



Chapter 7

Solving Quadratic Equations

Poh-Shen Loh proposed a method for solving quadratic equations that is based on a relation between the coefficients of the quadratic polynomial and its roots. Section 7.1 reviews the traditional methods for solving quadratic equations. Section 7.2 tries to convince the reader that Loh's method makes sense and then explains how to compute the roots. In Sect. 7.3 the computation is carried out for two quadratic polynomials and a similar computation for a quartic polynomial. Section 7.4 derives the traditional formula for the roots from Loh's formulas.

The introduction of algebra and modern algebraic notation is relatively recent. Previously, mathematicians used geometry almost exclusively, so it is interesting to look at al-Khwarizmi's geometric construction of the formula for the roots of quadratic equations (Sect. 7.5). Section 7.6 shows a clever geometric construction used by Cardano in developing the formula for the roots of cubic equations.

Section 7.8 presents other geometric methods for finding the roots of quadratic equations.¹ The chapter concludes with Sect. 7.9 which discusses numerical computation of the roots of quadratic equations.

7.1 Traditional Methods for Solving Quadratic Equations

Every student of mathematics memorizes the formula for obtaining the roots of a quadratic equation $ax^2 + bx + c = 0$:

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

¹ Chapter 11 is a prerequisite for a full understanding of these methods.

For now we will work with monic polynomials, $x^2 + bx + c = 0$, whose roots are:

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4c}}{2}. \quad (7.1)$$

Another method of solving quadratic equations is by factoring the polynomials more-or-less by trial-and-error. Sometimes it is easy to obtain the roots by factoring:

$$x^2 - 4x + 3 = (x - 1)(x - 3). \quad (7.2)$$

It is much harder to factor $x^2 - 2x - 24$ because there are many possible pairs of roots that must be considered:

$$(\pm 1, \mp 24), (\pm 2, \mp 12), (\pm 3, \mp 8), (\pm 4, \mp 6).$$

7.2 The Relation Between the Roots and the Coefficients

Theorem 7.1 *If r_1, r_2 are the roots of $x^2 + bx + c$ then:*

$$(x - r_1)(x - r_2) = x^2 - (r_1 + r_2)x + r_1 r_2 = x^2 + bx + c.$$

Therefore, even if we do not know the values of the roots, we do know that:

$$r_1 + r_2 = -b, \quad r_1 r_2 = c. \quad (7.3)$$

There is really nothing to prove because the result emerges from the computation. Consider some values of $-b, r_1, r_2$ and let m_{12} be the average of r_1, r_2 :

$-b$	r_1	r_2	m_{12}
33	12	21	$16\frac{1}{2}$
33	8	25	$16\frac{1}{2}$
33	1	32	$16\frac{1}{2}$
$-b$	r_1	r_2	m_{12}
-4	-16	12	-2
-4	-4	0	-2
-4	-3	-1	-2

For any quadratic equation the average of the two roots is constant:

$$m_{1,2} = \frac{r_1 + r_2}{2} = \frac{(-b - r_2) + r_2}{2} = -\frac{b}{2}.$$

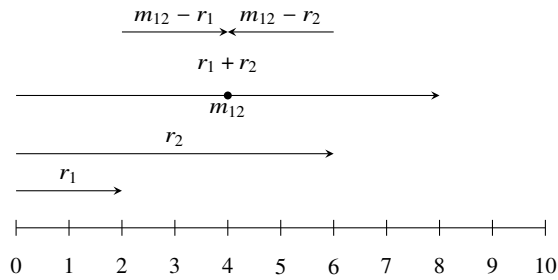


Fig. 7.1 Relation between the roots $r_1, r_2 = 2, 6$ and their average $m_{12} = 4$

Let s be any number. Then:

$$-b = -b + s + (-s) = \left(\frac{-b}{2} + s\right) + \left(\frac{-b}{2} - s\right) = r_1 + r_2.$$

If one root is at distance s from the average, the other root is at distance $-s$ from the average. For $r_1, r_2 = 2, 6$, where $m_{12} = 4, s = 2$, we have:

