Chapter 2
Simplicial Sets

Simplicial sets form a very convenient tool to study the homotopy theory of topological spaces. In this chapter we will present an introduction to the theory of simplicial sets. We assume some basic acquaintance with the language of category theory, but no prior knowledge of simplicial sets on the side of the reader. We present the basic definitions and constructions, including the geometric realization of a simplicial set, the nerve of a category, and the description of the product of two simplicial sets in terms of shuffles. The category of simplicial sets is an example of a category of presheaves, and we also take the opportunity to discuss Kan extensions and several other constructions for presheaves that will be used again later in this book. The chapter ends with some examples of other types of simplicial objects, such as bisimplicial sets and simplicial operads. The material in this chapter is quite classical, and different presentations each having their own virtues can be found in the books already mentioned in the introduction. Our particular way of selecting and presenting the material was mainly motivated by the need to prepare the ground for the extension of the theory to that of dendroidal sets in the next chapter.

2.1 The Simplex Category $\mathbf{\Delta}$

In this section we recall the definition of the category $\mathbf{\Delta}$ of finite linear orders, which lies at the basis of the theory of simplicial sets. In fact there are two equivalent definitions of this category, a skeletal and a non-skeletal one. The skeletal category $\mathbf{\Delta}$ has as its objects the natural numbers, which are denoted $[n]$ (for $n \geq 0$) and are thought of as linear orders

$$[n] = \{0 \leq 1 \leq 2 \leq \cdots \leq n\}$$

or equivalently as free categories

$$[n] = \big(0 \to 1 \to 2 \to \cdots \to n\big).$$

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The morphisms $\alpha : [n] \to [m]$ in $\mathbf{\Delta}$ are the non-decreasing functions or, from the second perspective, functors $[n] \to [m]$.

Sometimes it is convenient to consider a larger version of $\mathbf{\Delta}$, whose objects are finite non-empty linearly ordered sets and whose morphisms are non-decreasing functions between them. The difference between the two versions will not matter very much, but we will usually stick to the skeletal one described above for notational convenience.

There are some morphisms in $\mathbf{\Delta}$ for which we introduce additional notation. First, there is for each $0 \leq i \leq n$ the injective monotone function

$$\delta_i : [n-1] \longrightarrow [n]$$

which skips the value i. Also, for each $0 \leq j \leq n-1$ there is the surjective function

$$\sigma_j : [n] \longrightarrow [n-1]$$

which hits the value j twice and every other value once; in other words, it is given by $\sigma_j(k) = k$ for $k \leq j$ and $\sigma(k) = k-1$ for $k > j$. These morphisms are called the *elementary faces* and *elementary degeneracies* respectively.

Note that any injective function $[m] \to [n]$ can be written as a composition of elementary face maps (although not necessarily uniquely). Also, any surjective function factors as a composition of elementary degeneracies. Since any morphism $[m] \to [n]$ factors as a surjection $[m] \to [k]$ followed by an injection $[k] \to [n]$, this shows that the elementary faces and degeneracies generate all the morphisms of $\mathbf{\Delta}$. One easily figures out the relations satisfied by these generating maps. For example, if $0 \leq i < j \leq n$ then the composition

$$[n-2] \xrightarrow{\delta_i} [n-1] \xrightarrow{\delta_j} [n]$$

is the injective map skipping i and j in its image, as is

$$[n-2] \xrightarrow{\delta_{j-1}} [n-1] \xrightarrow{\delta_i} [n].$$

In other words, we have the relation

(1) $\delta_j \delta_i = \delta_i \delta_{j-1}$ for $i < j$.

The other relations are as follows:

(2) $\sigma_i \sigma_j = \sigma_{j-1} \sigma_i$ for $i < j$.

(3) $\sigma_i \delta_j = \begin{cases} \delta_{j-1}\sigma_i & \text{if } i < j-1 \\ \text{id} & \text{if } i = j-1 \text{ or } i = j \\ \delta_j \sigma_{i-1} & \text{if } i > j. \end{cases}$

These relations are called the *cosimplicial identities*. As a consequence, a functor F from $\mathbf{\Delta}$ into any other category $\mathbf{C}$ can be specified by giving the values $F([n])$ for all $n \geq 0$ together with the maps $F(\delta_i)$ and $F(\sigma_j)$ corresponding to the elementary faces and degeneracies, provided that these maps satisfy the cosimplicial identities.

The category $\mathbf{\Delta}$ has very few limits and colimits, but there are some which we wish to single out. Suppose we have inclusions $f : [k] \to [n]$ and $g : [k] \to [m]$ where f is an 'initial segment' and g is a 'terminal segment', i.e. they satisfy

$$f(i) = i \qquad \text{and} \qquad g(i) = i + m - k.$$

Then the pushout square

$$\begin{array}{ccc} [k] & \xrightarrow{f} & [n] \\ {\scriptstyle g}\downarrow & & \downarrow \\ [m] & \longrightarrow & [n] \cup_{[k]} [m] \end{array}$$

exists in $\mathbf{\Delta}$; indeed, the bottom right corner is the linear order $[m + n - k]$. The simplest example of this is the pushout square

$$\begin{array}{ccc} [0] & \xrightarrow{0} & [n] \\ {\scriptstyle m}\downarrow & & \downarrow \\ [m] & \longrightarrow & [m+n]. \end{array}$$

Iterating this type of pushout we can write $[n]$ as the colimit of a diagram involving only [0]'s and [1]'s:

$$[n] = [1] \cup_{[0]} [1] \cup_{[0]} \cdots \cup_{[0]} [1].$$

Here there are n copies of [1] and each [0] includes as the vertex 1 of the copy of [1] on its left and the vertex 0 of the copy of [1] to its right.

An example of a different kind is the pushout of two surjections

$$[k] \xleftarrow{p} [n] \xrightarrow{q} [l]$$

between linear orders. One can think of $[k]$ as obtained from $[n]$ by collapsing certain segments to points and similarly for $[l]$. When one collapses both families of (possibly overlapping) segments to points, one obtains a further quotient $[m]$ which is the pushout of p and q. For example, for $0 \le i < j < n$,

$$\begin{array}{ccc} [n] & \xrightarrow{\sigma_j} & [n-1] \\ {\scriptstyle \sigma_i}\downarrow & & \downarrow{\scriptstyle \sigma_i} \\ [n-1] & \xrightarrow[\sigma_{j-1}]{} & [n-2] \end{array}$$

is such a pushout. These pushouts in $\mathbf{\Delta}$ have a special property, expressed by the following proposition:

Proposition 2.1 (i) *In the square above with $i < j$ there exist sections $\alpha\colon [n-1] \to [n]$ of σ_i and $\beta\colon [n-2] \to [n-1]$ of σ_i, which are compatible in the sense that $\sigma_j \alpha = \beta \sigma_{j-1}$.*

(ii) *Consider a commutative square*

$$\begin{array}{ccc} A & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle r} \\ C & \xrightarrow[s]{} & D \end{array}$$

in a category **C**, *with p a split epimorphism (i.e., admitting a section) and q an epimorphism. If there exist compatible sections α of p and β of r (in the sense described in (i)), then the square is a pushout. In fact it is an* absolute pushout, *meaning any functor from* **Δ** *to another category sends the square to a pushout square.*

(iii) *Let $[k] \xleftarrow{p} [n] \xrightarrow{q} [l]$ be surjections in* **Δ**. *The pushout*

$$\begin{array}{ccc} [n] & \xrightarrow{q} & [l] \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle r} \\ [k] & \xrightarrow[s]{} & [m] \end{array}$$

exists in **Δ** *and is an absolute pushout.*

Proof (i) Define $\alpha = \delta_i\colon [n-1] \to [n]$ and $\beta = \delta_i\colon [n-2] \to [n-1]$. The equation $\sigma_j \delta_i = \delta_i \sigma_{j-1}$ is one of the cosimplicial identities discussed above.

(ii) If X is an object of **C** and $f : B \to X$, $g : C \to X$ are maps such that $fq = gp$, then one defines a corresponding map $h : D \to X$ by $h := f\beta$. We should check that $hs = g$ and $hr = f$. The first equality is clear from $f\beta s = fq\alpha = gp\alpha = g$. For the second equality, it suffices to prove $hrq = fq$ because q is epi. The left-hand side equals $hrq = f\beta sp = fq\alpha p = gp\alpha p = gp$, which equals fq by assumption. To see that our choice of extension $h : D \to X$ is uniquely determined by (f, g), one observes that r is an epimorphism. The conclusion that the square is an absolute pushout follows from the fact that our proof only uses structures (split epis, commutative diagrams) that are preserved by any functor.

(iii) The surjections p and q can both be factored as compositions of elementary degeneracies, so that the conclusion follows by repeatedly applying (i) and (ii). □

Finally, let us record the following two existence results:

Proposition 2.2 (i) *If $f\colon [m] \to [n]$ is a monomorphism, then the pullback of any morphism $g\colon [k] \to [n]$ along f exists, provided that the image of g intersects the image of f nontrivially.*

(ii) *If $f\colon [m] \to [n]$ is an epimorphism, then the pushout of any morphism $g\colon [m] \to [k]$ along f exists.*

Proof (i) is clear by restricting g to the preimage of $f([m])$. For (ii), it suffices to treat the case where f is an elementary degeneracy $\sigma_i : [m] \to [m-1]$. Then the pushout of g is the map which collapses the interval $[g(i), g(i+1)]$ to a single point.□

2.2 Simplicial Sets and Geometric Realization

Let $\mathcal{E}$ be a category. The reader should keep in mind the examples where $\mathcal{E}$ is the category of sets, of topological spaces, or of groups. A *simplicial object* in $\mathcal{E}$ is a functor

$$X\colon \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathcal{E}.$$

With natural transformations between such functors as morphisms, one obtains a category of simplicial objects in $\mathcal{E}$, which we denote by $\mathbf{s}\mathcal{E}$. One generally refers to a simplicial object in **Sets** as a *simplicial set*, and similarly for simplicial spaces, simplicial groups, simplicial schemes etc. We will soon see plenty of examples of such simplicial objects.

In more detail, a simplicial object X in $\mathcal{E}$ is given by a sequence of objects $X_n := X([n])$ in $\mathcal{E}$ ($n \geq 0$), together with maps $\alpha^* : X_n \to X_m$ for morphisms $\alpha : [m] \to [n]$ in $\mathbf{\Delta}$. These maps should be functorial, in the sense that

$$\begin{aligned} &\mathrm{id}^* = \mathrm{id} : X_n \to X_n, \\ &(\alpha\beta)^* = \beta^*\alpha^* : X_n \to X_k \quad \text{for} \quad [k] \xrightarrow{\beta} [m] \xrightarrow{\alpha} [n]. \end{aligned}$$

A morphism f between two such simplicial objects X and Y is then a sequence of morphisms $f : X_n \to Y_n$ in $\mathcal{E}$ compatible with all the α^*, in the sense that

$$f_m\alpha^* = \alpha^* g_n$$

for $\alpha : [m] \to [n]$. When $\mathcal{E} = \mathbf{Sets}$, we will often refer to the elements of the set X_n as the *n-simplices* of X.

By our description of the morphisms in $\mathbf{\Delta}$ in the previous section, one may equivalently describe a simplicial object by specifying the operations α^* only when α is an elementary face or degeneracy. These are usually denoted

$$\begin{aligned} d_i &= (\delta_i)^* : X_n \to X_{n-1} \qquad i = 0, \ldots, n, \\ s_j &= (\sigma_j)^* : X_{n-1} \to X_n \qquad i = 0, \ldots, n-1. \end{aligned}$$

These maps are called the *face maps* and *degeneracy maps* of the simplicial object X. To distinguish them from the corresponding elementary face and degeneracy maps in the category $\mathbf{\Delta}$, the latter are in the literature sometimes referred to as cofaces and codegeneracies. The functoriality requirement on the α^* is equivalent to the requirement that the d_i and s_j satisfy the following *simplicial identities* (dual to the cosimplicial identities of the previous section):

(i) $d_i d_j = d_{j-1} d_i$ for $i < j$
(ii) $s_j s_i = s_i s_{j-1}$ for $i < j$
(iii) $d_j s_i = \begin{cases} s_i d_{j-1} & \text{if } i < j-1 \\ \text{id} & \text{if } i = j-1 \text{ or } i = j \\ s_{i-1} d_j & \text{if } i > j. \end{cases}$

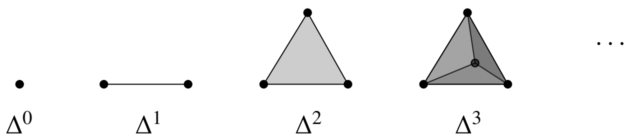
Similarly, a collection of maps $f : X_n \to Y_n$ determines a morphism of simplicial objects if and only if it is compatible with the face and degeneracy maps, as in

$$f_{n-1} d_i = d_i f_n \text{ for } n \geq 0, \quad i = 0, \ldots, n,$$
$$f_n s_j = s_j f_{n-1} \text{ for } n \geq 0, \quad j = 0, \ldots, n-1.$$

For the remainder of this section we will focus on the category **sSets** of simplicial sets. The main motivation for the concept of a simplicial set is to give a combinatorial procedure for building a topological space, as we will recall below, although the uses of simplicial sets and simplicial objects are now much more widespread.

Consider for each $n \geq 0$ the *standard topological n-simplex*

$$\Delta^n := \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid t_0 + \cdots + t_n = 1,\ t_i \geq 0 \quad \forall i\}.$$



This standard simplex has $n+1$ vertices $v_0, \ldots, v_n$, where

$$v_i = (0, \ldots, 0, 1, 0, \ldots, 0)$$

with the 1 in the ith entry. Thus, any function of sets $f : \{0, \ldots, m\} \to \{0, \ldots, n\}$ defines an affine map

$$f_* : \Delta^m \to \Delta^n$$

which is uniquely determined by the requirement $f(v_i) = v_{f(i)}$. In particular, this makes the family of standard simplices into a functor

$$\Delta^\bullet : \mathbf{\Delta} \longrightarrow \mathbf{Top}.$$

We write Δ^α as α_* as for f above. Explicitly, for $\alpha : [m] \to [n]$,

$$\alpha_*(t_0, \ldots, t_m) = (s_0, \ldots, s_n) \quad \text{with} \quad s_i = \sum_{\alpha(j)=i} t_j.$$

In particular, for an elementary face map $\delta_i : [n-1] \to [n]$, the map

$$(\delta_i)_* : \Delta^{n-1} \to \Delta^n$$

embeds Δ^{n-1} as the face opposite the vertex v_i. More generally, for an injective map $\alpha : [m] \to [n]$, the corresponding map α_* embeds the m-simplex Δ^m as a face of Δ^n of possibly high codimension. Also, for the elementary degeneracy $\sigma_j : [n] \to [n-1]$, the map

$$(\sigma_j)_* : \Delta^n \to \Delta^{n-1}$$

collapses Δ^n onto Δ^{n-1} by a projection parallel to the line connecting v_j and v_{j+1}.

