Chapter 7
Boundary Layer

In Chap. 6, we considered the model of incompressible inviscid fluid in plane irrotational flow. Through the application of the theory of complex potential around an airfoil profile, we discovered that the model can deduce the lift while the drag associated with the perfect fluid vanishes. This is in perfect contradiction to experimental observations that show that drag affects all flows of real fluids. As these fluids are viscous, in principle they stick to the walls and the tangential velocity component is zero if the wall is fixed. This last condition is never met by the inviscid fluid. Furthermore, the irrotationality condition is far from representing the reality as we observed in Chap. 4 that vorticity production occurs at the walls. In order to remedy the deficiencies of perfect fluid theory, one resorts to the theory of boundary layer that brings a necessary correction for the flows at high Reynolds numbers. The boundary layer theory is due to Ludwig Prandtl who was a distinguished aerodynamicist. Initially developed within the framework of fluid mechanics, the boundary layer theory knew a great success and today belongs to the corpus of applied mathematics methods. A complete presentation is given in the excellent book by Bender and Orszag [13]. On the fluid side, the reference monographs are the ones by Hermann Schlichting [85] and Louis Rosenhead [82].

The boundary layer is the flow zone close to the wall, either of an obstacle sitting in a uniform upstream flow, either of a container that confines an inner flow. Inside the boundary layer which is a very thin zone, one assumes that the viscous effects are of the same order of magnitude as the inertial effects (the *local* Reynolds number in the boundary layer is of the order of unity), whatever the value of the *global* Reynolds number characterizing the full flow is. The boundary layer is the location of intense vorticity generation that will diffuse and advect in the exterior region in the long run. Therefore we reach a very modern approach of the complete problem by decomposing it in two sub-domains: on the one hand the boundary layer where viscous effects will be handled by a simplified Navier-Stokes model, and on the other hand, the outer region where the theory of complex potential for the irrotational flow

M. O. Deville, *An Introduction to the Mechanics of Incompressible Fluids*,
https://doi.org/10.1007/978-3-031-04683-4_7

of perfect fluid will be used. Note that this outer region is characterized by velocities that are of the same order of magnitude as those of the incoming flow.

The boundary layer along an obstacle is consequently thin as the fluid flows over long distances downstream the leading edge during the time interval in which the vorticity diffuses slowly over a short orthogonal distance starting from the wall. The vorticity creation within the boundary layer allows for the physical realization of the fluid circulation around the profile. The circulation generates a wake in the region close to the trailing edge. The wake size will depend on the obstacle shape and the angle of attack of the upstream flow at the leading edge.

7.1 The Equations of the Laminar Boundary Layer

To set up these equations, we will carry out a dimensional analysis of the problem at hand. For the sake of facility, let us consider the uniform upstream flow of a viscous incompressible fluid with constant velocity U on a plane horizontal wall of characteristic length L. A laminar boundary layer develops from the leading edge of the plane wall. We use a Cartesian (x_1, x_2) coordinate system. Inside the boundary layer, in the x_2 direction orthogonal to the wall, the velocity goes from a zero value at the wall to a value of the order of U at the edge of the boundary layer. We conclude that the thickness of the boundary layer is always small with respect to the distances measured in the parallel direction to the wall along which it is developed.

7.1.1 Dimensional Analysis

Therefore we assume that the variation of every unknown in the x_2 direction across the boundary layer will be more important than the variation in x_1 direction. This allows us to write the following approximations on the next derivatives

$$\left| \frac{\partial v_1}{\partial x_1} \right| << \left| \frac{\partial v_1}{\partial x_2} \right|, \quad \left| \frac{\partial^2 v_1}{\partial x_1^2} \right| << \left| \frac{\partial^2 v_1}{\partial x_2^2} \right| . \tag{7.1}$$

Let us inspect the momentum equation in the x_1 direction taking the previous inequalities into account and neglecting the body forces. One writes

$$\rho \frac{Dv_1}{Dt} = -\frac{\partial p}{\partial x_1} + \mu \frac{\partial^2 v_1}{\partial x_2^2} . \tag{7.2}$$

If one neglects viscosity in (7.2), one recovers the Euler equation. This relation will be adequate in the region outside the boundary layer where the fluid viscosity does not supposedly affect the flow physics. As the local Reynolds number in the boundary layer is of the order of unity, one obtains

$$\mid \rho \frac{Dv_1}{Dt} \mid = O\left(\mid \mu \frac{\partial^2 v_1}{\partial x_2^2} \mid \right), \quad x_2 \to 0\,. \tag{7.3}$$

The order of magnitude of the left hand side may be evaluated through the term $\rho v_1 \partial v_1/\partial x_1$ for a steady state flow. We will observe that the term $\rho v_2 \partial v_1/\partial x_2$ is of the same order of magnitude. One replaces (7.3) by

$$\mid \rho v_1 \frac{\partial v_1}{\partial x_1} \mid = O\left(\mid \mu \frac{\partial^2 v_1}{\partial x_2^2} \mid \right), \quad x_2 \to 0 \tag{7.4}$$

Taking U as the reference velocity, L as reference length and δ_0 the average thickness of the boundary layer as reference length in the x_2 direction, we estimate the order of magnitude of the terms appearing in (7.4), namely

$$\rho \frac{U^2}{L} = O(\mu \frac{U}{\delta_0^2}), \quad x_2 \to 0\,, \tag{7.5}$$

or

$$\left(\frac{\delta_0}{L}\right)^2 = O\left(\frac{\mu}{\rho L U}\right) = O(Re^{-1}), \quad x_2 \to 0\,. \tag{7.6}$$

This last dimensionless number is the *global* Reynolds number. As per the definition of the Bachmann symbol O, the product $(\frac{\delta_0}{L})^2\, Re$ is bounded by a constant. If $Re \to \infty$, the ratio $\frac{\delta_0}{L}$ becomes very small, while remaining finite. One concludes the following estimate

$$\frac{\delta_0}{L} \sim Re^{-\frac{1}{2}}, \quad \text{for} \quad Re \to \infty\,. \tag{7.7}$$

For example, on a wing profile with a chord of the order of one meter and a Reynolds number $Re \sim 10^6$, the boundary layer thickness will be of the order of a few millimeters $(2-3)$.