$-b$	r_1	r_2	m_{12}	$m_{12} - r_1$	$m_{12} - r_2$
33	12	21	$16\frac{1}{2}$	$4\frac{1}{2}$	$-4\frac{1}{2}$
33	8	25	$16\frac{1}{2}$	$8\frac{1}{2}$	$-8\frac{1}{2}$
33	1	32	$16\frac{1}{2}$	$15\frac{1}{2}$	$-15\frac{1}{2}$
-4	-16	12	-2	14	-14
-4	-4	0	-2	2	-2
-4	-3	-1	-2	1	-1

Figure 7.1 visualizes this relation. If we use other values $r_1, r_2 = 3, 5$ for which $r_1 + r_2 = 8$ then $m_{12} = 4$ remains the same while s becomes 1 (**Fig. 7.2**).

The offset s seems to be arbitrary in:

$$r_1 = \left(\frac{-b}{2} + s\right), \quad r_2 = \left(\frac{-b}{2} - s\right),$$

but there is an additional constraint $r_1 r_2 = c$, where c is the constant term in the polynomial. By multiplying the two expressions we have derived for r_1, r_2 , we can determine s and then r_1, r_2 :

$$c = \left(\frac{-b}{2} + s\right) \left(\frac{-b}{2} - s\right) = \frac{b^2}{4} - s^2$$

$$s = \frac{\sqrt{b^2 - 4c}}{2}.$$

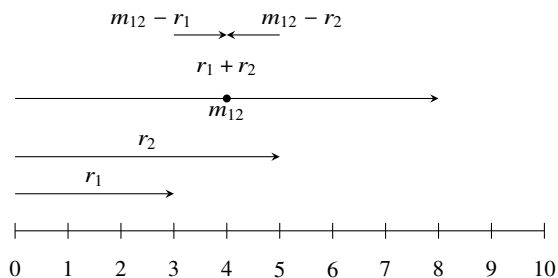


Fig. 7.2 Relation between the roots $r_1, r_2 = 3, 5$ and their average $m_{12} = 4$

7.3 Examples of Loh's Method

Example 7.1 Consider the polynomial $x^2 - 2x - 24$ where $b = -2, c = -24$:

$$\begin{aligned}
 c &= \left(-\frac{(-2)}{2} + s \right) \left(-\frac{(-2)}{2} - s \right) \\
 -24 &= (1 + s)(1 - s) \\
 s &= 5 \\
 r_1 &= 1 + 5 = 6 \\
 r_2 &= 1 - 5 = -4.
 \end{aligned}$$

Check: $(x - 6)(x - (-4)) = x^2 - 2x - 24$.

Example 7.2 Let us find the roots of $x^2 - 83x - 2310$:

$$\begin{aligned}
 -2310 &= \left(\frac{83}{2} + s \right) \left(\frac{83}{2} - s \right) \\
 s^2 &= \frac{6889}{4} + 2310 = \frac{16129}{4} \\
 s &= \frac{127}{2} \\
 r_1 &= \frac{83}{2} - \frac{127}{2} = -22 \\
 r_2 &= \frac{83}{2} + \frac{127}{2} = 105.
 \end{aligned}$$

Check: $(x + 22)(x - 105) = x^2 - 83x - 2310$.

Compare this computation with the computation using the traditional formula:

$$\begin{aligned}\frac{-b \pm \sqrt{b^2 - 4c}}{2} &= \frac{-(-83) \pm \sqrt{(-83)^2 - 4 \cdot (-2310)}}{2} \\ &= \frac{83 \pm \sqrt{16129}}{2} = \frac{83 \pm 127}{2} \\ r_1 &= \frac{83 - 127}{2} = -22 \\ r_2 &= \frac{83 + 127}{2} = 105.\end{aligned}$$

Example 7.3 Theorem 7.1 can be generalized to polynomials of higher degrees. Here is an interesting example for a *quartic equation* $x^4 - 10x^2 - x + 20 = 0$. As with quadratic equations there are formulas for solving cubic and quartic equations (though not equations of higher powers), but the formulas are quite complicated.

