Chapter 8
Instability

8.1 Transition

The transition between laminar flow and turbulence is generated by instability mechanisms. Those mechanisms do not occur in the same operational mode in every geometrical configuration. The associated dynamics. i. e. the temporal behavior of the flow field may be sometimes smooth, sometimes brutal.

We will distinguish two types of transition. The first one, named spectral transition, characterizes the passage to turbulence by numerous bifurcations. The most exemplary flow of this situation is the circular Couette flow between two concentric cylinders. Suppose that the outer cylinder of radius R_2 is fixed, while the inner cylinder of radius R_1 rotates with an angular constant velocity ω. The Reynolds number can be defined by the relation

$$Re = \frac{\omega R_1 (R_2 - R_1)}{\nu} \, . \tag{8.1}$$

For small values of the Reynolds number, the flow is laminar and the Navier-Stokes equations allow to calculate the azimuthal velocity profile (3.41)

$$v_\theta = Ar + B/r \, , \tag{8.2}$$

with $A = -\omega R_1^2/(R_2^2 - R_1^2)$ and $B = \omega R_1^2 R_2^2/(R_2^2 - R_1^2)$. If the rotational speed ω is increased, for a critical Reynolds number, a new configuration containing Taylor vortices appears, cf. Fig. 1.1. On top of the fundamental flow (8.2) is superimposed a secondary flow in such a way that a fluid particle moves on a toroidal axisymmetric material surface. At higher Reynolds numbers, the torus is deformed in the azimuthal direction, cf. Fig. 1.2. For high enough Reynolds numbers, turbulence is developing.

The second type of transition is catastrophic transition where the passage from laminar state to turbulence occurs suddenly and instantaneously without a cascade of transitions. This is the case of developed Poiseuille flow in a cylindrical pipe. This

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instability was first observed by O. Reynolds [77] who gave his name to the eponym dimensionless number.

Fluid instability is a very rich and subtle subject. It influences all flows and the investigations for a full understanding of those complex phenomena involve both experiments, theoretical developments and today numerical simulations. Therefore the reader is facing the challenge of selecting his preferences and choices. The recent books by Tapan Sengupta [87, 88] constitute an updated review of those crucial subjects. As far as I am concerned I will concentrate on parallel flows and the circular Couette flow.

8.2 Orr-Sommerfeld Equation

The theory of hydrodynamic stability is a complicated subject from the mathematical point of view. We will tackle it in the framework of parallel flows of homogeneous fluids. This theory consists in analysing a base flow on which perturbations of small amplitude are superimposed. The non-linear term of the Navier-Stokes equations will generate an interaction with these perturbations. If they grow with respect to time, then the flow is unstable. If, on the contrary, the viscosity is sufficient to prevent those perturbations to grow, then the flow is stable.

Let us resume the dimensionless Navier-Stokes equations (2.47)–(2.48) where the body force is neglected and the accents are omitted to alleviate the notation. Let us consider the steady-state plane base flow satisfying the Navier-Stokes equations such that

$$\boldsymbol{v} = (V_1(x_1, x_2), V_2(x_1, x_2), 0) \tag{8.3}$$

and let us suppose that it is almost parallel to x_1 axis, namely

$$\mid V_2 \mid \ll \mid V_1 \mid , \quad \mid \partial V_1/\partial x_1 \mid \ll \mid \partial V_1/\partial x_2 \mid \ . \tag{8.4}$$

In order to enforce the flow stability criterion, three-dimensional velocity $\boldsymbol{v}'$ and pressure p' perturbations are superimposed on the fundamental flow. We have

$$\boldsymbol{v} = (V_1(x_1, x_2) + v_1'(\boldsymbol{x}, t), V_2(x_1, x_2) + v_2'(\boldsymbol{x}, t), v_3'(\boldsymbol{x}, t)), \tag{8.5}$$

$$p = P + p' \ . \tag{8.6}$$

Inserting (8.5)–(8.6) in (2.47)–(2.48) and neglecting the quadratic perturbations terms, one obtains taking (8.4) into account the linearized equations

$$\nabla \cdot \boldsymbol{v}' = 0 \ , \tag{8.7}$$

$$\frac{\partial \boldsymbol{v}'}{\partial t} + V_1 \frac{\partial \boldsymbol{v}'}{\partial x_1} + v_2' \frac{\partial V_1}{\partial x_2} \boldsymbol{e}_1 = -\nabla p' + \frac{1}{Re} \Delta \boldsymbol{v}' \ . \tag{8.8}$$