By the incompressibility constraint (1.50), it is possible to obtain the order of magnitude of the vertical velocity component in the boundary layer. One has

$$v_2 \sim \frac{\delta_0}{L} U \sim U\, Re^{-\frac{1}{2}}\,. \tag{7.8}$$

For high Reynolds numbers, the vertical velocity will be several orders of magnitude less than the flow characteristic velocity.

7.1.2 Prandtl's Equations

As we mentioned in the preceding paragraphs, the boundary layer possesses essentially two normative space scales: the length L in the main flow direction and the thickness of the boundary layer we will now denote by δ. For this reason, we perform a multi-scale approach to write down the dimensionless form of the Navier-Stokes equations, using these normative scales in the plane case. Let us define the dimensionless variables

$$x_1' = \frac{x_1}{L}, \quad x_2' = \frac{x_2}{\delta} = \frac{x_2}{L} Re^{\frac{1}{2}}, \quad t' = \frac{tU}{L}, \tag{7.9}$$

$$v_1' = \frac{v_1}{U}, \quad v_2' = \frac{v_2}{U} Re^{\frac{1}{2}}, \quad p' = \frac{p - p_\infty}{\rho U^2}, \tag{7.10}$$

where p_∞ is the pressure at infinity upstream. With these variables, the Navier-Stokes equations become

$$\frac{\partial v_1'}{\partial t'} + v_1' \frac{\partial v_1'}{\partial x_1'} + v_2' \frac{\partial v_1'}{\partial y'} = -\frac{\partial p'}{\partial x_1'} + \frac{1}{Re} \frac{\partial^2 v_1'}{\partial x_1'^2} + \frac{\partial^2 v_1'}{\partial x_2'^2}, \tag{7.11}$$

$$\frac{1}{Re} \left(\frac{\partial v_2'}{\partial t'} + v_1' \frac{\partial v_2'}{\partial x_1'} + v_2' \frac{\partial v_2'}{\partial x_2'} \right) = -\frac{\partial p'}{\partial x_2'} + \frac{1}{Re^2} \frac{\partial^2 v_2'}{\partial x_1'^2} + \frac{1}{Re} \frac{\partial^2 v_2'}{\partial x_2'^2}, \tag{7.12}$$

$$\frac{\partial v_1'}{\partial x_1'} + \frac{\partial v_2'}{\partial x_2'} = 0. \tag{7.13}$$

Assuming that the velocities v_1', v_2' and pressure p' together with their derivatives with respect to space and time variables remain bounded, when $Re \to \infty$, one produces the reduced form of the equations of the laminar boundary layer

$$\frac{\partial v_1'}{\partial t'} + v_1' \frac{\partial v_1'}{\partial x_1'} + v_2' \frac{\partial v_1'}{\partial x_2'} = -\frac{\partial p'}{\partial x_1'} + \frac{\partial^2 v_1'}{\partial x_2'^2}, \tag{7.14}$$

$$0 = -\frac{\partial p'}{\partial x_2'}, \tag{7.15}$$

$$\frac{\partial v_1'}{\partial x_1'} + \frac{\partial v_2'}{\partial x_2'} = 0. \tag{7.16}$$

Equation (7.15) expresses that pressure remains constant over the boundary layer thickness. Therefore the boundary layer pressure will be equal to the pressure prevailing at its border, i.e. the one in the outer flow. The pressure gradient of the exterior region is thus the development engine of the boundary layer.

Coming back to dimensional variables, the boundary layer or Prandtl's equations are

$$\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} = -\frac{1}{\rho} \frac{\partial p_e(x_1, 0, t)}{\partial x_1} + \nu \frac{\partial^2 v_1}{\partial x_2^2} , \tag{7.17}$$

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0 , \tag{7.18}$$

with the following boundary conditions

$$v_1 = v_2 = 0, \quad \text{for} \quad x_2 = 0 \tag{7.19}$$

$$v_1 \to U_e(x, 0, t) \quad \text{for} \quad x_2 \to \infty . \tag{7.20}$$

The subscript e for p and U characterizes these variables in the exterior (inviscid) sud-domain. In the Eqs. (7.17) and (7.20), the variables p_e and U_e are evaluated by the Euler equation

$$\rho U_e \frac{dU_e}{dx_1} = -\frac{dp_e}{dx_1} . \tag{7.21}$$

Note that in (7.20), one imposes that at the location where the boundary layer meets the zone of perfect fluid, it is considered that the distance to the wall goes to infinity, namely $x_2 \to \infty$. For the perfect fluid region, everything occurs as if the boundary layer did not exist or at least is so thin that we use $U_e(x, 0, t)$ on the solid wall, while for the boundary layer, everything is treated as if it would ignore the presence of the outer zone, whence $x_2 \to \infty$. In Sect. 6.5.2 of [83], Ryhming writes "According to (7.20) one requires that when the inner solution approaches the outer solution, the value of the inner solution for $x_2 \to \infty$ be equal to the value of the outer solution for $x_2 \to 0$." A similar line of reasoning introduces in (7.17) the exterior pressure p_e evaluated at the wall $x_2 = 0$.

The link between the inner and outer solutions may be explained by a simple example proposed by von Mises and Friedrichs [116]. The problem is given by a second-order linear ordinary differential equation for a function $f(x_2)$ defined on the interval [0, 1] with two boundary conditions

$$\varepsilon f'' + f' = a , \tag{7.22}$$

$$f(0) = 0, \qquad f(1) = 1 , \tag{7.23}$$

where a is a positive constant and ε a positive coefficient such that $\varepsilon \ll 1$. This equation mimics the full Navier-Stokes equations. The small parameter ε that multiplies the second-order derivative corresponds to ν and is proportional to Re^{-1}. Therefore when the viscosity ν goes to zero, the mathematical nature of the problem changes. Instead of having a second-order problem with two boundary conditions, we are facing a first-order problem with the same conditions. This is typical of a singular perturbation problem.

