Chapter 5
Representations with a Unique Monomial Structure

In this chapter, we recall the notion of monomial structures (and their isomorphism) on a representation, show a natural monomial structure on induced representations, and introduce solitary characters (characters whose induced representation has a unique monomial structure up to isomorphism); these characters may be used to detect conjugacy of subgroups. We also recall a specific type of wreath product construction and state and prove Bart de Smit's theorem on the existence of solitary characters for these (and a follow-up result of Pintonello for characters of degree two)—these were previously formulated and used in the context of number theory, but we present them abstractly. We give an application to covering equivalence in a very specific setup of manifolds, and also count the number of required characters, based on a formula for the commutator of a wreath product.

5.1 Monomial Structures

Definition 5.1.1 Suppose $\rho\colon G \to \mathrm{Aut}(V)$ is a representation, and

$$V = \bigoplus_{x \in \Omega} \mathscr{L}_x$$

is a decomposition of V into one-dimensional spaces ("lines") $\mathscr{L}_x$ for $x \in \Omega$, with Ω some index set. If the action of G on V permutes the lines $\mathscr{L}_x$, we say that the G-set

$$L = \{\mathscr{L}_x : x \in \Omega\}$$

is a *monomial structure* on ρ.

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Equivalently, in a basis having precisely one element from each line $\mathscr{L}_x$, the action of any $g \in G$ is given by a matrix having exactly one non-zero entry in each row and column. Note that, contrary to the case of permutation matrices, the non-zero entry in the matrix need not be 1.

An *isomorphism of monomial structures* L and L' on two representation of the same group G is an isomorphism of L and L' as G-sets.

Example 5.1.2 An induced representation $\mathrm{Ind}_H^G \chi$ of a linear character $\chi \in \check{H}$ admits (by definition) a monomial structure where $\Omega = \{g_1, \ldots, g_n\}$ is such that $g_i H$ are the different cosets of H in G, and $\mathscr{L}_x = \mathbf{C} \cdot xH$. The corresponding matrices have as non-zero entries n-th roots of unity if χ is a character of order n. We call this monomial structure the *standard monomial structure* on $\mathrm{Ind}_H^G \chi$. This standard monomial structure is isomorphic to G/H as G-set. ◊

Definition 5.1.3 A linear character Ξ on a subgroup H of a group G is called G-*solitary* if $\mathrm{Ind}_H^G \Xi$ has a unique monomial structure up to isomorphism.

Lemma 5.1.4 *Let G denote a group with two subgroups H_1 and H_2, and suppose $\Xi \in \widecheck{H_1}$ is a G-solitary linear character. There exists a linear character $\chi \in \widecheck{H_2}$ for which there is an isomorphism of representations* $\mathrm{Ind}_{H_1}^G \Xi \cong \mathrm{Ind}_{H_2}^G \chi$ *if and only if H_1 and H_2 are conjugate subgroups of G.*

Proof In this situation, $\mathrm{Ind}_{H_2}^G \chi$ carries two monomial structures: the standard one and the one induced from the standard one on $\mathrm{Ind}_{H_1}^G \Xi$ through the isomorphism of representations. Hence these monomial structures have to be isomorphic. But as G-sets, they are G/H_1 and G/H_2, respectively (see Example 5.1.2). By Proposition 3.5.1(ii), this means precisely that H_1 and H_2 are conjugate in G. □

5.2 Wreath Product Construction

Definition 5.2.1 Let G denote a finite group and H a subgroup of index $n := [G : H]$ with cosets

$$\{g_1 H = H, g_2 H, \ldots, g_n H\}$$

of cardinality n. For a prime number ℓ, let $C = \mathbf{Z}/\ell\mathbf{Z}$ denote the cyclic group with ℓ elements, and let

$$\widetilde{G} := C^n \rtimes G$$

denote the *wreath product*; this is by definition the semidirect product where G acts on the n copies of C by permuting the coordinates in the same way as G permutes the cosets $g_i H$. In coordinates, this means that if we let $e_1, \ldots, e_n$ denote the standard basis vectors of C^n, and, as before, define the permutation $i \mapsto g(i)$ of $\{1, \dots, n\}$ by $gg_iH = g_{g(i)}H$, then the semidirect product is defined by the action

$$G \xrightarrow{\Phi} \mathrm{Aut}(C^n)\colon g \mapsto \Phi(g) = \left[\sum_{j=1}^{n} k_j e_j \mapsto \sum_{j=1}^{n} k_j e_{g(j)}\right] \tag{5.1}$$

where $k_j \in \mathbf{Z}/\ell\mathbf{Z}$. This is the (left) action of $g \in G$ on C^n given by

$$C^n \ni (k_1, \dots, k_n) \mapsto (k_{g^{-1}(1)}, \dots, k_{g^{-1}(n)}) \in C^n.$$