As stability theory is more developed for shear flows, we will restrict our attention to the fundamental parallel flow corresponding to $\boldsymbol{v} = (V_1(x_2), 0, 0)$. The stability problem is considerably simplified if Squire theorem [97] is evoked:

Theorem 8.1 (Squire theorem) *The behavior of three-dimensional perturbations can be deduced from the behavior of two-dimensional perturbations in a parallel flow of incompressible fluid.*

A proof is given in Rieutord [79]. Note that Squire's theorem pertains to disturbance growth in time, while in actual flow the disturbances actually grow spatio-temporally.

Therefore, for every three-dimensional unstable perturbation, there exists a corresponding two-dimensional perturbation that is still more unstable. This allows us to search the flow stability limit as a function of the Reynolds number, using a plane perturbation, while being assured that this procedure yields the inferior stability limit.

To verify the incompressibility constraint in a plane problem, it is natural to resort to the streamfunction. We get rid of the pressure variable by dealing with the dynamic vorticity equation. We have

$$v_1 = \frac{\partial \psi}{\partial x_2}, \quad v_2 = -\frac{\partial \psi}{\partial x_1}, \tag{8.9}$$

and consequently, by the vorticity definition (1.40)

$$\omega = -\Delta \psi \ . \tag{8.10}$$

Using (8.10) in (4.25), we obtain

$$\frac{\partial \Delta \psi}{\partial t} + \frac{\partial \psi}{\partial x_2}\frac{\partial \Delta \psi}{\partial x_1} - \frac{\partial \psi}{\partial x_1}\frac{\partial \Delta \psi}{\partial x_2} = \frac{1}{Re}\, \Delta\Delta\psi \ . \tag{8.11}$$

The boundary conditions to integrate (8.11) are no-slip conditions at the fixed lower and upper walls located in $x_2 = x_{2,lo}, x_{2,up}$ expressed in term of the streamfunction

$$\psi = const, \ \frac{\partial \psi}{\partial x_2} = 0, \quad \text{for } x_2 = x_{2,lo}, x_{2,up}. \tag{8.12}$$

Decomposing the streamfunction as $\psi = \Psi + \psi'$, where $\Psi = \int V_1(x_2)\, dx_2$ represents the fundamental flow and ψ' the perturbation, (8.11) is linearized

$$\frac{\partial \Delta \psi'}{\partial t} + V_1 \frac{\partial \Delta \psi'}{\partial x_1} - \frac{d^2 V_1}{dx_2^2}\frac{\partial \psi'}{\partial x_1} = \frac{1}{Re}\, \Delta\Delta\psi' \ . \tag{8.13}$$

The perturbation is approximated in normal modes

$$\psi' = \phi(x_2) e^{i\alpha(x_1 - ct)} \ . \tag{8.14}$$

Insertion of the normal mode (8.14) in (8.13) leads to the Orr-Sommerfeld equation [Orr [65]- Sommerfeld [93]]

$$(\frac{d^2}{dx_2^2} - \alpha^2)^2\phi = i\alpha\, Re\left[(V_1 - c)(\frac{d^2\phi}{dx_2^2} - \alpha^2\phi) - \phi\,\frac{d^2V_1}{dx_2^2}\right]. \tag{8.15}$$

This equation is the cornerstone of hydrodynamic stability. Numerical solutions have been given by Jordinson [41] and Orszag [66] more than half a century after its discovery. Equation (8.15) with the boundary conditions

$$\phi = \frac{d\phi}{dx_2} = 0 \quad \text{for } x_2 = x_{2,lo}, x_{2,up} \tag{8.16}$$

can be solved if the profile $V_1(x_2)$, the Reynolds number Re and the dimensionless wavenumber α are given. The equation will produce the eigenfunction $\phi(x_2)$, but also the complex wave velocity $c = c_R + i\,c_I$ as the associated eigenvalue. For fixed α and Re, the eigenvalue problem generates a discrete spectrum of eigenvalues $c_1, c_2, c_3, \ldots$. We consider that the eigenvalues are function of α and Re

$$\begin{aligned} c_R &= c_R(\alpha, Re) \\ c_I &= c_I(\alpha, Re)\,. \end{aligned} \tag{8.17}$$

