order coprime to $|G|$. By condition $(**)$, we have a decomposition of

$\mathbf{F}_\ell[G]$-modules

$$\mathrm{H}_1(M, \mathbf{F}_\ell) = \mathscr{N} \oplus V \cong \bigoplus \mathbf{F}_\ell \omega_i \oplus V$$

(for some $\mathbf{F}_\ell[G]$-submodule V), where the G-action on $\mathscr{N}$ is given in terms of the permutation of cosets as $g\omega_i = \omega_{g(i)}$, with the convention that $i = 1$ corresponds to the trivial H_1-coset in G. We also let

$$V' := \bigoplus_{i \geq 2} \mathbf{F}_\ell \omega_i$$

denote the vector space complement of $\mathbf{F}_\ell \omega_1$ in $\mathscr{N}$.

The quotient map $q_1 \colon M \to M_1 = H_1 \backslash M$ induces a surjective map

$$q_{1*} \colon \mathrm{H}_1(M, \mathbf{F}_\ell) \to \mathrm{H}_1(M_1, \mathbf{F}_\ell),$$

and we define $\omega_1' := q_{1*}(\omega_1)$.

Let Γ_1 denote the subgroup $\Gamma_1 \leq \Gamma_0$ for which $M_1 = \Gamma_1 \backslash \widetilde{M}$.

Lemma 7.3.1 *Suppose ℓ is coprime to $|G|$ and condition* $(*)$ (equivalently, $(**)$) *holds. Then we have a well-defined and commutative diagram:*

$$\begin{tikzcd}
\Gamma \arrow[r, two heads] \arrow[d, "\iota"] \arrow[rr, bend left, "\Psi"] & \mathrm{H}_1(M, \mathbf{F}_\ell) \arrow[r, two heads, "\varphi"'] \arrow[d, "q_{1*}"] & C^n \arrow[d, "r_1"] \\
\Gamma_1 \arrow[r, two heads] \arrow[rr, bend right, "\chi_0"'] & \mathrm{H}_1(M_1, \mathbf{F}_\ell) \arrow[r, two heads, "\varphi_0"] & C
\end{tikzcd} \tag{7.3}$$

where

- *ι is the embedding of* Γ *in* Γ_1*;*
- *$r_1 \colon C^n \to C$, $(k_1, \dots, k_n) \mapsto k_1$ is projection onto the first coordinate;*
- *φ_0 is defined by*

$$\varphi_0 \colon \mathrm{H}_1(M_1, \mathbf{F}_\ell) \xrightarrow{\cong} \mathbf{F}_\ell \omega_1' \oplus W \to \mathbf{F}_\ell \cong C$$
$$k_1 \omega_1' + w \mapsto k_1 \qquad (k_1 \in \mathbf{F}_\ell, w \in W).$$

with $W := q_{1*}(V \oplus V')$ a complementary vector space to $\mathbf{F}_\ell \omega_1'$ in $\mathrm{H}_1(M_1, \mathbf{F}_\ell)$.

Proof To see that this is well defined and the right square commutes, we need that ω_1' is linear independent of $W = q_{1*}(V \oplus V')$; so suppose that there are $a_1, a_2 \in \mathbf{F}_\ell$ such that $a_1 \omega_1' + a_2 q_{1*}(v) = 0$ for some $v \in V \oplus V'$. This means that

$$a_1\omega_1 + a_2 v \in \ker(q_{1*}). \tag{7.4}$$

By Lemma 7.2.1, the kernel of q_{1*} is equal to the kernel of $\underline{\mathrm{t}}_{\mathbf{F}_\ell}$, and by definition this kernel is spanned by elements $h_1(\omega_i) - \omega_i$ $(i = 1, \dots, n)$ and $h_1(v) - v$ for $v \in V$ and $h_1 \in H_1$. Now

- for any $h_1 \in H_1 \leq G$, $h_1(\omega_i) - \omega_i = \omega_{h_1(i)} - \omega_i$; if $i = 1$, this element is zero, since that index corresponds to the trivial conjugacy class of H_1 in G, whereas if $i \neq 1$, this element belongs to V', since then also $h_1(i) \neq 1$;
- since $\mathscr{N} \oplus V$ is a decomposition as $\mathbf{F}_\ell[G]$-modules, $h_1(v) - v \in V$ for all $v \in V$ and all $h_1 \in H_1$. □