In (7.22) the linearity of the problem allows for an exact solution of the problem comprising a particular solution and the general solution of the homogeneous ordinary differential equation. We obtain

$$f(x_2) = ax_2 + (1-a)\frac{1-e^{-\frac{x_2}{\varepsilon}}}{1-e^{-\frac{1}{\varepsilon}}} \; . \tag{7.24}$$

If we set the limit $\varepsilon \to 0$ for any fixed value of x_2, we get

$$\lim_{\varepsilon\to 0} f(x_2) = ax_2 + 1 - a = f_e(x_2) \; . \tag{7.25}$$

This exterior solution f_e satisfies the outer condition $f(1) = 1$ but not the inner condition as $f_e(0) \neq 0$. Let us inspect now the limit of $f(x_2)$ when $\varepsilon \to 0$ for a fixed value of x_2/ε. We obtain

$$\begin{aligned} \lim_{\substack{fixed\ \frac{x_2}{\varepsilon} \\ \varepsilon\to 0}} f(x_2) &= a\frac{x_2}{\varepsilon}\varepsilon + (1-a)(1-e^{-x_2/\varepsilon}) \\ &= (1-a)(1-e^{-x_2/\varepsilon}) = f_i(x_2) \; . \end{aligned} \tag{7.26}$$

The inner solution f_i satisfies the inner condition $f_i(0) = 0$ but fails to satisfy the exterior condition, $f_i(1) \neq 1$. However we note that the exterior limit $x_2/\varepsilon \to \infty$ of the inner solution is equal to the inner limit $x_2 \to 0$ of the exterior solution

$$f_i(\infty) = f_e(0) = 1 - a \; . \tag{7.27}$$

Therefore the solution of a singular perturbation problem is built by addition of the inner and outer solutions and by subtraction of their common quantity, which in our case is $1-a$. The approximate complete solution for $\varepsilon \ll 1$ reads

$$f_a = f_i + f_e - (1-a) = ax_2 + (1-a)(1-e^{-x_2/\varepsilon}) \; . \tag{7.28}$$

In Fig. 7.1 we observe that the approximate (red) f_a solution is not very far from the (green) analytical solution f.

The condition (7.20) with the use of $p_e(x_1, 0, t)$ at $x_2 = 0$ in Eq. (7.17) must be justified. As the thickness δ_0 given by (7.7) is small, we can imagine an intermediary length ζ small with respect to the dimension L and large with respect to δ_0 as is shown in Fig. 7.2.

Let us choose

$$\frac{\delta_0}{\zeta} \sim Re^{-1/4} \; . \tag{7.29}$$

This induces

$$\frac{\zeta}{L} \sim Re^{-1/4} \; . \tag{7.30}$$

The exterior solution will not vary much when x_2 will go from ζ to 0. Indeed, the exterior solution changes significantly over distances of the order of magnitude of L, much larger than ζ. Therefore we can approximate the following quantities

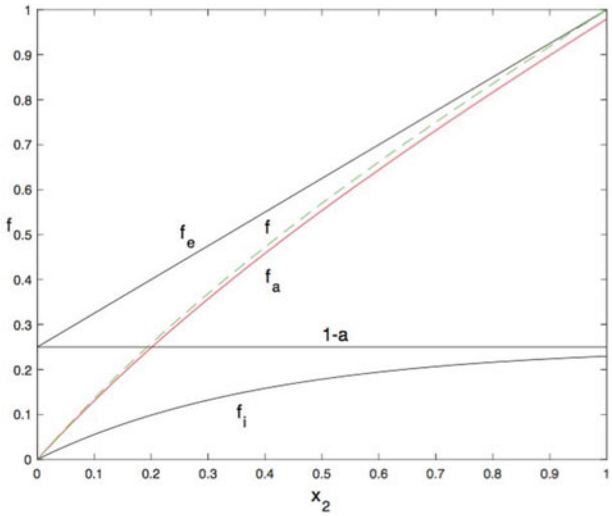

Fig. 7.1 Solutions of problem (7.22)

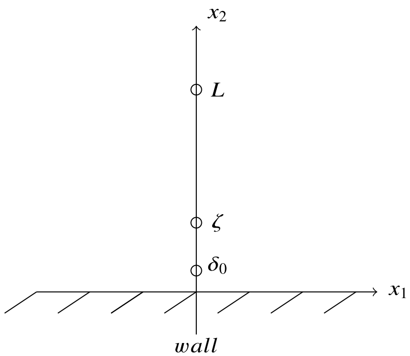

Fig. 7.2 Scales in the boundary layer theory

$$v_{1e(}x_1, \zeta, t) \simeq U_e(x_1, 0, t) \tag{7.31}$$

$$p_e(x_1, \zeta, t) \simeq p_e(x_1, 0, t) \; . \tag{7.32}$$

On the other hand, the interior solution is totally linked to the exterior solution at distances ζ from the wall which are larger than the thickness δ_0 of the boundary layer. Consequently, we write the following approximations

$$v_1(x_1, \zeta, t) \simeq U_e(x_1, \zeta, t) \tag{7.33}$$

$$p(x_1, \zeta, t) \simeq p_e(x_1, \zeta, t) \; . \tag{7.34}$$

Taking these various approximations into account, we conclude

$$v_1(x_1, \zeta, t) \simeq U_e(x_1, 0, t) \ . \tag{7.35}$$

If $Re \to \infty$, we can write the relation

$$\lim_{x_2 \to \infty} v_1(x_1, x_2, t) = U_e(x_1, 0, t) \ . \tag{7.36}$$

For the pressure, via a similar reasoning, we write

$$p(x_1, \zeta, t) \simeq p_e(x_1, 0, t) \ . \tag{7.37}$$

As the pressure is constant over the boundary layer thickness, we replace in the Prandtl equations the pressure $p(x_1, x_2, t)$ by $p(x_1, \zeta, t)$, leading to

$$p(x_1, x_2, t) \simeq p_e(x_1, 0, t), \quad x_2 \leq \zeta. \tag{7.38}$$

The pressure $p_e(x_1, 0, t)$ and the velocity $U_e(x_1, 0, t)$ will be computed by (7.21).