We will use these topological n-simplices to define for each simplicial set X its *geometric realization* $|X|$. This is a topological space defined as a quotient of the large disjoint sum of simplices

$$\coprod_{n\geq 0} X_n \times \Delta^n = \coprod_{n\geq 0} \coprod_{x\in X_n} \Delta^n,$$

the points of which we denote by

$$(x, t) \qquad \text{for} \quad x \in X_n, \quad t \in \Delta^n.$$

This quotient is formed by making the identification

$$(x, \alpha_* t) \sim (\alpha^* x, t)$$

for each morphism $\alpha\colon [m] \to [n]$ of $\mathbf{\Delta}$ and each $x \in X_n$, $t \in \Delta^m$. We write $x \otimes t$ for the equivalence class of a pair $(x, t) \in X_n \times \Delta^n$. This notation comes from the idea that X is a 'right module' over $\mathbf{\Delta}$ and $\Delta^\bullet$ is a 'left module', where left and right correspond to co- and contravariant functoriality respectively. There is a sense in which $|X|$ can be interpreted as a 'tensor product' $X \otimes_{\mathbf{\Delta}} \Delta^\bullet$ of such modules, but we will not elaborate on it here.

A map $f : X \to Y$ between simplicial sets induces an obvious continuous map

$$|f| : |X| \to |Y| : x \otimes t \mapsto f(x) \otimes t,$$

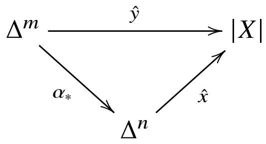
where we have suppressed the subscript n on f in the expression $f(x)$ for $x \in X_n$. This assignment makes geometric realization into a functor

$$|\cdot| : \mathbf{sSets} \longrightarrow \mathbf{Top}.$$

For a simplicial set, every n-simplex $x \in X_n$ defines a map

$$\hat{x} : \Delta^n \to |X| : t \mapsto x \otimes t.$$

The images of all these maps evidently cover all of $|X|$ and we will examine more closely how they overlap in the next section. For now, observe that by the equivalence relation imposed to form the geometric realization, these maps respect the simplicial structure of X, in the sense that for any $\alpha : [m] \to [n]$ and $y \in X_m$ such that $\alpha^* x = y$, the diagram



commutes.

If $C^\bullet : \mathbf{\Delta} \to \mathcal{E}$ is any functor, each object E of $\mathcal{E}$ defines a simplicial set $\mathrm{Sing}_{C^\bullet}(E)$ by the formula

$$\mathrm{Sing}_{C^\bullet}(E)_n = \mathcal{E}(C^n, E),$$

where for $\alpha : [m] \to [n]$ the map α^* is defined by precomposition with C^α, the image of α under the functor $C^\bullet$. This general way of constructing simplicial sets applies in particular to the standard simplices $\Delta^\bullet : \mathbf{\Delta} \to \mathbf{Top}$, so that any topological space T defines a simplicial set $\mathrm{Sing}_{\Delta^\bullet}(T)$. It is usually more briefly denoted $\mathrm{Sing}(T)$ and called the *singular complex* of T.

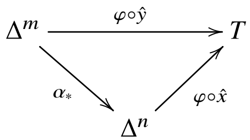
In this way we obtain a functor

$$\mathrm{Sing} : \mathbf{Top} \longrightarrow \mathbf{sSets}$$

which bears a special relation to geometric realization. Indeed, a continuous map $\varphi : |X| \to T$ of topological spaces is given by a family of continuous maps

$$\varphi \circ \hat{x} : \Delta^n \to |X| \to T \qquad x \in X_n$$

for which each diagram of the form



commutes. Thus, φ defines for each n a map of sets

$$\varphi_n : X_n \to \mathrm{Sing}(T)_n : x \mapsto \varphi \circ \hat{x}$$

which group together into a map of simplicial sets by the compatibility described above. In fact, this gives a natural bijective correspondence

$$\mathbf{Top}(|X|, T) \simeq \mathbf{sSets}(X, \mathrm{Sing}(T)),$$

so that the singular complex functor is right adjoint to geometric realization:

$$|\cdot| : \mathbf{sSets} \rightleftarrows \mathbf{Top} : \mathrm{Sing}.$$

2.3 The Geometric Realization as a Cell Complex

In this section we will examine the cellular structure of the geometric realization of a simplicial set X. Recall that we refer to the elements of X_n as the n-simplices of X. An n-simplex $x \in X_n$ is called *degenerate* if it lies in the image of one of the degeneracy operators $s_i : X_{n-1} \to X_n$ for $0 \leq i \leq n-1$. Equivalently, x is degenerate if there exists a surjection $\alpha : [n] \to [m]$ and $y \in X_m$ such that $x = \alpha^* y$. In fact, by choosing a further surjection in case y itself is degenerate, it is clearly possible to arrange that $x = \beta^* z$ for a surjection $\beta : [n] \to [k]$ and z a *non-degenerate* k-simplex of X. Furthermore, given $x \in X_n$, this choice of (β, z) with z non-degenerate is unique. Indeed, if (γ, w) was another such pair with $\gamma^* w = x$ and w non-degenerate, one forms the following pushout:

$$\begin{array}{ccc} [n] & \xrightarrow{\beta} & [k] \\ {\scriptstyle\gamma}\downarrow & & \downarrow \\ [l] & \longrightarrow & [j]. \end{array}$$

It is an absolute pushout by Proposition 2.1 and therefore the resulting square

$$\begin{array}{ccc} X_n & \xleftarrow{\beta^*} & X_k \\ {\scriptstyle\gamma^*}\uparrow & & \uparrow \\ X_l & \longleftarrow & X_j \end{array}$$

is a pullback. Thus there is an element $v \in X_j$ whose image is z (resp. w) in X_k (resp. X_l). By the assumption that w and z are non-degenerate, this can only happen if the maps $[k] \to [j]$ and $[l] \to [j]$ are identities. It follows that $\beta = \gamma$ and $z = w$. The reader should also note that any 0-simplex is non-degenerate.

Every point of $|X|$ can be represented in the form $y \otimes s$ with y a non-degenerate simplex of X. Indeed, for any $x \otimes t \in |X|$, choose y non-degenerate and $\alpha : [n] \to [m]$ so that $x = \alpha^* y$. Then $\alpha_* : \Delta^n \to \Delta^m$ is surjective, so that there is an $s \in \Delta^m$ with $\alpha_* s = t$, and $x \otimes t = y \otimes s$. This section serves to explain the much more precise statement formulated in the theorem below. Recall that every n-simplex $x \in X_n$ determines a map $\hat{x} : \Delta^n \to |X|$.

Theorem 2.3 *Let X be a simplicial set. Its geometric realization $|X|$ naturally has the structure of a CW complex with precisely one closed n-cell $\hat{x} : \Delta^n \to |X|$ for every non-degenerate n-simplex $x \in X_n$.*

We begin by describing the CW structure of the theorem in more detail. Recall that we write $x \otimes t$ for the point of $|X|$ determined by a pair $(x, t) \in X_k \times \Delta^k$. Denote by $|X|^{(n)}$ the subspace of $|X|$ consisting of points which can be represented as $x \otimes t$ for some $(x, t) \in X_k \times \Delta^k$ with $k \leq n$. This describes a filtration of X,

$$|X|^{(0)} \subseteq |X|^{(1)} \subseteq |X|^{(2)} \subseteq \cdots, \qquad \bigcup_n |X|^{(n)} = |X|,$$

and $|X|$ has the weak topology with respect to these subspaces. Indeed, the latter is clear from the definition of $|X|$ as a quotient of $\coprod_n X_n \times \Delta^n$. This filtration will serve as the skeletal filtration for the CW structure of $|X|$.

First, we claim that the space $|X|^{(0)}$ is discrete and is in fact given by $X_0 \times \Delta^0$, so that the elements of X_0 will serve as the 0-cells of $|X|$. Indeed, it is clear that the evident map $X_0 \times \Delta^0 \longrightarrow |X|^{(0)}$ is surjective. To see that is a bijection, we should argue that no two distinct 0-simplices of X are identified in $|X|$. If $x, y \in X_0$, then $x \otimes 1$ and $y \otimes 1$ represent the same point of $|X|$ only if there exists $z \otimes t$ with $z \in X_n$ and $t \in \Delta^n$, together with morphisms $\alpha, \beta : [0] \longrightarrow [n]$ such that $\alpha^* z = x$, $\beta^* z = y$ and $\alpha_* 1 = \beta_* 1 = t$. The last condition immediately implies that $\alpha = \beta$, from which it follows that $x = y$. This establishes our claim.

We should show that for $n \geq 1$ the space $|X|^{(n)}$ can be obtained from $|X|^{(n-1)}$ by attaching an n-cell for each non-degenerate n-simplex if X. More precisely, consider the square

$$\begin{array}{ccc}
\coprod_{x \in \mathrm{nd}(X_n)} \partial\Delta^n & \longrightarrow & |X|^{(n-1)} \\
\big\downarrow & & \big\downarrow \\
\coprod_{x \in \mathrm{nd}(X_n)} \Delta^n & \xrightarrow{\{\hat{x}\}} & |X|^{(n)},
\end{array}$$

where $\mathrm{nd}(X_n)$ denotes the subset of X_n consisting of non-degenerate n-simplices. The conclusion of the theorem is clear if we can show that this square is a pushout. Note that it is a pullback: indeed, a point $x \otimes t$ of $|X|^{(n)}$ with $x \in \mathrm{nd}(X_n)$ and $t \in \Delta^n$ is contained in $|X|^{(n-1)}$ if and only if t is contained in the boundary of Δ^n. (This conclusion would not hold if we replaced the collection of non-degenerate n-simplices by the collection of all n-simplices.)

To see that the square is a pushout, we should argue that if $x \otimes t = y \otimes s$ for points (x, t) and (y, s) of $\coprod_{x \in \mathrm{nd}(X_n)} \Delta^n$, then either $(x, t) = (y, s)$ or both (x, t) and (y, s) are contained in $\coprod_{x \in \mathrm{nd}(X_n)} \partial\Delta^n$. This will follow from:

Proposition 2.4 *Let $\xi \in |X|$. Choose $x \in X_n$ and $t \in \Delta^n$ with $\xi = x \otimes t$ and with n as small as possible. Then x is non-degenerate and if $n \geq 1$ then t is contained in the interior of Δ^n. Also, the pair (x, t) representing ξ with x non-degenerate and t in the interior of Δ^n is unique.*

Indeed, to conclude the theorem from this, suppose $x \otimes t = y \otimes s$, still with $x, y \in X_n$ non-degenerate as above. If both s and t are in the interior of Δ^n, then the proposition implies $(x, t) = (y, s)$. If one of them, say t, is on the boundary of Δ^n, then we can write $x \otimes t = z \otimes r$ for z of smaller dimension k and r in the interior of Δ^k, uniquely. But then s must also be on the boundary of Δ^n; if it were in the interior this would contradict the uniqueness of representatives expressed by the proposition.

Proof (of Proposition 2.4) First we show x is non-degenerate. If it were not then there would exist a nontrivial surjection $\alpha : [n] \longrightarrow [m]$ and $y \in X_m$ with $\alpha^* y = x$. But then $x \otimes t = \alpha^* y \otimes t = y \otimes \alpha_* t$, contradicting the minimality of n. It is also straightforward to see that t is in the interior of Δ^n (assuming $n \geq 1$); indeed, if it were on the boundary $\partial\Delta[n]$ then there would exist a nontrivial injection $\beta : [k] \longrightarrow [n]$ such that t is in the image of $\beta_* : \Delta^k \longrightarrow \Delta^n$, so we may write $t = \beta_* s$. In that case $x \otimes t = x \otimes \beta_* s = \beta^* x \otimes s$, which again contradicts the assumption that n is minimal.