It follows that $\ker(q_{1*}) \subseteq V \oplus V'$, and by (7.4), $a_1\omega_1 \in V \oplus V'$. Since ω_1 is linearly independent from $V \oplus V'$, we conclude that $a_1 = 0$, as desired. This guarantees that if $\omega = \sum k_i\omega_i + v \in \mathrm{H}_1(M, \mathbf{F}_\ell)$ with $v \in V$ (so $\varphi(\omega) = (k_1, \dots, k_n)$), then $q_{1*}(\omega) = k_1\omega_1' + w \in \mathrm{H}_1(M_1, \mathbf{F}_\ell)$ with $w \in W$, so

$$\varphi_0(q_{1*}(\omega)) = k_1 = r_1(\varphi(\omega)).$$

Just like we defined $\Gamma = \ker \Psi$ in (6.11), we now set

$$\Gamma_1' := \ker \chi_0 \lhd \Gamma_1 \text{ and } M_1' := \Gamma_1' \backslash \widetilde{M} \text{ with covering map } q_1' \colon M_1' \to M_1. \tag{7.5}$$

The following lemma describes the relationship between the group $\Gamma = \ker \Psi$ used in Chap. 6, and $\Gamma_1' := \ker \chi_0$, the group used in this chapter.

Lemma 7.3.2 *Suppose ℓ is coprime to $|G|$ and condition* $(*)$ (equivalently, $(**)$) *holds. The group $\Gamma' = \ker \Psi$ can be expressed in terms of the group $\Gamma_1' = \ker \chi_0$ and a set $\{\overline{g}_1, \dots, \overline{g}_n\}$ of lifts of $\{g_1, \dots, g_n\}$ to Γ_0, as $\Gamma' = \Gamma'_{\mathrm{new}}$, where*

$$\Gamma'_{\mathrm{new}} := \bigcap_{i=1}^{n} \overline{g}_i \Gamma_1' \overline{g}_i^{-1} \cap \Gamma = \bigcap_{i=1}^{n} (\Gamma \cap \overline{g}_i \Gamma_1' \overline{g}_i^{-1}) = \bigcap_{i=1}^{n} \overline{g}_i (\Gamma \cap \Gamma_1') \overline{g}_i^{-1}. \tag{7.6}$$

Proof The equalities in (7.6) follow since Γ is normal in Γ_0. It remains to prove $\Gamma'_{\mathrm{new}} = \ker \Psi$. Notice that it follows from diagram (7.3) that

$$\Gamma_1' \cap \Gamma = \ker \chi_0 \cap \Gamma = \{\gamma \in \Gamma \mid r_1 \circ \Psi(\gamma) = 0\} = \Psi^{-1}(\{0\} \times C^{n-1}). \tag{7.7}$$

Since Ψ is surjective, this implies $\Psi(\Gamma_1' \cap \Gamma) = \{0\} \times C^{n-1}$. Since by definition

$$\Phi(g_i)(\{0\} \times C^{n-1}) = C^{i-1} \times \{0\} \times C^{n-i},$$

from diagram (6.12), we conclude that

$$\Psi(\overline{g}_i(\Gamma'_1 \cap \Gamma)\overline{g}_i^{-1}) = \Phi(g_i)\Psi(\Gamma'_1 \cap \Gamma) = C^{i-1} \times \{0\} \times C^{n-i}, \tag{7.8}$$

and therefore

$$\Psi(\Gamma'_{\text{new}}) \subseteq \bigcap_i C^{i-1} \times \{0\} \times C^{n-i} = \{0\},$$

so $\Gamma'_{\text{new}} \subseteq \ker \Psi$.