7.2 Boundary Layer on a Flat Plate

In this section we analyze the development of the boundary layer on a semi-infinite flat plate located in steady-state uniform flow parallel to direction x_1 of a Cartesian coordinates system. We will solve Prandtl's equations and introduce several relevant considerations about boundary layer thicknesses and friction coefficient.

7.2.1 Solution of Prandtl's Equations

We will assume that the boundary layer thickness is sufficiently thin not to affect the velocity distribution in the exterior zone modeled by the perfect fluid. Then the exterior flow is such that $U_e = U$, $v_{2e} = 0$. The pressure distribution is obtained by the Euler equation (7.21) giving with $U_e = U$

$$\frac{dp_e}{dx} = 0 \ . \tag{7.39}$$

With (7.39), the Prandtl's equations simplify

$$v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} = \nu \frac{\partial^2 v_1}{\partial x_2^2} \tag{7.40}$$

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0 \, . \tag{7.41}$$

We position the origin of the axes at the leading edge of the plate. The boundary conditions are

$$v_1 = v_2 = 0, \quad \text{for} \quad x_2 = 0, \quad x_1 \geq 0, \tag{7.42}$$

$$v_1 = U, \quad \text{for} \quad x_2 \to \infty, \quad x_1 \geq 0 \, . \tag{7.43}$$

To impose a zero boundary layer thickness at the leading edge, we prescribe

$$v_1 = U, \quad \text{for} \quad x_1 = 0, \quad x_2 \neq 0 \, . \tag{7.44}$$

The boundary layer thickness will grow with the distance x_1 and therefore

$$\delta_0 = \delta_0(x_1) \, . \tag{7.45}$$

As the semi-infinite plate has no normative scale that can be derived from the geometry, the abscissa x_1 will fulfill this purpose. The Eq. (7.7) becomes

$$\frac{\delta_0(x_1)}{x_1} = \left(\frac{U x_1}{\nu}\right)^{-1/2} = Re_{x_1}^{-\frac{1}{2}} \tag{7.46}$$

with the definition of a local Reynolds number Re_{x_1} based on the distance to the leading edge. We conclude

$$\delta_0(x_1) = \left(\frac{\nu x_1}{U}\right)^{1/2} \, . \tag{7.47}$$

Let us write the system (7.40)–(7.41) in dimensionless form (cf. Chap. 2). We search a solution of these equations such that

$$\frac{v_1}{U} = F\left(\frac{U x_1}{\nu}, \frac{U x_2}{\nu}\right) \, . \tag{7.48}$$

Relation (7.48) can be simplified by writing it in self-similar form. Indeed, if x_1, x_2, v_1, v_2 are replaced respectively by $\alpha^2 x_1, \alpha x_2, \alpha^2 v_1, \alpha v_2$, the Eqs. (7.40)–(7.41) are not modified. This suggests to write relation (7.48) as

$$\frac{v_1}{U} = g(s) \tag{7.49}$$

with the variable

$$s = \frac{Ux_2/\nu}{(Ux_1/\nu)^{1/2}} = x_2\sqrt{\frac{U}{\nu x_1}} = \frac{x_2}{\delta_0(x_1)} \,. \tag{7.50}$$

This type of solution is self-similar as the function that will describe the velocity profile at various stations x_1 will be the same with respect to the variable x_2, normalized by the local boundary layer thickness $\delta_0(x_1)$, this thickness taking the variation of the solution in direction x_1 into account.

The incompressibility constraint (7.41) is trivially satisfied by the introduction of the stream function $\psi(x_1, x_2)$. One has

$$\psi = \int_0^{x_2} v_1\, dx_2' = U\delta_0 \int_0^{\infty} g(s')\, ds' = U\delta_0 f(s) = (U\nu x_1)^{1/2} f(s) \,, \tag{7.51}$$

with $f'(s) = df/ds = g(s)$. The component v_2 is obtained easily

$$v_2 = -\frac{\partial\psi}{\partial x_1} = \frac{1}{2}\sqrt{\frac{\nu U}{x_1}}[sf'(s) - f(s)] \,. \tag{7.52}$$

With the help of relations (7.49) and (7.52), Eq. (7.40) produces the third-order non linear ordinary differential equation, named Blasius equation

$$f'''(s) + \frac{1}{2} f''(s) f(s) = 0 \tag{7.53}$$

with the boundary conditions

$$f(s) = f'(s) = 0, \quad \text{for} \quad s = 0, \tag{7.54}$$

$$f'(s) = 1, \quad \text{for} \quad s \to \infty \,. \tag{7.55}$$

This equation is numerically integrated by a fourth-order Runge-Kutta method for example. As we have only two initial conditions in $s = 0$, we must calculate a third condition, namely $f''(0)$, in an iterative fashion. This is nicely performed by a shooting method [39].

One notes that v_1 goes quickly towards the value U of the perfect fluid flow as numerical results yield

$$\frac{v_1}{U} = 0.99 \quad \text{for} \quad s = 4.99 \,, \tag{7.56}$$

$$\frac{v_1}{U} = 0.999 \quad \text{for} \quad s = 5.99 \,. \tag{7.57}$$

We observe in Fig. 7.3 the excellent concordance of the theory and the experimental results represented by small circles. The boundary layer theory constitutes one of the major achievements of the twentieth century fluid mechanics.