Does this polynomial of degree four factor into two quadratic polynomials with integer coefficients? If so, the coefficients of the x terms must be *equal and of opposite signs* since the coefficient of the x^3 term is zero. Therefore, the form of the quadratic factors is:

$$f(x) = (x^2 - nx + k_1)(x^2 + nx + k_2).$$

Carrying out the multiplication results in:

$$\begin{aligned}f(x) &= x^4 + nx^3 + k_2x^2 \\ &\quad - nx^3 - n^2x^2 - nk_2x \\ &\quad + k_1x^2 + nk_1x + k_1k_2.\end{aligned}$$

Equating the coefficients gives three equations in the three unknowns n, k_1, k_2 gives:

$$\begin{aligned}(k_1 + k_2) - n^2 &= -10 \\ n(k_1 - k_2) &= -1 \\ k_1k_2 &= 20.\end{aligned}$$

Since we are looking for factors with integer coefficients, from the last two equations it is clear that:

$$n = 1, k_1 = 4, k_2 = 5 \quad \text{or} \quad n = 1, k_1 = -5, k_2 = -4.$$

Only $n = 1, k_1 = -5, k_2 = -4$ satisfy the first equation for the coefficient of x^2 :

$$f(x) = (x^2 - x - 5)(x^2 + x - 4).$$

Solving these quadratic equations gives four solutions of the quartic equation:

$$x = \frac{1 \pm \sqrt{21}}{2} \quad \text{or} \quad x = \frac{-1 \pm \sqrt{17}}{2}.$$

7.4 Derivation of the Traditional Formula

For an arbitrary monic polynomial $x^2 + bx + c$, Loh's formulas are:

$$\begin{aligned} c &= r_1 r_2 = \left(\frac{-b}{2} + s \right) \left(\frac{-b}{2} - s \right) = \left(\frac{b^2}{4} - s^2 \right) \\ s &= \sqrt{\left(\frac{b^2}{4} \right) - c} \\ r_{1,2} &= \frac{-b}{2} \pm \sqrt{\left(\frac{b^2}{4} \right) - c} = \frac{-b \pm \sqrt{b^2 - 4c}}{2}, \end{aligned}$$

the traditional formula for obtaining the roots of a monic quadratic polynomial. If the polynomial is not monic divide it by a , substitute in the equation and simplify:

$$\begin{aligned} x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\ r_{1,2} &= \frac{-(b/a) \pm \sqrt{(b/a)^2 - 4(c/a)}}{2} \\ &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \end{aligned}$$

7.5 Al-Khwarizmi's Geometric Solution of Quadratic Equations

Let us write a monic quadratic polynomial as $x^2 + bx - c$. The roots can be found by *completing the square*:

$$\begin{aligned} x^2 + 2\left(\frac{b}{2}\right)x + \left(\frac{b}{2}\right)^2 &= c + \left(\frac{b}{2}\right)^2 \\ \left(x + \frac{b}{2}\right)^2 &= c + \left(\frac{b}{2}\right)^2 \\ x &= -\frac{b}{2} \pm \sqrt{c + \left(\frac{b}{2}\right)^2} = \frac{-b \pm \sqrt{b^2 + 4c}}{2}. \end{aligned}$$

This is the familiar formula for finding the roots of a quadratic equation, except that $4c$ has the opposite sign since the coefficient of the constant term was $-c$.

Completing the square was developed in the 8th century by Muhammad ibn Musa al-Khwarizmi in a geometric context. Given the equation $x^2 + bx = c$, assume that there is a square whose side is x so that its area is x^2 . To the area x^2 add bx by adding four rectangles of area $bx/4$ whose sides are $b/4$ and x (Fig. 7.3a). Now complete the diagram to a square by adding the four little squares of area $(b/4)^2$ (Fig. 7.3b).