Define

$$\widetilde{H} := C^n \rtimes H$$

to be the subgroup of $\widetilde{G}$ corresponding to H. The cosets of $\widetilde{H}$ in $\widetilde{G}$ are of the form

$$\{\widetilde{g}_1\widetilde{H} = \widetilde{H}, \widetilde{g}_2\widetilde{H}, \dots, \widetilde{g}_n\widetilde{H}\},$$

where for $g_i \in G$, we have a corresponding element $\widetilde{g}_i := (0, g_i) \in \widetilde{G}$.

Remark 5.2.2 Recall that $\mathrm{Ind}_H^G \mathbf{1}$ is the $\mathbf{Z}[G]$-module corresponding to the permutation representation of G acting on the G-cosets of H. Thus, if we identify C with the additive group of the finite field $\mathbf{F}_\ell$, the action of G on $C^n \cong \mathbf{F}_\ell^n$ corresponds to the $\mathbf{F}_\ell[G]$-module $(\mathrm{Ind}_G^H \mathbf{1}) \otimes_{\mathbf{Z}} \mathbf{F}_\ell$.

Proposition 5.2.3 (Bart de Smit [28, §10]) *For all $\ell \geq 3$, there exists a $\widetilde{G}$-solitary character of order ℓ on $\widetilde{H}$.*

Proof Define Ξ by

$$\Xi\colon \widetilde{H} \to \mathbf{C}^*\colon (k_1, \dots, k_n, g) \mapsto e^{2\pi i k_1/\ell}. \tag{5.2}$$

Let $L = \{\mathscr{L}_x\}$ and $L' = \{\mathscr{L}'_x\}$ denote two monomial structures on $\rho := \mathrm{Ind}_{\widetilde{H}}^{\widetilde{G}} \Xi$, where L is the standard one (see Example 5.1.2). The action of $G \leq \widetilde{G}$ on L is that of G on G/H and (after rearranging) the action of $C^n \leq \widetilde{G}$ is given by

$$(k_1, \dots, k_n) \cdot \mathscr{L}_j = e^{2\pi i k_j/\ell} \cdot \mathscr{L}_j, \tag{5.3}$$

where we used the simplified notation $\mathscr{L}_j := \mathscr{L}_{g_j\widetilde{H}}$. The character ψ of ρ can be computed using as basis any set of vectors from the lines in L or L'. From the above,

$$|\psi((1, 0, \dots, 0))| = |e^{2\pi i/\ell} + \underbrace{1 + \cdots + 1}_{n-1}| > n - 2,$$

where the last inequality is strict since $\ell \geq 3$. On the other hand, computing the same trace using a basis from L', we get a sum of some number, say, m, of ℓ-th roots of unity, where m is the number of lines in L' that are mapped to itself by $(1, 0, \ldots, 0)$. If there is a line not mapped to itself (a zero diagonal entry in the corresponding matrix), then there are at least two (since every row/column has precisely two non-zero entries), so $m = n$ or $m \leq n - 2$. In the latter case, $|\psi((1, 0, \ldots, 0))| \leq n - 2$, which is impossible. Since C^n is generated by G-conjugates of $(1, 0, \ldots, 0)$, we find that C^n fixes all lines in L'. Hence $L' \subseteq L$, but since $|L| = |L'| = [\widetilde{G} : \widetilde{H}]$, we have $L = L'$. □

Pintonello [80, Theorem 3.2.2] has shown that for $\ell = 2$, there does not always exist a solitary character as in Proposition 5.2.3. However, he also proved the following result, of which we give a self-contained proof.

Proposition 5.2.4 (Pintonello [80, Theorem 2.3.1]) *Given a group G with two subgroups H_1 and H_2, consider the corresponding wreath products $\widetilde{G}$, $\widetilde{H}_1$ and $\widetilde{H}_2$ with $C = \mathbf{Z}/2\mathbf{Z}$. Set $\Xi \colon \widetilde{H}_1 \to \mathbf{C}^* \colon (k_1, \ldots, k_n, g) \mapsto (-1)^{k_1}$, and assume that both*

$$\operatorname{Ind}_{\widetilde{H}_1}^{\widetilde{G}} \mathbf{1} \cong \operatorname{Ind}_{\widetilde{H}_2}^{\widetilde{G}} \mathbf{1} \text{ and} \tag{5.4}$$

$$\operatorname{Ind}_{\widetilde{H}_1}^{\widetilde{G}} \Xi \cong \operatorname{Ind}_{\widetilde{H}_2}^{\widetilde{G}} \chi, \tag{5.5}$$

for some linear character χ on $\widetilde{H}_2$. Then $\widetilde{H}_1$ and $\widetilde{H}_2$ are conjugate in G.