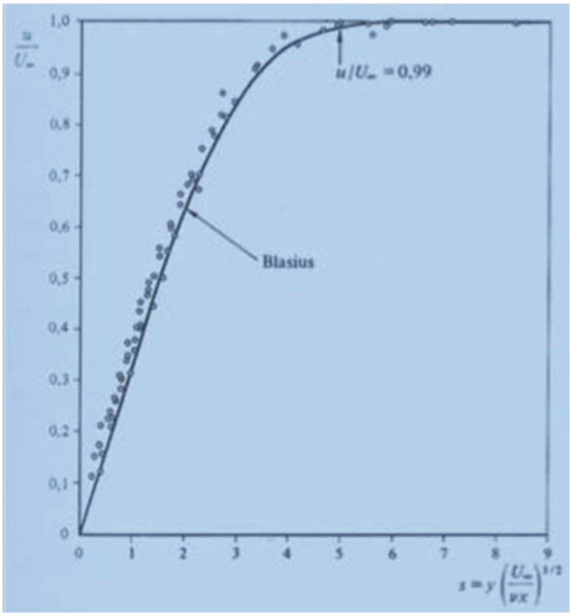

Fig. 7.3 Blasius profile and experimental data [83] where $u = v_1, x = x_1, y = x_2$

7.2.2 Boundary Layer Thicknesses

There exists several ways to define with accuracy the boundary layer thickness. We know that the component v_1 tends asymptotically towards the value U when one approaches this thickness. A first definition giving the sensitive thickness δ_∞ is the value of the ordinate x_2 where v_1 is equal to 99% of U_e. For the flat plate, one obtains

$$\delta_\infty = \delta_{0.99} \simeq 5\sqrt{\frac{\nu x_1}{U}} \,. \tag{7.58}$$

We notice that this thickness depends on the chosen accuracy indicated by the index of δ_∞. Note also that the thickness grows along the obstacle according to $\sqrt{x_1}$.

The concept of displacement thickness δ^* is based on the effective displacement of the exterior flow generated by the reduction of the mass flow rate inside the boundary layer close to the wall, because of the viscous effects as can be seen in Fig. 7.4. It is defined by the relation

$$\delta^* = \int_0^\infty \left(1 - \frac{v_1}{U}\right) dx_2 \,. \tag{7.59}$$

This notion corresponds to the idea of a thickened obstacle, the displacement thickness being defined in such a way to maintain the flow rate conservation between transverse sections made in the flow. We write the relation

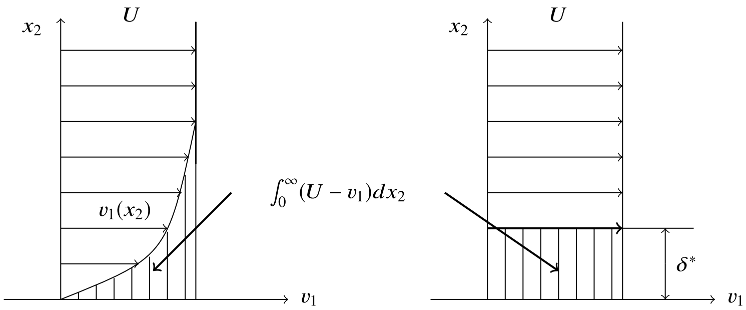

Fig. 7.4 Displacement thickness δ^* in relation with the velocity v_1 at position x_1

$$U\delta^* = \int_0^\zeta (U - v_1)\, dx_2 \tag{7.60}$$

or

$$\int_0^\zeta v_1 dx_2 = U(\zeta - \delta^*) \tag{7.61}$$

expressing the flow rate conservation over the height ζ introduced by (7.29). This length is large with respect to the boundary layer thickness so that it can stretch to infinity. In this case, (7.61) becomes (7.59). For the flat plate, the computation gives

$$\delta^* = 1.721\sqrt{\frac{\nu x_1}{U}} \ . \tag{7.62}$$

Last but not least, the momentum thickness θ defined by the relation

$$\theta = \int_0^\infty \frac{v_1}{U}\left(1 - \frac{v_1}{U}\right) dx_2 \tag{7.63}$$

is linked to the loss of momentum in the boundary layer with respect to the one in the exterior flow. We calculate it as follows

$$\rho \int_0^\infty v_1(U - v_1)\, dx_2 = \rho U^2 \theta \ . \tag{7.64}$$

For the flat plate, one has

$$\theta = 0.664\sqrt{\frac{\nu x_1}{U}} \ . \tag{7.65}$$

These various thicknesses are in the ratio

$$\frac{\delta^*}{\delta_\infty} = \frac{1}{3}, \quad \frac{\theta}{\delta_\infty} = \frac{1}{8} \ . \tag{7.66}$$

7.2.3 Friction and Drag Coefficients

The knowledge of the velocity profile in the vicinity of the flat plate allows the calculation of the wall shear stress and the drag on the plate in the flow. The wall shear stress is given by the relation

$$\tau_w = \mu \frac{\partial v_1}{\partial x_2} \mid_{x_2=0} \ . \tag{7.67}$$

With the stream function (7.51), we obtain

$$v_1 = \frac{\partial \psi}{\partial x_2} = U f'(s) \tag{7.68}$$

$$\frac{\partial v_1}{\partial x_2} = U f''(s) \frac{1}{\delta_0} \ . \tag{7.69}$$

Consequently,

$$\tau_w = \mu U f''(s) \mid_{s=0} \frac{1}{\delta_0} = 0.33 \mu U \sqrt{\frac{U}{\nu x_1}} \ . \tag{7.70}$$

The drag on a plate of length L and of unit width is worth

$$2F_{x_1} = 2 \int_0^L \tau_w \, dx_1 \ . \tag{7.71}$$

The factor of 2 takes into account the existence of the two faces of the plate. By (7.70), one gets

$$F_{x_1} = 0.33 \mu U \sqrt{\frac{U}{\nu}} \int_0^L x_1^{-1/2} dx_1 = 0.66 \sqrt{\mu \rho U^3 L} \ . \tag{7.72}$$

In practice, one expresses the wall shear stress and the drag by dimensionless quantities: the local friction coefficient and the drag coefficient

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2}, \quad C_{x_1} = \frac{F_{x_1}}{\frac{1}{2}\rho U^2 L} \tag{7.73}$$

this last expression corresponding to a plate with unit width.