We can't construct the diagram in Fig. 7.3a because we don't know what x is, but the area of the larger square in Fig. 7.3b is:

$$x^2 + bx + \frac{b^2}{4} = c + \frac{b^2}{4},$$

which we do know since we are given the coefficients b, c . By constructing the diagram and erasing the small squares whose sides are $(b/4)$ —another known quantity—we obtain the line segment of length x .

Example 7.4 Let $x^2 + 12x = 64$. Then $c + (b^2/4) = 64 + 36 = 100$. It is easy to construct a square of area 100 since each side has length 10. Now subtract $(b/4) + (b/4) = 6$, the sides of the smaller squares, to get $x = 10 - 6 = 4$.

7.6 Cardano's Construction for Solving Cubic Equations

The formula for the roots of cubic equations was first published in the 16th century by Gerolamo Cardano. We will not develop the formula here, but it is interesting that the central idea is based on a geometric construction similar to al-Khwarizmi's.

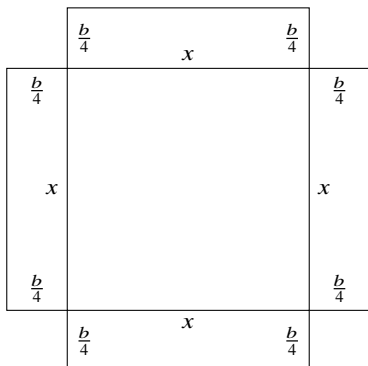


Fig. 7.3a The area is $x^2 + 4(b/4)x = x^2 + bx$

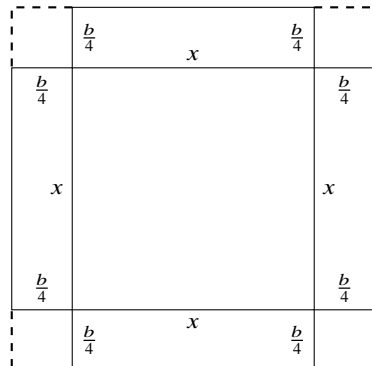


Fig. 7.3b The area is $x^2 + 4(b/4)x + 4(b/4)^2 = x^2 + bx + (b^2/4)$

The construction can be obtained very simply using algebra. By multiplication:

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = (a^3 + b^3) + 3ab(a + b). \quad (7.4)$$

Geometrically, we start with a cube whose side is $a + b$ so that its volume is $(a + b)^3$. The cube is decomposed into five pieces. The first two are cubes whose sides are a and b with volumes a^3 (blue) and b^3 (red), respectively (Fig. 7.4).

The other three parts are boxes (the technical term is *cuboid*) each with one side of length $a + b$ coinciding with a side of the cube, one side of length a and one side of length b , so that the volume of each of the three boxes is $ab(a + b)$. In Fig. 7.5, there is one box at the left side of the cube (blue), one at the back of the cube (red) and one at the top of the cube (green). By combining the five solids in Fig. 7.4 and Fig. 7.5 we obtain Eq. 7.4.

