Chapter 4
Proportionality

A key difference among ABC rules is how they treat minorities of voters, i.e., small groups with preferences different from larger groups. Let us illustrate this issue with the following simple example.

Example 4.1 Consider the approval-based preference profile with 60 voters approving $A = \{a_1, \ldots, a_{10}\}$, 20 voters approving $B = \{b_1, \ldots, b_6\}$, 10 voters approving $C = \{c_1, c_2\}$, 8 voters approving $D = \{d_1, d_2, d_3, d_4\}$, and 2 voters approving $E = \{e_1, e_2, e_3\}$; assume our goal is to pick a committee of ten candidates. Given this instance AV returns committee A, and in some cases this is a reasonable choice (e.g., when the goal of the election is to select finalists of a contest). Yet, when the goal is to select a representative body that should reflect voters' preferences in a proportional fashion, this committee violates very basic principles of fairness. Indeed, the voters who approve committee A constitute 60% of the population, yet effectively they decide about the whole committee; at the same time the group of 20% who approve B is ignored. A committee that consists of six candidates from A, two candidates from B, one candidate from C, and one candidate from D is, for example, a much more proportional choice.

In Example 4.1, picking an outcome that is intuitively proportional is easy due to a very specific structure of voters' approval sets—each two approval sets are either the same or disjoint. Finding a proportional committee in the general case, when any two approval sets can arbitrarily overlap, is by far less straightforward, and to some extent ambiguous. Several approaches that allow one to formally reason about proportionality have been proposed in the literature.

The goal of this chapter is to discuss the many faces of proportional representation. Proportionality, at its core, is a notion of fairness that grants smaller and larger groups of voters a fair consideration of their preferences.[1] The concrete definitions

[1] The concept of proportionality also finds application beyond voting, such as proportional clustering in machine learning [22, 49].

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Table 4.1 Proportionality of ABC rule. There are three rules which perform particularly well in terms of proportionality: PAV, Phragmén's sequential rule, and the Method of Equal Shares. The mark † means that the result holds only when the number of voters n is divisible by the committee size k. References of the form (A.x) refer to propositions in Appendix A

	Proportionality degree	EJR	PJR	JR	Laminar prop.	Priceability	Apportionment
AV	0 [68]						None
PAV	$\ell - 1$ [4]	✓ [3]	✓ [3]	✓ [3]			D'Hondt [13]
seq-PAV	$\approx 0.7\ell - 1$ (for $k \leq 200$) [68]						D'Hondt [13]
rev-seq-PAV	?						D'Hondt [13]
CC	≤ 1 (Example 4.6)			✓ [3]			None
seq-CC	≤ 1 (Example 4.6)			✓ [3]			None
seq-Phragmén	$(\ell-1)/2$ [68]		✓ [12]	✓ [12]	✓ [56]	✓ [56]	D'Hondt [13]
M. Equal Shares	$(\ell\pm1)/2$ (A.10)	✓ [56]	✓ [56]	✓ [56]	✓ [56]	✓ [56]	D'Hondt [56]
leximax-Phragmén	1 [68]		✓ [12]	✓ [12]		✓ [56]	D'Hondt [13]
Monroe	≤ 1 (Ex. 4.6)		† [66]	✓ [3]			LRM † [13]
Greedy Monroe	≤ 1 (Ex. 4.6)		† [66]	✓ (A.7)			LRM † (A.5)
MAV	0 (A.10)						None
SAV	0 (A.10)						None

of what proportionality exactly means, however, differ. In this chapter, we review the main approaches to proportionality and identify ABC rules which can be considered proportional. Table 4.1 and Fig. 4.1 provide an overview of this analysis; the corresponding concepts are explained in this chapter.

But before we delve into this topic, let us answer the question why proportionality has such a prominent place in this book. The main reason is that this reflects the attention this topic has received. Since 2015, when Aziz et al. [3] first introduced (extended) justified representation (Sect. 4.2), there has been rapid progress in the understanding of proportionality in ABC elections. This progress has been along two trajectories: (i) defining stronger and stronger proportionality properties and (ii) finding (computationally tractable) ABC rules satisfying these properties. In many situations, a proportional committee corresponds to a fair selection of candidates. Thus, this line of research can be viewed as the search for a maximally fair ABC voting rule. The following sections (Sects. 4.1–4.4) provide an overview of this exciting endeavour.

However, non-proportional rules are certainly also relevant and even necessary in many applications. For example, when shortlisting candidates for a prize, we may want to select the "best" candidates without considerations of a proportional selection. Or if we want to form a group that deliberates a topic, we would like to

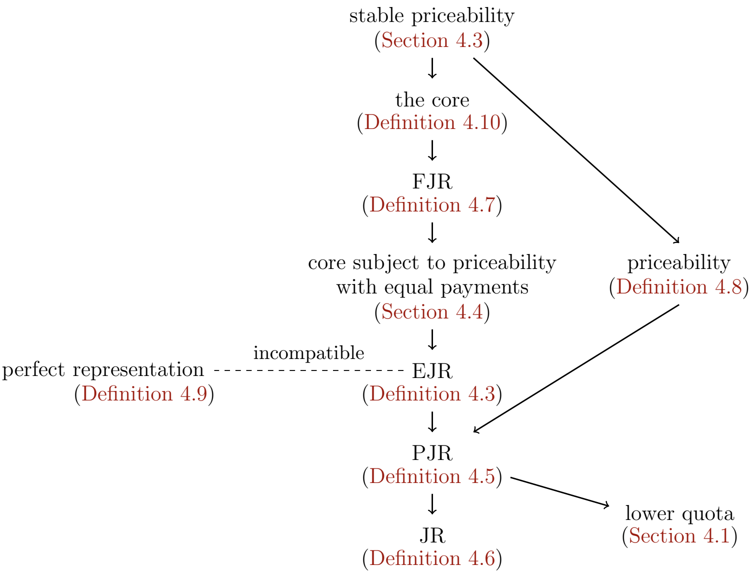

Fig. 4.1 The relation between different proportionality axioms. An arrow from property A to B means that A implies B

include as many diverse opinions as possible and thus we do not give a higher weight to popular opinions. In general, much less work has been done on analysing and understanding non-proportional rules and this topic deserves much more attention. In Sect. 4.5, we summarise the existing literature and discuss concepts of "non-proportionality".

The two final sections of this chapter are dedicated to the interplay of proportionality and strategyproofness (Sect. 4.7) and considerations of proportionality when candidates have external attributes (Sect. 4.6).

4.1 Apportionment

One approach to reasoning about proportionality of voting rules is to first identify a class of well-structured preference profiles where the concept of proportionality can be intuitively captured, and then to examine the behaviour of voting rules on such well-structured profiles. We focus here on so-called *party-list profiles*, which are election instances of the form as we have seen in Example 4.1.

Definition 4.1 (*Party-list profiles*) We say that an approval profile $A = (A(1), \ldots, A(n))$ is a *party-list profile* if for each two voters $i, j \in N$ we have that either $A(i) = A(j)$ or that $A(i) \cap A(j) = \emptyset$. We say that an election instance (A, k) is a *party-list instance* if (i) A is a party-list profile, and (ii) for each voter $i \in N$ we have that $|A(i)| \geq k$.

Party-list profiles closely resemble political elections with political parties, hence the name of the domain. In such elections, voters are typically asked to vote for exactly one party. To see the connection to party-list profiles, note the following: If A is a party-list profile, then the sets of voters and candidates can be divided into p disjoint groups each, $N = N_1 \cup \dots \cup N_p$ and $C \supseteq C_1 \cup \dots \cup C_p$, so that all voters from group N_i, $i \in [p]$, approve exactly the candidates from C_i (and no others). The candidates from C_i can be thought of as members of some (virtual) party, and the voters from N_i are those who cast their vote on party C_i.

In such elections, where the voters do not vote for individual candidates but rather each voter casts a single vote for one political party, the problem of distributing seats to political parties is called the *apportionment problem*. The concept of proportionality in the apportionment setting has been extensively studied in the literature and is well understood—for a detailed overview we refer the reader to the comprehensive books by Balinski and Young [5] and by Pukelsheim [62].

We see from Definition 4.1 that the apportionment problem can be viewed as a strict subdomain of approval-based multi-winner elections, and consequently ABC rules can be viewed as functions that extend apportionment methods to the more general setting of approval profiles. This connection was already known and referred to by Thiele [73] and Phragmén [59]. In a more systematic fashion, Brill et al. [13] showed such relations between various ABC rules and methods of apportionment. To properly explain this relation, let us first define three prominent apportionment methods, used in parliamentary elections all over the world.

In the following, we assume that there are p political parties, consisting of the candidate sets $C_1, \dots, C_p$. By n_i we denote the number of votes cast on party C_i. Further, in line with our usual notation, k denotes the number of committee seats that we want to distribute among the parties.

Apportionment Rule 1 (D'Hondt method[2]) *The D'Hondt method proceeds in k rounds, in each round allocating one seat to some party. Consider the r-th round, and let $s_i(r)$ be the number of seats that are currently assigned to party C_i; thus, $\sum_{i \in [p]} s_i(r) = r - 1$. The D'Hondt method assigns the r-th seat to the party C_i with the highest ratio $\frac{n_i}{s_i(r)+1}$ (using a tiebreaking order between parties if necessary).*

[2] Victor D'Hondt (1841–1901) was a Belgian professor of law and active proponent of proportional representation [24, 25]. The D'Hondt method is also known as the Jefferson method. Thomas Jefferson (1743–1826) was president of the United States, and proposed this method to allocate seats in the House of Representatives to states. D'Hondt's proposal was specifically meant for proportional representation in parliaments. D'Hondt developed this method independently of Jefferson, even though Jefferson's proposal was earlier and largely similar. The name "Jefferson method" is typically used in the U.S., while "D'Hondt method" is prevalent in Europe.

Apportionment Rule 2 (Sainte-Laguë[3] method) *The Sainte-Laguë method is defined analogously to the D'Hondt method, but in the r-th round it allocates the r-th seat to the party* C_i *which maximises the ratio* $\frac{n_i}{2s_i(r)+1}$.

Both the D'Hondt and the Sainte-Laguë method belong to the class of divisor methods. Divisor methods differ in the formula for the ratio used to distribute seats to parties. The aforementioned books by Balinski and Young [5] and by Pukelsheim [62] discuss this important class of apportionment methods in much more detail.