For the flat plate, one has

$$C_f = 0.66 \frac{\nu}{U} \sqrt{\frac{U}{\nu_1 x}} = \frac{0.66}{Re_{x_1}^{1/2}} \ , \tag{7.74}$$

with the Reynolds number defined by (7.46). The drag coefficient is

$$C_{x_1} = \frac{1.33}{Re_{x_1}^{1/2}} \ . \tag{7.75}$$

7.3 von Kármán Integral Equation

Prandtl's equations become difficult to integrate in closed form as soon as we do not deal with thin profiles. Of course we may always resort to numerical integration; however in this monograph we like to use the analytical tools as much as possible. Therefore we will use another approach based on the integral equation of the momentum relation for the boundary layer. This method is valid for an obstacle shape placed in a moving fluid inasmuch the considered ordinates are small with respect to the local radius of curvature of the obstacle profile.

We integrate Eq. (7.17) in x_2 direction between $x_2 = 0$ and $\zeta(x_1) > \delta_0$. This procedure will generate an ordinary differential equation for θ, called the von Kármán integral equation. The detailed development goes as follows. The first term of (7.17) yields

$$\int_0^{\zeta(x_1)} v_1 \frac{\partial v_1}{\partial x_1} dx_2 = \frac{1}{2} \int_0^{\zeta(x_1)} \frac{\partial v_1^2}{\partial x_1} dx_2 \ . \tag{7.76}$$

An analytical formula that will be most helpful is Leibnitz relation

$$\frac{d}{dx_1} \int_0^{\zeta(x_1)} f(x_2, x_1) dx_2 = \int_0^{\zeta(x_1)} \frac{\partial f}{\partial x_1} dx_2 + f(x_2 = \zeta(x_1), x_1) \frac{d\zeta}{dx_1} \ . \tag{7.77}$$

Using (7.77), we obtain

$$\int_0^{\zeta(x_1)} \frac{\partial v_1^2}{\partial x_1} dx_2 = \frac{d}{dx_1} \int_0^{\zeta(x_1)} v_1^2 dx_2 - v_1^2(\zeta(x_1)) \frac{d\zeta}{dx_1} \ . \tag{7.78}$$

The second term of (7.17) reads

$$\int_0^{\zeta(x_1)} v_2 \frac{\partial v_1}{\partial x_2} dx_2 = v_1 v_2 \Big|_0^{\zeta(x_1)} - \int_0^{\zeta(x_1)} v_1 \frac{\partial v_2}{\partial x_2} dx_2 \ . \tag{7.79}$$

At the wall $x_2 = 0$, we have $v_1 = v_2 = 0$. We assume that at $x_2 = \zeta(x_1)$, $v_1(\zeta(x_1)) = U_e(x_1)$. Furthermore by the incompressibilty constraint, $v_2(\zeta(x_1))$ is expressed by the integral

$$v_2(\zeta(x_1)) = \int_0^{\zeta(x_1)} \frac{\partial v_2}{\partial x_2} dx_2 = -\int_0^{\zeta(x_1)} \frac{\partial v_1}{\partial x_1} dx_2 \,. \tag{7.80}$$

Consequently we write

$$\int_0^{\zeta(x_1)} v_2 \frac{\partial v_1}{\partial x_2} dx_2 = -U_e(x_1) \int_0^{\zeta(x_1)} \frac{\partial v_1}{\partial x_1} dx_2 + \frac{1}{2} \int_0^{\zeta(x_1)} \frac{\partial}{\partial x_1} \left(v_1^2\right) dx_2 \,. \tag{7.81}$$

Applying Leibnitz formula to both integrals on the right-hand side of (7.81), one gets

$$\int_0^{\zeta(x_1)} \left(v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} \right) dx_2 = \frac{d}{dx_1} \int_0^{\zeta(x_1)} v_1^2 dx_2 - U_e \frac{d}{dx_1} \int_0^{\zeta(x_1)} v_1 dx_2 \,. \tag{7.82}$$

Using Eq. (7.21) the pressure gradient becomes

$$\int_0^{\zeta(x_1)} -\frac{1}{\rho} \frac{dp_e}{dx_1} dx_2 = \int_0^{\zeta(x_1)} U_e \frac{dU_e}{dx_1} dx_2 \,. \tag{7.83}$$

The viscous term yields

$$\frac{1}{\rho} \int_0^{\zeta(x_1)} \frac{\partial}{\partial x_2} \left(\mu \frac{\partial v_1}{\partial x_2} \right) dx_2 = \frac{1}{\rho} \left[\mu \frac{\partial v_1}{\partial x_2} \right] \Big|_0^{\zeta(x_1)} = -\frac{\tau_w}{\rho} \,, \tag{7.84}$$

as in the exterior region, we have a perfect fluid (no shear stress involved). At this stage, Prandtl's equation is transformed into the relation

$$\frac{d}{dx_1} \int_0^{\zeta(x_1)} v_1^2 dx_2 - U_e \frac{d}{dx_1} \int_0^{\zeta(x_1)} v_1 dx_2 - \int_0^{\zeta(x_1)} U_e \frac{dU_e}{dx_1} dx_2 = -\frac{\tau_w}{\rho} \,. \tag{7.85}$$