To prove the reverse inclusion, assume that $\Psi(\gamma) = 0$ for some $\gamma \in \Gamma$. Then by diagram (6.12) we also have $\Psi(\gamma_0^{-1}\gamma\gamma_0) = 0$ for any $\gamma_0 \in \Gamma_0$, so

$$\gamma_0^{-1}\gamma\gamma_0 \in \Psi^{-1}(0) \subseteq \Psi^{-1}(\{0\} \times C^{n-1}) \overset{(7.7)}{=} \Gamma'_1 \cap \Gamma.$$

Therefore $\gamma \in \gamma_0(\Gamma'_1 \cap \Gamma)\gamma_0^{-1}$ for all γ_0, showing that $\gamma \in \Gamma'_{\text{new}}$, so $\ker \Psi \subseteq \Gamma'_{\text{new}}$. □

Remark 7.3.3 Standard expressions for the kernel of the restriction and induction of representations (see, e.g., [54, Lemma 5.11]) allow one to give a representation-theoretic description of Γ'_{new}. Namely, let $\widetilde{\chi}_0$ denote the linear character on Γ_1 given by $\widetilde{\chi}_0(\gamma) = e^{2\pi i \chi_0(\gamma)/\ell}$ where χ_0 is as in diagram (7.3). Then, with $\ker \widetilde{\chi}_0 = \ker \chi_0 = \Gamma'_1$, we have

$$\ker \operatorname{Res}^{\Gamma_0}_{\Gamma} \operatorname{Ind}^{\Gamma_0}_{\Gamma_1} \widetilde{\chi}_0 = \Gamma \cap \ker \operatorname{Ind}^{\Gamma_0}_{\Gamma_1} \widetilde{\chi}_0 = \Gamma \cap \bigcap_{\gamma_0 \in \Gamma_0} \gamma_0 \ker(\widetilde{\chi}_0)\gamma_0^{-1} = \Gamma'_{\text{new}}.$$

We now perform the following 2-step geometric construction:

(a) For $g \in G$, "twist" the cover $q_1 \colon M \to M_1$ by defining $q_1^g \colon M \to M_1$ by $x \mapsto q_1(g^{-1}x)$, and set

$$M''_g := M'_1 \times_{M_1, q_1^g} M;$$

corresponding to the following diagram:

$$\begin{tikzcd}
& M''_g \arrow[dl] \arrow[dr] & & \\
M'_1 \arrow[dr, "q'_1", "C"'] & & M \arrow[dl, "q_1^g"'] \arrow[r, "g^{-1}\cdot"] & M \arrow[dll, "q_1"] \\
& M_1 & &
\end{tikzcd} \tag{7.9}$$

The two different M in the diagram are in fact identical, but the maps to M_1 are different.

(b) Iteratively construct the fiber product

$$M'_{\mathrm{new}} := M''_{g_1} \times_M M''_{g_2} \times_M \cdots \times_M M''_{g_n}, \tag{7.10}$$

where $\{g_1, \dots, g_n\}$ is the chosen set of representatives for G/H_1; this is presented in the following diagram:

$$\begin{tikzcd}
 & & M'_{\mathrm{new}} \arrow[dll] \arrow[dl] \arrow[drr] & & \\
M''_{g_1} \arrow[drr, "C"'] & M''_{g_2} \arrow[dr, "C"] & & \cdots & M''_{g_n} \arrow[dll, "C"] \\
 & & M & &
\end{tikzcd} \tag{7.11}$$

We will prove that this manifold M'_{new} is the same as M', the one constructed in the previous chapter.

Proposition 7.3.4 *Suppose ℓ is coprime to $|G|$ and condition* $(*)$ (equivalently, $(**)$) *holds.*

(i) *The fiber product M'_{new} in* (7.10) *is represented as*

$$M'_{\mathrm{new}} = \Big\{(x_1, \dots, x_n, x) \in M'_1 \times \cdots M'_1 \times M \mid \\ q'_1(x_i) = q_1(g_i^{-1}x),\ i = 1, \dots, n\Big\}, \tag{7.12}$$

and in these coordinates, the projection $M'_{\mathrm{new}} \to M''_{g_i}$ is given by

$$M'_{\mathrm{new}} \ni (x_1, x_2, \dots, x_n, x) \mapsto (x_i, x) \in M''_{g_i}.$$

M'_{new} is a connected manifold and corresponds to the subgroup Γ'_{new}, so that in fact $M'_{\mathrm{new}} = M'$.