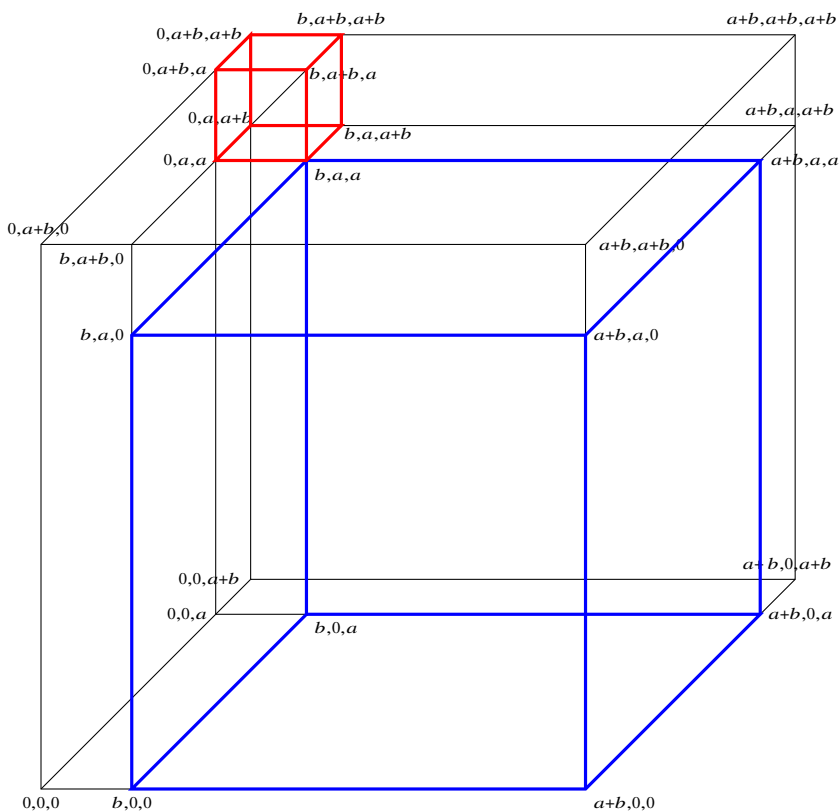


Fig. 7.4 $(a^3 + b^3) = (a^3 + b^3) + \dots$

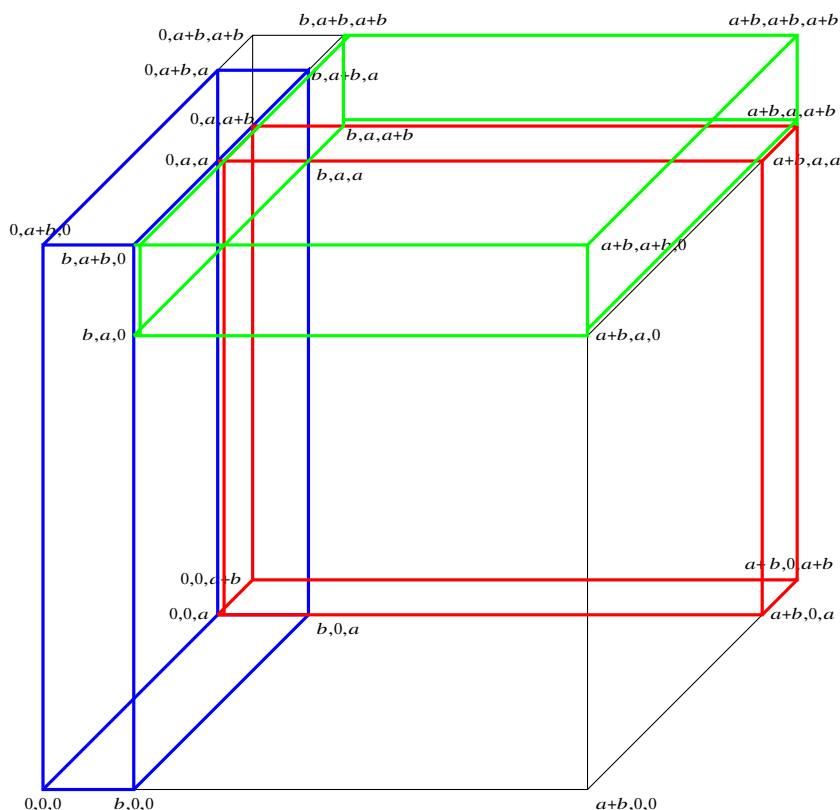


Fig. 7.5 $(a^3 + b^3) = \cdots + 3ab(a + b)$

7.7 They Weren't Intimidated by Imaginary Numbers

The history of mathematics demonstrates a progression of concepts that were initially considered to be meaningless, but were eventually understood, accepted and proved to be useful. “Obviously,” since numbers count things, -1 , a negative number, is meaningless. “Obviously,” since numbers are ratios of integers (rational numbers), $\sqrt{2}$, which can easily be proved to be irrational, is meaningless. “Obviously,” $\sqrt{-1}$, the square root of a negative number, is meaningless since there is no number—integer, rational or real—whose square is -1 .