Using Leibnitz formula again, let us compute the following term with $U_e = U_e(x_1)$

$$
\begin{aligned}
\frac{d}{dx_1}\int_0^{\zeta(x_1)} v_1 U_e dx_2 &= \int_0^{\zeta(x_1)} \frac{\partial}{\partial x_1}(v_1 U_e) dx_2 + v_1 U_e(\zeta(x_1))\frac{d\zeta}{dx_1} \\
&= \int_0^{\zeta(x_1)} U_e \frac{\partial v_1}{\partial x_1} dx_2 + \int_0^{\zeta(x_1)} v_1 \frac{dU_e}{dx_1} dx_2 + v_1 U_e(\zeta(x_1))\frac{d\zeta}{dx_1} \\
&= U_e \int_0^{\zeta(x_1)} \frac{\partial v_1}{\partial x_1} dx_2 + \frac{dU_e}{dx_1}\int_0^{\zeta(x_1)} v_1 dx_2 + v_1 U_e(\zeta(x_1))\frac{d\zeta}{dx_1} \\
&= U_e \frac{d}{dx_1}\int_0^{\zeta(x_1)} v_1 dx_2 - v_1 U_e(\zeta(x_1))\frac{d\zeta}{dx_1} + v_1 U_e(\zeta(x_1))\frac{d\zeta}{dx_1} \\
&+ \frac{dU_e}{dx_1}\int_0^{\zeta(x_1)} v_1 dx_2 \, . \qquad (7.86)
\end{aligned}
$$

With (7.86) the second term of relation (7.85) is rewritten

$$
U_e \frac{d}{dx_1}\int_0^{\zeta(x_1)} v_1 dx_2 = \frac{d}{dx_1}\int_0^{\zeta(x_1)} v_1 U_e dx_2 - \frac{dU_e}{dx_1}\int_0^{\zeta(x_1)} v_1 dx_2 \, . \qquad (7.87)
$$

The left hand side of (7.85) leads to the relationship

$$
\begin{aligned}
&\frac{d}{dx_1}\int_0^{\zeta(x_1)} (v_1^2 - v_1 U_e) dx_2 + \frac{dU_e}{dx_1}\int_0^{\zeta(x_1)} (v_1 - U_e) dx_2 \\
&= -\frac{d}{dx_1}\left[U_e^2 \int_0^{\zeta(x_1)} \frac{v_1}{U_e}\left(1 - \frac{v_1}{U_e}\right) dx_2\right] - U_e \frac{dU_e}{dx_1}\int_0^{\zeta(x_1)} \left(1 - \frac{v_1}{U_e}\right) dx_2 \, . \qquad (7.88)
\end{aligned}
$$

As $\zeta(x_1)$ is still arbitrary, we impose now that $\zeta(x_1) \to \infty$. Therefore, with (7.88) and the definitions (7.59) and (7.63), the relation (7.85) reads

$$
\frac{d}{dx_1}(U_e^2 \theta) + U_e \frac{dU_e}{dx_1}\delta^* = \frac{\tau_w}{\rho} \, . \qquad (7.89)
$$

Dividing through by U_e^2 and carrying the algebra, we find the classical form of von Kármán equation

$$\frac{d\theta}{dx_1} = \frac{\tau_w}{\rho U_e^2} - \frac{1}{U_e}\frac{dU_e}{dx_1}(2\theta + \delta^*)$$
$$= \frac{C_f}{2} - \frac{\theta}{U_e}(H+2)\frac{dU_e}{dx_1}\,, \tag{7.90}$$

where H is the shape factor defined by

$$H = \frac{\delta^*}{\theta}\,. \tag{7.91}$$

The increase of the momentum thickness θ depends on the wall shear stress. If the exterior flow is accelerated, i. e. $dU_e/dx_1 > 0$, the θ growth is penalized. On the contrary, if $dU_e/dx_1 < 0$, the thickness θ will grow. If the deceleration is strong enough, the boundary layer will separate from the wall involving that $\tau_w = 0$ and therefore the separation criterion is

$$\frac{\partial v_1}{\partial x_2}|_{x_2=0} = 0\,. \tag{7.92}$$

7.4 von Kármán-Pohlhausen Approximate Method

When the boundary layer theory was proposed by Prandtl, the computational resources of today did not exist. The integral equation of the previous section was very much in use. The von Kármán-Pohlhausen method is a solution procedure that consists in assuming the approximate shape of the velocity profile. Pohlhausen considered a fourth degree polynomial. For the sake of simplicity we will restrict ourselves to a third degree case and we then write the velocity as follows

$$\frac{v_1}{U_e(x_1,0)} = a(x_1) + b(x_1)s + c(x_1)s^2 + d(x_1)s^3,\ s = x_2/\delta_\infty,\ 0 \le s \le 1\,. \tag{7.93}$$

This profile is subjected to several boundary conditions. On the obstacle, the fluid sticks to the wall

$$v_1 = 0 \quad \text{for} \quad s = 0\,. \tag{7.94}$$

Furthermore along the wall, the velocity profile must satisfy Prandtl's equation (7.17). We write

$$\frac{1}{\rho}\frac{dp}{dx_1} = \nu\frac{\partial^2 v_1}{\partial x_2^2} \quad \text{for} \quad s = 0. \tag{7.95}$$

The velocity v_1 for $x_2 = \delta_\infty$ where the boundary layer meets the exterior flow, must be very close to $U_e(x_1, 0)$. Thus we have

$$v_1 = U_e(x_1, 0), \ \frac{\partial v_1}{\partial x_2}\Big|_{x_2=\delta_\infty} = 0 \quad \text{for} \quad s = 1. \tag{7.96}$$

The variable $U_e(x_1, 0)$ will be computed from the theory of perfect fluids taking the shape of the obstacle into account. Then the conditions (7.94) and (7.95) with the integral equation (7.89) will provide the evaluation of the four coefficients introduced in (7.93) and the thickness for the matching condition. When the shape of the profile is more complex, one utilizes a polynomial of higher degree. We thus impose that derivatives of degree equal or higher than two are zero at $s = 1$ and expressions obtained by successive derivatives of Prandtl's equation with respect to the x_2 variable vanish at $s = 1$.