(ii) *Geometrically, the action of $\widetilde{G}$ on M'_{new} is expressed as follows in the coordinates used in* (7.12)*: there exists an isometry $\iota\colon M'_{\mathrm{new}} \to M'_{\mathrm{new}}$ that conjugates the action of $\widetilde{G}$ into*

- *$\underline{c} = (c_i) \in C^n \le \widetilde{G}$ acts componentwise on each factor M''_{g_i}, i.e.,*

$$\iota^{-1}\underline{c}\iota \cdot (x_1, x_2, \dots, x_n, x) = (c_1x_1, \dots, c_nx_n, x); \tag{7.13}$$

- *$g \in G \le \widetilde{G}$ acts on M'_{new} by*

$$\iota^{-1}g\iota \cdot (x_1, x_2, \dots, x_n, x) = (x_{g^{-1}(1)}, x_{g^{-1}(2)}, \dots, x_{g^{-1}(n)}, gx), \tag{7.14}$$

where $g^{-1}(i)$ is defined, as before, via $g^{-1}g_i \in g_{g^{-1}(i)}H_1$. Colloquially, this means that, up to an isometry, in diagram (7.11), g act naturally on the "base" manifold M, while the points in the various M''_{g_j} above a given point in M are permuted across these different manifolds in the same way as g^{-1} permutes the cosets G/H_1.

Proof Since the group homomorphism $\chi_0 \colon \Gamma_1 \to C$ in diagram (7.3) is surjective, $\Gamma'_1 := \ker \chi_0 \lhd \Gamma_1$ is of index ℓ in Γ_1, and $q'_1 \colon M'_1 \to M_1$ is a C-Galois cover.

(a) Since M_1 is a manifold, the compositum is described as

$$M''_g = \{(x_1, x) \in M'_1 \times M \colon q'_1(x_1) = q_1(g^{-1}x)\}.$$

Since the degrees of the covers $q'_1 \colon M'_1 \to M_1$ and $q_1 \colon M \to M_1$ are coprime, Lemma 2.3.2 implies that M''_g is connected and equal to the compositum. As in (6.5), the action of g^{-1} on M_0 and M_1 corresponds to the action on Γ_0 and the subgroup Γ'_1 by conjugation with $\overline{g}^{-1}$, where $\overline{g}$ is a lift of g to Γ_0. Hence the corresponding group is the intersection $\Gamma''_g := \Gamma \cap \overline{g}\Gamma'_1\overline{g}^{-1}$, i.e., $M''_g = \Gamma''_g \backslash \widetilde{M}$. By Lemma 2.3.3 and coprimality of the degree, the covering $M''_g \to M$ is C-Galois.

(b) Since M is a manifold, the underlying set of the fiber product is indeed the set theoretic fiber product in (7.12). We next argue that M'_{new} is connected, agrees with the compositum, and indeed corresponds to the group Γ'_{new} (and hence Γ') in (7.6), i.e., $M'_{\text{new}} = M'$. This will finish the proof of (i). To see the connectedness, we use induction with respect to the number of factors. So suppose we have already proven that $M''_{g_1} \times_M \dots M''_{g_{N-1}} \to M$ is a connected C^{N-1}-cover corresponding to the group $\bigcap_{i=1}^{N-1} \overline{g}_i(\Gamma \cap \Gamma'_1)\overline{g}_i^{-1}$. By Lemma 2.3.2, the product with the next factor M''_{g_N} is connected if and only if

$$\Gamma = \langle \bigcap_{i=1}^{N-1} \overline{g}_i(\Gamma \cap \Gamma'_1)\overline{g}_i^{-1}, \overline{g}_N(\Gamma \cap \Gamma'_1)\overline{g}_N^{-1} \rangle. \tag{7.15}$$

To prove this, we notice that is true after applying Ψ, using (7.8): the image of left hand side is C^n, and the image of the right hand side is the subgroup of C^n spanned by $\{0\}^{N-1} \times C^{n-N}$ and $C^{N-1} \times \{0\} \times C^{n-N}$, which equals the whole of C^n. Hence Eq. (7.15) is true up to $\ker \Psi$, and from Lemma 7.3.2, it follows that $\ker \Psi$ is contained in both the left hand side and the right hand side of the equality, proving that (7.15) holds on the nose.