It remains to argue that the representative pair (x, t) of the proposition is unique. So suppose $\xi = x \otimes t = y \otimes s$ where both x and y are non-degenerate and s, t are interior points of Δ^n. By the equivalence relation involved in the definition of $\otimes$, this means that there is a zigzag in $\mathbf{\Delta}$ of the form

$$\begin{array}{ccccccccccccc} [n] & & & & [m_1] & & & & [m_2] & & \cdots & & & & [n] \\ & \nwarrow^{\alpha_1} & & \nearrow^{\beta_1} & & \nwarrow^{\alpha_2} & & \nearrow^{\beta_2} & & & & \nwarrow^{\alpha_N} & & \nearrow^{\beta_N} & \\ & & [k_1] & & & & [k_2] & & & \cdots & & & [k_N] & & \end{array}$$

and elements $(a_i, u_i) \in X_{k_i} \times \Delta^{k_i}$, $(b_i, v_i) \in X_{m_i} \times \Delta^{m_i}$ for which

$$\alpha_i^* b_{i-1} = a_i, \quad (\alpha_i)_* u_i = v_{i-1},$$
$$\beta_i^* b_i = a_i, \quad (\beta_i)_* u_i = v_i.$$

Here we have written $(x, t) = (b_0, v_0)$ and $(y, s) = (b_N, v_N)$. Since $t = v_0$ and $s = v_N$ are interior points, the maps α_1 and β_N must be surjective. We now reason by induction on the length of the zig-zag. If $N = 1$, then the pushout of the surjections $[n] \leftarrow [k_1] \rightarrow [n]$ exists in $\mathbf{\Delta}$ and is absolute, so that X turns it into a pullback

$$\begin{array}{ccc} X_l & \xrightarrow{\gamma^*} & X_n \\ \downarrow & & \downarrow{\scriptstyle \alpha_1^*} \\ X_n & \xrightarrow{\beta_1^*} & X_k \end{array}$$

for the appropriate value of $l \leq n$. But then there is a $z \in X_l$ with $\gamma^* z = x$, meaning that l must equal n (otherwise we reach a contradiction with the minimality of n) and $\alpha_1 = \beta_1 = \text{id}$. Thus $(x, t) = (y, s)$. If $N > 1$, factor β_1 as

$$[k_1] \xrightarrow{\varepsilon} [m_1'] \xrightarrow{\delta} [m_1]$$

with δ a monomorphism and ε surjective. Then one can apply the same argument as above to the pushout of the degeneracies

$$[n] \xleftarrow{\alpha_1} [k_1] \xrightarrow{\varepsilon} [m_1']$$

and the elements $(x, t) \in X_n \times \Delta^n$ and $(\delta^* b_1, \varepsilon_* u_1) \in X_{m'_1} \times \Delta^{m'_1}$ to conclude that ε must be the identity. Hence $\beta_1 = \delta$ is a monomorphism. But then the pullback of β_1 and α_2 exists in $\mathbf{\Delta}$ (cf. Proposition 2.2(i)) and such pullbacks are easily checked to be preserved by the functor $\Delta^\bullet$:

$$\begin{array}{ccc} [k'_1] & \xrightarrow{\theta} & [k_2] \\ \downarrow{\eta} & & \downarrow{\alpha_2} \\ [k_1] & \xrightarrow{\beta_1} & [m_1], \end{array} \qquad \begin{array}{ccc} \Delta^{k'_1} & \xrightarrow{\theta_*} & \Delta^{k_2} \\ \downarrow{\eta_*} & & \downarrow{(\alpha_2)_*} \\ \Delta^{k_1} & \xrightarrow{(\beta_1)_*} & \Delta^{m_1}. \end{array}$$

So we can shorten the zigzag by replacing the first two spans by the single span

$$[n] \xleftarrow{\alpha_1 \eta} [k'_1] \xrightarrow{\beta_2 \theta} [m_2]$$

and using the element $(c, w) \in X_{k'_1} \times \Delta^{k'_1}$, with $c = \eta^* a_1 = \theta^* a_2$ and w the unique point in $\Delta^{k'_1}$ satisfying $\eta_* w = u_1$ and $\theta_* w = u_2$. This completes the inductive step. □

The filtration of the geometric realization $|X|$ by subspaces $|X|^{(n)}$ has a counterpart in the theory of simplicial sets, called the *skeletal filtration* of X. It is a fundamental tool when one proves properties of X 'simplex by simplex'. In fact, our filtration of $|X|$ above simply arises as the geometric realization of the skeletal filtration of the simplicial set X.

We define $\mathrm{sk}_n X$ to be the simplicial subset of X generated by its simplices of dimension at most n. In other words, it is the smallest subobject $\mathrm{sk}_n X \subseteq X$ which contains every simplex $x \in X_k$ for $k \leq n$. Clearly $\cup_n \mathrm{sk}_n X = X$. The crucial property is that $\mathrm{sk}_n X$ can be built from $\mathrm{sk}_{n-1} X$ by 'cell attachments' as follows:

Proposition 2.5 *The evident square*

$$\begin{array}{ccc} \coprod_{x \in \mathrm{end}(X_n)} \partial\Delta[n] & \longrightarrow & \mathrm{sk}_{n-1} X \\ \downarrow & & \downarrow \\ \coprod_{x \in \mathrm{end}(X_n)} \Delta[n] & \longrightarrow & \mathrm{sk}_n X, \end{array}$$

is a pushout. As before, the coproduct is over the set $\mathrm{nd}(X_n)$ *of non-degenerate n-simplices of X.*

Proof For the length of this proof we write P for the pushout in the square above and $p : P \to \mathrm{sk}_n X$ for the evident map. We should demonstrate that P is an isomorphism of simplicial sets.

To see that p is surjective, consider an n-simplex $x \in X_n$. Then we can write $x = \alpha^* y$ for some degeneracy $\alpha : [n] \to [m]$ and a unique non-degenerate simplex $y \in X_m$. If $m < n$, then y (and hence also x) is already contained in $\mathrm{sk}_{n-1} X$ and hence in the image of p. If $m = n$ then α is the identity and x is non-degenerate, so that x occurs in the coproduct in the lower left corner of the square of the proposition. Again, x is therefore in the image of p.

It remains to argue that p is injective. There are two things to check:

(a) If $x \in \mathrm{nd}(X_n)$, then the pullback of the corresponding span

$$\Delta[n] \xrightarrow{x} \mathrm{sk}_n X \leftarrow \mathrm{sk}_{n-1} X$$

is precisely the boundary $\partial\Delta[n]$.

(b) For two distinct non-degenerate simplices $x, y \in \mathrm{nd}(X_n)$, consider the pullback square

$$\begin{array}{ccc} Q & \xrightarrow{v} & \Delta[n] \\ \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle x} \\ \Delta[n] & \xrightarrow{y} & \mathrm{sk}_n X. \end{array}$$

Then v and w factor through the boundary inclusion $\partial\Delta[n] \subseteq \Delta[n]$.

Indeed, (a) and (b) express the idea that all identifications to be made when adding non-degenerate n-simplices to $\mathrm{sk}_{n-1}X$ concern only the boundary of those simplices.

Proof of (a): Say $[k] \xrightarrow{\alpha} [n]$ is a map so that $\alpha^* x$ is contained in $\mathrm{sk}_{n-1}X$. We should show that α is not surjective, so that it factors through $\partial\Delta[n]$. We reason by contradiction; suppose α *is* surjective. By definition of the $(n-1)$-skeleton we can write $\alpha^* x = \beta^* y$ for some non-degenerate m-simplex $y \in X_m$ (with $m < n$) and a surjective map $\beta : [k] \to [l]$. Form the absolute pushout square

$$\begin{array}{ccc} [k] & \xrightarrow{\alpha} & [n] \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \delta} \\ [m] & \xrightarrow{\gamma} & [l]. \end{array}$$

Since X turns it into a pullback, there exists a $z \in X_l$ with $\gamma^* z = y$ and $\delta^* z = x$. Now, δ is surjective and x non-degenerate, so we must have that δ is the identity. But then $k = m$, contradicting the fact that $m < n \leq k$.

Proof of (b): Consider maps $\alpha, \beta : [k] \to [n]$ so that $\alpha^* x = \beta^* y$. We should show that both α and β are *not* surjective. Factor these maps as

$$[k] \xrightarrow{\alpha_-} [m_1] \xrightarrow{\alpha_+} [n], \quad [k] \xrightarrow{\beta_-} [m_2] \xrightarrow{\beta_+} [n],$$

with α_-, β_- surjective and α_+, β_+ injective. Now form the absolute pushout

$$\begin{array}{ccc} [k] & \xrightarrow{\alpha_-} & [m_1] \\ \downarrow{\scriptstyle \beta_-} & & \downarrow{\scriptstyle \delta} \\ [m_2] & \xrightarrow{\gamma} & [l]. \end{array}$$

As before it follows that there is a $z \in X_l$ with $(\gamma\alpha_-)^* z = x$ and $(\delta\beta_-)^* z = y$. The non-degeneracy of x and y then implies that all the surjections in the above square are in fact identities. We conclude that $\alpha = \alpha_+$ and $\beta = \beta_+$ are both injective. It remains to argue that neither can be the identity. But if one of them was, then $[k] = [n]$ and clearly both of them are identities. It follows that $x = y$, contradicting our assumption. □

2.4 Simplicial Sets as a Category of Presheaves

For a small category **C**, a functor

$$X : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$$

is called a *presheaf* (of sets) on **C**. Together with the natural transformations between them, these presheaves form a category which we denote by

$$\mathrm{PSh}(\mathbf{C}).$$

(Other common notations are $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ and $\widehat{\mathbf{C}}$.) Thus, the category **sSets** of simplicial sets is the category $\mathrm{PSh}(\mathbf{\Delta})$ of presheaves on $\mathbf{\Delta}$ and as such enjoys the general properties of such categories of presheaves. In this section we review several of those properties which will be relevant to us.

First some notation: for a presheaf X as above and a morphism $\alpha : c \to d$ in **C**, its value under X is denoted

$$\alpha^* : X(d) \to X(c).$$

If $f : X \to Y$ is a morphism between presheaves, consisting of a natural family of morphisms $f_c : X(c) \to Y(c)$ for c ranging through the objects of **C**, we often abbreviate f_c by f again if no confusion can arise.

Representable presheaves. Each object $c \in \mathbf{C}$ determines a so-called representable presheaf $\mathbf{y}(c)$, defined on objects by

$$\mathbf{y}(c)(d) = \mathbf{C}(d, c)$$

and with the evident action of morphisms in **C** by precomposition. It can also be denoted $\mathbf{C}(-, c)$. This construction is also functorial in c and determines a functor

$$\mathbf{y} : \mathbf{C} \to \mathrm{PSh}(\mathbf{C})$$

called the *Yoneda embedding*. The basic *Yoneda lemma* states that for any presheaf X there is a natural bijective correspondence between morphisms of presheaves $f : \mathbf{y}(c) \to X$ and elements $x \in X(c)$. This correspondence is given by $x = f(\mathrm{id}_c)$ and $f(\alpha) = \alpha^* x$. We write $\overline{x}$ for this morphism f corresponding to x.

Standard simplices. For the special case of simplicial sets, the representable presheaf $y([n])$ is denoted by $\Delta[n]$ and referred to as the (simplicial) standard n-simplex. It mirrors the topological n-simplex Δ^n in the sense that

$$|\Delta[n]| = \Delta^n,$$

as one easily checks. The Yoneda lemma gives a correspondence between n-simplices $x \in X_n$ and maps $\overline{x} : \Delta[n] \longrightarrow X$ and the geometric realization of the latter is precisely the map we denoted by $\hat{x} : \Delta^n \longrightarrow |X|$ in previous sections.

Limits and colimits. Each presheaf category PSh(**C**) has all small limits and colimits and these are all computed 'pointwise'. To be precise, if

$$X : I \longrightarrow \mathrm{PSh}(\mathbf{C}) : i \mapsto X_i$$

is a diagram of presheaves indexed by a small category I, then

$$(\varinjlim_I X_i)(c) \simeq \varinjlim_I X_i(c),$$

the colimit on the left being computed in PSh(**C**), the one on the right in **Sets**. The same applies to limits. To give a simple example, the product of simplicial sets X and Y is constructed as

$$(X \times Y)_n = X_n \times Y_n,$$

with simplicial operators (e.g. faces and degeneracies) defined componentwise, as $d_i(x, y) = (d_i x, d_i y)$, etc.

A similar observation applies to epimorphisms, monomorphisms and images: a map $f : X \longrightarrow Y$ between presheaves is epi (resp. mono) if and only if each of its components $f : X(c) \longrightarrow Y(c)$ is. For a general $f : X \longrightarrow Y$, its image $f(X) \subseteq Y$ is constructed as $f(X)(c) = f(X(c))$ for each object c of **C**. A monomorphism $A \longrightarrow Y$ for which each component $A(c) \longrightarrow Y(c)$ is the inclusion of a subset is referred to as a *subpresheaf* of Y. Sometimes we will also use this terminology to refer to an isomorphism class of monos $A \longrightarrow Y$, secretly identifying them with their common image.

Colimits of representables. Every presheaf X on a category **C** is canonically isomorphic to a colimit of representable presheaves. To see this, one first constructs the *category of elements* of X, variously denoted $\mathrm{El}(X)$, $\int_{\mathbf{C}} X$ or $\mathbf{C}/X$ in the literature. We will use the latter notation. The objects of $\mathbf{C}/X$ are pairs (c, x) with $c \in \mathbf{C}$ and $x \in X(c)$. A morphism $(c, x) \longrightarrow (d, y)$ is a morphism $\alpha : c \longrightarrow d$ with the property that $\alpha^* y = x$. There is an evident projection

$$\pi_X : \mathbf{C}/X \longrightarrow \mathbf{C} : (c, x) \mapsto c$$

and an isomorphism

$$\theta_X : \varinjlim_{\mathbf{C}/X} \mathbf{y} \circ \pi_X \longrightarrow X.$$

This natural transformation θ_X is induced by the morphisms

$$\overline{x} : \mathbf{y}(c) \to X$$

for (c, x) ranging over the objects of $\mathbf{C}/X$.

Kan extension. The category $\mathrm{PSh}(\mathbf{C})$ is the *free* category with all small colimits generated by $\mathbf{C}$. What this means is that for any category $\mathcal{E}$ with all small colimits, any functor $F : \mathbf{C} \to \mathcal{E}$ extends (uniquely up to natural isomorphism) to a functor $F_! : \mathrm{PSh}(\mathbf{C}) \to \mathcal{E}$, such that $F_!$ preserves all small colimits. To make sense of the word 'extends' here, one should regard $\mathbf{C}$ as a subcategory of $\mathrm{PSh}(\mathbf{C})$ via the Yoneda embedding. In other words, there is a natural isomorphism $F_! \circ \mathbf{y} \simeq F$. Another common notation for $F_!$ is $\mathrm{Lan}_{\mathbf{y}}F$, indicating that it is the *left Kan extension* of F along $\mathbf{y}$. The functor $F_!$ can be constructed explicitly by writing every presheaf as a colimit of representables:

$$F_!(X) = \varinjlim_{\mathbf{C}/X} F \circ \pi_X.$$

More informally, one might also write

$$F_!(X) = \varinjlim_{c \in \mathbf{C}, x \in F(c)} F(c).$$

To check that $F_!$ is indeed a functor, one observes that the construction of the category of elements is itself functorial in X. To see that $F_!$ extends F, one observes that the category of elements $\mathbf{C}/\mathbf{y}(c)$ is isomorphic to the slice category $\mathbf{C}/c$. The latter has a terminal object, namely id_c, so that the colimit over this category may be computed by evaluation at this object.