A full understanding of the square roots of negative numbers, to this day called *imaginary numbers* although they are no less real than real numbers, was not achieved until the nineteenth century. Therefore, it is surprising that already in the sixteenth century, Geralamo Cardano and Rafael Bombelli refused to be intimidated by the concept, and took the first small steps towards understanding these numbers.

Consider the quadratic equation:

$$x^2 - 10x + 40 = 0. \quad (7.5)$$

By the familiar formula (Eq. 7.1):

$$r_1, r_2 = \frac{10 \pm \sqrt{100 - 160}}{2} = 5 \pm \sqrt{-15}.$$

Well, we don't know anything about the square roots of negative numbers and we don't know what these values are, but like Cardano we do know by Thm 7.1 that:

$$\begin{aligned} r_1 + r_2 &= (5 + \sqrt{-15}) + (5 - \sqrt{-15}) = 10 = -b \\ r_1 r_2 &= (5 + \sqrt{-15})(5 - \sqrt{-15}) = 25 - 5\sqrt{-15} + 5\sqrt{-15} - (-15) = 40 = c. \end{aligned}$$

which correspond with the coefficients of the quadratic equation Eq. 7.5. It is rather intuitive that $\sqrt{-15} + (-\sqrt{-15}) = 0$ even if we know nothing about $\sqrt{-15}$, and, similarly, it is rather intuitive that $\sqrt{-15} \cdot -(\sqrt{-15}) = -(-15) = 15$ even if we don't know what $\sqrt{-15}$ is.

Consider now the cubic equation:

$$x^3 - 15x - 4 = 0. \quad (7.6)$$

It is not hard to observe that 4 is a root, but how can it be computed? Cardano's formula gives the root:

$$r = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}}, \quad (7.7)$$

a quite complicated formula that bears no obviously relation to 4.

Bombelli courageously performed the following computation (see Eq. 7.4):

$$\begin{aligned} (2 + \sqrt{-1})^3 &= 8 + 3 \cdot 4\sqrt{-1} + 3 \cdot 2(-1) + (-1\sqrt{-1}) = 2 + 11\sqrt{-1} \\ (2 - \sqrt{-1})^3 &= 8 - 3 \cdot 4\sqrt{-1} + 3 \cdot 2(-1) - (-1\sqrt{-1}) = 2 - 11\sqrt{-1}, \end{aligned}$$

and by Eq. 7.7:

$$\begin{aligned} r &= \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}} \\ &= \sqrt[3]{(2 + \sqrt{-1})^3} + \sqrt[3]{(2 - \sqrt{-1})^3} \\ &= (2 + \sqrt{-1}) + (2 - \sqrt{-1}) = 4. \end{aligned}$$

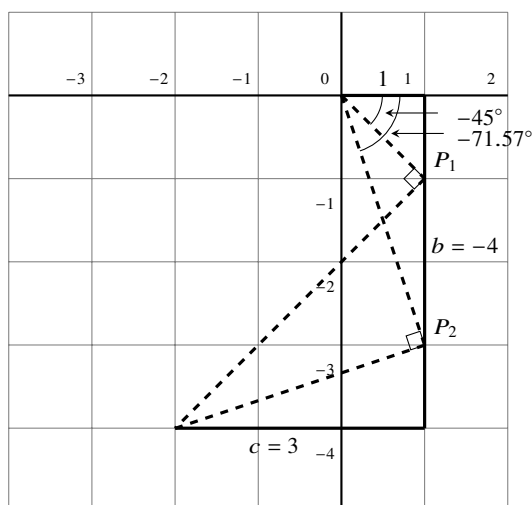


Fig. 7.6 Lill's method on $x^2 - 4x + 3$

7.8 Lill's Method and Carlyle's Circle

Lill's method can be applied to solve quadratic equations.² As an example we use Eq. 7.2 which gives the roots of a quadratic equation obtained by factorization:

$$x^2 + bx + c = x^2 - 4x + 3 = (x - 1)(x - 3).$$

Applying Lill's method results in the paths shown in Fig. 7.6.