Apportionment Rule 3 (Largest remainder method, LRM[4]) *The largest remainder method first assigns to each party* $\lfloor k \cdot \frac{n_i}{n} \rfloor$ *seats—this way at least* $k - p + 1$ *seats are assigned. Second, it assigns the remaining* $r < p$ *seats to the r parties with the largest remainders* $k \cdot \frac{n_i}{n} - \lfloor k \cdot \frac{n_i}{n} \rfloor$, *assigning each party at most one seat.*

Example 4.2 Consider a party-list representation of the profile from Example 4.1. We have five parties, A, B, C, D, and E, each getting, respectively, 60, 20, 10, 8, and 2 votes; the committee size is $k = 10$. The computation of the D'Hondt method can be followed in the left table below:

	A	B	C	D	E
n_i	**60**	**20**	**10**	8	2
$n_i/2$	**30**	**10**	5	4	1
$n_i/3$	**20**	6 2/3	3 1/3	2 2/3	2/3
$n_i/4$	**15**	5	2 1/2	2	1/2
$n_i/5$	**12**	4	2	1 3/5	2/5
$n_i/6$	**10**	3 1/3	1 2/3	1 1/3	1/3
$n_i/7$	**8 4/7**	2 6/7	1 3/7	1 1/7	2/7
$n_i/8$	7 1/2	2 1/2	1 1/4	1	1/4

	A	B	C	D	E
n_i	**60**	**20**	**10**	**8**	2
$n_i/3$	**20**	**6 2/3**	3 1/3	2 2/3	2/3
$n_i/5$	**12**	4	2	1 3/5	2/5
$n_i/7$	**8 4/7**	2 6/7	1 3/7	1 1/7	2/7
$n_i/9$	**6 2/3**	2 2/9	1 1/9	1 8/9	2/9
$n_i/11$	**5 5/11**	1 9/11	10/11	8/11	2/11
$n_i/13$	4 8/13	1 7/13	10/13	8/13	2/13
$n_i/15$	4	1 1/3	2/3	8/15	2/15

In the subsequent rounds the D'Hondt method allocates seats to parties A, A, A (by tie-breaking), B, A, A, A (by tie-breaking), B (by tie-breaking), C, and A. For example, in the fourth round, when A is already allocated 3 seats and B is allocated none, the rule will give the next seat to B rather than to A, because $\frac{20}{0+1} > \frac{60}{3+1}$. Summarising, seven seats will be allocated to party A, two seats to party B, and one seat to party C; the remaining parties will get no seats. In the diction of ABC rules, winning committees are exactly those that consist of seven candidates from A, two candidates from B and one candidate from C.

[3] As it is the case with the D'Hondt/Jefferson method, this rule has been developed independently in Europe and in the U.S. and goes by different names: Sainte-Laguë is used in Europe (in particular in the context of proportional representation in parliaments) and Webster is the name used in the U.S. literature. Sainte-Laguë (1882–1950) was a French mathematician and proposed this method in 1910 [65]. Daniel Webster (1782–1852) was a U.S. statesman and proposed this method in 1832 [5].

[4] The largest remainder method is also known as the Hamilton method, as it was proposed in the U.S. by Alexander Hamilton (1755–1804). His proposal was abandoned in favour of Jefferson's method [5].

The computation of the Sainte-Laguë method is illustrated in the above right table. It will allocate six seats to A, two seats to B, one seat to C, and one seat to D.

The largest remainder method first assigns to parties A, B, C, D, and E—respectively—6, 2, 1, 0, and 0 seats. Then, the remainders are considered:

	A	B	C	D	E
n_i	60	20	10	8	2
$\lfloor k \cdot \frac{n_i}{n} \rfloor$	6	2	1	0	0
Remainder	0	0	0	0.8	0.2
Seats	6	2	1	1	0

There is one unassigned seat which will be given to the party with the largest remainder, namely to D. Thus, LRM will allocate six seats to A, two seats to B, one seat to C, and one seat to D.

The D'Hondt method, the Sainte-Laguë method, and LRM exhibit particularly appealing properties. For example, the D'Hondt method satisfies *lower quota*, which means that a party i which receives n_i out of n votes must be allocated at least $\lfloor k \cdot {}^{n_i}/_{n} \rfloor$ committee seats. The largest remainder method satisfies not only lower quota but also *upper quota*: a party i with n_i out of n votes must not receive more than $\lceil k \cdot {}^{n_i}/_{n} \rceil$ seats. However, the largest remainder method fails an important axiom called population monotonicity, which states that an increase in support must not harm a party. In contrast, population monotonicity is satisfied by D'Hondt and Sainte-Laguë. For further details, we refer the interested reader to the aforementioned books on apportionment methods [5, 62].

We are now ready to formulate the main results of Brill et al. [13]:

Theorem 4.1 (Brill et al. [13]) *PAV, sequential PAV, seq-Phragmén, and leximax-Phragmén extend the D'Hondt method of apportionment. Phragmén's variance-minimising rule*[5] *extends the Sainte-Laguë method of apportionment. If n is divisible by k, then Monroe's rule extends the largest remainders method.*

Theorem 4.1 lists ABC rules that behave proportionally on party-list profiles and thus these rules can be considered good contenders for being proportional in the general ABC model. In addition, we show in the appendix that also Greedy Monroe extends the largest remainder method when n is divisible by k (Proposition A.5), but both Monroe's rule and Greedy Monroe do not if n is not divisible by k (Proposition A.6).

Lackner and Skowron [44] strengthened the results of Brill et al. [13], providing a strong argument in favour of PAV:

Theorem 4.2 (Lackner and Skowron [44]) *PAV is the unique extension of the D'Hondt method that satisfies neutrality, anonymity, consistency, and continuity.*

[5] This rule is similar to leximax-Phragmén but minimises the variance of loads instead of the maximum load, see [12, 36] for a precise definition.

Lackner and Skowron [44] further show that this result can be generalised to arbitrary divisor-based apportionment methods. For example, the Sainte-Laguë method yields the w-Thiele method with $w(x) = \sum_{j=1}^{x} \frac{1}{2j-1}$.

4.2 Cohesive Groups

In party-list profiles (Definition 4.1), voters can be arranged in groups with identical preferences. Then, proportionality requires that a large-enough group of voters with identical preferences deserves a certain number of representatives in the elected committee (proportional to the size of the group). This approach can be generalised to groups with non-identical but similar preferences. We now discuss axioms that relax the requirements for groups of voters to be entitled to representatives. These axioms are based on the concept of ℓ-cohesiveness:

Definition 4.2 For $\ell \geq 1$, a group $V \subseteq N$ is *ℓ-cohesive* if:

(i) $|V| \geq \ell \cdot \frac{n}{k}$, and
(ii) $\left|\bigcap_{i \in V} A(i)\right| \geq \ell$.

An ℓ-cohesive group consists of an ℓ/k-th fraction of voters, thus, intuitively, such a group should be able to control at least $\ell/k \cdot k = \ell$ committee seats. Further, an ℓ-cohesive group agrees on ℓ candidates, so one can ensure each member of the group gets ℓ representatives by selecting only ℓ candidates. It is, hence, tempting to require that for each ℓ-cohesive group V, each voter from V should be given at least ℓ representatives in the elected committee. Unfortunately, this would be too strong—there exists no rule that would satisfy this property.

Example 4.3 (Aziz et al. [4]) Consider a profile A with four candidates (a, b, c, d) and 12 voters, with the following approval sets:

$A(1)\colon \{a, d\}$	$A(4)\colon \{a, b\}$	$A(7)\colon \{b, c\}$	$A(10)\colon \{c, d\}$
$A(2)\colon \{a\}$	$A(5)\colon \{b\}$	$A(8)\colon \{c\}$	$A(11)\colon \{d\}$
$A(3)\colon \{a\}$	$A(6)\colon \{b\}$	$A(9)\colon \{c\}$	$A(12)\colon \{d\}$.

Let $k = 3$. The group $\{1, 2, 3, 4\}$ is 1-cohesive, as it has a commonly approved candidate (a) and is of size $\frac{12}{3} = 4$. If we want to give each voter in this group a representative, candidate a has to be in the winning committee (voters 2 and 3 only approve a). Now observe that also the groups $\{4, 5, 6, 7\}$, $\{7, 8, 9, 10\}$, and $\{10, 11, 12, 1\}$ are 1-cohesive. Thus, also candidates b, c, and d have to be in every winning committee. This is impossible as we are interested in committees of size 3. We see that it is impossible to satisfy *every* voter in 1-cohesive groups.

We see from this example that the requirement that each voter from an ℓ-cohesive group should have at least ℓ representatives in the elected committee is simply too strong.[6] However, it can be weakened a bit without losing much of its intuitive appeal. We start our discussion with *extended justified representation (EJR)* [3] and *proportionality degree* [4, 66, 68, 69].[7] The former concept is formulated as an axiom, the latter as a proportionality guarantee specified by a function.

Definition 4.3 (*Extended justified representation, EJR*) An ABC rule $\mathcal{R}$ satisfies *extended justified representation (EJR)* if for each election instance $E = (A, k)$, each winning committee $W \in \mathcal{R}(E)$, and each ℓ-cohesive group of voters V there exists a voter $i \in V$ with at least ℓ representatives in W, i.e., $|A(i) \cap W| \geq \ell$.

Example 4.4 Let us revisit Example 4.3. The committee $\{a, b, c\}$ satisfies the condition of EJR: every 1-cohesive group contains at least one voter with one representative in $\{a, b, c\}$. For example, for the 1-cohesive group $\{10, 11, 12, 1\}$, the voters 10 and 1 have a representative in the committee. Note that in this example actually all size-3 committees satisfy the EJR condition; also there are no ℓ-cohesive groups for $\ell \geq 2$.