The perturbation growth goes like $e^{\alpha c_I t}$. The flow is unstable for $c_I > 0$. A neutral stability mode is obtained for $c_I = 0$. Consequently, imposing $c_I = 0$ in (8.17) gives the neutral stability curve $c_I(\alpha, Re) = 0$. If, moreover, it is possible to show that c_I changes sign when we cross the neutral curve, then this curve is also that of marginal stability. The marginal stability curve separates the stable and unstable regions in the domain (α, Re).

Figure 8.1 shows the stability diagram in the plane (α, Re) for the plane Poiseuille flow (3.19). Instability occurs for parameter values inside the marginal stability curve. The critical Reynolds number Re_{crit} is obtained at the lowest value of the Reynolds number defining the stability curve and corresponds to the vertical tangent line in Fig. 8.1. The critical Reynolds number is 5772 for $\alpha = 1.02$.

The numerical solution of the Orr-Sommerfeld equation by S. A. Orszag used the Chebyshev Tau method [36, 66]. The Fortran program as coded in the 1970s is given in the Appendix D.

8.3 Stability of the Circular Couette Flow

Referring to the radial Navier-Stokes equation (3.39) for the circular Couette flow, we observe that the centrifugal force is balanced by the radial pressure gradient. Therefore the destabilizing physical force of the laminar Couette flow is this centrifugal force that is no longer in equilibrium with pressure and viscous forces.

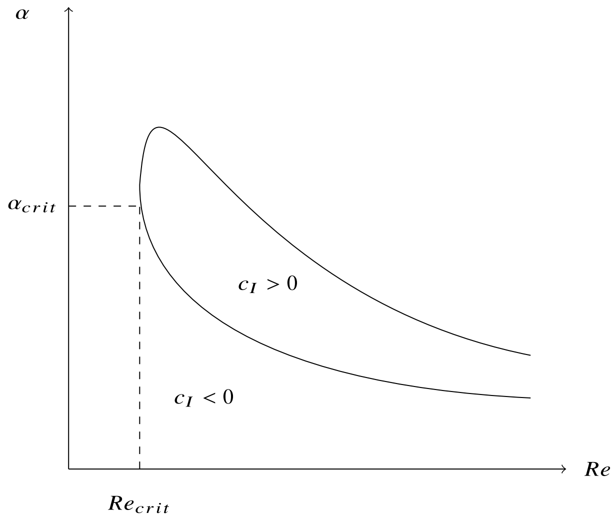

Fig. 8.1 Example of the marginal stability curve for the plane Poiseuille flow

8.3.1 Rayleigh's Criterion

The first analysis of this unstable phenomenon was carried out by Rayleigh [76] in the context of inviscid fluids. Suppose that the streamlines are circles and the disturbances are axisymmetric in a cylindrical coordinate system. As the pressure is independent of θ, the momentum equation in the azimuthal direction is

$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} = 0 \, . \tag{8.18}$$

If we multiply (8.18) by r we obtain the relation

$$\frac{\partial (r v_\theta)}{\partial t} + v_r \frac{\partial (r v_\theta)}{\partial r} + \frac{v_\theta}{r}\frac{\partial (r v_\theta)}{\partial \theta} + v_z \frac{\partial (r v_\theta)}{\partial z} = \frac{D\Gamma}{Dt} = 0 \, , \tag{8.19}$$

showing that the circulation $\Gamma = 2\pi r v_\theta$ should remain constant for a fluid element.