The functor $F_!$ just constructed admits a right adjoint F^*. Indeed, for $E \in \mathcal{E}$ we simply define the presheaf F^*E by

$$F^*E(c) = \mathcal{E}(F(c), E).$$

Functoriality of F^* is clear; to see it is indeed right adjoint, consider a presheaf $X \in \mathrm{PSh}(\mathbf{C})$ and observe the sequence of natural isomorphisms

$$\begin{aligned}
\mathcal{E}(F_!X, E) &\simeq \varprojlim_{\mathbf{C}/X} \mathcal{E}(F \circ \pi_X, E) \\
&\simeq \varprojlim_{\mathbf{C}/X} F^*E \circ \pi_X \\
&\simeq \varprojlim_{\mathbf{C}/X} \mathrm{PSh}(\mathbf{C})(\mathbf{y} \circ \pi_X, F^*E) \\
&\simeq \mathrm{PSh}(\mathbf{C})(X, F^*E).
\end{aligned}$$

Here we have applied the Yoneda lemma to go from the second to the third line.

Geometric realization. The category **Top** of topological spaces has all small colimits. Therefore the left Kan extension explained above applies to the functor of standard topological simplices

$$\Delta^\bullet : \boldsymbol{\Delta} \to \mathbf{Top}.$$

The resulting functor from **sSets** to **Top** is precisely the geometric realization discussed in previous sections. Indeed, geometric realization preserves colimits and the composition $|\cdot| \circ \mathbf{y}$ is (isomorphic to) the functor $\Delta^\bullet$. Therefore geometric realization is the left Kan extension of $\Delta^\bullet$ to the category of simplicial sets. In this specific example, the right adjoint discussed in the previous paragraph yields the singular complex functor

$$\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSets}.$$

The nerve of a category. An important construction which is analogous to the adjoint pair $|\cdot|$ and Sing is the following. Consider the category **Cat** of small categories. It contains the categories of partially ordered and linearly ordered sets as full subcategories and in particular there is a fully faithful functor

$$\iota \colon \boldsymbol{\Delta} \to \mathbf{Cat}$$

sending an object $[n]$ to the corresponding linear order $(0 \to 1 \to \cdots \to n)$. The left Kan extension of i defines a functor which is usually denoted τ in the literature,

$$\tau = \iota_! \colon \mathbf{sSets} \to \mathbf{Cat}.$$

Following the general pattern explained above, this functor τ has a right adjoint called the *nerve functor* and usually written

$$N \colon \mathbf{Cat} \to \mathbf{sSets}.$$

Spelling out the general formula for the right adjoint in this specific case, we see that for a small category $\mathbf{C}$, its nerve can be described as follows: the set of 0-simplices $(N\mathbf{C})_0$ is the set of objects of $\mathbf{C}$ and the set of n-simplices $(N\mathbf{C})_n$ is the set of strings of n composable morphisms

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} c_n.$$

The simplicial operators $d_i : (N\mathbf{C})_n \to (N\mathbf{C})_{n-1}$ and $s_j : (N\mathbf{C})_{n-1} \to (N\mathbf{C})_n$ can somewhat cryptically be described by 'd_i deletes c_i' and 's_j inserts the identity $c_j = c_j$'. To be more precise, for $0 < i < n$ we have:

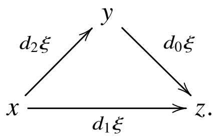
$$\begin{aligned}
d_0(c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} c_n) &= c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} c_n,\\
d_n(c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} c_n) &= c_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} c_{n-1},\\
d_i(c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} c_n) &= c_0 \xrightarrow{f_1} \cdots c_{i-1} \xrightarrow{f_{i+1} \circ f_i} c_{i+1} \cdots \xrightarrow{f_n} c_n,\\
s_j(c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} c_{n-1}) &= c_0 \xrightarrow{f_1} \cdots \to c_j \xrightarrow{\mathrm{id}_{c_j}} c_j \cdots \xrightarrow{f_n} c_{n-1}.
\end{aligned}$$

One easily checks that the nerve functor $N\colon \mathbf{Cat} \longrightarrow \mathbf{sSets}$ is fully faithful. In other words, a functor $\mathbf{C} \longrightarrow \mathbf{D}$ is essentially the same thing as a morphism of simplicial sets $N\mathbf{C} \longrightarrow N\mathbf{D}$. An equivalent statement is that the counit $\tau N\mathbf{C} \longrightarrow \mathbf{C}$ is an isomorphism. This can also easily be checked using the following explicit description of τ.

For a simplicial set X, the category τX has X_0 as its set of objects. Any 1-simplex $f \in X_1$ defines a morphism $x \longrightarrow y$ in τX, with $x = d_1 f$ and $y = d_0 f$. These morphisms generate all morphisms in τX, in the sense that any arrow $x \longrightarrow y$ in τX can be represented by a finite string of 'composable' 1-simplices $(f_1, \ldots, f_k)$, i.e. these 1-simplices satisfy $d_1 f_1 = x$, $d_0 f_k = y$ and $d_0 f_i = d_1 f_{i+1}$. The relations satisfied by these generators are of two kinds: each degenerate 1-simplex is identified with an identity morphism in τX and each 2-simplex $\xi \in X_2$ describes a composition relation, namely

$$d_1\xi = d_0\xi \circ d_2\xi.$$

More graphically, the 2-simplex ξ imposes that the following be a commutative diagram in τX:



One may wonder why this explicit description indeed describes τ. This can be proved by showing directly that the functor we just described is left adjoint to the nerve functor.

For later use, we note that the functor $\tau\colon \mathbf{Cat} \longrightarrow \mathbf{sSets}$ preserves products. Indeed, for representable simplicial sets $\Delta[n]$ and $\Delta[m]$, this follows from the chain of natural isomorphisms

$$\tau(\Delta[n] \times \Delta[m]) \cong \tau N(\iota[n] \times \iota[m]) \cong \iota[n] \times \iota[m].$$

General simplicial sets are colimits of representables and the assertion follows since the functors involved preserve colimits in each variable separately.

The classifying space. Composing the nerve functor with geometric realization, one recovers the well-known and important construction of a space out of a category $\mathbf{C}$, namely its *classifying space*, denoted

$$B\mathbf{C} := |N\mathbf{C}|.$$

This classifying space functor is particularly useful in relating the (co)homology of categories to the (co)homology of spaces; in the case where $\mathbf{C}$ is a group G (i.e., $\mathbf{C}$ has a single object and all its morphisms are isomorphisms), then the (co)homology of G as defined by homological algebra coincides with the singular (co)homology of its classifying space BG. This is related to the fact that the geometric realization and singular complex functors are homotopy inverse to each other in an appropriate sense, a fact we will discuss extensively in the second part of this book.

Internal hom or exponential. Any presheaf category PSh(**C**) is *cartesian closed*, meaning that for any object $X \in \mathrm{PSh}(\mathbf{C})$, the product functor

$$- \times X : \mathrm{PSh}(\mathbf{C}) \to \mathrm{PSh}(\mathbf{C})$$

has a right adjoint. This right adjoint is referred to as the *internal hom* or *exponential* and accordingly denoted $\mathbf{hom}(X, -)$ or $(-)^X$ respectively. The construction of this adjoint can actually be viewed as another example of an adjoint pair $F_!$ and F^* obtained by Kan extension as discussed above. Indeed, the functor $- \times X$ preserves colimits, because the same is true in the category of sets and colimits of presheaves are computed objectwise. So $- \times X$ is the left Kan extension of its restriction to representables,

$$F : \mathbf{C} \to \mathrm{PSh}(\mathbf{C}) : c \mapsto y(c) \times X.$$

Therefore the right adjoint F^* exists and gives the exponential alluded to above. The adjointness of these functors is the usual exponential relation

$$\mathrm{PSh}(\mathbf{C})(Z \times X, Y) \simeq \mathrm{PSh}(\mathbf{C})(Z, Y^X).$$

For the special case of simplicial sets, we thus have the formula

$$(Y^X)_n = \mathbf{sSets}(\Delta[n] \times X, Y).$$

Some of the functors we have discussed in this section behave well with respect to exponentials. For left adjoints this is rarely the case, but right adjoints are generally better. Explicitly, consider an adjoint pair $\varphi_! : \mathbf{D} \leftrightarrows \mathbf{E} : \varphi^*$ between categories with finite products and exponentials. Then the exponential law gives, for objects $X, Y \in \mathbf{E}$, a canonical map

$$\gamma : \varphi^*(Y^X) \to (\varphi^* Y)^{\varphi^* X}.$$

Indeed, we have natural maps

$$\varphi_!(Z \times \varphi^* X) \to \varphi_!(Z) \times \varphi_!(\varphi^* X) \to \varphi_!(Z) \times X.$$

The first one derives from the universal property of the product, the second uses the counit of the adjoint pair $(\varphi_!, \varphi^*)$. Write p for the composition of these two maps. Then we can form the sequence of natural maps

$$\begin{aligned} \mathbf{D}(Z, \varphi^*(Y^X)) &\simeq \mathbf{E}(\varphi_! Z \times X, Y) \\ &\xrightarrow{p^*} \mathbf{E}(\varphi_!(Z \times \varphi^* X), Y) \\ &\simeq \mathbf{D}(Z \times \varphi^* X, \varphi^* Y) \\ &\simeq \mathbf{D}(Z, (\varphi^* Y)^{\varphi^* X}). \end{aligned}$$

Applying this to the case $Z = \varphi^*(Y^X)$ and its identity map this gives the promised comparison map γ. At the same time, we conclude that γ is an isomorphism for all $X, Y \in \mathbf{E}$ if and only if p is an isomorphism for all $X \in \mathbf{E}$ and $Z \in \mathbf{D}$. For example, this applies to the adjoint pair $\tau : \mathbf{sSets} \leftrightarrows \mathbf{Cat} : N$. Indeed, it is straightforward to verify that τ commutes with products. We already noted that the counit $\tau N\mathbf{C} \to \mathbf{C}$ is an isomorphism, so that the map $p : \tau(X \times N\mathbf{C}) \to \tau X \times \tau N\mathbf{C} \to \tau X \times \mathbf{C}$ is an isomorphism for any simplicial set X and small category $\mathbf{C}$. Thus, the nerve functor preserves exponentials.

The case of the adjoint pair $|\cdot|$ and Sing is different; a similar argument to the above would apply if **Top** had exponentials, if geometric realizations would preserve products and if Sing was fully faithful. However, all three of these statements are in general false. The first two can be corrected by replacing the category of spaces by 'a convenient category of spaces', such as the category of compactly generated weak Hausdorff spaces. Still, not much can be done about the third: for a topological space X, the map $|\mathrm{Sing}(X)| \to X$ is generally not a homeomorphism. It is, however, a weak homotopy equivalence, as we shall discuss in Section 8.6. We will come back to the relation between geometric realization and products in the next section.

Dependence on $\mathbf{C}$. We include a general remark on how the presheaf category $\mathrm{PSh}(\mathbf{C})$ depends on $\mathbf{C}$. Consider a functor $\varphi : \mathbf{C} \to \mathbf{D}$ between small categories. It induces an obvious restriction functor

$$\varphi^* : \mathrm{PSh}(\mathbf{D}) \to \mathrm{PSh}(\mathbf{C}), \quad \varphi^* Y(c) = Y(\varphi c).$$

Since colimits in presheaf categories are computed pointwise, the functor φ^* preserves colimits. Therefore it is the left Kan extension of its restriction to representables and by the same logic as before, it must admit a right adjoint for which we write

$$\varphi_* : \mathrm{PSh}(\mathbf{C}) \to \mathrm{PSh}(\mathbf{D}), \quad \varphi_* X(d) = \mathrm{PSh}(\mathbf{C})(\varphi^*(y(d)), X).$$

But $\varphi : \mathbf{C} \to \mathbf{D}$ also induces an obvious functor

$$y \circ \varphi : \mathbf{C} \to \mathbf{D} \to \mathrm{PSh}(\mathbf{D})$$

resulting in another pair of adjoint functors, which we should for now denote by $(y \circ \varphi)_!$ and $(y \circ \varphi)^*$, in accordance with our earlier discussion of Kan extensions. But

$$(y \circ \varphi)^* : \mathrm{PSh}(\mathbf{D}) \to \mathrm{PSh}(\mathbf{C})$$

is given by the formula

$$(y \circ \varphi)^* Y(c) = \mathrm{PSh}(\mathbf{D})((y \circ \varphi)(c), Y) \simeq Y(\varphi(c)),$$

the latter by the Yoneda lemma. In other words, $(y \circ \varphi)^* Y$ is (isomorphic to) the presheaf $\varphi^* Y$ consider before, so we need not distinguish between φ^* and $(y \circ \varphi)^*$. Similarly, we will abbreviate the notation $(y \circ \varphi)_!$ to $\varphi_!$. Up to natural isomorphism it is the unique functor

$$\varphi_! : \mathrm{PSh}(\mathbf{C}) \to \mathrm{PSh}(\mathbf{D})$$

which is left adjoint to the restriction functor φ^*. Also, it is up to natural isomorphism the unique functor preserving small colimits and agreeing with φ on representables, in the precise sense that there is an isomorphism $\varphi_! y(c) \simeq y(\varphi c)$, natural in c.

As an example, consider the inclusion

$$\iota : \mathbf{\Delta}_{\leq n} \to \mathbf{\Delta}$$

of the full subcategory $\mathbf{\Delta}_{\leq n}$ of $\mathbf{\Delta}$ on the objects $[k]$ for $k \leq n$. It gives rise to three functors

$$\mathrm{PSh}(\mathbf{\Delta}_{\leq n}) \;\substack{\xrightarrow{\;\iota_!\;} \\ \xleftarrow{\;\iota^*\;} \\ \xrightarrow[\;\iota_*\;]{}}\; \mathbf{sSets},$$

with $\iota_!$ and ι_* the left and right adjoint of ι^* respectively. For a simplicial set X, the counit of the first and unit of the second adjunction give rise to maps

$$\iota_!\iota^*X \to X \to \iota^*\iota_*X.$$

The simplicial set $\iota_!\iota^*X$ is precisely the n-skeleton $\mathrm{sk}_n X$ discussed at the end of Section 2.3. Dually, the simplicial set $\iota^*\iota_*X$ is called the *n-coskeleton* of X and usually denoted $\mathrm{cosk}_n X$.