Definition 4.4 (*Proportionality degree*) Fix a function $f : \mathbb{N} \to \mathbb{R}$. An ABC rule $\mathcal{R}$ has a *proportionality degree* of f if for each election instance $E = (A, k)$, each winning committee $W \in \mathcal{R}(E)$, and each ℓ-cohesive group of voters V, the average number of representatives that voters from V get in W is at least $f(\ell)$, i.e.,

$$\frac{1}{|V|} \cdot \sum_{i \in V} |A(i) \cap W| \geq f(\ell).$$

At first, it might appear that even for large cohesive groups, EJR gives a guarantee only to a single voter within this group. However, the EJR property applies to any group of agents: Let V be an ℓ-cohesive group. If we remove a voter with ℓ representatives (who, by EJR, is guaranteed to exist), the resulting group will be at least $(\ell - 1)$-cohesive. Consequently, in such a group there must exist a voter with at least $\ell - 1$ representatives, etc. As a consequence of this argument, EJR implies a proportionality degree of at least $f_{\mathcal{R}}(\ell) = \frac{\ell-1}{2}$ [66]. The other direction does not hold: even an ABC rule with a proportionality degree of $f_{\mathcal{R}}(\ell) = \ell - 1$ may fail EJR (cf. Proposition A.8).

Example 4.3 also shows that there exists no rule with a proportionality degree of $f(\ell) = \ell$:

Example 4.5 Consider again the profile of Example 4.3. Assume, there exists a rule $\mathcal{R}$ with a proportionality degree of $f_{\mathcal{R}}(\ell) = \ell$ and let $k = 3$. The group $\{1, 2, 3, 4\}$ is 1-cohesive, so in order to ensure that these voters get on average one

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[6] In a very recent work, Brill et al. [16] explore this intuitive (but unachievable) requirement—called individual representation—in much more depth. In particular, they show that all ABC rules presented in this book sometimes fail individual representation even for elections where such a committee exists. In addition, they study conditions under which individual representation can be satisfied.

[7] The concept of proportionality degree was initially referred to as *average satisfaction of ℓ-cohesive groups* [4, 66]. Skowron et al. [69] called an almost equivalent property κ-group representation.

representative, candidate a must be selected. By applying the same reasoning to $\{4, 5, 6, 7\}$ we infer that b must be selected. Analogously, we conclude that c and d must be selected. However, there are only three seats in the committee, a contradiction.

Aziz et al. [4] generalise the above example and prove that there exists no rule with a proportionality degree of $f(\ell) = \ell - 1 + \epsilon$ for $\epsilon > 0$. PAV matches this bound, and thus has an optimal proportionality degree. Below we include the proof of this result, since a similar idea is often used in the analysis of proportionality properties of Thiele methods.

Theorem 4.3 (Aziz et al. [3, 4]) *PAV has a proportionality degree of* $\ell - 1$. *It also satisfies EJR.*

Proof Consider an election $E = (A, k)$ and let W be a winning committee according to PAV. Let N and C denote the sets of voters and candidates in E, respectively. We will show that for each ℓ-cohesive group of voters V it holds that $\frac{1}{|V|} \cdot \sum_{i \in V} |A(i) \cap W| > \ell - 1$. This proves that PAV has the proportionality degree of $\ell - 1$. We can further conclude that there exists a voter $i \in V$ with $|A(i) \cap W| > \ell - 1$, and hence PAV also satisfies EJR.

Towards a contradiction assume there exists an ℓ-cohesive group of voters V with $\frac{1}{|V|} \cdot \sum_{i \in V} |A(i) \cap W| \leq \ell - 1$. We will show that there exists a pair of candidates, $c \in W$ and $c' \notin W$, such that $\mathrm{score}_{\mathrm{PAV}}(A, (W \cup \{c'\}) \setminus \{c\}) > \mathrm{score}_{\mathrm{PAV}}(A, W)$. This would indicate that we can replace one member of W with another not-selected candidate so that the new winning committee has a higher PAV-score than W. This would contradict the fact that W is a winning committee.

For convenience, for a set of candidates X and a candidate y we will use the notation:

$$\Delta(X, y) = \mathrm{score}_{\mathrm{PAV}}(X \cup \{y\}) - \mathrm{score}_{\mathrm{PAV}}(X),$$

i.e., $\Delta(X, y)$ is the marginal contribution of y given X.

Since $\frac{1}{|V|} \cdot \sum_{i \in V} |A(i) \cap W| \leq \ell - 1$ and V is ℓ-cohesive, there exists a not-selected candidate $c' \in C$ that is approved by all the voters from V. If we add this candidate to the committee W, the PAV-score will increase by:

$$\Delta(W, c') = \sum_{i \in N(c')} \frac{1}{|A(i) \cap W| + 1} \geq \sum_{i \in V} \frac{1}{|A(i) \cap W| + 1}.$$

From the inequality between the arithmetic and harmonic means we further get that:

$$\Delta(W, c') \geq \frac{|V|^2}{\sum_{i \in V} (|A(i) \cap W| + 1)} \geq \frac{|V|^2}{|V|(\ell - 1) + |V|} = \frac{|V|}{\ell} \geq \frac{n}{k}.$$

The last inequality follows from ℓ-cohesiveness.

Now, consider a committee $W' = W \cup \{c'\}$, and observe that

$$\sum_{c \in W'} \Delta(W' \setminus \{c\}, c) = \sum_{c \in W'} \sum_{i \in N(c)} \frac{1}{|A(i) \cap W'|} = \sum_{i \in N} \sum_{c \in A(i) \cap W'} \frac{1}{|A(i) \cap W'|}$$
$$= \sum_{i \in N:\ A(i) \cap W' \neq \emptyset} |A(i) \cap W'| \cdot \frac{1}{|A(i) \cap W'|} \leq n.$$

As a result, there exists $c \in W'$ such that $\Delta(W' \setminus \{c\}, c) \leq \frac{n}{k+1}$. Consequently:

$$\text{score}_{\text{PAV}}(A, (W \cup \{c'\}) \setminus \{c\}) = \text{score}_{\text{PAV}}(A, W) + \Delta(W, c') - \Delta(W' \setminus c, c)$$
$$\geq \text{score}_{\text{PAV}}(A, W) + \frac{n}{k} - \frac{n}{k+1} > \text{score}_{\text{PAV}}(A, W).$$

This yields a contradiction and completes the proof. □

In contrast to PAV, the two sequential variants of PAV, seq-PAV and rev-seq-PAV, do not satisfy EJR. However, the proportionality guarantees of Theorem 4.3 also hold for a local-search variant of PAV [4], which—in contrast to PAV itself—runs in polynomial time. Thus, EJR and a proportionality degree of $\ell - 1$ are achievable in polynomial time. Aziz et al. [4] also construct a second polynomial-time computable (but rather involved) rule that satisfies EJR. More recently, Peters and Skowron [56] prove that the Method of Equal Shares, which is also computable in polynomial time, satisfies EJR. Among the rules introduced in Chap. 2, PAV and the Method of Equal Shares are the only ones that satisfy EJR. An overview of the proportionality degree of rules can be found in Table 4.1.

Let us now consider two properties that are weaker than EJR.

Definition 4.5 (*Proportional justified representation, PJR* [66]) An ABC rule $\mathcal{R}$ satisfies *proportional justified representation (PJR)* if for each election $E = (A, k)$, each winning committee $W \in \mathcal{R}(E)$, and each ℓ-cohesive group of voters V it holds that $\left|W \cap \left(\bigcup_{i \in V} A(i)\right)\right| \geq \ell$.

Definition 4.6 (*Justified representation, JR* [3]) An ABC rule $\mathcal{R}$ satisfies *justified representation (JR)* if for each election $E = (A, k)$, each $W \in \mathcal{R}(E)$, and each 1-cohesive group of voters V there exists a voter $i \in V$ who is represented by at least one member of W, i.e., $|W \cap A(i)| \geq 1$.

PJR and JR are much weaker properties than EJR; in particular EJR implies PJR, which in turn implies JR. Example 4.6, below, illustrates that the stronger of the two axioms, PJR, can be satisfied even by rules that could be considered very bad from the perspective of proportionality degree (and, thus, also from the perspective of approximating EJR). On the other hand, there exist rules with good proportionality degree that do not satisfy even JR—this happens, e.g., when a rule does not provide sufficient guarantees for 1-cohesive groups (although it might satisfy EJR for $\ell \geq 2$). Generally, justified representation cannot be viewed as a proportionality axiom as it grants even large group only a single representative in the selected committee. In contrast, PJR can be viewed as a moderate proportionality requirement, significantly weaker than EJR but stronger than, e.g., lower quota on party-list profile. We refer to Table 4.1 for an overview which rules satisfy JR and PJR.

Example 4.6 Fix k and consider the following instance:

<table>
<tr><td colspan="5" align="center">c_{2k}</td></tr>
<tr><td colspan="5" align="center">$\dots$</td></tr>
<tr><td colspan="5" align="center">c_{k+2}</td></tr>
<tr><td colspan="5" align="center">c_{k+1}</td></tr>
<tr><td>c_1</td><td>c_2</td><td>c_3</td><td>$\dots$</td><td>c_k</td></tr>
<tr><td>V_1</td><td>V_2</td><td>V_3</td><td>$\dots$</td><td>V_k</td></tr>
</table>

There are $2k$ candidates. The voters can be divided into k equal-size groups so that the voters from the ith group, in the diagram denoted as V_i, approve c_i and $\{c_{k+1}, \dots, c_{2k}\}$. Committee $\{c_1, \dots, c_k\}$ (marked blue) satisfies PJR, but clearly, $\{c_{k+1}, \dots, c_{2k}\}$ (marked green) is a much better choice from the perspective of proportionality degree. Also, $\{c_{k+1}, \dots, c_{2k}\}$ satisfies the EJR condition while $\{c_1, \dots, c_k\}$ does not. This example shows that PJR implies no better proportionality degree than $f(\ell) = 1$.

Given that there are rather few rules satisfying EJR, Bredereck et al. [11] performed computer simulations for several distributions of voters' preferences and verified how hard it is *on average* to find a committee that satisfies the condition imposed by EJR. They concluded that ℓ-cohesive groups for $\ell \geq 2$ are quite rare, and that a random committee among those that satisfy the much weaker condition of JR is quite likely to satisfy EJR as well. Their second conclusion was that JR, PJR, and EJR, while highly desired, do not guarantee on their own a sensible selection of committees, and one needs to put forward additional criteria. Specifically, they showed that there are often many committees satisfying these conditions, and these committees may vary significantly. Bredereck et al. [11] derived their conclusions from the analysis of specific distributions of voters' preferences; it would be desirable to analyse this phenomenon more broadly, e.g., for other types of distributions.