Let us consider two fluid annular elements of radii r_1 and r_2, respectively, with $r_1 < r_2$. The kinetic energy of these rings are $E_k = \frac{1}{2}\rho v_\theta^2 = \rho \Gamma^2/8\pi^2 r^2$. If we exchange these two elements, their mass and angular momentum will be conserved, while the kinetic energy will not. The total initial kinetic energy is

$$E_{k,init} = \frac{\rho}{8\pi^2}\left[\left(\frac{\Gamma_1}{r_1}\right)^2 + \left(\frac{\Gamma_2}{r_2}\right)^2\right] \, . \tag{8.20}$$

As this exchange is a perturbation with constant circulation, the final kinetic energy reads

$$E_{k,fin} = \frac{\rho}{8\pi^2}\left[\left(\frac{\Gamma_1}{r_2}\right)^2 + \left(\frac{\Gamma_2}{r_1}\right)^2\right] . \tag{8.21}$$

The difference between these two energies is

$$E_{k,fin} - E_{k,init} = \frac{\rho}{8\pi^2}(\Gamma_2^2 - \Gamma_1^2)(\frac{1}{r_1^2} - \frac{1}{r_2^2}) . \tag{8.22}$$

If $E_{k,fin}$ is larger than $E_{k,init}$, the perturbation needs receiving a definite quantity of energy. This cannot be provided by the flow at hand and therefore the physical situation is stable. On the contrary, if $E_{k,fin}$ is smaller than $E_{k,init}$, this means that the disturbance releases energy and it can be captured to feed the disturbance itself. Consequently, for a stable flow, we have Rayleigh's condition

$$\Gamma_2^2 > \Gamma_1^2 \quad \text{for} \quad r_2 > r_1 . \tag{8.23}$$

Rephrasing the condition with the velocity profile, we write

$$(r_2 v_{\theta 2})^2 > (r_1 v_{\theta 1})^2 . \tag{8.24}$$

With the solution of the circular Couette flow (3.41), we have

$$r v_\theta = A r^2 + B . \tag{8.25}$$

If the outer cylinder rotates in the positive direction $\omega_2 > 0$, the quantity (8.25) will increase with r provided the following condition obtained from the constant A in (3.41) holds

$$\omega_2 \geq \left(\frac{R_1}{R_2}\right)^2 \omega_1 . \tag{8.26}$$

If in (8.26) we use the equal sign, we obtained the Rayleigh line. Figure 8.2 exhibits the stability diagram for the perfect fluid in the (ω_1, ω_2) plane.

If the inner cylinder is fixed and the outer one rotates, the flow is stable. If the outer cylinder is fixed with the inner rotating, the flow is unstable according to the inviscid analysis. However, in the real world, viscosity is always present and damps all disturbances if the Reynolds number is not too high.

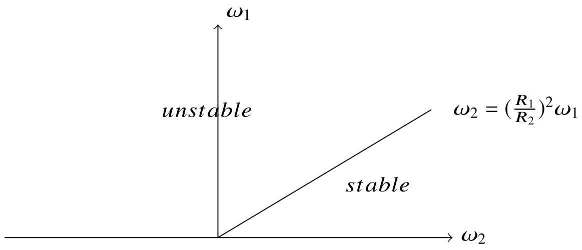

Fig. 8.2 For the inviscid fluid, the Rayleigh line $\omega_2 = (R_1/R_2)^2\omega_1$ separates the stable and unstable regions in the (ω_1, ω_2) plane

8.3.2 Linear Stability of Viscous Circular Couette Flow

Let us start from the Navier-Stokes equations written in cylindrical coordinates (A.21)–(A.23) with the incompressibility constraint (A.20). Assuming axial symmetry and neglecting the body force, with the simplification $\partial/\partial\theta = 0$, the equations read

$$\rho(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}) = -\frac{\partial p}{\partial r} + \mu(\Delta v_r - \frac{v_r}{r^2}) \tag{8.27}$$

$$\rho(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}) = \mu(\Delta v_\theta - \frac{v_\theta}{r^2}) \tag{8.28}$$

$$\rho(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z}) = -\frac{\partial p}{\partial z} + \mu\Delta v_z \tag{8.29}$$