To prove (ii), note that $M'_{\text{new}} \to M$ is a C^n-Galois cover by Lemma 2.3.3, with one copy of C acting componentwise on each factor M''_{g_i}, and this is the same as the action of C^n on M'. The claim about the action of $g \in G \leq \widetilde{G}$ can be proven as follows: the action of G on M' is given by considering G as a subgroup of $\widetilde{G}$, and as such it acts by isometries on $M_{\text{new}} = M'$. We know that, in the geometric representation (7.12) for M'_{new},

$$g\underline{x} = \underline{y} \text{ with } \underline{x} = (x_1, \ldots, x_n, x), \underline{y} = (y_1, \ldots, y_n, y)$$

for some *unique* $y_i \in M'_1$ and $y \in M$ with $q'_1(y_i) \overset{\text{(I)}}{=} q_1(g_i^{-1}y)$. We only need to determine what y_i and y are. Since the action of G on M is as given, $y \overset{\text{(II)}}{=} gx$. Recall also that $g^{-1}g_i = g_{g^{-1}(i)}h_{g,i}$ for some $h_{g,i} \in H_1$. In particular, with $q_1 \colon M \to M_1$ the covering with group H_1, for $x \in M$ we have $q_1((g^{-1}g_i)^{-1}x) \overset{\text{(III)}}{=} q_1(g_{g^{-1}(i)}^{-1}x)$. We collect this information to compute

$$q'_1(y_i) \overset{\text{(I)}}{=} q_1(g_i^{-1}y) \overset{\text{(II)}}{=} q_1(g_i^{-1}gx) = q_1((g^{-1}g_i)^{-1}x) = q_1(g_{g^{-1}(i)}^{-1}x) \overset{\text{(III)}}{=} q'_1(x_{g^{-1}(i)}).$$

Since $q'_1 \colon M'_1 \to M_1$ is a C-cover, this shows that $y_i = c_i x_{g^{-1}(i)}$ for some $\underline{c} = (c_i) \in C^n$, that a priori depends on g and $\underline{x}$, i.e., it is a map

$$\underline{c} \colon G \times M' \to C^n.$$

Let us first prove that it does not depend $x \in M'$. Denote the dependency on $\underline{x}$ by $\underline{c}(\underline{x})$. Let $d(\cdot, \cdot)$ denote the distance on a manifold induced from the Riemannian metric. Since C acts properly discontinuously on M'_1 there is a $\delta > 0$ such that, for any two elements $c, c' \in C$ and $x \in M'_1$, if $d(cx, c'x) < \delta$, then $c' = c$. If $\underline{x}'$ is at distance ε from $\underline{x}$ in M', then so is $g\underline{x}$ from $g\underline{x}'$, and hence so is $c_i(\underline{x})x_{g^{-1}(i)}$ from $c_i(\underline{x}')x'_{g^{-1}(i)}$ for all i. Hence

$$\begin{aligned} &d(c_i(\underline{x})x_{g^{-1}(i)}, c_i(\underline{x}')x_{g^{-1}(i)}) \\ &\qquad \le d(c_i(\underline{x})x_{g^{-1}(i)}, c_i(\underline{x}')x'_{g^{-1}(i)}) + d(c_i(\underline{x}')x'_{g^{-1}(i)}, c_i(\underline{x}')x_{g^{-1}(i)}) \\ &\qquad = d(c_i(\underline{x})x_{g^{-1}(i)}, c_i(\underline{x}')x'_{g^{-1}(i)}) + d(x'_{g^{-1}(i)}, x_{g^{-1}(i)}) \le 2\varepsilon, \end{aligned}$$

(the equality in the above formula holds since $c_i(\underline{x}')$ is an isometry) and thus $c_i(\underline{x}) = c_i(\underline{x}')$ as soon as $\underline{x}$ and $\underline{x}'$ are at distance $< \delta/2$. We conclude that $\underline{c}(\underline{x})$ is locally constant in $\underline{x}$, and since M' is connected, $\underline{c}$ is actually independent of $\underline{x}$. so that we have a map

$$\underline{c} \colon G \to C^n. \tag{7.16}$$

Now denote the dependence on g by $\underline{c}(g)$. We will prove that this is a cocycle; note that we write the group operation on C^n multiplicatively. We observe that for two elements $g, h \in G$,

$$
\begin{aligned}
(c_i(gh)x_{(gh)^{-1}(i)}, ghx) = gh\underline{x} &= g(c_i(h)x_{h^{-1}(i)}, hx) \\
&= (c_i(g)c_{g^{-1}(i)}(h)x_{h^{-1}g^{-1}(i)}, ghx),
\end{aligned}
$$