Recently, Peters et al. [57] introduced an even stronger axiom, called *fully justified representation (FJR)*, where the precondition of ℓ-cohesiveness is relaxed. In EJR we say that a group of voters V is ℓ-cohesive if $|V| \geq \ell \cdot {}^n\!/_k$ and if there exists a set T of ℓ candidates such that each voter from V approves all ℓ candidates from T. In the definition of FJR, on the other hand, we only require that there must exist an integer β such that each voter from V approves at least β candidates from T. FJR enforces that at least one member of V must have at least β representatives in the elected committee. Note that EJR corresponds to FJR with a fixed value of $\beta = \ell$.

Definition 4.7 (*Fully justified representation* [57]) Given an integer value β and a subset of candidates $T \subseteq C$, we say that a group of voters V is weakly (β, T)-cohesive if $|V| \geq |T| \cdot n/k$ and if for each voter $i \in V$ it holds that $|A(i) \cap T| \geq \beta$. An ABC rule $\mathcal{R}$ satisfies *fully justified representation (FJR)* if for each election $E = (A, k)$, each winning committee $W \in \mathcal{R}(E)$, each integer β and $T \subseteq C$, and each weakly (β, T)-cohesive group of voters V, there exists a voter $i \in V$ such that $|W \cap A(i)| \geq \beta$.

For the time being, the only known rule that satisfies FJR is rather artificial and specifically tailored to the definition of the axiom [57]. It is an open question whether there exists a natural ABC rule which satisfies FJR together with other desirable properties.

All proportionality concepts discussed in this section ensure that cohesive groups are guaranteed a certain representation in the elected committee. Cevallos and Stewart [19] argue that in some contexts—for example when using ABC rules for selecting validators in the blockchain protocol—it is equally important to ensure that groups are not over-represented. To the best of our knowledge formal axioms capturing this intuitive requirement are still missing.[8]

To sum up, when considering proportionality axioms based on cohesive groups, PAV stands out as the most proportional rule. The Method of Equal Shares comes at a close second (its proportionality degree is lower) but it is computable in polynomial time. If we desire a committee monotone rule, then seq-Phragmén is a very good choice: it has a proportionality degree of $f_{\text{Phrag}}(\ell) = \frac{\ell-1}{2}$ [68], i.e., the proportionality degree that is implied by EJR, and satisfies PJR [12]. Also seq-PAV is a good choice: for reasonable sizes of committees seq-PAV has a better proportionality degree than seq-Phragmén; on the other hand, it satisfies neither PJR nor JR.

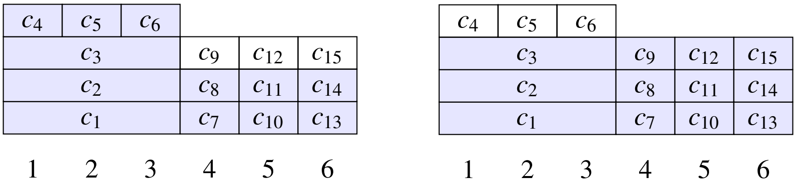
4.3 Laminar Proportionality and Priceability

The properties that we discussed in Sect. 4.2 (extended justified representation and the proportionality degree) and the axiomatic characterisation given in Theorem 4.2 all indicate that PAV provides particularly strong proportionality guarantees. Specifically, one could interpret these results as suggesting that PAV is a better rule—in terms of proportionality—than Phragmén's sequential rule and the Method of Equal Shares. However, drawing such a conclusion based on the so-far presented results would be too early. In the following we explain that proportionality can be understood in at least two different ways and that the axioms we discussed so far capture and formalise only one specific form of proportionality. We explain that Phragmén's sequential rule and the Method of Equal Shares provide very strong proportionality guarantees, but with respect to an interpretation of proportionality that is not captured by properties based on cohesive groups, and which is—to some extent—incomparable with the type of proportionality guaranteed by PAV.

[8] We note that the upper quota axiom in the apportionment setting can be viewed as such an axiom.

Let us start by illustrating the difference in how PAV and Phragmén's sequential rule (and the Method of Equal Shares) operate with the following example.

Example 4.7 ([56]) There are 15 candidates and 6 voters—the voters' approval sets are depicted in the diagram below. The committee shaded in blue in the left-hand side picture is the one that is selected by the Phragmén's sequential rule and by the Method of Equal Shares. The committee shaded in the right-hand side picture is chosen by PAV.



(a) seq-Phragmén and Equal Shares (b) PAV

The approval sets of voters 1, 2, and 3 are disjoint from those of voters 4, 5, and 6. It seems intuitive that the first three voters, who together form half of the society, should be able to decide about half of the elected candidates. Phragmén's sequential rule and the Method of Equal Shares select committees where the first three voters approve in total half of the members, thus the behaviour of these rules is consistent with the aforementioned understanding of proportionality. PAV follows a different principle: In the committee depicted in (a), each of the first three voters approves 4 candidates; each of the remaining three voters approves only 2 committee members. PAV notices that this is the case, and tries to reduce the societal inequality of voters' satisfaction by removing one representative of voter 1 and adding one to 4; similarly, PAV considers that it is more fair to remove the representatives of 2 and 3, and add the candidates liked by 5 and 6. On the one hand, PAV prefers to pick a committee that minimises the societal inequality in the voters' satisfactions (measured as the number of approved committee members). On the other hand, it punishes voters 1, 2, and 3 for being agreeable and "easy to satisfy" with fewer committee members—PAV allows them to decide only about one quarter of the committee.

Example 4.7 illustrates that PAV and Phragmén's sequential rule (and the Method of Equal Shares) follow two different types of proportionality. PAV implements a *welfarist* type of proportionality which is primarily concerned with the welfare (satisfaction) of the voters. This type of proportionality is captured, e.g., by the properties discussed in Sect. 4.2. PAV also satisfies the Pigou–Dalton principle of transfers, which says that given an election (A, k) and two committees, W and W', which in total get the same numbers of approvals ($\text{score}_{\text{AV}}(A, W) = \text{score}_{\text{AV}}(A, W')$), the one which minimises the societal inequality should be preferred [56]. Phragmén's sequential rule and the Method of Equal Shares, on the other hand, implement proportionality with respect to power, which—informally speaking—says that a group consisting of an α fraction of voters should be given a voting power that enables to decide about an α fraction of the committee. In other words, the type of proportionality of Phragmén-like rules is not mainly concerned with the welfare of groups but with the justification of welfare, achieved by endowing each voter with the same amount of virtual budget that represents the voting power.

Peters and Skowron [56] discuss two properties—laminar proportionality and priceability—which aim at formally capturing the high-level idea of proportionality with respect to power.[9] The first of the two properties—*laminar proportionality*—is very similar in spirit to proportionality on party-list profiles. The corresponding axiom identifies a class of well-structured election instances—called *laminar elections*—and specifies how a laminar proportional rule should behave on these profiles. Laminar profiles are more general than party-list profiles and are defined by a recursive structure, similar to the election from Example 4.7.

The second property, which we will discuss in more detail, is *priceability*. Intuitively, we say that a committee W is priceable if we can endow each voter with the same fixed budget and if for each voter there exists a payment function such that: (1) voters do not spend more than their allotted budget, (2) voters pay only for the candidates they approve, (3) each elected candidate gets a total payment of 1; candidates that are not elected receive no payments, and (4) there is no group of voters who approve a non-elected candidate, and who in total have more than one unit of unspent budget. Priceability is a notion of proportionality as it distributes power to groups of sufficient size; a large enough group receives enough collective budget to afford one or more candidates in the committee.

Formally, we obtain the following definition:

Definition 4.8 (*Priceability*) Given an election instance (A, k), a committee W is priceable if there exists a per-voter budget $p \in \mathbb{R}_+$ and $p_i : C \to [0, 1]$ for each voter $i \in N$ such that:

(1) $\sum_{c \in C} p_i(c) \le p$ for each $i \in N$,
(2) $p_i(c) = 0$ for each $i \in N$ and $c \notin A(i)$,
(3) $\sum_{i \in N} p_i(c) = \begin{cases} 1 & \text{if } c \in W, \\ 0 & \text{otherwise.} \end{cases}$
(4) $\sum_{i \in N(c)} \left(p - \sum_{c' \in W} p_i(c')\right) \le 1$ for each $c \notin W$.

An ABC rule is priceable if it returns only priceable committees.

Example 4.8 Consider the election instance from Example 4.7. The committees returned by Phragmén's sequential rule and by the Method of Equal Shares are priceable. For example, consider $W_1 = \{c_1, \dots, c_6, c_7, c_8, c_{10}, c_{11}, c_{13}, c_{14}\}$ (the committee shaded blue in the left figure in Example 4.7). This committee is priceable as witnessed by the following price system: the voters' budget is $p = 2$, and the payment functions are as follows (we only specify the non-zero payments): $p_1(c_i) = p_2(c_i) =$

[9] Laminar proportionality and priceability are similar in spirit but are logically independent (neither implies the other).

$p_3(c_i) = 1/3$ for $i \in \{1, 2, 3\}$ and $p_1(c_4) = p_2(c_5) = p_3(c_6) = p_4(c_7) = p_4(c_8) = p_5(c_{10}) = p_5(c_{11}) = p_6(c_{13}) = p_6(c_{14}) = 1$. Each voter fully spends their budget of 2.

On the other hand, the committee $W_2 = \{c_1, c_2, c_3, c_7, \dots, c_{15}\}$ returned by PAV (the one shaded blue in the right figure in Example 4.7) is not priceable. Indeed, if the voters' budget p were ≤ 2, then the voters 4, 5, 6 could not afford to pay for 9 candidates $c_7, \dots, c_{15}$. If $p > 2$, then some of the voters 1, 2, 3, say voter 1, would have a remaining budget of more than 1. Hence, this voter would have more budget than needed to buy a candidate outside of W_2 (e.g., c_4), which contradicts condition (4) in Definition 4.8.

Peters and Skowron [56] generalised Example 4.8 and showed that no welfarist rule (see Definition 2.1) is priceable. This shows that priceability is inherently not a welfarist concept. The same is true for laminar proportionality.