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0\,. \tag{8.30}$$

The stability analysis requires that the base flow be perturbed by three-dimensional axisymmetric velocity and pressure disturbances, denoted for ease of notation by (u, v, w) and p_p, respectively,

$$\boldsymbol{v} = (u, V + v, w) \quad \text{and} \quad p = P + p_p\,, \tag{8.31}$$

with V being the Couette flow solution (3.41). Inserting (8.31) in (8.27)–(8.30), we proceed to a linearization keeping only the first-order terms in the perturbations, while discarding higher order terms

$$\rho\left(\frac{\partial u}{\partial t} - \frac{(V+v)^2}{r}\right) = -\frac{\partial(P+p_p)}{\partial r} + \mu(\Delta u - \frac{u}{r^2}) \tag{8.32}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{dV}{dr} + \frac{u}{r}(V+v)\right) = \mu(\Delta v - \frac{v}{r^2}) \tag{8.33}$$

$$\rho\frac{\partial w}{\partial t} = -\frac{\partial(P+p_p)}{\partial z} + \mu\Delta w \tag{8.34}$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \,. \tag{8.35}$$

With V satisfying the Navier-Stokes equations and carrying through a further step of the linearization process, we obtain

$$\rho\left(\frac{\partial u}{\partial t} - \frac{2Vv}{r}\right) = -\frac{\partial p_p}{\partial r} + \mu(\triangle u - \frac{u}{r^2}) \tag{8.36}$$

$$\rho\left(\frac{\partial v}{\partial t} + u(\frac{V}{r} + \frac{dV}{dr})\right) = \mu(\triangle v - \frac{v}{r^2}) \tag{8.37}$$

$$\rho\frac{\partial w}{\partial t} = -\frac{\partial p_p}{\partial z} + \mu\Delta w \tag{8.38}$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \,. \tag{8.39}$$

This set of linear partial differential equations has coefficients depending only on the radial direction r. Therefore it is usual to search solutions in the form of normal axisymmetric modes such that

$$(u, v, w, p_p) = (\hat{u}(r), \hat{v}(r), \hat{w}(r), \hat{p}_p(r))e^{ikz+\sigma t} \,, \tag{8.40}$$

where k is the real wavenumber associated with the axial direction and σ the growth rate.

Let us set

$$D = \frac{d}{dr} \quad \text{and} \quad D_* = \frac{d}{dr} + \frac{1}{r} \,. \tag{8.41}$$

The stability equations become

$$\sigma\hat{u} - \frac{2V\hat{v}}{r} = -\frac{1}{\rho}D\hat{p}_p + \nu(DD_* - k^2)\hat{u} \tag{8.42}$$

$$\sigma\hat{v} + \hat{u}D_*V = \nu(DD_* - k^2)\hat{v} \tag{8.43}$$

$$\sigma\hat{w} = -ik\frac{\hat{p}_p}{\rho} + \nu(D_*D - k^2)\hat{w} \tag{8.44}$$

$$D_*\hat{u} + ik\hat{w} = 0 \,. \tag{8.45}$$

From Eq. (8.45) we extract $\hat{w}$ in terms of $\hat{u}$. This expression is then inserted in Eq. (8.44) that yields the pressure

$$\frac{\hat{p}_p}{\rho} = \frac{1}{k^2}\left(\nu(D_*D - k^2) - \sigma\right) D_*\hat{u} \,. \tag{8.46}$$

Taking the derivative D of (8.46) and using it in (8.42) we are left with the relations

$$\nu\left(DD_* - k^2\right)^2 \hat{u} - 2k^2\frac{V}{r}\hat{v} = \sigma(DD_* - k^2)\hat{u} \tag{8.47}$$

$$\nu\left(DD_* - k^2\right) \hat{v} - (D_*V)\hat{u} = \sigma\hat{v} \tag{8.48}$$

$$\hat{u} = \hat{v} = D\hat{u} = 0 \qquad \text{for} \quad r = R_1, R_2\,. \tag{8.49}$$