so $\underline{c}(gh) = \underline{c}(g)\underline{c}(h)^g$, where the action of g on $\underline{c} = (c_i)$ is given by $\underline{c}^g := (c_{g^{-1}(i)})$. This shows that the map $\underline{c}$ in Eq. (7.16) is a cocycle from G to C^n, and the corresponding first group cohomology class lies in $\mathrm{H}^1(G, C^n)$. Since $|G|$ and $|C^n| = \ell^n$ are coprime, the latter cohomology group is zero [21, III.(10.1)], proving that $\underline{c}$ is a coboundary, i.e., there exists $\underline{v} \in C^n$ (independent of g) such that $\underline{c}(g) = \underline{v}^{-1}\underline{v}^g = (v_i^{-1}v_{g^{-1}(i)})$. Consider the isometry

$$
\iota\colon M'_{\mathrm{new}} \to M'_{\mathrm{new}}\colon \underline{x} = (x_i, x) \mapsto (v_i^{-1}x_i, x).
$$

Now

$$
\iota^{-1}g\iota(\underline{x}) = \iota^{-1}(c_i(g)v^{-1}_{g^{-1}(i)}x_{g^{-1}(i)}, gx) = \iota^{-1}(v_i^{-1}x_{g^{-1}(i)}, gx) = (x_{g^{-1}(i)}, gx),
$$

as was claimed. Note also that conjugating by ι commutes with the action of C^n, so it does not change that action. □

Remark 7.3.5 The action of $\widetilde{G}$ on M'_{new} ties up with the group theoretical construction from the previous chapter, as follows. The group $\widetilde{G} \cong \Gamma_0/\Gamma'$ acts naturally on $M' = \Gamma'\backslash\widetilde{M}$ via

$$
(\gamma_0\Gamma')\cdot(\Gamma'\widetilde{x}) = \Gamma'\cdot(\gamma_0\widetilde{x}). \tag{7.17}
$$

The explicit identification between M' and M'_{new} is given by the map

$$
M' = \Gamma'\backslash\widetilde{M} \ni \Gamma'\widetilde{x} \mapsto (\Gamma'_1\overline{g}_1^{-1}\widetilde{x}, \dots, \Gamma'_1\overline{g}_n^{-1}\widetilde{x}, \Gamma\widetilde{x}) =: (x_1, \dots, x_n, x) \in M'_{\mathrm{new}}, \tag{7.18}
$$

where $\{g_iH_1\}$ represent the cosets of H_1 in G and $G \to \Gamma_0 : g \mapsto \overline{g}$ is a section such that we have $\overline{e}_G = e_{\Gamma_0}$, $\overline{g^{-1}} = \overline{g}^{-1}$ and $\overline{g}\Gamma' = \jmath(g)$ with the homomorphism $\jmath : G \to \Gamma_0/\Gamma'$ representing the splitting of (6.14). The action of $\widetilde{G} \cong \Gamma_0/\Gamma'$, transferred from M' to M'_{new} is then

$$
(\gamma_0\Gamma')\cdot(\Gamma'_1\overline{g}_1^{-1}\widetilde{x}, \dots, \Gamma'_1\overline{g}_n^{-1}\widetilde{x}, \Gamma\widetilde{x}) = (\Gamma'_1\overline{g}_1^{-1}\gamma_0\widetilde{x}, \dots, \Gamma'_1\overline{g}_n^{-1}\gamma_0\widetilde{x}, \Gamma\gamma_0\widetilde{x}). \tag{7.19}
$$

Let $c \in \Gamma$ be an element satisfying $\Psi(c) = e_1$ and set $c_i := \overline{g}_ic\overline{g}_i^{-1} \in \Gamma$, as in Sect. 6.3. Utilising the diagrams (6.12) and (7.3), we see that $c_i \in \overline{g}_j\Gamma'_1\overline{g}_j^{-1}$ for all $j \neq i$. Thus, (7.19) implies

$$
\begin{aligned}
(c_i\Gamma')\cdot(x_1, \dots, x_n, x) &= (\Gamma'_1\overline{g}_1^{-1}c_i\widetilde{x}, \dots, \Gamma'_1\overline{g}_n^{-1}c_i\widetilde{x}, \Gamma c_i\widetilde{x}) \\
&= (\Gamma'_1\overline{g}_1^{-1}\widetilde{x}, \dots, \underbrace{\Gamma'_1c\overline{g}_i^{-1}\widetilde{x}}_{i\text{-th entry}}, \Gamma'_1\overline{g}_n^{-1}\widetilde{x}, \Gamma\widetilde{x}).
\end{aligned} \tag{7.20}
$$