Theorem 4.4 (Peters and Skowron [56]) *Phragmén's sequential rule and the Method of Equal Shares are laminar proportional and priceable. No welfarist rule is laminar proportional nor priceable. No rule satisfying the Pigou–Dalton principle of transfers is laminar proportional nor priceable.*

While priceability is not a welfarist concept, it implies proportional justified representation. Further, all priceable rules must be equivalent to the D'Hondt method of apportionment on party-list profiles (cf. Theorem 4.1). A price system provides an explicit and easily verifiable evidence explaining that the voters can use their power (represented through virtual money) to ensure that the candidates from the committee are selected. This intuitively explains that priceability captures the idea of proportionality with respect to power—proportionality follows from the fact that each voter is initially endowed with the same amount of virtual money.

Priceability itself puts rather mild constraints on the payment functions $\{p_i\}_{i \in N}$. Recently, Peters et al. [58] introduced a stronger version of the axiom: we say that a price system $(p, \{p_i\}_{i \in N})$ is *stable* if it satisfies conditions (1)–(3) from Definition 4.8 and the following strengthening of condition (4):

(4*) Condition for Stability: There exists no non-empty group of voters $V \subseteq N$, no subset $W' \subseteq C \setminus W$, and no collections $\{p'_i\}_{i \in V}$ ($p'_i \colon W' \to [0, 1]$) and $\{R_i\}_{i \in V}$ (with $R_i \subseteq W$ for all $i \in V$) such that all the following conditions hold:

1. For each $c \in W'$: $\sum_{i \in V} p'_i(c) > 1$.
2. For each $i \in V$: $p_i(W \setminus R_i) + p'_i(W') \leq p$.
3. Each voter $i \in V$ approves more candidates in $W \setminus R_i \cup W'$ than in W, or i approves as many candidates in $W \setminus R_i \cup W'$ as in W but $\sum_{c \in W \setminus R_i} p_i(c) + \sum_{c \in W'} p'_i(c) < \sum_{c \in W} p_i(c)$.

In words, it should not be possible for the voters from V to propose a set of candidates W' such that if each voter $i \in V$ transferred her money from $R_i \subseteq W$ to the candidates from W', then these candidates would garner more than enough money to be elected, and each voter from $i \in V$ would be happier with $W \setminus R_i \cup W'$ than with W.

Stable priceability is a strong condition: stable-priceable committees do not always exist, and if so, they belong to the core (see Sect. 4.4). On the other hand, one can check in a polynomial time whether a committee is stable-priceable, and such committees often exist in practice. Peters et al. [58] also introduced the concept of *balanced stable-priceability*, which additionally requires that each two voters must pay the same amount of virtual money for the same candidate. They proved that balanced stable-priceable committees can be characterised as outputs of slightly modified version of the Method of Equal Shares.

We mention one more property—*perfect representation* [66]—which is loosely related to priceability. It also requires an explanation how voters can distribute their support/power in a way that justifies electing a committee; however, the axiom applies only in very specific situations.

Definition 4.9 (Sánchez-Fernández et al. [66]) We say that a committee W satisfies *perfect representation* if the set of voters can be divided into k equally-sized disjoint groups $N = N_1 \cup \ldots \cup N_k$ ($|N_i| = {}^{n}/_{k}$ for each $i \in k$) and if we can assign a distinct candidate from W to each of these groups in a way that for each $i \in k$ the voters from N_i all approve their assigned candidate. An ABC rule $\mathcal{R}$ satisfies perfect representation if $\mathcal{R}$ returns only committees satisfying perfect representation whenever such committees exist.

Perfect representation is incompatible with EJR [66] and with weak (and strong) Pareto efficiency (Proposition A.9), and it is not implied by (nor implies) priceability. Among the rules considered in this paper, only Monroe [66] and leximax-Phragmén [12] satisfy perfect representation, as does the variance-based rule by Phragmén mentioned in Theorem 4.1 [12].

To sum up, if we are mainly interested in the welfarist interpretation of proportionality, as captured by axioms that specify how cohesive groups of voters should be treated, then PAV is the best among the considered rules. Yet, sequential PAV, seq-Phragmén, and the Method of Equal Shares perform also reasonably well with respect to these criteria, and they are computable in polynomial time. Sequential PAV does not satisfy JR, and so it might discriminate small cohesive groups of voters. On the other hand, for reasonably small committees sequential PAV has better proportionality degree than seq-Phragmén, and the Method of Equal Shares. The axioms that well describe the welfarist type of proportionality are EJR and proportionality degree, and to a lesser extent PJR and JR. If we are interested in proportionality with respect to power, then we shall also consider the axioms of priceability and laminar proportionality. In this case the Method of Equal Shares and Phragmén's sequential rule are the two superior rules. It is not entirely clear which one of the two rules is better. On the one hand, the Method of Equal Shares satisfies the appealing axiom of EJR; on the other hand, Phragmén's sequential rule is committee monotone (see Sect. 3.3). In Table 4.1, we highlighted the three rules that—with the current state of knowledge—we consider the best ABC rules in terms of proportionality.

4.4 The Core

An important concept of group fairness that has been extensively studied in the context of ABC rules is the *core*. This notion of proportionality is adopted from cooperative game theory,[10] and was first introduced in the context of multi-winner voting by Aziz et al. [3].

Definition 4.10 Given an instance (A, k) we say that a committee W is in the *core* if for each non-empty $V \subseteq N$ and each $T \subseteq C$ with

$$\frac{|T|}{k} \leq \frac{|V|}{n}, \tag{4.1}$$

there exists a voter $i \in V$ such that $|A(i) \cap T| \leq |A(i) \cap W|$, i.e., voter i is at least as satisfied with W as with T. We say that an ABC rule $\mathcal{R}$ satisfies the core property if for each instance (A, k) each winning committee $W \in \mathcal{R}(A, k)$ is in the core.

Informally speaking, the core property requires that a group V constituting an α fraction of voters should be able to control an α fraction of the committee. If such a group can propose a set T of $\lfloor \alpha k \rfloor$ candidates such that each voter in V is more satisfied with the proposed set T than with the winning committee W, then the group V would have an incentive to deviate, hence would witness that committee W is not stable (and, in some sense, also not fair). If a winning committee is in the core, then no such deviation is possible.

The core property implies extended justified representation (Definition 4.3): Assume an ABC rule $\mathcal{R}$ satisfies the core property and consider an instance (A, k), a winning committee W, and an ℓ-cohesive group of voters V. Let T be the set of ℓ candidates that are approved by all the voters in V (such candidates exist because V is ℓ-cohesive). Since W is in the core, there must exist a voter $i \in V$ such that $|A(i) \cap W| \geq |A(i) \cap T| = \ell$, hence the condition of EJR must be satisfied. While the notion of core strictly generalises EJR and thus implies strong satisfaction guarantees for cohesive groups, it can also be viewed as a concept formalising the idea of proportionality with respect to power (cf. Sect. 4.3).

It is an important open question whether there exists an ABC rule that satisfies the core property, or—equivalently—whether the core is always non-empty. For the time being only partial answers to this intriguing question are known:

1. None of the rules mentioned in Chap. 2 satisfies the property. Since a rule satisfying the core must satisfy EJR, only PAV and the Method of Equal Shares come into consideration. However, counterexamples for both are known [3, 56]. For PAV, the instance from Example 4.7 shows a violation of the core. A simple example for the Method of Equal Shares can be found in [60, Example 4].
2. No welfarist rule (Definition 2.1) can satisfy the core property [56].

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[10] Specifically, the definition used in the literature on multi-winner voting is based on the definition of the core for cooperative games with non-transferable payoffs [20, 52].

3. If one restricts the attention to a special subclass of approval profiles, so-called *approval-based party-list profiles* as introduced by Brill et al. [15], the situation changes. Approval-based party-list profiles are approval profiles where each candidate appears with at least k copies, i.e., for every candidate c it holds that $\left|\{c' \in C : N(c) = N(c')\}\right| \geq k$. Approval-based party-list profiles are thus *more general* than party-list profiles (cf. Definition 4.1)—intuitively each voter can approve one or more parties. Brill et al. [15] prove that PAV satisfies the core property on approval-based party-list profiles. As mentioned before, PAV does not satisfy the core property in the general case.
4. It is known that the core can be empty in settings that are related to the ABC model but are more expressive. This is the case, e.g., in committee elections with ranking-based preferences [23, 60] and in participatory budgeting with additive utilities [30, Appendix C]; these two settings are discussed Sect. 6.1 and in Sect. 6.4, respectively.

As it remains unclear whether an ABC rule satisfying the core property is an achievable goal, several works in the most recent literature analysed relaxed notions of the core. We review these notions in the following.

4.4.1 Relaxation by Randomisation

The first type of relaxation that we consider is a probabilistic variant of the notion, i.e., the question becomes: "can core-like properties be guaranteed in expectation (ex-ante)?" Cheng et al. [23] prove that there always exists a lottery over committees that satisfies the core property in expectation. Let $\mathbb{E}_{X\sim\Delta}(X)$ denote the expected value of a random variable X distributed according to a lottery (probability distribution) Δ.

Theorem 4.5 (Cheng et al. [23]) *For each election instance (A, k) there exists a lottery over committees Δ such that for each group of candidates $T \subseteq C$ it holds that*

$$\frac{|T|}{k} > \frac{\mathbb{E}_{W\sim\Delta}\left(N(T, W)\right)}{n}, \tag{4.2}$$

where $N(T, W)$ is the set of voters who prefer T over W:

$$N(T, W) = \{i \in N : |A(i) \cap T| > |A(i) \cap W|\}.$$

Note that Eq. (4.2) is indeed a negated, probabilistic version of Eq. (4.1), showing that in expectation there are too few voters to propose a different committee. While it is not known whether such a lottery Δ can be found in a polynomial time, Cheng et al. [23] prove that if we restrict our attention only to sets T of size bounded by a constant, then for each $\epsilon > 0$ there is a polynomial-time algorithm that computes Δ such that $(1+\epsilon) \cdot \frac{|T|}{k} > \frac{\mathbb{E}_{W\sim\Delta}(N(T,W))}{n}$.

4.4.2 Relaxation by Deterministic Approximation

Another approach is to ask whether the core property can be well approximated. A few notions of approximation have been proposed; Definition 4.11 below unifies the definitions considered in the literature.