Constant presheaves. There is an evident notion of *constant presheaf* on a category $\mathbf{C}$. Indeed, the constant presheaf F with value a set S is the functor which satisfies $F(c) = S$ for every object c of $\mathbf{C}$ and which sends every morphism of $\mathbf{C}$ to the identity map of S. With the notation of the previous paragraph, one can consider the functor $\varphi : \mathbf{C} \to 1$, where 1 denotes the trivial category with one object and only the identity morphism. Then under the obvious isomorphism $\mathrm{PSh}(1) \cong \mathbf{Sets}$, the constant presheaf with value S is precisely φ^*S. The left adjoint $\varphi_!$ (resp. the right adjoint φ_*) is now the functor which takes the colimit (resp. the limit) of a presheaf F over the category $\mathbf{C}^{\mathrm{op}}$.

In the context of simplicial sets we introduce some terminology and notation for this situation. We will say that a simplicial set X is *discrete* if it is constant as a presheaf on $\mathbf{\Delta}^{\mathrm{op}}$. The reason for this terminology is the relation to topology; for a space Y with the discrete topology, the singular complex $\mathrm{Sing}(Y)$ is a discrete simplicial set. The functor which assigns to a set S the corresponding discrete simplicial set admits a left adjoint (called $\varphi_!$ in the previous paragraph) for which we will write

$$\pi_0 : \mathbf{sSets} \to \mathbf{Sets}.$$

As the notation suggests, we will refer to $\pi_0 X$ as the set of *connected components* of X. Again the reason is the analogy with topology. To be precise, $\pi_0 X$ is exactly the set of connected components of the geometric realization $|X|$. Indeed, the inclusion

$$\mathrm{dis} : \mathbf{Sets} \to \mathbf{CW}$$

which equips a set with the discrete topology (thought of as a CW-complex) admits a left adjoint (also denoted π_0), sending a CW-complex to its set of connected components. The composition of right adjoints Sing $\circ$ dis sends a set to the corresponding discrete simplicial set. Hence the composition of left adjoints $\pi_0 \circ |\cdot|$ agrees with the functor π_0 we defined above.

The interested reader may wish to verify that to compute $\pi_0 X$ in practice, one can simply take the coequalizer

$$X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0 \longrightarrow \pi_0 X,$$

rather than the colimit of the entire diagram $X_\bullet$ on $\mathbf{\Delta}^{\mathrm{op}}$.

2.5 Products of Simplicial Sets and Shuffle Maps

The goal of this section is twofold: first, we discuss the product $X \times Y$ of two simplicial sets X and Y, in particular in the case where X and Y are representable. The reason for the latter is that for simplicial sets, as for any cartesian closed category, the product as a functor

$$X \times - \quad \text{or} \quad - \times Y : \mathbf{sSets} \to \mathbf{sSets}$$

admits a right adjoint and hence preserves colimits. In other words, the product preserves colimits in each variable separately. Since every simplicial set is canonically a colimit of representables, many properties of the product can be deduced from those of the product of two standard simplices. We will discuss these products and their description in terms of *shuffle maps* in some detail, since an analysis of shuffle maps and generalizations thereof will return at various places in this book. In the second part of this section we examine the behaviour of the geometric realization functor with respect to products and, more generally, finite limits.

Consider a p-simplex σ of the simplicial set $\Delta[n] \times \Delta[m]$, the binary product of two standard simplices. It corresponds to a pair of maps

$$\sigma_1 : \Delta[p] \to \Delta[n], \quad \sigma_2 : \Delta[p] \to \Delta[m]$$

or more simply a morphism (still denoted by the same symbol)

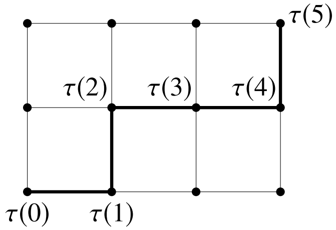
$$\sigma : [p] \to [n] \times [m]$$

in the category of partially ordered sets. We will call the simplex σ *non-degenerate* if this morphism is injective. Moreover, such a simplex σ is a face of another non-degenerate simplex τ precisely if the map of partially ordered sets σ can be extended to an injective map τ as follows:

$$\begin{array}{ccc} [p] & \xrightarrow{\quad\sigma\quad} & [n]\times[m]. \\ & \searrow \quad \nearrow_{\tau} & \\ & [q] & \end{array}$$

Let us say such an injective map $\tau : [q] \to [n] \times [m]$ is *maximal* if it cannot be factored further in this way. Such maximal simplices τ correspond precisely to the injective maps $[n+m] \to [n] \times [m]$. Any non-degenerate simplex σ of the product $\Delta[n] \times \Delta[m]$ is clearly a face of some maximal non-degenerate simplex τ, although this τ need not be unique.

An injective map $\tau : [n+m] \to [n] \times [m]$ of partially ordered sets can be pictured as a staircase. The following is an example with $(n, m) = (3, 2)$:

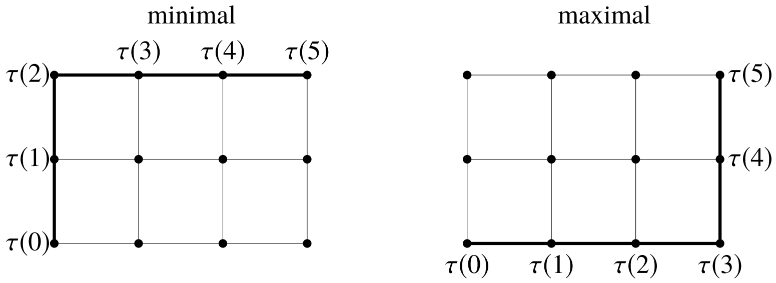


Indeed, the values of τ trace out a path through the rectangle starting at $\tau(0) = (0, 0)$ and ending at $\tau(n + m) = (n, m)$. We will refer to such a maximal injective map τ as a *shuffle* of $[n]$ and $[m]$. The reason for this terminology is that such a shuffle is uniquely described by specifying the 'steps' in this staircase. Indeed, observe that the staircase consists of $n + m$ edges, of which n are horizontal and m are vertical. Those vertical edges are specified by a strictly increasing map $v_\tau : \{1, \ldots, m\} \to \{1, \ldots, n + m\}$. Equivalently, one can specify the horizontal edges by a strictly increasing map $h_\tau : \{1, \ldots, n\} \to \{1, \ldots, n + m\}$, whose image is the complement of the previous map. In this sense, the staircase above corresponds to a 'shuffle' of the linearly ordered sets $\{1, \ldots, n\}$ and $\{1, \ldots, m\}$. Observe that there are $\binom{n+m}{n}$ such shuffles.

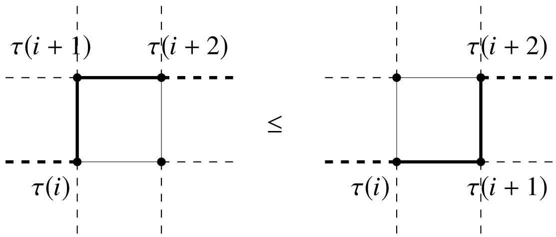
For later use, we note that there is a natural partial ordering on these shuffles. Indeed, for shuffles τ_1 and τ_2 with associated maps

$$v_{\tau_1}, v_{\tau_2} : \{1, \ldots, m\} \to \{1, \ldots, n + m\}$$

as above, one sets $\tau_1 \leq \tau_2$ if $v_{\tau_1}(i) \leq v_{\tau_2}(i)$ for each $1 \leq i \leq m$. This partial order has a minimal and a maximal element. These are pictured below:



Also, the following illustrates a typical relation between two shuffles:



The conclusion of our discussion is that one can write

$$\Delta[n] \times \Delta[m] = \bigcup_{\tau} \Delta[n+m],$$

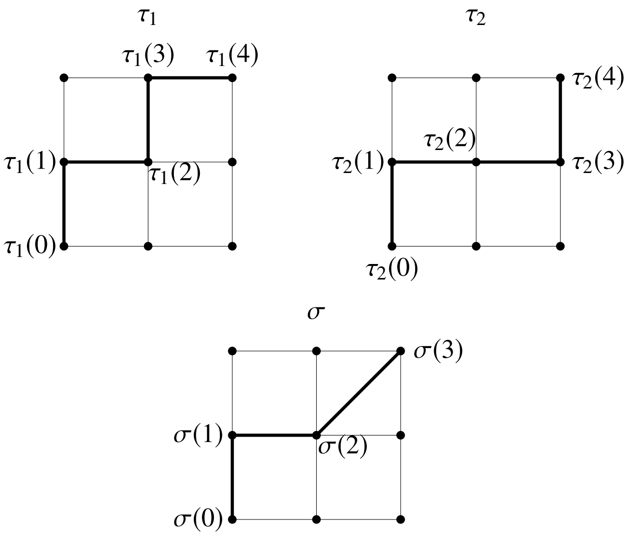
where the union is over all monomorphisms

$$\Delta[n+m] \to \Delta[n] \times \Delta[m]$$

corresponding to shuffles τ. These simplices overlap in a way which is easily expressed in terms of shuffles: for two shuffles τ_1 and τ_2, the pullback

$$\begin{array}{ccc} \Delta[k] & \longrightarrow & \Delta[n+m] \\ \downarrow & & \downarrow{\scriptstyle \tau_2} \\ \Delta[n+m] & \xrightarrow[\tau_1]{} & \Delta[n] \times \Delta[m] \end{array}$$

corresponds to the map of partially ordered sets $\sigma : [k] \to [n] \times [m]$ enumerating the common values of τ_1 and τ_2. It always satisfies $\sigma(0) = (0, 0)$ and $\sigma(k) = (n, m)$. A typical example is as follows:



The remainder of this section will concern the behaviour of the geometric realization functor with respect to products (and more generally finite limits) of simplicial sets. First, a remark on the kind of topological spaces we consider is in order. As we have alluded to before, the category of topological spaces is *not* cartesian closed; in particular, products of topological spaces do not in general behave well with respect to colimits, in contrast to the case of simplicial sets. To remedy this, one works in a 'convenient category of spaces'. For us this will be the category of compactly generated weak Hausdorff spaces, which includes all CW complexes and *is* cartesian closed. The product of two such spaces X and Y agrees with the usual product of topological spaces in the case that both are compact. For general compactly generated weak Hausdorff spaces X and Y, one retopologizes the product $X \times Y$ with the *compactly generated topology*, for which a subset A is open precisely if its intersection with every compact subset $K \subset X \times Y$ is open in K. From now **Top** will always refer to this category of compactly generated weak Hausdorff spaces. Geometric realization obviously takes values in these compactly generated weak Hausdorff spaces. Hence with this new interpretation of **Top**, geometric realization is still left adjoint to the functor Sing. We will prove the following result:

Proposition 2.6 *The geometric realization functor*

$$|\cdot| : \mathbf{sSets} \to \mathbf{Top}$$

preserves finite limits.

To prove that a functor preserves finite limits it suffices to show it preserves finite products and equalizers. The fact that geometric realization preserves equalizers is rather easy to show (see Lemma 2.7) and clearly it preserves the empty product, i.e. the terminal object, because $|\Delta[0]| \simeq \Delta^0$. Lemma 2.8 will show that it preserves binary products of simplices. From this it follows that $|X \times Y| \simeq |X| \times |Y|$ for general simplicial sets X and Y; indeed, one expresses X and Y as colimits of simplices, uses that the left adjoint functor geometric realization preserves colimits and finally the fact that the product in (our new interpretation of) **Top** preserves colimits in each variable separately. This last step is why using a convenient category of spaces is necessary.

Lemma 2.7 *The geometric realization functor preserves equalizers.*

Proof If X is a simplicial set and $E \subseteq X$ a simplicial subset, then $|E|$ is a subcomplex of $|X|$, considered as a CW complex as in Theorem 2.3. In particular, the topology of $|E|$ is the subspace topology inherited from $|X|$. Thus it suffices to show that if

$$E \longrightarrow X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

is an equalizer of simplicial sets, then the resulting diagram

$$|E| \longrightarrow |X| \underset{|g|}{\overset{|f|}{\rightrightarrows}} |Y|$$

is an equalizer of sets. It suffices to show that if $x \otimes t$ is a point of $|X|$ such that $f(x) \otimes t = g(x) \otimes t$ in $|Y|$, then $f(x) = g(x)$. We may assume that $x \otimes t$ is in the form described in Proposition 2.4, so that x is a non-degenerate n-simplex and t is in the interior of Δ^n when $n \geq 1$. As explained at the beginning of Section 2.3, there is a unique non-degenerate k-simplex y of Y and a surjection $\alpha : [n] \longrightarrow [k]$ such that $f(x) = \alpha^* y$. Similarly, there is a unique non-degenerate l-simplex z and a surjection $\beta : [n] \longrightarrow [l]$ with $g(x) = \beta^* z$. Then we have

$$f(x) \otimes t = y \otimes \alpha_* t = z \otimes \beta_* t = g(x) \otimes t$$

and by the uniqueness of representatives in Proposition 2.4 this implies $y = z$ (in particular $k = l$) and $\alpha_* t = \beta_* t$. But one easily checks that if the two surjective maps $\alpha_*, \beta_* : \Delta^n \longrightarrow \Delta^k$ agree on an interior point t, then $\alpha = \beta$. It follows that $f(x) = \alpha^* y = \beta^* z = g(x)$, as was to be shown. □

Lemma 2.8 *The natural map*

$$|\Delta[n] \times \Delta[m]| \longrightarrow |\Delta[n]| \times |\Delta[m]|$$

is a homeomorphism.

Proof We write the points of $\Delta^n = |\Delta[n]|$ as convex linear combinations of its vertices:

$$t_0 v_0 + \cdots + t_n v_n, \quad \sum_i t_i = 1, \quad t_i \geq 0.$$

Similarly we denote the points of Δ^m by

$$s_0 w_0 + \cdots + s_m w_m.$$

Then $\Delta^n \times \Delta^m$ is the space of all such pairs

$$(t_0 v_0 + \cdots + t_n v_n, s_0 w_0 + \cdots + s_m w_m).$$

On the other hand, each shuffle map $\tau : [n+m] \to [n] \times [m]$ defines an embedding

$$|\tau| : \Delta^{n+m} \to \Delta^n \times \Delta^m$$

sending $(r_0, \ldots, r_{n+m})$ to the convex combination

$$\sum_i r_i (v_{\tau_1(i)}, v_{\tau_2(i)})$$

where τ_1 and τ_2 are the components of τ. Now $|\Delta[n] \times \Delta[m]|$ is the colimit of these embeddings, glued together along their intersections as discussed above. It suffices to check that the resulting map

$$T : \bigcup_\tau \Delta^{n+m} \to \Delta^n \times \Delta^m$$

is a bijection, because the spaces involved are compact Hausdorff.