Proof Equality (5.5) induces two monomial structures L_1 and L_2 on $\rho := \operatorname{Ind}_{\widetilde{H}_1}^{\widetilde{G}} \Xi$, where L_i is isomorphic to $\widetilde{G}/\widetilde{H}_i$. As in (5.3), $\varepsilon := (1, 0, \ldots, 0) \in C^n \leq \widetilde{G}$ fixes all lines in L_1. Note that the number of lines in L_i fixed by ε is the value of the character of $\operatorname{Ind}_{\widetilde{H}_i}^{\widetilde{G}} \mathbf{1}$ at ε, given in (3.3), and by (5.4), these are equal for $i = 1$ and $i = 2$. Hence all lines in L_2 are fixed by ε, and as in the previous proof, we conclude that C^n fixes all lines in L_2. Hence $L_2 \subseteq L_1$, but since $|L_1| = |L_2| = [\widetilde{G} : \widetilde{H}_i]$, we have $L_1 = L_2$. □

5.3 Application to Manifolds

We deduce the following intermediate result.

Corollary 5.3.1 *Suppose we have a diagram (1.2). Let $C := \mathbf{Z}/\ell\mathbf{Z}$ denote a cyclic group of prime order $\ell \geq 3$. Let $\widetilde{G}$ and $\widetilde{H}_1$ denote the wreath products as in Definition 5.2.1 (with $H = H_1$) and $\widetilde{H}_2 := C^n \rtimes H_2$ (with the same action defined via the H_1-cosets), and assume that there exists a diagram of Riemannian coverings*

$$\begin{tikzcd}
& M' \arrow[d, "C^n"] \arrow[ddl, bend right, "\widetilde{H}_1"'] \arrow[ddd, bend left=60, "\widetilde{G}"] \\
& M \arrow[dl, "H_1"'] \arrow[dd, "G"] \\
M_1 \arrow[dr, "p_1"'] & \\
& M_0
\end{tikzcd} \tag{5.6}$$

Then M_1 and M_2 are equivalent Riemannian covers of M_0 if and only if for a $\widetilde{G}$-solitary character Ξ on $\widetilde{H}_1$ and for some linear character χ on $\widetilde{H}_2$, the multiplicity of zero is equal in the two spectra

$$\sigma_{M_1}(\overline{\Xi} \otimes \mathrm{Res}^{\widetilde{G}}_{\widetilde{H}_1} \mathrm{Ind}^{\widetilde{G}}_{\widetilde{H}_1} \Xi) \textit{ and } \sigma_{M_2}(\overline{\chi} \otimes \mathrm{Res}^{\widetilde{G}}_{\widetilde{H}_2} \mathrm{Ind}^{\widetilde{G}}_{\widetilde{H}_1} \Xi)$$

and in the two spectra

$$\sigma_{M_1}(\overline{\Xi} \otimes \mathrm{Res}^{\widetilde{G}}_{\widetilde{H}_1} \mathrm{Ind}^{\widetilde{G}}_{\widetilde{H}_2} \chi) \textit{ and } \sigma_{M_2}(\overline{\chi} \otimes \mathrm{Res}^{\widetilde{G}}_{\widetilde{H}_2} \mathrm{Ind}^{\widetilde{G}}_{\widetilde{H}_2} \chi).$$

Proof First of all, since $\ell \geq 3$, a $\widetilde{G}$-solitary character Ξ on H_1 exists, by Proposition 5.2.3. By Proposition 4.1.1, the equalities of multiplicities of zero is equivalent to $\mathrm{Ind}^{\widetilde{G}}_{\widetilde{H}_1} \Xi \cong \mathrm{Ind}^{\widetilde{G}}_{\widetilde{H}_2} \chi$. Since Ξ is $\widetilde{G}$-solitary, we conclude by Lemma 5.1.4 that $\widetilde{H}_1$ and $\widetilde{H}_2$ are conjugate in $\widetilde{G}$. As C^n is normal in $\widetilde{H}_2$ with quotient H_2, we find that $\widetilde{H}_2 \backslash M' = H_2 \backslash M = M_2$ and hence this conjugacy defines an isometry from M_1 to M_2 that is the identity on M_0. □

Since χ runs over linear characters of $\widetilde{H}_2$, the "less abelian" the extension is, the less spectra need to be compared. A more precise statement is the following, where we use the abelianisation H_2^{ab} of H_2, defined as the quotient of H_2 by the subgroup generated by commutators (equivalently, the largest abelian quotient of H_2; equivalently, $H_2^{\mathrm{ab}} \cong \mathrm{Hom}(H_2, \mathbf{C}^*)$). The two extremes are then: if H_2 is abelian, H_2^{ab} is as large as H_2; but if H_2 is non-abelian simple, then $|H_2^{\mathrm{ab}}| = 1$.