Definition 4.11 We say that an ABC rule $\mathcal{R}$ provides a γ-multiplicative-η-additive-satisfaction β-group-size approximation to the core if for each instance (A, k), each winning committee $W \in \mathcal{R}(A, k)$, each non-empty subset of voters $V \subseteq N$, and each subset of candidates $T \subseteq C$ with

$$\beta \cdot \frac{|T|}{k} \leq \frac{|V|}{n}$$

there exists a voter $i \in V$ such that $|A(i) \cap T| \leq \gamma \cdot |A(i) \cap W| + \eta$.

There are two components in Definition 4.11: The satisfaction-approximation component says that a voter i has an incentive to deviate towards T only if her gain in satisfaction is sufficiently large, that is, if i's satisfaction in T is greater at least by a multiplicative factor of γ *and* an additive factor of η than her satisfaction in W. The group-size-approximation component prohibits deviations towards sets T which are (by a multiplicative factor of β) smaller than $k \cdot \frac{|V|}{n}$, as imposed by the core. If $\gamma = 1$, then we omit the term "γ-multiplicative" from the name of the property. Similarly, if $\eta = 0$ we omit the term "η-additive", and if $\beta = 1$, then we omit the term "β-group-size". The satisfaction-approximation and the group-size approximation are incomparable.

When considering the problem of approximating the core, we distinguish two classes of algorithms. The first class contains dedicated approximation algorithms, which are mostly based on dependent rounding of fractional committees. The second class consists of established rules, such as PAV or the Method of Equal Shares, which can be shown to approximate the core (to some degree).

Jiang et al. [38] present an algorithm that provides 32-group-size approximation to the core. Their approach is based on dependent rounding of lotteries that are in expectation in the core (the existence of such lotteries is guaranteed by Theorem 4.5). Notably, the approach of Jiang et al. [38] extends much beyond the approval-based preferences; for cardinal utilities they round a lottery that in expectation 2-approximates the core and obtain a discrete committee with the 32-group-size approximation guarantee.

Fain et al. [30] present a family of algorithms based on dependent rounding of fractional committees (returned by a linear program that closely resembles the formulation of PAV as an integer linear program). For each $\lambda \in (1, 2]$ they provide an algorithm that guarantees a λ-multiplicative-η-additive-satisfaction $\frac{1}{2-\lambda}$-group-size approximation to the core, where $\eta = O\left(\frac{1}{\lambda^4} \log\left(\frac{k}{\lambda}\right)\right)$. Their algorithm naturally extends to a more general model related to participatory budgeting.

The result of Fain et al. [30] has recently been improved. Munagala et al. [51] presented a polynomial time algorithm that guarantees 67.37-multiplicative-1-additive-satisfaction approximation to the core. They also presented an algorithm that offers a 9.27-multiplicative-1-additive-satisfaction approximation to the core, yet running in exponential time. These algorithms, which are based on dependent rounding, can be also applied to more general types of voters' preferences.

For commonly known rules the following results are known: Cheng et al. [23] prove that PAV does not guarantee β-group-size approximation to the core even for $\beta = \Theta(\sqrt{k})$. On the other hand, Peters and Skowron [56] prove that PAV gives 2-multiplicative-satisfaction approximation to the core. Further, for each $\epsilon > 0$ no rule that satisfies the Pigou–Dalton principle can provide a $(2 - \epsilon)$-multiplicative-satisfaction approximation to the core. Thus, PAV can be viewed as giving the strongest multiplicative-satisfaction approximation to the core subject to satisfying the Pigou–Dalton principle of transfers. Finally, they show that the Method of Equal Shares provides $O(\log(k))$-multiplicative-1-additive-satisfaction approximation to the core.

4.4.3 Relaxation By Constraining the Space of Deviations

Yet another approach to relaxing the core property is to prohibit only certain types of deviations. As we have already explained at the beginning of this section, EJR can be viewed as a restricted variant of the core property: It prohibits the deviations of groups of voters towards outcomes T on which the deviating voters unanimously agree. Intuitively, if a group V agrees on all candidates from T, then it is easier for such a group to synchronise and to deviate, thus EJR can be viewed as the minimal restricted variant of the core. Motivated by the same arguments, Peters and Skowron [56] considered other restricted variants of the core property.

A *committee property* is a set of triples (A', k', W'), where (A', k') is an election instance and W' is a size-k' committee. We write $A|_V$ for profile A restricted to voters in $V \subseteq N$.

Definition 4.12 (Peters and Skowron [56]) Let $\mathcal{P}$ be a committee property. Given an instance (A, k), we say that a pair (V, T), with $V \neq \emptyset$, $V \subseteq N$, $T \subseteq C$, is an *allowed deviation* from a committee W if (1) $\frac{|T|}{k} \leq \frac{|V|}{n}$, (2) $|A(i) \cap T| > |A(i) \cap W|$ for each $i \in V$, and (3) T has property $\mathcal{P}$, i.e., $(A|_V, |T|, T) \in \mathcal{P}$. An ABC rule $\mathcal{R}$ satisfies the *core subject to* $\mathcal{P}$ if for each instance (A, k) and each winning committee $W \in \mathcal{R}(A, k)$ there exists no allowed deviation.

For example, let $\mathcal{P}_{\text{coh}}$ be a committee property such that $(A', k', W') \in \mathcal{P}_{\text{coh}}$ if and only if $W' \subseteq A'(i)$ for all voters i in the domain of A'; we call $\mathcal{P}_{\text{coh}}$ cohesiveness (cf. Definition 4.2). Then, EJR can be equivalently defined as the core subject to cohesiveness.

The Method of Equal Shares satisfies core subject to priceability with equal payments, which is a variant of priceability that additionally requires that voters must pay the same amount of virtual budget for the same candidate (cf. Definition 4.8);

priceability with equal payments is thus stronger than priceability, yet weaker than cohesiveness [56]. It is an open question whether the core subject to weaker (yet still natural) types of constraints is always non-empty.

4.5 Degressive and Regressive Proportionality

The notions of proportionality that we discussed in Sects. 4.1–4.4 aimed at capturing the following intuitive idea: An α fraction of voters should be able to decide about an α fraction of the committee—in this approach the relation between the size of the group and its eligibility is linear. In this section we discuss two alternative concepts: degressive and regressive proportionality. These two concepts should be viewed more as high-level ideas than formal properties. We first explain them intuitively, providing an illustrative example, and next we will discuss a few formal approaches to reasoning about degressive and regressive proportionality.

According to *degressive* proportionality, smaller groups of voters should be favoured, i.e., be eligible to more representatives in the elected committee than in the case of linear proportionality.[11] An extreme form of degressive proportionality is *diversity* [32]—there, if possible, each voter should be represented by at least one candidate in the elected committee. At the other end is the idea of *regressive* proportionality, where the emphasis is put on well-representing large groups. An extreme form of regressive proportionality is *individual excellence* [32], where it is assumed that only the candidates with the highest total support from the voters should be elected. In fact, these two notions—diversity and individual excellence—are extreme to the extent that they can no longer be considered notions of proportionality. Example 4.9, below, illustrates the ideas of degressive and regressive proportionality, and the two extreme variants of them—diversity and individual excellence.

Example 4.9 Consider the approval-based preference profile from Example 4.1:

$$\begin{aligned} &60 \text{ voters}: && \{a_1, \ldots, a_{10}\} && 20 \text{ voters}: && \{b_1, \ldots, b_6\} && 10 \text{ voters}: && \{c_1, c_2\} \\ &8 \text{ voters}: && \{d_1, \ldots, a_4\} && 2 \text{ voters}: && \{e_1, e_2, e_3\}. \end{aligned}$$

A linearly-proportional committee W_1 could consist of six candidates from A, two candidates from B, one candidate from C, and one candidate from D (this is the committee selected by the Sainte-Laguë apportionment method). Another linearly-proportional committee could consist of seven candidates candidates from A, two from B, one from C, but none from D (this is the committee selected by the D'Hondt apportionment method).

[11] Degressive proportional apportionment is often used for distributing parliamentary seats among geographical regions, e.g., in the division of the European Parliament seats among EU countries (see the book of Rose [64] for a discussion of arguments and negotiations that resulted in a degressive apportionment rule being used for assembling the European Parliament).

Table 4.2 Flavors of (dis)proportionality

# Votes	60	20	10	8	2
Example of linear proportionality (Sainte-Laguë)	6	2	1	1	0
A different example of linear proportionality (D'Hondt)	7	2	1	0	0
An example of degressive proportionality	4	3	2	1	0
Another example of degressive proportionality	3	3	3	1	0
An example of diversity	4	3	1	1	1
Another example of diversity	2	2	2	2	2
An example of regressive proportionality	8	2	0	0	0
Individual excellence	10	0	0	0	0

In contrast, a degressive-proportional committee W_2 could, for example, consist of four candidates from A, three candidates from B, two candidates from C, and one candidate from D. Another example of a degressive-proportional committee would be W_3 with three candidates from each of the sets A, B, and C, and one from D. Committees W_2 and W_3, however, are not diverse, since two voters who support $E = \{e_1, e_2, e_3\}$ are not represented at all. A diverse committee could consist of, e.g., four candidates from A, three candidates from B, one candidates from C, one candidate from D, and one candidate from E. A regressive-proportional committee would include more candidates from the set $A = \{a_1, \dots, a_{10}\}$ at the cost of groups supported by less voters. For example, a committee that consists of eight candidates from A and two candidates from B would be regressive-proportional. Table 4.2 shows the example relations between a size of a group and its number of representatives for different forms of proportionality:

The arguments in favour of degressive proportionality usually come from the analysis of probabilistic models describing how the decisions made by the elected committee map to the satisfaction of individual voters participating in the process of electing the committee (for party-list preferences, an excellent exposition is given by Koriyama et al. [41]; see also [47, 48]). An interesting concrete example of degressive proportionality is square-root proportionality devised by Penrose [53] (see also [70]), where the idea is that the groups of voters should be represented proportionally to the square-roots of their sizes.[12] Further, degressive proportionally in general, and diversity in particular, are particularly appealing ideas in the context of deliberative democracy—there, the goal is to select a committee that should discuss and deliberate on various issues rather than make majoritarian decisions. It is argued that for deliberative democracy it is particularly important to represent as many

[12] This method has been proposed for the United Nations Parliamentary Assembly [17] and for allocating voting weights in the Council of the European Union [71].

various opinions in the committees as possible [21, 50], which can be achieved by maximising the number of voters who are represented in the elected committee.