To see that T is injective, let τ and σ be two shuffle maps and suppose that

$$|\tau|(r_0, \ldots, r_{n+m}) = |\sigma|(r'_0, \ldots, r'_{n+m}),$$

so that

$$r_i(v_{\tau_1(i)}, w_{\tau_2(i)}) = r'_i(v_{\sigma_1(i)}, w_{\sigma_2(i)})$$

for each $i = 0, \ldots, n+m$. Since the vertices $(v_i, w_j) \in \Delta^n \times \Delta^m$ are linearly independent this can only happen if $r_i = r'_i$ for all i and $\tau(i) = \sigma(i)$ whenever $r_i \neq 0$. Let $\rho : [k] \to [n+m]$ enumerate those i for which $r_i \neq 0$. Then $\tau \circ \rho = \sigma \circ \rho$ and

$$|\tau|(r_0, \ldots, r_{n+m}) = |\tau \circ \rho|(r_{\rho(0)}, \ldots, r_{\rho(k)}) = |\sigma|(r'_0, \ldots, r'_{n+m}).$$

Therefore the points $\tau \otimes (r_0, \ldots, r_{n+m})$ and $\sigma \otimes (r'_0, \ldots, r'_{n+m})$ are already identified in $|\Delta[n] \times \Delta[m]|$, proving injectivity.

To prove surjectivity of T, we should check that every point of $\Delta^n \times \Delta^m$ lies in an $n+m$-simplex spanned by the vertices $(v_{\tau_1(i)}, w_{\tau_2(i)})$ enumerated by a shuffle map $\tau : [n+m] \to [n] \times [m]$. We work by induction on $n+m$, noting that the cases $n+m \leq 1$ are trivial. Consider a point

$$(x, y) = (t_0 v_0 + \cdots + t_n v_n, s_0 w_0 + \cdots + s_m w_m)$$

of $\Delta^n \times \Delta^m$. If $t_n \leq s_m$, we can write it as

$$t_n(v_n, w_m) + (t_0 v_0 + \cdots + t_{n-1} v_{n-1}, s_0 w_0 + \cdots + (s_m - t_n) w_m) = t_n(v_n, w_m) + (x', y').$$

In this case we set $\tau(n+m) = (n, m)$ and $\tau(n+m-1) = (n-1, m)$. Note that if $t_n = s_m = 1$ then $(x, y) = (v_n, w_m)$, which is clearly in the image of φ. Therefore assume $t_n < 1$. The point

$$(x'', y'') := \frac{1}{1-t_n}(x', y')$$

lies in the product of simplices $\Delta^{n-1} \times \Delta^m$ spanned by the vertices (v_i, w_j) for $i = 0, \ldots, n-1$ and $j = 0, \ldots, m$. By the inductive hypothesis there is a shuffle map $\tau' : [n+m-1] \to [n-1] \times [m]$ such that the image of the map $|\tau'|$ contains (x', y'). In other words, there exist coefficients r_i with $\sum_i r_i = 1$ such that

$$(x'', y'') = \sum_{i=0}^{n+m-1} r_i (v_{\tau_1'(i)}, w_{\tau_2'(i)}).$$

Extend the definition of τ by $\tau(i) = \tau'(i)$ for $i \leq n+m-1$. Then

$$\begin{aligned}(x, y) &= t_n(v_n, w_m) + (1-t_n) \sum_{i=0}^{n+m-1} r_i (v_{\tau_1'(i)}, w_{\tau_2'(i)}) \\ &= \sum_{i=0}^{n+m} r_i' (v_{\tau_1(i)}, w_{\tau_2(i)})\end{aligned}$$

with $r_i' = (1-t_n) r_i$ for $i < n+m$ and $r_{n+m} = t_n$, proving that (x, y) is in the image of φ. If $t_n > s_m$ then one sets $\tau(n+m-1) = (n, m-1)$ and proceeds similarly. □

We conclude this section with a well-known consequence of the fact that geometric realization preserves products.

Corollary 2.9 *Consider functors $F, G : \mathbf{C} \to \mathbf{D}$ between small categories and a natural transformation $\nu : F \to G$ between them. Then ν induces a homotopy*

$$|N\nu| : \Delta^1 \times B\mathbf{C} \to B\mathbf{D}$$

between the corresponding maps of classifying spaces.

Proof The natural transformation ν can be seen as functor

$$\nu : (0 \to 1) \times \mathbf{C} \to \mathbf{D},$$

which upon applying the nerve functor (which preserves products, being a right adjoint) gives

$$N\nu : \Delta[1] \times N\mathbf{C} \to N\mathbf{D}.$$

Applying geometric realization and using that it preserves products gives a map

$$|N\nu| : \Delta^1 \times B\mathbf{C} \to B\mathbf{D}.$$ □

In particular, considering the unit and counit of an adjoint pair of functors gives the following:

Corollary 2.10 *An adjoint pair of functors between categories* **C** *and* **D** *induces a homotopy equivalence between the classifying spaces* $B\mathbf{C}$ *and* $B\mathbf{D}$.

2.6 Simplicial Spaces and Bisimplicial Sets

As mentioned at the start of Section 2.2, one can define simplicial objects in any category $\mathcal{E}$ as functors $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{E}$. The cases where $\mathcal{E}$ is the category of spaces or of simplicial sets itself frequently occur in the literature and we will need some general facts and constructions for these.

2.6.1 Simplicial Spaces

A *simplicial space* is a functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Top}$. In other words, X is a simplicial set equipped with a topology on each X_n for which all the face maps $d_i : X_n \to X_{n-1}$ and degeneracy maps $s_j : X_{n-1} \to X_n$ are continuous. With natural transformations between them, they form a category for which we write **sTop**.

For example, if X is a simplicial set and T a topological space, one can define a simplicial space $X \otimes T$ by setting

$$(X \otimes T)_n = X_n \times T,$$

giving X_n the discrete topology and $X_n \times T$ the product topology, while defining the face and degeneracy maps in the obvious way.

A second example, perhaps one of the most important, is the following. If G is a topological group, its nerve NG is naturally a simplicial space. Recall from Section 2.4 that as a simplicial set it is defined by $(NG)_n = G^n$, which we can now equip with the product topology. Its face and degeneracy maps are

$$d_i(g_1, \dots, g_n) = \begin{cases} (g_2, \dots, g_n) & \text{if } i = 0, \\ (g_1, \dots, g_{i+1}g_i, \dots, g_n) & \text{if } 0 < i < n, \\ (g_0, \dots, g_{n-1}) & \text{if } i = n, \end{cases}$$

and

$$s_j(g_1, \dots, g_{n-1}) = (g_1, \dots, g_j, 1, g_{j+1}, \dots, g_{n-1}).$$

For a simplicial space X one can define its *geometric realization* $|X|$ by exactly the same formula as for simplicial sets, again describing it as a quotient space of

$$\coprod_{n \geq 0} X_n \times \Delta^n,$$

except that now each X_n is a topological space (rather than just a set). In this way one obtains a functor

$$|\cdot| : \mathbf{sTop} \to \mathbf{Top},$$

which again preserves colimits. In the first example above, for a simplicial set X and a topological space T, one has

$$|X \otimes T| \simeq |X| \times T.$$

For a topological group G as in our second example one defines

$$BG := |N(G)|,$$

the *classifying space* of G.

The homotopical properties of this realization functor have been widely discussed in the literature and we will address some aspects of this in Chapter 8.

2.6.2 Bisimplicial Sets

A *bisimplicial set* is a simplicial object in the category of simplicial sets, i.e. a functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{sSets}$ or equivalently a functor

$$X : (\mathbf{\Delta} \times \mathbf{\Delta})^{\mathrm{op}} \to \mathbf{Sets}.$$

Thus, such an X is given by a collection of sets $X_{p,q}$ for $p, q \geq 0$ together with 'horizontal' and 'vertical' face and degeneracy maps:

$$\begin{array}{ccc} X_{p,q} & \overset{d_i^h}{\underset{s_j^h}{\rightleftarrows}} & X_{p-1,q} \\ d_i^v \downarrow\uparrow s_j^v & & \\ X_{p,q-1}. & & \end{array}$$

These horizontal and vertical operations should satisfy the simplicial identities, while 'horizontal' and 'vertical' commute with one another. For instance,

$$d_i^h d_j^h = d_{j-1}^h d_i^h \quad (i < j), \quad d_i^h d_j^v = d_j^v d_i^h, \quad \text{etc.}$$

Bisimplicial sets and natural transformations between them form a category denoted **bisSets**. Applying the general facts about presheaf categories from Section 2.4, we find that the diagonal functor

$$\delta : \mathbf{\Delta} \to \mathbf{\Delta} \times \mathbf{\Delta}$$

induces a triple of adjoint functors

$$\mathbf{sSets} \underset{\delta_*}{\overset{\delta_!}{\underset{\longleftarrow}{\overset{\longrightarrow}{\xleftarrow{\delta^*}}}}} \mathbf{bisSets}.$$

The functor δ^* is usually referred to as the diagonal and for a bisimplicial set X it gives

$$(\delta^* X)_n = X_{n,n}.$$

The functor $\delta_!$ is essentially uniquely determined by the fact that it preserves colimits and sends representables to representables according to δ. If we write

$$\mathbf{\Delta}[p,q] = (\mathbf{\Delta} \times \mathbf{\Delta})(-, ([p],[q]))$$

for the bisimplicial set represented by $([p],[q]) \in \mathbf{\Delta} \times \mathbf{\Delta}$, then

$$\delta_! \Delta[n] = \Delta[n,n].$$

Any two simplicial sets X and Y define a bisimplicial set $X \boxtimes Y$, their *external product*, by

$$(X \boxtimes Y)_{p,q} := X_p \times Y_q.$$

For example, the representable bisimplicial set $\Delta[p,q] = \Delta[p] \boxtimes \Delta[q]$ is such an external product. Notice that δ^* maps the external product to the ordinary product, i.e.

$$\delta^*(X \boxtimes Y) = X \times Y.$$

Any bisimplicial set X gives rise to two simplicial spaces, obtained by geometric realization in the horizontal and vertical direction respectively. The horizontal one

$$|X|_q^{(h)} = |X_{\bullet,q}|$$

is obtained by taking the geometric realization of the qth row for each fixed q and the vertical one

$$|X|_p^{(v)} = |X_{p,\bullet}|$$

is similarly obtained by realizing the columns. One can next realize these simplicial spaces to obtain spaces

$$||X|_\bullet^{(h)}| \quad \text{and} \quad ||X|_\bullet^{(v)}|.$$

It is a simple and very useful observation that these two spaces are naturally homeomorphic and furthermore coincide with a third way of 'realizing' X, namely by taking the geometric realization of the diagonal. We record this result as follows:

Proposition 2.11 *There are natural homeomorphisms*

$$||X|_\bullet^{(h)}| \simeq |\delta^* X| \simeq ||X|_\bullet^{(v)}|.$$

Proof Observe that the three functors involved are colimit preserving functors **bisSets** $\to$ **Top**, so it suffices to show that they are naturally isomorphic on representables, i.e. that there are natural homeomorphisms

$$||\Delta[p,q]|_\bullet^{(h)}| \simeq |\delta^*\Delta[p,q]| \simeq ||\Delta[p,q]|_\bullet^{(v)}|.$$

Recall that $\Delta[p,q] = \Delta[p] \boxtimes \Delta[q]$. But for general simplicial sets Y and Z one has natural identifications

$$|Y \boxtimes Z|^{(v)} = Y \otimes |Z| \quad \text{and} \quad |Y \boxtimes Z|^{(h)} = |Y| \otimes Z$$

and so by the observation at the beginning of this section also

$$||Y \boxtimes Z|^{(v)}| = |Y| \times |Z| \quad \text{and} \quad ||Y \boxtimes Z|^{(h)}| = |Y| \times |Z|.$$

Finally, we have

$$|\delta^*(Y \boxtimes Z)| = |Y \times Z| \simeq |Y| \times |Z|$$

since geometric realization preserves products. We conclude by setting $Y = \Delta[p]$ and $Z = \Delta[q]$. □

For a bisimplicial set X we will sometimes write $||X||$ for $|\delta^* X|$ and refer to it as *the* geometric realization of X. The observations above are useful when studying the classifying spaces of *simplicial groups*. Such a simplicial group is of course a functor $G : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Gp}$ or equivalently a simplicial set G with a group structure on each G_n such that the face and degeneracy maps of G are group homomorphisms between them. Taking the nerve of each G_n (where one regards a group as a category with one object) yields a bisimplicial set,

$$NG_{p,q} = N(G_p)_q = \underbrace{G_p \times \cdots \times G_p}_{q \text{ times}}$$

and its *classifying space* is the geometric realization,

$$BG = ||NG||.$$

In particular, by the proposition above, BG coincides with the classifying space of the topological group $|G|$. (Note that $|G|$ indeed inherits a group structure from G, because geometric realization preserves products.)

2.7 Simplicial Categories and Simplicial Operads

We encountered some examples of topological categories and topological operads in Chapter 1. These come up naturally when describing algebraic structures in homotopy theory (e.g. the $\mathbf{E}_n$-operads for n-fold loop spaces) and when one wishes to describe structures 'up to coherent homotopy' (by means of the W-resolution, for example). In this section we replace topological spaces by simplicial sets and discuss the resulting notions.