The first analytical investigation of this eigenvalue problem was carried out by G. I. Taylor [103] with the narrow-gap approximation where the gap $d = R_2 - R_1$ is smaller than the mean radius $R_m = (R_1 + R_2)/2$. With this assumption, the operator D_* is approximated by D and the set of equations simplifies. The reader is referred to Chandrasekhar [18] and Drazin and Reid [27] for the detailed analysis. The stability problem is characterized by the dimensionless Taylor number Ta

$$Ta = 4\left(\frac{\omega_1 R_1^2 - \omega_2 R_2^2}{R_2^2 - R_1^2}\right)\frac{\omega_1 d^4}{\nu^2}\,. \tag{8.50}$$

Another definition of the Taylor number when the outer cylinder is fixed is

$$Ta = \left(\frac{\omega_1 R_1 d}{\nu}\right)^2 \frac{d}{R_m}\,. \tag{8.51}$$

The critical Taylor number is obtained for the fixed outer cylinder $\omega_2 = 0$, $Ta_{crit} = 1706$. Figure 8.3 exhibits the curve of marginal stability ($\sigma = 0$) for the apparatus used by Taylor [103] where the full dots are observed data while open circles are computational results. Above the marginal curve the flow is unstable. Note that the dotted line in the right quadrant is the Rayleigh stability line for inviscid perturbation. The axial wavenumber corresponding to the critical Taylor number, $k_{crit} = 3.12$, corresponds to a wavelength $\lambda_{crit} = 2\pi d/k_{crit} \approx 2d$. This means that the Taylor vortices have a height that is approximately equal to d so that each vortex has a shape close to a square.

To solve the set of Eqs. (8.45)–(8.49) high-order numerical discretizations are used like Chebyshev collocation or Tau methods or spectral elements [23]. The resulting discrete equations form a generalized eigenvalue problem that is solved using appropriate routines to be found in Lapack.

8.3.3 Non-linear Axisymmetric Taylor Vortices

The next step in analyzing the stability of Taylor vortices has been carried out by Davey [22] (see also Koschmieder [44]) who developed the velocity and pressure disturbances in Fourier series of k modes as non linearities excite high-order harmonics. Therefore we write

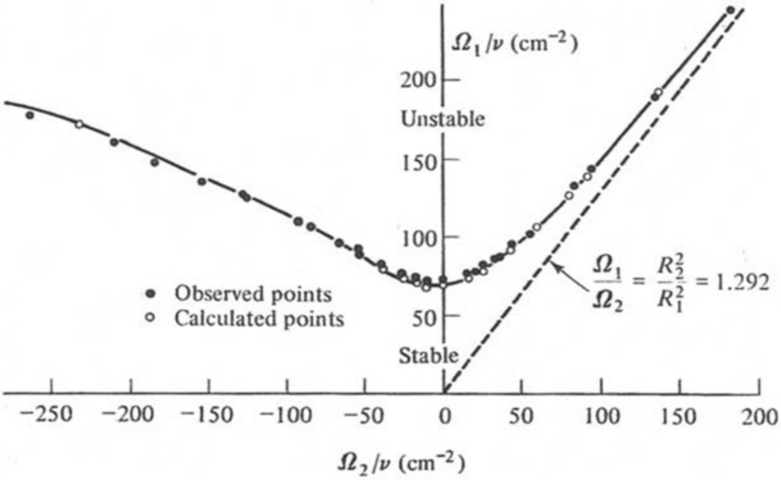

Fig. 8.3 Marginal stability curve for the Taylor-Couette flow with water. Experimental results and narrow-gap calculations. $R_1 = 3.55$ cm and $R_2 = 4.035$ cm. Reproduced from [27]

$$u = \sum_{n=1}^{\infty} u_n(r,t) \cos nkz \ , \tag{8.52}$$

$$v = \overline{v}(r,t) + \sum_{n=1}^{\infty} v_n(r,t) \cos nkz \ , \tag{8.53}$$

$$w = \sum_{n=1}^{\infty} w_n(r,t) \sin nkz \ , \tag{8.54}$$

where $\overline{v}$ is the mean azimuthal velocity averaged in the z periodic direction. The elimination of the pressure p and the vertical velocity w leads to two differential equations.