On the other hand, the idea of regressive proportionality is particularly appealing when the goal is to select a committee of candidates based on their individual merits, for example when the goal of an election is to select finalists in a contest or to choose a set of grants that should be funded (then, the voters act as judges/experts).

In the remaining part of this section we discuss two approaches to formalising the ideas of degressive and regressive proportionality: axiomatic approaches and a quantitative approach.

4.5.1 Axiomatic Approaches to Diversity and Individual Excellence

The axiomatic approach generally applies only to the extreme forms of the degressive and regressive proportionality, i.e., to diversity and individual excellence, respectively. This approach is similar to the one we discussed in Sect. 4.1: by formalising the concepts of diversity and individual excellence on party-list profiles (Definition 4.1), we obtain axiomatic characterisations for the more general domain of ABC rules.

Intuitively, disjoint diversity requires that in party-list profiles as many voters as possible have at least one representative in the elected committee. Disjoint equality says that each approval carries the same strength, and so all candidates that are approved once have the same right for being elected.

Definition 4.13 (*Disjoint diversity*) An ABC rule $\mathcal{R}$ satisfies *disjoint diversity* if for each party-list instance (A, k) with voter sets $(N_1, \ldots, N_p)$ and $|N_1| \geq |N_2| \geq \ldots \geq |N_p|$, there exists a winning committee $W \in \mathcal{R}(A, k)$ that contains one candidate for each of the k largest parties, i.e., for each $r \leq \min(p, k)$ and each $i \in N_r$ we have that $A(i) \cap W \neq \emptyset$.

Definition 4.14 (*Disjoint equality*) An ABC rule $\mathcal{R}$ satisfies *disjoint equality* if for each election instance (A, k) where each candidate is approved at most once and the number of approved candidates is at least k (i.e., $|\bigcup_{i \in N} A(i)| \geq k$), a committee W is winning if and only if it contains only approved candidates, $W \subseteq \bigcup_{i \in N} A(i)$.

Intuitively, disjoint equality is aimed at capturing the idea of individual excellence—the candidates that are approved exactly once are virtually indistinguishable from the perspective of the support coming from the voters; thus all such candidates should have equal rights to be selected.

The following theorems show that, similarly to the case of D'Hondt proportionality (Theorem 4.2), the concepts of disjoint diversity and disjoint equality uniquely extend to the full domain of approval-based preferences if one assumes the natural axioms of anonymity, neutrality, and consistency (and a few more technical axioms).

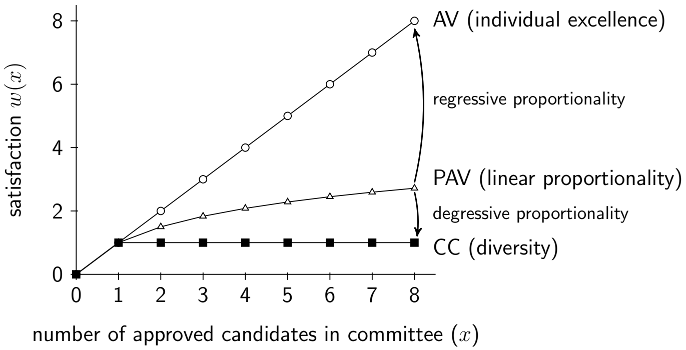

Fig. 4.2 A diagram illustrating the relation between defining w-functions of Thiele methods and the type of proportionality these Thiele rules implement

Theorem 4.6 (Lackner and Skowron [44]) *The Approval Chamberlin–Courant rule is the only non-trivial ABC ranking rule that satisfies anonymity, neutrality, consistency, weak efficiency, continuity, and disjoint diversity. Multi-Winner Approval Voting is the only ABC ranking rule that satisfies anonymity, neutrality, consistency, weak efficiency, continuity, and disjoint equality.*

Lackner and Skowron [44] provided a similar analysis for intermediate notions of degressive and regressive proportionality. They conclude that w-Thiele methods based on w-scoring functions that have a larger slope than the w-function of PAV are more oriented towards regressive proportionality, whereas w-functions that have a smaller slope are closer in spirit to the idea of degressive proportionality. This relation is symbolically visualised in Fig. 4.2.

Jaworski and Skowron [37] constructed a class of rules that generalise Phragmén's rule. Intuitively, a degressive variant of seq-Phragmén is obtained by assuming that the voters who already have more representatives earn money at a slower rate than those that have fewer. Regressive proportionality is implemented by assuming that the candidates who are approved by more voters cost less than those that garnered fewer approvals.

Faliszewski et al. [33] discuss three specific classes of rules that span the spectrum between individual excellence and diversity. They analyse these rules in the ranking-based model, that is when voters rank the candidates instead of approving some of them (see Sect. 6.1). These classes of rules can be analogously defined for approval ballots. Brill et al. [14], Faliszewski and Talmon [31] extend Monroe's rule so that it can implement the idea of regressive proportionality; this is also done in the ranking-based framework. It would be interesting to see whether their techniques can be successfully applied to the ABC model.

Finally, Subiza and Peris [72] propose an axiom called α-unanimity (parameterized with $\alpha \in [0, 1]$), which can be seen as a strong diversity axiom. The authors propose a voting rule (Lexiunanimous Approval Voting) that satisfies this axiom; this rule is a refined version of CC. Thiele methods (including CC itself) do not satisfy this axiom for any α.

4.5.2 Quantifying Degressive and Regressive Proportionality

The second approach to formally reason about degressive and regressive proportionality is quantitative in nature. Lackner and Skowron [43] define two measures—the utilitarian guarantee and the representation guarantee—that can be used to quantify how well a given rule performs in terms of individual excellence and diversity.

Recall that $\text{score}_{\text{AV}}(A, W)$ denotes the total number of approvals a given committee receives in profile A and $\text{score}_{\text{CC}}(A, W)$ denotes the number of voters who approve at least one member of W.

Definition 4.15 (*Utilitarian and Representation Guarantee* [43]) The utilitarian guarantee of an ABC rule $\mathcal{R}$ is a function $\kappa_{\text{AV}} \colon \mathbb{N} \to [0, 1]$ that takes as input an integer k, representing the committee size, and is defined as:

$$\kappa_{\text{AV}}(k) = \inf_{A} \frac{\min_{W \in \mathcal{R}(A,k)}(\text{score}_{\text{AV}}(A, W))}{\max_{W \colon |W|=k}(\text{score}_{\text{AV}}(A, W))}.$$

The representation guarantee of an ABC rule $\mathcal{R}$ is a function $\kappa_{\text{CC}} \colon \mathbb{N} \to [0, 1]$ defined as:

$$\kappa_{\text{CC}}(k) = \inf_{A} \frac{\min_{W \in \mathcal{R}(A,k)}(\text{score}_{\text{CC}}(A, W))}{\max_{W \colon |W|=k}(\text{score}_{\text{CC}}(A, W))}.$$

Note that the utilitarian and the representation guarantee of an ABC rule $\mathcal{R}$ measure how well rule $\mathcal{R}$ approximates Multi-Winner Approval Voting and the Approval Chamberlin–Courant rule, respectively. These two rules embody the principles of diversity and individual excellence (cf. Theorem 4.6).

Lackner and Skowron [43] show that the utilitarian guarantee of PAV, sequential PAV, and seq-Phragmén is $\Theta(1/\sqrt{k})$; their representation guarantee is $1/2 + \Theta(1/k)$. CC and seq-CC achieve a better representation guarantee (of 1 and $1 - 1/e$, respectively), but their utilitarian guarantee is only $\Theta(1/k)$. In that sense, these three proportional rules (PAV, sequential PAV, and seq-Phragmén) can be viewed as a desirable compromise between the two guarantees. On the other, the authors also show that proportional rules are never an *optimal* compromise. Finally, p-geometric rules—the Thiele rules defined by $w_{p\text{-geom}}(x) = \sum_{i=1}^{x} (1/p)^i$—for different values of the parameter p span the whole spectrum from AV to CC. By adjusting the parameter p, one can obtain any desired compromise between the utilitarian and representation goals.

Elkind et al. [29] extend this work by considering the "price" of justified representation axioms: what are the optimal utilitarian and representation guarantees when requiring justified representation (Definition 4.6) or extended justified representation (Definition 4.3)? Their results show that already justified representation implies a utilitarian guarantee of no better than $2/\sqrt{k}$; the same holds for EJR. The consequences for the representation guarantee are less pronounced: JR does not restrict the representation guarantee (e.g., CC satisfies JR and has a representation guarantee of 1) and EJR is compatible with a representation guarantee of $\frac{3}{4}$.

4.5.3 An Experimental View on Degressive and Regressive Proportionality

Godziszewski et al. [35] visualised the structure of the committees produced by various ABC rules on histograms. They performed computer simulations in which the candidates and the voters were represented as points in the two-dimensional Euclidean space. Intuitively, a point corresponding to a voter or a candidate might represent their position in the spectrum of possible opinions regarding various issues. In each simulation the candidates and the voters were drawn from a given distribution, and a preference profile was constructed from the positions of the voters and the candidates. The main idea was that the voters are more likely to approve candidates whose corresponding points are closer to them, since their opinions resemble views of such candidates. Given a preference profile, a specific ABC rule was used to find a winning committee, and the points corresponding to the selected candidates were marked with red dots on the histogram of the respective rule. The experiment was repeated multiple times, and each time the dots were put on the same histogram. Thus, the density of red dots in a given area represent the probabilities that the candidates from this area are chosen for the winning committee. This idea was first proposed by Elkind et al. [28] in the context of ranking-based elections.

Such histograms give valuable insights into the nature of voting rules. We depict several of them in Fig. 4.3. In the left column of the figure, we depict distributions of the points representing the voters and the candidates: red areas correspond to the candidates, green areas to the voters, and olive areas correspond to both. The subsequent columns depict distributions of the elected candidates for six ABC rules. These histograms already illustrate the very different natures of the considered rules. For example, the distributions obtained for PAV and the sequential Phragmén's rule closely resemble distributions of the voters (which is exactly what one would expect from proportional rules), CC puts more emphasis on representing as diverse a spectrum of voters as possible, AV selects candidates that are in the centres of the distributions—the choice that corresponds to individual excellence. The Method of Equal Shares induces histograms that are in some sense between PAV and AV. Finally, the behaviour of Minimax AV (MAV) is inconsistent with our intuitive interpretation of proportionality in the Euclidean model.