2.7.1 Internal Versus Enriched Categories and Operads

For a category $\mathcal{E}$ with pullbacks there is a notion of *internal category in* $\mathcal{E}$ or a *category object in* $\mathcal{E}$. Such an internal category $\mathbf{C}$ is given by two objects ob($\mathbf{C}$) and ar($\mathbf{C}$), the 'object of objects' of $\mathbf{C}$ and the 'object of arrows' of $\mathbf{C}$, with structure maps for domain, codomain, identity morphisms and composition in $\mathbf{C}$, together making up a diagram in $\mathcal{E}$ of the form

$$\mathrm{ar}(\mathbf{C}) \times_{\mathrm{ob}(\mathbf{C})} \mathrm{ar}(\mathbf{C}) \longrightarrow \mathrm{ar}(\mathbf{C}) \rightleftarrows \mathrm{ob}(\mathbf{C})$$

satisfying the usual equations for a category. A morphism $f : \mathbf{C} \to \mathbf{D}$ between two such internal categories, also called an *internal functor*, consists of two morphisms $f_{\mathrm{ob}} : \mathrm{ob}(\mathbf{C}) \to \mathrm{ob}(\mathbf{D})$ and $f_{\mathrm{ar}} : \mathrm{ar}(\mathbf{C}) \to \mathrm{ar}(\mathbf{D})$ satisfying the usual equations. In this way we obtain a category of internal categories in $\mathcal{E}$.

Each such internal category $\mathbf{C}$ gives rise to a simplicial object $N\mathbf{C}$ in $\mathcal{E}$, its nerve, by means of pullbacks in $\mathcal{E}$. One sets

$$N\mathbf{C}_n := \underbrace{\mathrm{ar}(\mathbf{C}) \times_{\mathrm{ob}(\mathbf{C})} \cdots \times_{\mathrm{ob}(\mathbf{C})} \mathrm{ar}(\mathbf{C})}_{n \text{ times}}$$

for $n \geq 1$ and

$$N\mathbf{C}_0 := \mathrm{ob}(\mathbf{C}).$$

A closely related notion is that of a category *enriched in* $\mathcal{E}$, or briefly an $\mathcal{E}$-*category*. An $\mathcal{E}$-category $\mathbf{C}$ consists of a collection of objects ob($\mathbf{C}$) and for any two objects x, y of $\mathbf{C}$ an object $\mathbf{C}(x, y)$ of $\mathcal{E}$, the 'object of arrows' from x to y. Furthermore, there are structure maps

$$\mathbf{C}(y, z) \times \mathbf{C}(x, y) \to \mathbf{C}(x, z)$$

for composition and $1 \to \mathbf{C}(x, x)$ for identities, where 1 denotes the terminal object of $\mathcal{E}$. A morphism between such $\mathcal{E}$-categories $f : \mathbf{C} \to \mathbf{D}$ consists of a map $f : \mathrm{ob}(\mathbf{C}) \to \mathrm{ob}(\mathbf{D})$ and for each $x, y \in \mathrm{ob}(\mathbf{C})$ a morphism $\mathbf{C}(x, y) \to \mathbf{D}(f(x), f(y))$ of $\mathcal{E}$, compatible with composition and identities. In this way one obtains a category of $\mathcal{E}$-categories.

As we have just described it the notion of enriched category is more restrictive than that of internal category: if $\mathcal{E}$ has coproducts which distribute over products, one can view an enriched category $\mathbf{C}$ as an internal category in which the object of objects is a coproduct of copies of the terminal object indexed by the set ob($\mathbf{C}$). On the other hand, the definition of enriched category only uses products, not pullbacks, and in fact makes sense if $\mathcal{E}$ is any monoidal category: in the definition one simply replaces the product and terminal object by the tensor and unit of the monoidal structure.

Exactly the same dichotomy applies to operads. For $\mathcal{E}$ with pullbacks, there is a notion of internal operad $\mathbf{P}$ in $\mathcal{E}$, given by an object of colours C and for each $n \geq 0$ an object of operations $\mathbf{P}_n$. The structure maps are $\mathbf{P}_n \to C^n \times C$ (for domain and codomain), an action of the symmetric group Σ_n on $\mathbf{P}_n$, and maps for unit and composition

$$\mathbf{P}_n \times_{C^n} (\mathbf{P}_{k_1} \times \cdots \times \mathbf{P}_{k_n}) \to \mathbf{P}_{k_1+\cdots+k_n},$$
$$C \to \mathbf{P}_1,$$

all satisfying equations we leave for the reader to spell out.

On the other hand, an operad $\mathbf{P}$ *enriched in* $\mathcal{E}$ (or more simply an $\mathcal{E}$-*operad*) consists of a *set* of colours C and for each sequence of colours $(c_1, \ldots, c_n; c)$ an object $\mathbf{P}(c_1, \ldots, c_n; c)$ of $\mathcal{E}$, all equipped with structure maps for composition, symmetries and units, much like the definition of operad we gave in Chapter 1.

Thus, when one speaks of topological or simplicial operads or categories, it is a priori not clear whether one is referring to the internal or the enriched notion. In this book *we will always mean the enriched notion*, unless explicitly stated otherwise, which is also fairly standard in the literature. This is consistent with our use of the phrases *topological category* and *topological operad* in Chapter 1. Note that the ambiguity disappears when the object of objects is the terminal object of $\mathcal{E}$, as is the case for topological or simplicial groups, monoids, or operads with a single colour.

Remark 2.12 Another valid interpretation of the phrase 'simplicial category' would have it be a simplicial object in **Cat**. We leave it to the reader to verify that this is essentially the same thing as a category internal to simplicial sets.

2.7.2 Simplicial Categories

According to the convention above, a simplicial category $\mathbf{C}$ is given by a set of objects ob($\mathbf{C}$) and for any two $x, y \in$ ob($\mathbf{C}$) a simplicial set $\mathbf{C}(x, y)$ of arrows from x to y. Furthermore, there are identity elements $\mathrm{id}_x \in \mathbf{C}(x, x)_0$ and maps of simplicial sets $\mathbf{C}(y, z) \times \mathbf{C}(x, y) \to \mathbf{C}(x, z)$ for composition. The equations for these structure maps come down to the requirement that for each fixed n, one has a category $\mathbf{C}_n$, all having *the same* set of objects ob($\mathbf{C}$), with the face and degeneracy operators between the various $\mathbf{C}_n$ being the identity on objects. With morphisms of simplicial categories as defined above we obtain a category which we denote by **sCat**.

The nerve of a simplicial category gives a bisimplicial set $N\mathbf{C}$ with

$$N\mathbf{C}_{p,q} = N(\mathbf{C}_p)_q = \coprod_{x_0,\ldots,x_q} \mathbf{C}_p(x_0, x_1) \times \cdots \times \mathbf{C}_p(x_{q-1}, x_q)$$

for $q > 0$ and

$$N\mathbf{C}_{p,0} = \mathrm{ob}(\mathbf{C}_p) = \mathrm{ob}(\mathbf{C}),$$

i.e. this bisimplicial set is *constant* (or *discrete*) in its bottom row. Another relevant construction is the following: one can take the geometric realizations $|\mathbf{C}(x, y)|$ of all the objects of arrows to obtain a topological category $|\mathbf{C}|$. (The reason that this is still naturally a category is, again, that geometric realization preserves products.) We can define the *classifying space* of $\mathbf{C}$ in the evident way and express it in the following two ways, cf. Proposition 2.11:

$$B\mathbf{C} = ||N\mathbf{C}|| \simeq |N(|\mathbf{C}|)|.$$

2.7.3 Boardman–Vogt Resolution

For a simplicial category $\mathbf{C}$ one can construct its *Boardman–Vogt resolution* $W\mathbf{C}$ exactly as in the topological case, now using the representable 1-simplex $\Delta[1]$ instead of the topological unit interval $[0, 1]$. Indeed, the construction essentially only uses the elements 0 and 1 of $[0, 1]$ together with the supremum operation $\vee : [0, 1] \times [0, 1] \to [0, 1]$. These are now replaced by the vertices $0, 1 : \Delta[0] \to \Delta[1]$ (which correspond to ∂_1 and ∂_0 respectively) and the map

$$\vee : \Delta[1] \times \Delta[1] \to \Delta[1]$$

corresponding to the map of partially ordered sets $\vee : [1] \times [1] \to [1]$ taking the supremum of a pair. Since this is really all the structure we need, we will discuss the W-resolution slightly more generally, with respect to an arbitrary *interval object*. For us, this will be a simplicial set I together with maps

$$+ : \Delta[0] \to I \quad \text{and} \quad - : \Delta[0] \to I$$

and an associative operation $\vee : I \times I \to I$ for which $-$ is a unit and $+$ is absorbing, in the sense that $x \vee + = + = + \vee x$. One could define such intervals in more general monoidal categories as well (and carry out much of what we do in this section), but we will remain in the relatively explicit setting of simplicial sets. The most important examples of intervals we have in mind are the following:

Example 2.13 (i) The representable $\Delta[1]$, with $- = 0$ and $+ = 1$ and $\vee$ being the supremum, as described above.

(ii) The opposite of the previous example, with again $I = \Delta[1]$ but now $- = 1$ and $+ = 0$, with $\vee$ taking the infimum rather than supremum.

(iii) The nerve J of the 'free isomorphism'. To be precise, write $\mathbf{J}$ for the category with objects $-$ and $+$ together with morphisms $t : - \to +$ and $s : + \to -$ satisfying the relations $ts = \mathrm{id}_+$ and $st = \mathrm{id}_-$. Thus, $\mathbf{J}$ consists of two objects and an isomorphism between them. One then defines $J = N\mathbf{J}$. The geometric realization of J is homeomorphic to the infinite-dimensional sphere. Indeed, J has two nondegenerate simplices of dimension n for every $n \geq 0$, given by the alternating sequences $sts\cdots$ and $tst\cdots$ of length n. Thus, $|J|$ has a CW structure with precisely two cells in each dimension. It is therefore clear that $|J|^{(0)} = S^0$ and more generally (by induction) one sees that $|J|^{(n)}$ is the n-sphere S^n with its 'equatorial' cell structure, i.e. the one consisting of two closed n-cells glued along their common boundary S^{n-1}. One makes J into an interval object in the same way as for $\Delta[1]$, setting $- = 0$ and $+ = 1$ and taking the supremum operation. This makes the evident inclusion $\Delta[1] \to J$ into a morphism of intervals.

Let us now describe the simplicial category $W\mathbf{C}$ in detail. Its objects are the same as those of $\mathbf{C}$. For any two objects x, y of $\mathbf{C}$ we construct $W\mathbf{C}(x, y)$ by induction as a filtered simplicial set,

$$W\mathbf{C}(x, y)^{(0)} \subseteq W\mathbf{C}(x, y)^{(1)} \subseteq \cdots, \qquad \bigcup_n W\mathbf{C}(x, y)^{(n)} = W\mathbf{C}(x, y).$$

We start with $W\mathbf{C}(x, y)^{(0)} := \mathbf{C}(x, y)$. At each stage, $W\mathbf{C}(x, y)^{(n)}$ will come equipped with a map

$$\coprod_{x_1,\ldots,x_n} I^n \times \mathbf{C}(x, x_1) \times \cdots \times \mathbf{C}(x_n, y) \xrightarrow{\xi^{(n)}} W\mathbf{C}(x, y)^{(n)}$$

which is the identity for $n = 0$. One constructs $W\mathbf{C}(x, y)^{(n)}$ and the map $\xi^{(n)}$ from $W\mathbf{C}(x, y)^{(n-1)}$ and $\xi^{(n-1)}$ by means of the pushout square of simplicial sets

$$\begin{array}{ccc} A & \xrightarrow{\quad\zeta\quad} & W\mathbf{C}(x, y)^{(n-1)} \\ \downarrow & & \downarrow \\ \coprod_{x_1,\ldots,x_n} I^n \times \mathbf{C}(x, x_1) \times \cdots \times \mathbf{C}(x_n, y) & \xrightarrow[\xi^{(n)}]{} & W\mathbf{C}(x, y)^{(n)}. \end{array}$$

Here A is the simplicial subset of the bottom left corner consisting of those elements which have $-$ in one of the n entries of I^n or an identity arrow of $\mathbf{C}$ in one of the slots $\mathbf{C}(x_i, x_{i+1})$. More explicitly, A is the coproduct of simplicial subsets of two types; the first are of the form

$$I^{i-1} \times \Delta[0] \times I^{n-i} \times \mathbf{C}(x, x_1) \times \cdots \times \mathbf{C}(x_n, y),$$

which are included by means of the map $- : \Delta[0] \to I$ and which map to $W\mathbf{C}(x, y)^{(n-1)}$ by using the composition

$$\mathbf{C}(x_{i-1}, x_i) \times \mathbf{C}(x_i, x_{i+1}) \to \mathbf{C}(x_{i-1}, x_{i+1})$$

and the evident identification

$$I^{i-1} \times \Delta[0] \times I^{n-i} \simeq I^{n-1},$$

and then composing these with $\xi^{(n-1)}$. The subsets of the second type are of the form

$$I^n \times \mathbf{C}(x, x_1) \times \cdots \times \{\mathrm{id}_{x_i}\} \times \mathbf{C}(x_{i+1}, x_{i+2}) \times \cdots \times \mathbf{C}(x_n, y),$$

which map to

$$I^{n-1} \times \mathbf{C}(x, x_1) \times \cdots \times \mathbf{C}\widehat{(x_i, x_{i+1})} \times \cdots \times \mathbf{C}(x_n, y)$$

by using the map

$$I^n \simeq I^{i-1} \times I^2 \times I^{n-i-1} \xrightarrow{\mathrm{id}\times\vee\times\mathrm{id}} I^{i-1} \times I \times I^{n-i-1} \simeq I^{n-1}$$

and then to $W\mathbf{C}(x, y)^{(n-1)}$ by composing with $\xi^{(n-1)}$ again.

This construction mimics the identifications made in the topological case and gives a well-defined simplicial category $W\mathbf{C}$. The composition in $W\mathbf{C}$ is uniquely determined by the fact that each diagram of the form

$$\begin{array}{ccc}
(I^n \times \prod_{i=0}^{n} \mathbf{C}(x_i, x_{i+1})) \times (I^m \times \prod_{j=0}^{m} \mathbf{C}(y_j, y_{j+1})) & \longrightarrow & W\mathbf{C}(x, y)^{(n)} \times W\mathbf{C}(y, z)^{(m)} \\
\downarrow & & \downarrow \\
I^n \times I^m \times \prod_{k=0}^{n+m+1} \mathbf{C}(z_k, z_{k+1}) & & W\mathbf{C}(x, z)^{(n+m+1)} \\
{\scriptstyle u\times\mathrm{id}}\downarrow & \nearrow{\scriptstyle \xi^{(n+m+1)}} & \\
I^{n+m+1} \times \prod_{k=0}^{n+m+1} \mathbf{C}(z_k, z_{k+1}) & &
\end{array}$$

commutes, where $x_0 = x$, $x_{n+1} = y = y_0$, $y_{m+1} = z$. Also, the sequence of z_k's is the sequence $x_0, \ldots, x_{n+1} = y_0, \ldots, y_{m+1}$ and

$$u : I^n \times I^m \simeq I^n \times \Delta[0] \times I^m \to I^n \times I \times I^m \simeq I^{n+m+1}$$

inserts $+$ in the $(n + 1)$st coordinate.