Proposition 5.3.2 *In the setup of Corollary 5.3.1, the dimension of the representations of which the spectra are being compared is the index $[G : H_2]$. Furthermore, the number of spectral equalities to be checked in Corollary 5.3.1 by using all possible linear characters on $\widetilde{H}_2$ is bounded above by $2\ell \cdot |H_2^{\mathrm{ab}}|$.*

In Corollary 5.3.1 and Proposition 5.3.2, one may interchange the roles of H_1 and H_2, which could lead to tighter results.

Proof The dimension of the representations we are considering, as induced representations, is the index $[\widetilde{G} : \widetilde{H}_2] = [G : H_2]$.

The spectral criterion in the proposition requires testing of 2 equalities of spectra for each linear character on $\widetilde{H}_2$, so there are at most $2|\widetilde{H}_2^{\mathrm{ab}}|$ equalities to be checked.

The commutator subgroup of a wreath product $\widetilde{H}_2 = C^n \rtimes H_2$ is computed in [69, Cor. 4.9], and we find that in our case, with $\Omega = \{g_1, \ldots, g_n\}$ a set of representatives for the cosets,

$$|[\widetilde{H}_2, \widetilde{H}_2]| = |[H_2, H_2]| \cdot |\{f : \Omega \to C : \sum_{y \in \Omega} f(y) = 0\}|;$$

where, with $|C| = \ell$, the second factor is $\ell^{|\Omega|-1}$. Hence we find $|\widetilde{H}_2^{\mathrm{ab}}| = |H_2^{\mathrm{ab}}| \cdot \ell$, and the result follows. □

Remark 5.3.3 Using Proposition 4.1.1 to reformulate spectrally the extra assumption in Proposition 5.2.4 (where $\ell = 2$), we find that in this case, the number of equalities to check is at most $2 + 4|H_2^{\mathrm{ab}}|$.

Remark 5.3.4 By Lemma 3.9.1, the multiplicity of zero in the spectrum $\sigma_M(\rho)$ can be computed purely representation theoretically as the multiplicity of the trivial representation in ρ, which is in principle possible by Mackey theory (cf. Remark 4.2.6), but this would be going in reverse (from spectra to group theory instead of the other way around). Knowing the group G and its subgroups H_1 and H_2, Riemannian equivalence of M_1 and M_2 over M_0 can be checked by a finite computation, verifying that H_1 and H_2 are conjugate in G. Corollary 5.3.1 translates this into a spectral statement (in the special setup where the group $\widetilde{G}$ is realised as indicated there).

Remark 5.3.5 One may strip all geometric analysis from the results so far, and formulate the following purely group theoretical result. *Given a finite group G and two subgroups H_1 and H_2, then*

$$H_1 \text{ and } H_2 \text{ are conjugate in } G \text{ if and only if } \mathrm{Ind}_{\widetilde{H}_1}^{\widetilde{G}} \Xi = \mathrm{Ind}_{\widetilde{H}_2}^{\widetilde{G}} \chi$$

for some linear character χ on $\widetilde{H}_2$. Here, $\widetilde{G}$ denotes the wreath product corresponding to the action of G on the G-cosets of H_1, and Ξ denotes a solitary character of order 3 on $\widetilde{H}_1$ (which exists by Proposition 5.2.3). The proof is immediate from Lemma 5.1.4 and the final sentence in the proof of Proposition 5.3.1. Observe that the construction of the wreath products and of Ξ is completely explicit, and the linear characters on $\widetilde{H}_2$ can be described in terms of those on H_2 via the results used in the proof of Proposition 5.3.2.

In the next chapters, we study under which circumstances we have a cover as in Corollary 5.3.1, i.e., we deal with the realisation problem for the wreath product as isometry group of a cover, given an isometric free action of G on a closed manifold

M. This is analogous to the inverse problem of Galois theory, realising the wreath product as Galois group of a number field. In manifolds, some condition is necessary on M for such an extension to be possible at all.