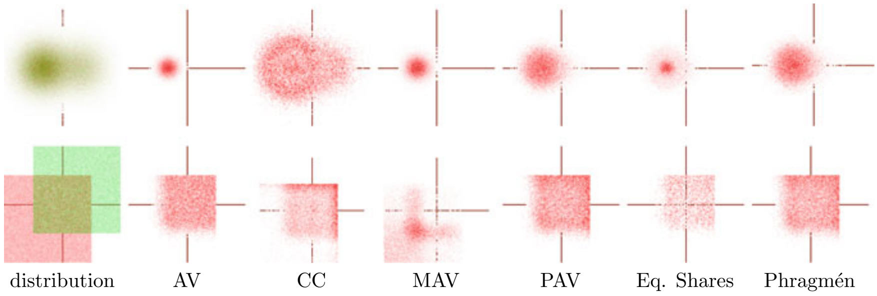

Fig. 4.3 Visualising the outcomes of some selected ABC rules (from [35])

The conclusions from this experimental exercise are to a large extent consistent with the conclusions coming from the axiomatic analysis. For a more detailed discussion we refer to the original work [35].

4.6 Proportionality and Strategic Voting

The ABC rules that we have considered in the context of proportionality are all prone to manipulations (cf. Sect. 3.6). In this section we explain that this is not a coincidence—achieving proportionality and strategyproofness at the same time is inherently impossible. This impossibility was first proven by Peters [54, 55] for resolute rules (rules that always return a single winning committee), even for very weak formulations of the desired axioms. (Earlier work by Aziz et al. [2] and Janson [36] already showed that certain proportional rules—such as PAV, seq-PAV, and seq-Phragmén—are not strategyproof.)

Theorem 4.7 (Peters [54, 55]) *Suppose $k \geq 3$, the number n of voters is divisible by k, and $m \geq k + 1$. Then there exists no resolute ABC rule $\mathcal{R}$ which satisfies the following three axioms:*

1. *weak proportionality: for each party-list election (A, k) where some singleton ballot $\{c\}$ appears at least n/k times ($|\{i : A(i) = \{c\}\}| \geq n/k$), candidate c must belong to the winning committee, i.e., $c \in \mathcal{R}(A, k)$,*
2. *weak efficiency: a candidate who is approved by no voter may not be part of the winning committee, unless fewer than k candidates receive at least one approval,*
3. *inclusion-strategyproofness[13] (as defined in Sect. 3.6).*

Kluiving et al. [40] prove a similar result for irresolute rules (i.e., when rules are allowed to output multiple tied winning committees), using cardinality-strategy-proofness and Pareto efficiency. Further, Duddy [27] proves a related impossibility

[13] This axiom can be further weakened to allow voters only to manipulate by reporting subsets of their true approval sets.

result for irresolute rules using slightly different axioms; this result also requires a form of Pareto efficiency.

Lackner and Skowron [42] showed that AV is the only ABC scoring rule (Sect. 3.5) that satisfies SD-strategyproofness; this result can also be seen as an impossibility result concerning proportionality and strategyproofness within the class of ABC scoring rules. Further, they quantified the trade-off between strategyproofness and proportionality. For various ABC rules they empirically measured their level of strategyproofness by assessing the fraction of profiles, for which there exists a voter who has an incentive to misreport her approval set. They concluded that rules which are more similar to AV (i.e., rules that follow the principle of regressive proportionality) are less manipulable than proportional rules. The rules that follow the principle of degressive proportionality are the most manipulable. A similar conclusion was obtained by Barrot et al. [6], but there the authors analysed a different class of rules—namely those based on the Hamming distance, and spanning the spectrum from AV to Minimax Approval Voting.

Since in the general case, there exist no proportional strategyproof ABC rule, Botan [9] restricted the analysis to three specific types of manipulations: (1) subset manipulations, where a voter can manipulate only by submitting a subset of her true approval set, (2) superset manipulations, where each voter can only send a superset of her true preferences, and (3) disjoint manipulations, where a manipulation can be performed only by submitting a subset of candidates disjoint from the true approval set of the voter. They showed that for party-list preference profiles (see Definition 4.1) all Thiele methods are cardinality-strategyproof[14] against subset, superset, and disjoint manipulations.

4.7 Proportionality with Respect to External Attributes

In Sects. 4.1–4.6, we have considered formal concepts that capture, in various ways, what it means that the structure of the elected committee proportionally reflects the (approval-based) preferences of the voters. In other words, we have considered proportionality with respect to the preferences given by the voters. In this section, we briefly consider a framework that approaches the concept of proportionality quite differently: we analyse proportionality with respect to external attributes of the candidates.[15]

Let us start by recalling the apportionment setting that we discussed in Sect. 4.1. In the apportionment model we are given a set of candidates, each candidate belonging to a single political party; for each political party we are given a desired fraction

[14] Formally, Botan [9] defines strategyproofness for irresolute rules and states their results for the general class of Gärdenfors preference extensions [34]. These extensions define preference relations over sets of winning committees and thus can be applied to irresolute rules.

[15] A noteworthy real-world example is the Lebanese Parliament, where an equal representation of Christians and Muslims (64 seats each) is mandated [26].

of seats the party should ideally get in the elected committee (typically, this is the fraction of votes cast on the party). The goal is to pick the committee that matches the desired fractions as closely as possible. Thus, one can say that in the apportionment setting there is one external attribute, which is the party affiliation, each candidate has a certain value of this attribute, and the goal is to pick the committee where the different values of the attribute are represented proportionally to the given desired fractions.

Now, assume that there are two attributes—each candidate has a political affiliation and a geographic region that she represents. For each value of each attribute we are given a desired fraction of seats that the candidates with this attribute value should get. This setting is called bi-apportionment, and it is discussed in detail in a book chapter by Pukelsheim [61] (several articles study the bi-apportionment setting from a computational perspective [46, 63, 67]). The model of bi-apportionment has been further extended to an arbitrary number of attributes by Lang and Skowron [45].[16] There, the authors analysed axiomatically and algorithmically two rules that extend the D'Hondt method and the largest remainder method to the multi-attribute apportionment.

The desired fractions in the (multi-attribute) apportionment model can be based on the voters preferences, or they might be given exogenously, e.g., by imposing certain quotas, specifying how many candidates with given attribute values should be included in the winning committee. Taking one specific interpretation, namely assuming the voters are asked to approve attribute values, Kagita et al. [39] proposed several other rules for selecting committees. They formulated axioms, requiring that the selected committee should consist of candidates whose attribute values proportionally represent voters' preferences. Unfortunately, none of the rules they propose satisfies any of these axioms. In general our axiomatic understanding of the multi-attribute apportionment model is still not well-advanced.

In the final part of this section we will consider a model which takes into account both the voters' preferences over candidates, and external constraints based on attributes of the candidates. Instead of defining this model formally, we provide an illustrative example.

Example 4.10 Assume we want to select a representative committee. Such a committee should be gender-balanced, containing 50% of male (M) and 50% of female (F) committee members. Additionally, the committee should represent people from different educational backgrounds: at least 25% and at most 50% of its members should have a bachelor's degree (B), between 40% and 60% should have an upper-secondary education (U), and between 10% and 25%—a primary or lower-secondary education (P). Finally, the selected committee should contain at least 25% young people (Y)

[16] The multi-attribute model finds its application, e.g., in the process of sortition. In sortition one needs to select a committee of ordinary people who will discuss certain controversial matters, and come up with recommendations helping the governments make decisions. In this process it is important to select a committee consisting of people who are representative for the whole society. Currently, randomised algorithms are mostly used for such selections [8]. The multi-attribute model provides alternative methods that take advantage of information regarding attributes of the potential committee members.

and at least 50% senior people (S). The pool of candidates from which we can select members of such a committee is given in the table below. Additionally, seven voters express their preferences via the following approval ballots.

Name	Gender	Education	Age
c_1	F	B	Y
c_2	M	U	Y
c_3	M	U	S
c_4	F	P	S
c_5	M	U	Y
c_6	M	U	Y
c_7	M	U	Y
c_8	F	B	S

$A(1) = \{c_1, c_2, c_3\}$

$A(2) = \{c_3, c_5\}$

$A(3) = \{c_7, c_8\}$

$A(4) = \{c_3, c_4, c_5, c_7\}$

$A(5) = \{c_1, c_8\}$

$A(6) = \{c_6\}$

$A(7) = \{c_1, c_2, c_6\}$

Assume we want to select $k = 4$ committee members. The winning committee according to AV would be $W_1 = \{c_1, c_3, c_7, c_8\}$ (for simplicity, we assume the ties are broken lexicographically $c_8 \succ c_7 \succ \cdots \succ c_1$), and according to PAV, the winning committee would be $W_2 = \{c_1, c_3, c_6, c_8\}$. However, each of these two committees violates the attribute-level constraints. The committee maximising the AV-score and the PAV-score subject to these constraints would be, respectively, $W_3 = \{c_1, c_3, c_4, c_7\}$ and $W_4 = \{c_3, c_4, c_6, c_8\}$.

As can be seen in Example 4.10, score-based ABC rules (in particular Thiele methods) are suitable for this approach: the winning committee is the one with the highest score that satisfies all external constraints. Following this approach, Bredereck et al. [10] and Celis et al. [18] considered the model of multi-winner elections with external constraints, but where the qualities of the committees are assessed via a general set function f. The function f may in particular depend on the voters' ballots, for example we can set $f(W) = \text{score}_{\text{AV}}(A, W)$. Aziz [1] studied a similar model, but assuming there is a global ranking over C that represents the objective qualities of the candidates. There, the goal is to select the lexicographically best committee subject to the multi-attribute constraints, which are treated more softly than in case of Bredereck et al. [10] and Celis et al. [18]. Let us also mention that Bei et al. [7] studied a related model, but there the goal is to select the committee of maximal cardinality that satisfies the attribute-level constraints. We will consider algorithmic aspects of these and related approaches in Sect. 5.3.

Note that this approach is not compatible with rules that do not naturally provide a ranking of committees by scores (e.g., seq-Phragmén or the Method of Equal Shares). It is an interesting question how to adapt these rules to the model with external constraints.

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