Instead of solving the general initial value problem, Davey assumes that the spatial shape of the disturbance is unaltered and characterized in time by an unknown amplitude function $A(t)$. Because of the non-linear terms, the fundamental disturbance will grow according to the linear theory for $Ta > Ta_{crit}$, and will constrain the harmonics $\cos nkz$ and a mean motion term. As $\cos^2 kz = (1 + \cos 2kz)/2$, the first harmonic $\cos 2kz$ and the mean motion will be $O(A^2)$. Afterwards the fundamental mode interacting with the first harmonic $\cos kz \cos 2kz$ will generate a second harmonic $\cos 3kz$ and a correction to the fundamental $\cos kz$ of order $O(A^3)$. Consequently one writes

$$u_n(r,t) = A^n \left[u_n(r) + \sum_{n=1}^{\infty} A^{2m} u_{nm}(r) \right] , \quad n \geq 1 \tag{8.55}$$

$$v_n(r,t) = A^n \left[v_n(r) + \sum_{n=1}^{\infty} A^{2m} v_{nm}(r) \right] , \quad n \geq 1 \tag{8.56}$$

$$\overline{v} = \overline{v_{lam}} + \sum_{n=1}^{\infty} A^{2m} f_m(r) , \tag{8.57}$$

with $\overline{v_{lam}}$ the mean of the laminar state. With these expansions, the time dependent amplitudes are governed by the equation

$$\frac{dA}{dt} = f(A) = \sum_{m=0}^{\infty} a_m A^{2m+1} , \quad a_0 = \sigma . \tag{8.58}$$

We invoke the symmetry of the dynamical system following the line of reasoning in Rieutord [79]. If the system is invariant under the symmetry $A \to -A$ such that if A is a solution, $-A$ is also a solution, and we thus have $f(-A) = -f(A)$ because of the linearity of d/dt. We then conclude that all even derivatives in f are zero in (8.58). As the analysis is performed for small amplitudes, we discard all terms after the first two terms in f reducing (8.58) to the Landau equation (cf. Sect. 26 of Landau and Lifshitz [48] and Sect. 49.1 of Drazin and Reid [27])

$$\frac{dA}{dt} = \sigma A + a_1 A^3 . \tag{8.59}$$

Let us divide (8.59) by A^3 and solve for A^{-2}. Setting $x = A^{-2}$ we have

$$\frac{dx}{dt} + 2\sigma x = -2a_1 . \tag{8.60}$$

The integration of (8.60) gives

$$x = -\frac{a_1}{\sigma} + Ce^{-2\sigma t} . \tag{8.61}$$

Imposing the initial condition $A_0^{-2} = C - a_1/\sigma$ we find

$$\frac{1}{A^2} = \frac{a_1}{\sigma}(e^{-2\sigma t} - 1) + \frac{1}{A_0^2} e^{-2\sigma t} . \tag{8.62}$$

Eventually we obtain

$$A^2 = \frac{A_0^2 e^{2\sigma t}}{1 + \frac{a_1 A_0^2}{\sigma}(1 - e^{2\sigma t})} , \tag{8.63}$$

to be compared with Davey's solution

$$A^2 = \frac{A_0^2 e^{2\sigma t}}{1 - \frac{a_1 A_0^2}{\sigma} e^{2\sigma t}} \,. \tag{8.64}$$

Let us inspect Landau equation written a bit differently

$$\frac{dA}{dt} = (\sigma + a_1 A^2) A \,. \tag{8.65}$$

The equilibrium amplitude corresponding to an effective growth rate going to zero is such that

$$A_{eq}^2 = -\frac{\sigma}{a_1} \,. \tag{8.66}$$

Depending on $a_1 > 0$ or $a_1 < 0$ we have subcritical or supercritical disturbances that decay or amplify up to their equilibrium values, respectively. For the Taylor-Couette flow, experimental evidence shows that the instability is supercritical. Davey calculated the equilibrium amplitude for various cases, the wide and small gaps, and found that the equilibrium amplitude A_e^2 was proportional to $1 - Ta_{crit}/Ta$. This means that the amplitude increases as $\varepsilon^{1/2}$ with $\varepsilon = (Ta - Ta_{crit})/Ta_{crit}$.

To show the wealth of physical phenomena that can be generated in a Couette flow, Fig. 8.4 [2] offers the various regimes produced in a small-gap apparatus.