2.7.4 Homotopy-Coherent Nerve

As for the topological Boardman–Vogt resolution, the case where $\mathbf{C}$ is the free category

$$[n] = (0 \to 1 \to \cdots \to n)$$

(regarded as a discrete simplicial category) is much easier to describe. In this section we take $I = \Delta[1]$ with $- = 0$ and $+ = 1$, as in Example 2.13(i). Explicitly, $W[n]$ has the same objects as $[n]$ and

$$W[n](i, j) = \Delta[1]^{j-i-1}$$

for $i \leq j$, with the convention that $\Delta[1]^{-1} = \Delta[0]$. (As before, we heuristically think of an arrow $i \to j$ in $W[n]$ as a sequence of waiting times on the objects $i+1, \ldots, j-1$ in a virtual composition $i \to i+1 \to \cdots \to j-1 \to j$, so there are $j-i-1$ such waiting times.) Composition

$$W[n](i,j) \times W[n](j,k) \to W[n](i,k)$$

is given by the map which inserts 1 in the appropriate slot, as in the following diagram:

$$\begin{array}{ccc} \Delta[1]^{j-i-1} \times \Delta[1]^{k-j-1} & = & \Delta[1]^{j-i-1} \times \Delta[0] \times \Delta[1]^{k-j-1} \\ \downarrow & & \downarrow {\scriptstyle \mathrm{id}\times\partial_0\times\mathrm{id}} \\ \Delta[1]^{k-i-1} & = & \Delta[1]^{j-i-1} \times \Delta[1] \times \Delta[1]^{k-j-1}. \end{array}$$

This construction gives a functor

$$W : \mathbf{\Delta} \to \mathbf{sCat}.$$

By the general procedure of left Kan extension it induces an adjoint pair

$$w_! : \mathbf{sSets} \rightleftarrows \mathbf{sCat} : w^*.$$

Explicitly, for a simplicial category $\mathbf{C}$, the simplicial set $w^*\mathbf{C}$ is defined by

$$w^*\mathbf{C}_n = \mathrm{Hom}(W[n], \mathbf{C}),$$

where Hom is the set of functors of simplicial categories. This simplicial set $w^*\mathbf{C}$ is called the *homotopy-coherent nerve* of $\mathbf{C}$. If $\mathbf{C}$ happens to be a discrete simplicial category, it agrees with the usual nerve $N\mathbf{C}$. The left adjoint $w_!$ is uniquely determined (up to isomorphism) by the fact that it preserves colimits and agrees with W on representables, i.e.

$$w_!(\Delta[n]) = W[n].$$

In the second part of this book we will explain in what sense this adjoint pair gives an equivalence of homotopy theories between simplicial sets and simplicial categories.

Remark 2.14 The adjoint pair $(w_!, w^*)$ is commonly denoted $(\mathfrak{C}, N)$ in the literature, but we will not use this notation.

2.7.5 Simplicial Operads

Our discussion of simplicial categories has a parallel for simplicial operads. A simplicial operad $\mathbf{P}$ is given by a set of colours C and for each sequence $c_1, \ldots, c_n, c$ of colours a simplicial set $\mathbf{P}(c_1, \ldots, c_n; c)$, thought of as the simplicial set of operations from $c_1, \ldots, c_n$ to c. Furthermore, there are composition maps and symmetric group actions, as well as units $q \in \mathbf{P}(c, c)_0$ for each colour c, similar to the cases of operads in **Sets** and **Top** discussed in the first chapter. In particular, for each simplicial degree q, there is an operad $\mathbf{P}_q$ in **Sets** with the same set of colours C and the sets $\mathbf{P}(c_1, \ldots, c_n; c)_q$ as operations. Moreover, the simplicial face and degeneracy operators give morphisms of operads $\mathbf{P}_q \to \mathbf{P}_{q-1}$ and $\mathbf{P}_{q-1} \to \mathbf{P}_q$ respectively. A morphism of simplicial operads $\mathbf{P} \to \mathbf{Q}$ consists of a function $\varphi : C \to D$ between the respective sets of colours of $\mathbf{P}$ and $\mathbf{Q}$ and a family of morphisms of simplicial sets (all denoted φ again)

$$\varphi : \mathbf{P}(c_1, \ldots, c_n; c) \to \mathbf{Q}(\varphi(c_1), \ldots, \varphi(c_n); \varphi(c)),$$

one such for each sequence of colours $c_1, \ldots, c_n, c$ of $\mathbf{P}$. These morphisms are required to be compatible with composition, symmetries and units. A different way of describing such a morphism is as a collection $\varphi_q : \mathbf{P}_q \to \mathbf{Q}_q$ of morphisms of operads in **Sets**, natural in q with respect to face and degeneracy operators. In this way one obtains a category of simplicial operads, which we denote by **sOp**.

Since geometric realization preserves products, each such simplicial operad $\mathbf{P}$ yields a topological operad with the same set of colours C and spaces of operations $|\mathbf{P}(c_1, \ldots, c_n; c)|$. In fact, this defines a functor

$$|\cdot| : \mathbf{sOp} \to \mathbf{Op}_{\mathbf{Top}}$$

between the categories of simplicial and topological operads. When compared to the discussion of simplicial categories above, the parallel seems to stop here, since we do not have a nerve functor for operads at our disposal which parallels the nerve functor for categories. This gap will be filled in the next chapter and is in fact a major theme of this book.

Finally, we note that any simplicial operad $\mathbf{P}$ of course 'contains' a simplicial category $j^*\mathbf{P}$ with the set C as its set of objects and the simplicial sets $\mathbf{P}(c; d)$ as morphisms from c to d. Conversely, any simplicial category $\mathbf{C}$ can be regarded as a simplicial operad (which we denote $j_!\mathbf{C}$) with only unary operations, i.e. one for which the simplicial sets $j_!\mathbf{C}(c_1, \ldots, c_n; c)$ are empty unless $n = 1$. This procedure is easily seen to define an adjoint pair

$$j_! : \mathbf{sCat} \rightleftarrows \mathbf{sOp} : j^*.$$

2.7.6 The Barratt–Eccles Operad

In this section we discuss an important example of a simplicial operad, the so-called *Barratt–Eccles operad*. As before we write Σ_n for the symmetric group on n letters, for $n \geq 0$. Every group G (or in fact every monoid) gives rise to a category usually denoted EG, whose objects are the elements of G and where an arrow $g \to h$ is an element $k \in G$ with $g = hk$. Of course for groups this element k is unique, namely $k = h^{-1}g$. Alternatively, viewing G as a category with one object $*$ and the elements of G as morphisms, EG is simply the slice category $G/*$. Then the identity of the object $*$ is a terminal object of EG, so that its classifying space $|N(EG)|$ is contractible. Moreover G acts on EG from the left in the obvious way, using multiplication of elements of G. Clearly this action is free.

The simplicial sets $N(E\Sigma_n)$ fit together to form an operad with one colour, as we will now explain. We will define the operadic composition maps by first specifying group homomorphisms

$$\gamma : \Sigma_n \times \Sigma_{k_1} \times \cdots \times \Sigma_{k_n} \to \Sigma_k$$

for each $n, k_1, \ldots, k_n \geq 0$ and $k = k_1 + \cdots + k_n$. One way to do this is to view the elements $\sigma \in \Sigma_n$ as permutation matrices and defining $\gamma(\sigma, \tau_1, \ldots, \tau_n)$ by replacing the 1 in the ith column (and $\sigma(i)$th row) of σ by the permutation matrix τ_i. Another way is to combine the embedding

$$\text{block} : \Sigma_{k_1} \times \cdots \times \Sigma_{k_n} \to \Sigma_k,$$

given by letting Σ_{k_i} act on the ith 'block' of length k_i in the set $\{1, \ldots, k\}$, with the homomorphism $\Sigma_n \to \Sigma_k$ permuting the n blocks. If we write

$$f : \{1, \ldots, k\} \to \{1, \ldots, n\}$$

for the map sending every element of the ith block to the number i, we denote this homomorphism by $f^* : \Sigma_n \to \Sigma_k$. Then the map γ can be described as a product in the group Σ_k:

$$\begin{aligned}\gamma(\sigma, \tau_1, \ldots, \tau_n) &= f^*\sigma \cdot \text{block}(\tau_1, \ldots, \tau_n) \\ &= \text{block}(\tau_1, \ldots, \tau_n) \cdot f^*\sigma.\end{aligned}$$

Now the maps $N(E\gamma)$ give the desired operadic composition. The action of Σ_n on $E\Sigma_n$ is as described above and one easily verifies that these are compatible with the operadic composition in the desired way. In this way we obtain a simplicial and a topological operad $\mathrm{BE}_{\boldsymbol{\Delta}}$ and $\mathrm{BE}_{\mathbf{Top}}$ with

$$\begin{aligned}\mathrm{BE}_{\boldsymbol{\Delta}}(n) &= N(E\Sigma_n) \\ \mathrm{BE}_{\mathbf{Top}}(n) &= |N(E\Sigma_n)|\end{aligned}$$

which are called the (simplicial and topological) Barratt–Eccles operad. As we already observed, it is an operad whose spaces are contractible and have a free Σ_n-action. In this respect it resembles the little disks operad $\mathbf{E}_\infty$ of Example 1.9.

2.7.7 The Simplicial Boardman–Vogt Resolution of an Operad

For a simplicial operad $\mathbf{P}$, one can mimic the topological Boardman–Vogt resolution for topological operads (as in Section 1.7) and construct a simplicial operad $W\mathbf{P}$ equipped with a map to $\mathbf{P}$ which is the identity on colours. The modification to the simplicial case is done completely analogously to the W-construction for simplicial categories of Section 2.7.3, replacing the topological unit interval by the simplex $\Delta[1]$. In particular, for a sequence of colours $c_1, \ldots, c_n, c$ the simplicial set $W\mathbf{P}(c_1, \ldots, c_n; c)$ is a quotient of a coproduct of simplicial sets $W\mathbf{P}^{(T)}$ indexed over planar trees T with numbered leaves and edges labelled by colours of $\mathbf{P}$, so that the leaves of T are labelled by $c_1, \ldots, c_n$ (not necessarily in that order) and the root by c. For such a labelled tree T, the simplicial set $W\mathbf{P}^{(T)}$ is the product

$$\prod_{v \in V(T)} \mathbf{P}(v) \times \prod_{e \in \mathrm{in}(T)} \Delta[1],$$

where v ranges over the vertices of $\mathbf{P}$ and e over the inner edges, while $\mathbf{P}(v) = \mathbf{P}(e_1, \ldots, e_n; e)$ with $e_1, \ldots, e_n$ the incoming edges of v and e its outgoing edge.

We will return to this simplicial W-construction many times in this book. For now, we note that if $\mathbf{P}$ has only unary operations (i.e. is a simplicial category), this W-construction agrees with the one given in Section 2.7.3. Recall that in that case the description of $W[n]$ was much simpler than the general case. The same is true for the free operads $\Omega(T)$ corresponding to trees, see Section 1.3. Recall that the colours of $\Omega(T)$ are the edges of T and its operations are generated by the vertices of T. The Boardman–Vogt resolution $W\Omega(T)$ has the edges of T as colours again. For a sequence $e_1, \ldots, e_n, e$ of such edges, the simplicial set $W\Omega(T)(e_1, \ldots, e_n; e)$ is empty unless there exists a subtree S of T whose leaves are $e_1, \ldots, e_n$ and whose root is e. In this case

$$W\Omega(T)(e_1, \ldots, e_n; e) = \prod_{s \in \mathrm{in}(S)} \Delta[1],$$

the product now ranging over the inner edges s of S. Operadic composition in $W\Omega(T)$ is defined by grafting subtrees, now inserting length 1 for the edges along which the grafting takes place, exactly as in the topological case.

One would hope that this construction gives rise to a ‘homotopy-coherent nerve’ for simplicial operads. This is indeed the case, once we have found a suitable context for nerves of operads in the next chapter.

Historical Notes

Simplicial sets were introduced as a tool to describe the homotopical properties of topological spaces in a combinatorial way. The first definition of simplicial sets (then called complete semi-simplicial complexes) appears in a 1950 paper of Eilenberg–Zilber [54]. Much of the basic theory was developed soon after; Moore's lectures in the Séminaire Henri Cartan of 1954–1955, his overview paper [118], and the early works of Kan [96, 97, 98] are classic references. These include the fundamentals of homotopy theory (such as homotopy groups and fibrations) from the simplicial point of view; much of this material will appear in Chapters 5 and 7 of this book. Standard textbook references on simplicial sets were written in the 1960s by Gabriel–Zisman [61], Lamotke [101], and May [111]. Another very useful survey of work from this time is given by Curtis [47].

All of the above references focus on the application of simplicial sets to the homotopy theory of topological spaces. The shift in focus to general simplicial objects and their applications came a bit later. The application of simplicial techniques to problems of algebra is perhaps most famously promoted in the works of Quillen [123, 126] on the (co)homology of commutative rings and on higher algebraic K-theory. Simplicial categories, as defined in this chapter, rose to importance in the work of Dwyer–Kan [51, 50] from the 1980s. The more recent textbook on simplicial homotopy theory by Goerss–Jardine [69] takes some of these developments into account.

In this chapter we have also introduced the homotopy-coherent nerve, which produces a simplicial set out of a simplicial category. This construction first arose in the work of Vogt [139] on homotopy-coherent diagrams of spaces, building on his earlier work on homotopy-coherent algebraic structures with Boardman [21]. Cordier [45] systematically studied the simplicial version of the homotopy-coherent nerve and made the connection to the work of Dwyer–Kan on simplicial categories.

The Barratt–Eccles operad we described just above originates in [10].