Check that the angles are correct:

$$-\tan(-45^\circ) = -1, \quad -\tan(-71.57^\circ) \approx -3.$$

For quadratic equations we can find the points P_1, P_2 as the intersections of the line representing the coefficient b and the circle whose diameter is the line connecting the starting point and the end point of the paths (Fig. 7.7). In order for a point on the line b to be a root, the reflection of the line must be 90° and therefore the inscribed angle is subtended by a diameter.

This can also be checked by computation. The center of the circle is the midpoint of the diameter $(-1, -2)$. The length of the diameter is:

$$\sqrt{(-2)^2 + (-4)^2} = \sqrt{20},$$

² This section assumes that you have read about Lill's method in Chap. 11.

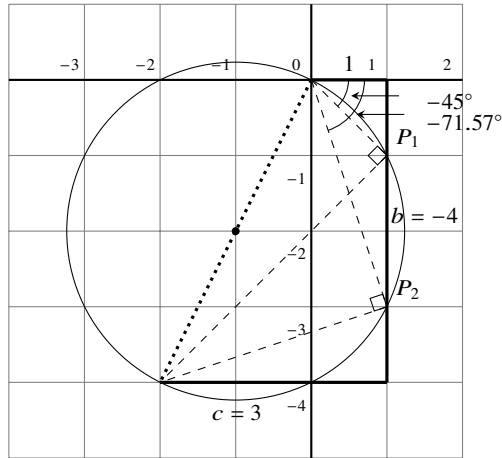


Fig. 7.7 Constructing a circle to find the roots

so the square of the length of the radius is $\left(\sqrt{20}/2\right)^2 = 5$. We need the intersection of this circle and the line $x = 1$:

$$\begin{aligned}(x - (-1))^2 + (y - (-2))^2 &= r^2 \\(x^2 + 2x + 1) + (y^2 + 4y + 4) &= 5 \\y^2 + 4y + 3 &= 0 \\y &= -1, -3.\end{aligned}$$

A similar method for solving quadratic equations is the Carlyle circle which predates Lill's method. Given a quadratic equation $x^2 - bx + c$ (note the minus sign on the linear term), construct points at $(0, 1)$ and (b, c) . Construct a circle whose diameter is the line connecting the two points (Fig. 7.8). Its intersections (if any) with the x -axis are the roots of the equation.

In the general case, the center of the circle is $(b/2, (c - (-1))/2)$ and the length of the diameter is $\sqrt{b^2 + (c - 1)^2}$, so the equation of the circle is:

$$\left(x - \frac{b}{2}\right)^2 + \left(y - \frac{c+1}{2}\right)^2 = \frac{b^2 + (c-1)^2}{4}.$$

For the example, substituting $b = 4$, $c = 3$ and $y = 0$, we see that $x = 1$ and $x = 3$ are the roots of the quadratic equation.

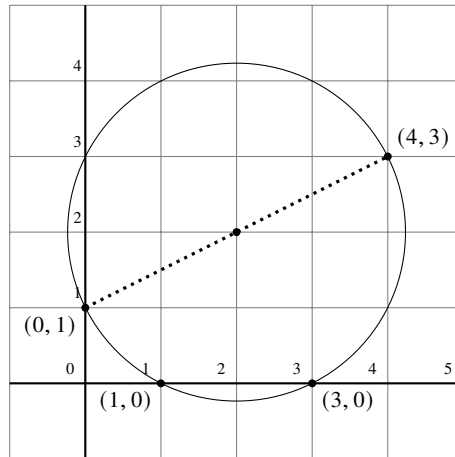


Fig. 7.8 Carlyle circle for $x^2 - 4x + 3$

7.9 Numerical Computation of the Roots

Students learn symbolic computation of roots, derivatives and so on. Today, most computation is performed by computers so symbolic computation is less important. *Numerical analysis* is the branch of mathematics and computer science that develops accurate and efficient computational methods. The main challenge is to deal with the finiteness of values stored in the computer's memory. The computation:

$$0.12 \times 0.14 = 0.0168$$

is easy to do, but:

$$0.123456789 \times 0.123456789$$

needs eighteen digits to be accurately represented and this cannot be done in a memory word that stores sixteen digits. This error is called a *round-off error*.