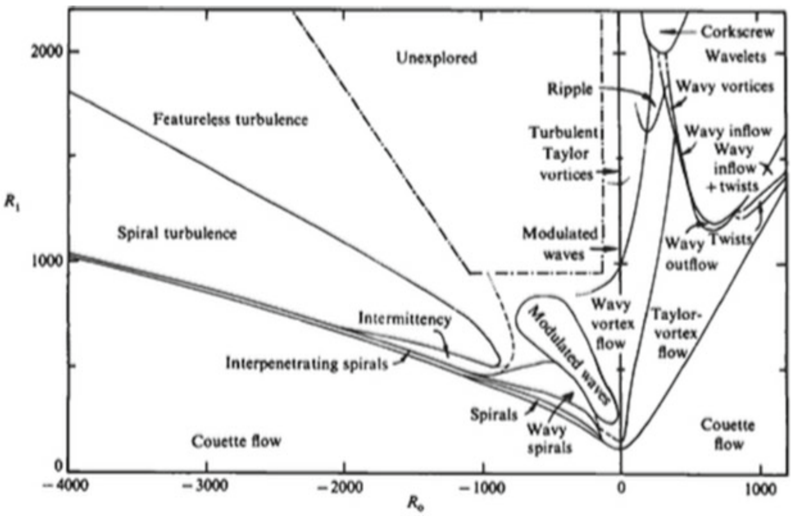

Fig. 8.4 Flow regimes in a Couette apparatus with long co-rotating cylinders. Reproduced from [2]

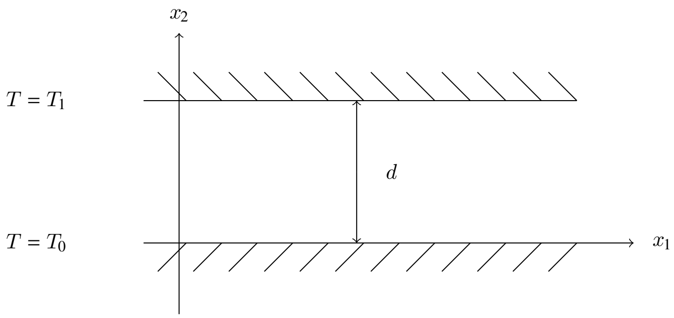

Fig. 8.5 Convection in a fluid layer between two parallel plates

Exercises

8.1 Obtain the linearized equations for the stability analysis of the spiral flow made of the combined circular Couette and Poiseuille flows under axisymmetric perturbations.

8.2 Rayleigh-Bénard instability A heat conducting fluid is at rest in between two parallel rigid plates, the lower one at temperature $T = T_0$, the upper one at $T = T_1$ such that $T_0 > T_1$, as shown in Fig. 8.5. The layer height is denoted by d. Using the Boussinesq approximation, the question is "When does the layer become unstable and show the presence of convection rolls?". The question makes sense since there are two stabilizing mechanics impeding the flow to start, namely the fluid viscosity and its thermal conductivity.

- Compute the base solution of the rest state: velocity, pressure p_{hyd} and temperature T_C.
- As this is a conduction/convection problem, the thermal diffusivity is used to define the reference velocity in the dimensionless equations (10.27)–(10.29). These relations are then linearized assuming that the perturbations are

$$\boldsymbol{v} = \boldsymbol{0} + \boldsymbol{v}' \;, \quad p = p_{hyd} + p' \;, \quad T = T_C + T' \;. \tag{8.67}$$

- Eliminate the pressure variable by taking the **curl** of the linearized momentum equation as **curl** $\nabla = 0$. The result is a governing relation for the vorticity.
- To recover the velocity perturbation as the main variable, the **curl** operator is applied to the vorticity equation. The velocity pops up as **curl** $\omega =$ **curl curl**$\boldsymbol{v} = -\triangle\triangle\boldsymbol{v}$.
- Write the governing equation for the x_2 perturbation velocity component.
- Obtain the fourth-order equation for the velocity perturbation by elimination of the temperature in the former governing relation and write the boundary conditions.
- Carry out the normal modes analysis of the perturbations system.