Let $g \in G$; by definition, we have $\overline{g}^{-1}\overline{g}_i\Gamma'_1 = \overline{g}_{g^{-1}(i)}c^{-k_i(g)}\Gamma'_1$ for some $k_i(g)$ modulo ℓ. This implies that

$$\begin{aligned}
(\overline{g}\Gamma') \cdot (x_1, \ldots, x_n, x) &\overset{(7.19)}{=} (\Gamma'_1\overline{g}_1^{-1}\overline{g}\widetilde{x}, \ldots, \Gamma'_1\overline{g}_n^{-1}\overline{g}\widetilde{x}, \Gamma\overline{g}\widetilde{x}) \\
&= (\Gamma'_1 c^{k_1(g)}\overline{g}_{g^{-1}(1)}\widetilde{x}, \ldots, \Gamma'_1 c^{k_n(g)}\overline{g}_{g^{-1}(n)}\widetilde{x}, \Gamma\overline{g}\widetilde{x}) \\
&\overset{(7.20)}{=} (c_1^{k_1(g)} \cdots c_n^{k_n(g)}\Gamma') \cdot (x_{g^{-1}(1)}, \ldots, x_{g^{-1}(n)}, g \cdot x).
\end{aligned}$$

Now $g \mapsto (c_i^{k_i(g)}\Gamma')_{i=1}^n$ is a cocycle from G to $C^n = \Gamma/\Gamma'$, and since $\mathrm{H}^1(G, C^n) = 0$, there exists $(m_1, \ldots, m_n)$ with $k_i(g) = m_{g^{-1}(i)} - m_i$ (modulo ℓ). Using the commutativity of the elements $c_i\Gamma'$, this implies that if we set $c_0 := \prod_{i=1}^n c_i^{m_i}$, then

$$(c_0 \jmath(g) c_0^{-1}) \cdot (x_1, \ldots, x_n, x) = (x_{g^{-1}(1)}, \ldots, x_{g^{-1}(n)}, g \cdot x)$$

for all $g \in G$ and all $(x_1, \ldots, x_n, x) \in M'_{\mathrm{new}}$. This shows that a copy of G in $\widetilde{G} \cong \Gamma_0/\Gamma'$, namely $c_0\jmath(G)c_0^{-1}$, acts on M'_{new} via permutation of the first n entries. In other words, it is possible to conjugate the subgroup G in $\widetilde{G}$ to realise the specific action (7.14) on M'_{new}.

7.4 Universal Property of the Wreath Product

The appearance of the wreath product in our constructions becomes less of a surprise given the following universal property, showing that the minimal Galois cover that "contains" a G-cover and a C-cover as in our situation arises from this wreath product (the analogous result in the theory of field extensions is well known, compare [37, 13.7]).

Proposition 7.4.1 *Let G and C denote finite groups with C cyclic of prime order ℓ not dividing the order of G. Suppose that we are given Riemannian manifolds M, M_1, M'_1 and a developable Riemannian orbifold M_0 such that $M \to M_0$ is G-Galois with subcover $M_1 \to M_0$, and $M'_1 \to M_1$ is C-Galois. If $N \to M_0$ is a Galois cover of minimal degree admitting Riemannian covers $N \to M$ and $N \to M'_1$, then the Galois group G' of N over M_0 is the wreath product $\widetilde{G} := C^n \rtimes G$, where n is the degree of the cover $M_1 \to M_0$ (see Figure (7.21).)*

$$\begin{tikzcd}
& N \arrow[dl, dashed] \arrow[dr, dashed] \arrow[ddd, dashed, bend left=70, "G'"] & \\
M_1' \arrow[dr, "C"'] & & M \arrow[dl] \arrow[ddl, bend left, "G"] \\
& M_1 \arrow[d, "n"'] & \\
& M_0 &
\end{tikzcd} \tag{7.21}$$