We illustrate the Pohlhausen method for the uniform flow over a flat plate. Conditions (7.94) and (7.95) show immediately that coefficients a and c are zero. Imposing conditions (7.96), the two other coefficients must satisfy the relationships

$$\begin{aligned} b + d &= 1\,, \\ b + 3d &= 0\,. \end{aligned} \tag{7.97}$$

The polynomial reads

$$\frac{v_1}{U_e} = \frac{s}{2}\left(3 - s^2\right)\,. \tag{7.98}$$

We are able now to calculate the approximate displacement thickness

$$\delta^* = \delta_\infty \int_0^1 (1 - \frac{v_1}{U_e}) ds = \frac{3}{8}\delta_\infty \tag{7.99}$$

and the approximate momentum thickness

$$\theta = \delta_\infty \int_0^1 \frac{v_1}{U_e}\left(1 - \frac{v_1}{U_e}\right) ds = \frac{117}{840}\delta_\infty\,. \tag{7.100}$$

The wall shear stress is given by

$$\frac{\tau_w}{\rho} = \nu \frac{\partial v_1}{\partial x_2}\Big|_{x_2=\infty} = \nu \frac{3}{2}\frac{U_e}{\delta_\infty}\,. \tag{7.101}$$

To compute the matching thickness δ_∞ we use Prandtl's equation which is simplified as the variable $U_e(x_1, 0)$ is equal to the constant velocity U

$$\frac{\tau_w}{\rho} = U^2 \frac{d\theta}{dx_1}\,. \tag{7.102}$$

Combining (7.100)–(7.102), we obtain

$$\frac{3\nu U}{2\delta_\infty} = U^2 \frac{117}{840} \frac{d\delta_\infty}{dx_1} \; . \tag{7.103}$$

Integrating (7.103) we eventually get

$$\delta_\infty = \sqrt{\frac{840}{39}} \sqrt{\frac{\nu x_1}{U}} = 4.64 \sqrt{\frac{\nu x_1}{U}} \; . \tag{7.104}$$

This last equation shows that the approximate method is not that far from the exact solution (7.58).

Exercises

7.1 Evaluate by the von Kármán-Pohlhausen method the velocity profile for the plane plate case. The profile is approximated by the fourth degree polynomial

$$\frac{v_1}{U_e(x_1)} = a(x_1) + b(x_1)s + c(x_1)s^2 + d(x_1)s^3 + e(x_1)s^4, \; s = \frac{x_2}{\delta_\infty}, 0 \le s \le 1 \; . \tag{7.105}$$

where x_2 is the normal distance to the wall and δ_∞ the thickness of the boundary layer.

Deduce the boundary layer thicknesses, the displacement thickness δ^*, the momentum thickness θ and the wall shear stress τ_w.

7.2 The velocity profile of a laminar boundary layer over a flat plate can be approximated by the relation

$$v_1(x_2) = A \sin(Bx_2) + C \; . \tag{7.106}$$

- Enumerate the boundary conditions for the problem and obtain the values of the constants A, B, C.
- Compute the expression of the boundary layer thickness $\delta_0 = \delta_0(x_1)$.

  The method goes through the following steps

  1. Calculate θ the momentum thickness as

  $$\theta = \int_0^{\delta_0} \frac{v_1}{U}\left(1 - \frac{v_1}{U}\right) dx_2 \; . \tag{7.107}$$

  2. Show that the equation for θ is

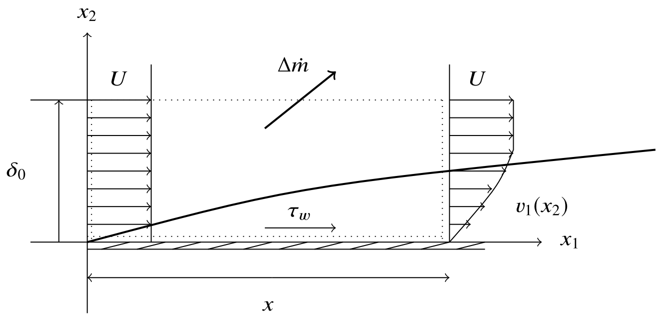

Fig. 7.5 Boundary layer over a flat plate

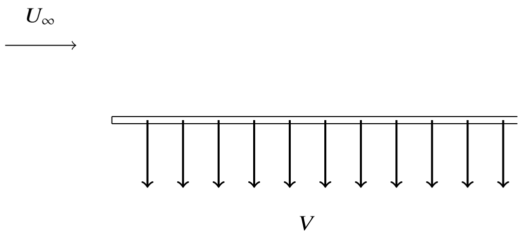

Fig. 7.6 Boundary layer over a porous wall

$$\frac{\partial \theta}{\partial x_1} = \frac{\tau_w}{\rho U^2} \, , \tag{7.108}$$

using the momentum conservation and continuity on the control volume (dotted lines) displayed in Fig. 7.5.

- Calculate the wall shear stress τ_w and deduce δ_0.

7.3 We consider the plane flow of an incompressible fluid over a porous plate. Through evenly distributed small holes, part of the laminar boundary layer is sucked. Let us denote by V the normal component of the velocity to the plate surface at $x_2 = 0$. Assuming V is constant, it is possible to demonstrate that for very small values of the aspiration velocity, i.e. $V/U_\infty \ll 1$, the boundary layer thickness becomes constant and the shape of the velocity profile invariable at long distance from the leading edge (Fig. 7.6).

At a long distance from the leading edge, it is requested to calculate

- from Prandtl's equations, the velocity profile $v_1(x_2)/U_\infty$.
- the value of the wall shear stress τ_w.
- the displacement and momentum thicknesses. Compute the shape factor (7.91). Compare this factor to the one of the laminar boundary layer over a non porous plate.