An even more serious problem is encountered when *floating point arithmetic* is performed. Clearly:

$$(0.12 \times 10^{-10}) \times (0.14 \times 10^{-8})$$

would not be computed by writing out all the zero digits. Instead, we multiply the mantissas and add the exponents to obtain 0.0168×10^{-18} , which is normalized to 0.168×10^{-19} so that the most significant digit appears after the decimal point, ensuring maximum precision given the fixed size of the mantissa. If the maximum exponent that can be represented is -16 the result simply cannot even be stored. This error is called *floating-point underflow*.

The formula for finding the roots of the quadratic equation $x^2 + bx + c$ is:

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4c}}{2}. \quad (7.8)$$

Consider what happens if $b = 1000$ and $c = 4$. The roots are:

$$r_1, r_2 = \frac{-1000 \pm \sqrt{1000000 - 16}}{2}.$$

Depending on the precision of the arithmetic, it is possible that one of the roots is so close to zero that the value stored is zero. Evaluating the quadratic equation gives the surprising result $0^2 + b \cdot 0 + 4 = 4 = 0$.

Can we do better? By Eq. 7.3:

$$r_1 + r_2 = -b, \quad r_1 r_2 = c.$$

If r_2 is much less than r_1 , written $r_2 \ll r_1$, then $r_1 \approx -b$ and $r_2 = c/b$. Table 7.1, computed by a computer program, compares the values of the roots computed by these formulas with the values obtained from the traditional formula Eq. 7.8. The value of c is fixed at 4 and the roots for increasing values of b are shown.

Initially, the true values computed by the traditional formula for r_2 are more accurate ($r_2 - r_{2v}$ is negative) but from $b = 100000$, the computation based upon Eq. 7.3 is more accurate. Such are the surprises of numerical analysis.

Table 7.1 Two computations of the roots of a quadratic equation. r_1, r_2 are the roots computed by Eq. 7.8. r_{1v}, r_{2v} are the roots computed using Eq. 7.3. The errors are $r_i - r_{iv}$. The values are truncated to four decimal places. Floating-point numbers are written $-4e-5$ in place of 4×10^{-5} because computer programs are normally written as linear sequences of characters.

b	r_1	r_{1v}	Error ₁	r_2	r_{2v}	Error ₂
100	-99.9599	-100	0.0400	-0.04001	-0.04	-1.6012e-05
1000	-999.9959	-1000	0.0040	-0.0040	-0.004	-1.6000e-08
10000	-9999.9996	-10000	0.0004	-0.0004	-0.0004	-1.6270e-11
100000	-99999.9999	-100000	3.9999e-5	-3.9999e-5	-4e-5	1.0104e-12
1000000	-999999.9999	-1000000	4.0000e-6	-3.9999e-6	-4e-6	2.7749e-11
10000000	-10000000.0	-10000000	3.9860e-7	-3.9953e-7	-4e-7	4.6261e-10

What Is the Surprise?

Poh-Shen Loh's approach provides a new way of looking at the relation between the coefficients and the roots that one doesn't see simply by memorizing the traditional formula. What is surprising is that this relation is fundamental in Gauss's algebraic proof of the constructibility of a regular heptadecagon (Chap. 16).

With the modern dominance of algebraic methods in geometry it is important to be reminded that the reverse once held. As shown by the constructions of Al-Khwarizmi and Cardano, geometric methods were used to obtain results in algebra. Lill and Carlyle both developed geometric methods for solving quadratic equations. Considerations of numerical computation on computers will surprise students who have not experienced it before.

Sources

Poh-Shen Loh's method is from [28, 29]. Al-Khwarizmi's construction is from [6, Chapter 1] and [32]. Cardano's construction can be found in [6, Chap. 1]. For the colorful history of the development of Cardano's formula see [52]. The early attempts at computing with imaginary numbers are from [6, Chapter 2]. Lill's method and Carlyle's circle can be found in [61] together with a discussion of numerical computation of the roots.

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