Proof Writing the manifolds M_0, M, M_1, M_1', N as quotients of he universal cover $\widetilde{M}_0$ of M_0 by the respectively group Γ_0, Γ, Γ_1, Γ_1', Γ_N, the defining properties of N imply that it is the normal closure of the compositum of M_1' and M over M_0, and hence

$$\Gamma_N = \bigcap_{\gamma_0 \in \Gamma_0} \gamma_0(\Gamma \cap \Gamma_1')\gamma_0^{-1}.$$

First of all, for $g \in G$, choose one element $\overline{g} \in \Gamma_0$ that maps to $g \in \Gamma_0/\Gamma \cong G$. We claim that

$$\Gamma_N = \bigcap_{i=1}^{n} \Gamma_{g_i} \text{ where } \Gamma_{g_i} := \overline{g}_i(\Gamma \cap \Gamma_1')\overline{g}_i^{-1},$$

for $\{g_i\}$ a set of coset representatives for H_1 in G. Indeed, for any $\gamma_0 \in \Gamma_0$ we can write $\gamma_0 = \overline{g}_i\gamma_1$ for some $i \in \{1, \ldots, n\}$ and some $\gamma_1 \in \Gamma_1$, since the cosets of $H_1 \cong \Gamma_1/\Gamma$ in $G \cong \Gamma_0/\Gamma$ are $g_1H_1, \ldots, g_nH_1$ and the cosets of Γ_1 in Γ_0 are therefore $\overline{g}_1\Gamma_1, \ldots, \overline{g}_n\Gamma_1$. The statement now follows from the fact that $\gamma_0 \in \Gamma_0$ must lie in one of these cosets $\overline{g}_i\Gamma_1$; since both Γ and Γ_1' are normal in Γ_1, we have

$$\begin{aligned}
\gamma_0(\Gamma \cap \Gamma_1')\gamma_0^{-1} &= (\overline{g}_i\gamma_1)(\Gamma \cap \Gamma_1')(\overline{g}_i\gamma_1)^{-1} = \overline{g}_i\gamma_1(\Gamma \cap \Gamma_1')\gamma_1^{-1}\overline{g}_i^{-1} \\
&= \overline{g}_i(\Gamma \cap \Gamma_1')\gamma_1\gamma_1^{-1}\overline{g}_i^{-1} = \overline{g}_i(\Gamma \cap \Gamma_1')\overline{g}_i^{-1}.
\end{aligned}$$

Now since Γ is normal in Γ_0, $\Gamma \geq \Gamma_N$, and we find an exact sequence

$$1 \to \Gamma/\Gamma_N \to \Gamma_0/\Gamma_N \to \Gamma_0/\Gamma \cong G \to 1.$$

The natural map $\varphi\colon \Gamma \to \prod_{i=1}^{n} \Gamma/\Gamma_{g_i}$ has kernel $\bigcap_{i=1}^{n} \Gamma_{g_i} = \Gamma_N$. Next, $\Gamma/\Gamma_{g_i} \cong C$ since the index is the prime number ℓ. Finally, we claim that φ is surjective. For this, it suffices to find for every i an element $\gamma_i \in \Gamma$ with

$$\varphi(\gamma_i) = e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in C^n.$$

Since C is cyclic of prime order, every non-zero element is a generator, and it suffices to choose $\gamma_i \in (\bigcap_{j \neq i} \Gamma_{g_j}) \setminus \Gamma_{g_i}$. This is possible since the reasoning in the first paragraph of this proof shows that the latter set is non-empty. In the end, we find a sequence

$$1 \to C^n \to \Gamma_0 / \Gamma_N \to G \to 1$$

where G acts on C^n by permuting the factors like it permutes the cosets of H_1, and this finishes the proof. □

Remark 7.4.2 In our setup, the universality property says the following: if we search for the "easiest possible" twisted Laplace operator on M_1, meaning associated to the Laplace operator on some prime order cyclic cover of M_1, we necessarily arrive at a diagram of the form (5.6).

Project

Assuming condition ($**$), one can now give the following alternative construction of diagram (5.6) used in the main Theorem 6.4.1: perform the above two step construction of M'_{new} and *define* an action of $\widetilde{G}$ on M'_{new} using the right hand side of Eqs. (7.13) and (7.14). Prove directly that this manifold satisfies the required properties.