Chapter 6
Plane Irrotational Flows of Perfect Fluid

In this chapter we consider the steady state two-dimensional irrotational flows of inviscid incompressible fluid. The monograph by L. M. Milne-Thomson [59] is a major contribution to the subject of this chapter.

In order to introduce the theory of complex variables, we will leave the index notation and use the standard coordinates, namely $x = x_1$, $y = x_2$. In that case the velocity components are $u = v_1$, $v = v_2$. We follow closely the book by Rhyming [83] that has been a cornerstone in the fluid mechanics courses of the Mechanical Engineering Department at the Swiss Institute of Technology Lausanne.

The assumption of irrotational flow induces the existence of a velocity potential such that

$$\boldsymbol{v} = \boldsymbol{\nabla}\varphi \,. \tag{6.1}$$

The plane flow satisfies two relations

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{6.2}$$

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \,. \tag{6.3}$$

The incompressibility constraint is trivially satisfied by the stream function ψ

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \,. \tag{6.4}$$

On the other hand by the velocity potential, one has

$$u = \frac{\partial \varphi}{\partial x}, \quad v = \frac{\partial \varphi}{\partial y} \,. \tag{6.5}$$

M. O. Deville, *An Introduction to the Mechanics of Incompressible Fluids*,
https://doi.org/10.1007/978-3-031-04683-4_6

Let us show that the stream function and velocity potential are conjugate harmonic functions. Indeed, combining (6.1) and (6.2), one obtains $\Delta\varphi = 0$. Inserting (6.4) in (6.3), one gets $\Delta\psi = 0$. With (6.4) and (6.5), one writes

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x} \ . \tag{6.6}$$

These relations are the Cauchy-Riemann conditions. If the partial derivatives $\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}$ are continuous in a simply connected domain, relations (6.6) are necessary and sufficient conditions for the existence of a holomorphic (continuous, analytic and uniform) function $f(z)$ called the complex potential of the flow. It is defined as

$$f(z) = \varphi(x, y) + i\,\psi(x, y) \tag{6.7}$$

and depends on the complex variable $z = x + iy$, $i = \sqrt{-1}$. If the domain is multiply connected, the function given by (6.7) is analytic, although not necessarily uniform.

The complex analytic potential generates two sets of orthogonal curves: the equipotentials $\varphi = cst$ and the streamlines $\psi = cst$. From the definitions of the velocity potential and the stream function, it appears that the differentials $d\varphi$ and $d\psi$ are exact differentials

$$d\varphi = u\,dx + v\,dy \ , \tag{6.8}$$

$$d\psi = -v\,dx + u\,dy \ . \tag{6.9}$$

For an irrotational flow, the circulation of the velocity vector along a closed contour surrounding neither obstacle nor singularity is zero. We have using Stokes theorem

$$\Gamma = \int_C \boldsymbol{v} \cdot d\boldsymbol{\tau} = \int_S \mathbf{curl}\,\boldsymbol{v} \cdot \boldsymbol{n}\,dS = 0 \ , \tag{6.10}$$

with $\boldsymbol{\tau}$ the unit tangent vector to the curve C. In a simply connected domain, for any curvy segment located between points A and B, the circulation is independent on the followed path. One has

$$\Gamma = \int_A^B \boldsymbol{v} \cdot d\boldsymbol{\tau} = \int_A^B (u\,dx + v\,dy) = \int_A^B d\varphi = \varphi_B - \varphi_A \ . \tag{6.11}$$

The flow rate Q across the segment AB is also independent on the path. One has

$$Q = \int_A^B \boldsymbol{v} \cdot \boldsymbol{n}\,d\tau = \int_A^B (un_x + vn_y)d\tau = \int_A^B (udy - vdx) = \int_A^B d\psi = \psi_B - \psi_A \ . \tag{6.12}$$

6.1 Complex Velocity

The use of complex variables enables the introduction of the velocity vector in the complex plane, namely $u + iv$. The conjugate velocity is called the *complex velocity*

$$w = u - iv \, , \tag{6.13}$$

which is evaluated by the derivative of the complex potential f by the relation

$$w = \frac{df}{dz} = u - iv \, . \tag{6.14}$$

As the function $f(z)$ is analytic, its derivative $f'(z)$ is independent on the way the limit is taken, i.e. on how the increment Δz goes to zero. As a reminder, by analogy with the derivative of a real function, one has

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \, . \tag{6.15}$$

It is easy to verify that using $\Delta z = \Delta x$ or $\Delta z = i\Delta y$, the same limit is obtained and one has

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = \frac{\partial \varphi}{\partial x} + i\frac{\partial \psi}{\partial x} = u - iv \, , \tag{6.16}$$

$$\frac{df}{dz} = \frac{1}{i}\frac{\partial f}{\partial y} = -i\left(\frac{\partial \varphi}{\partial y} + i\frac{\partial \psi}{\partial y}\right) = u - iv \, . \tag{6.17}$$

The polar representation of complex numbers $z = re^{i\theta}$ gives the relation

$$dz = e^{i\theta}\, dr \, , \tag{6.18}$$

leading to the definition

$$f'(z) = \frac{1}{e^{i\theta}}\frac{\partial f}{\partial r} = w \, . \tag{6.19}$$

Examination of Fig. 6.1 shows that

$$v_r - iv_\theta = (u - iv)e^{i\theta} \, . \tag{6.20}$$

Combining Eqs. (6.19) and (6.20), one obtains

$$\frac{\partial f}{\partial r} = v_r - iv_\theta \, . \tag{6.21}$$

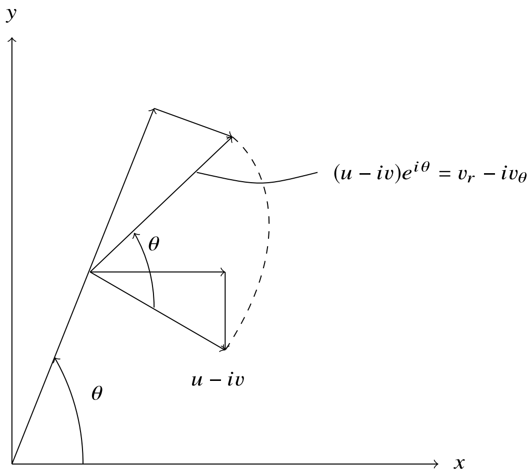


Fig. 6.1 Rotation of the complex velocity w by the angle θ

6.2 Complex Circulation Γ

Integration of the complex potential along an arbitrary path AB yields the following result

$$\int_A^B df(z) = \int_A^B (u - iv)\, dz = \int_A^B (udx + vdy) + i \int_A^B (udy - vdx)$$
$$= \Gamma + iQ = f_B - f_A \ . \tag{6.22}$$

The complex velocity w must be defined in a univocal manner at each point of the complex plane for obvious physical reasons. Thus it is a holomorphic function. If this velocity were a multiform (or multivalued) function, cuts must be used in order to make w uniform and therefore integrable. As far as the complex potential is concerned, the situation is different as it results from the integration of (6.19) and may be a multiform function.

Recall that every point, where $w(z)$ is holomorphic in a circle centered in z_0 and may be developed in Taylor series, is an ordinary or regular point. If this development is not possible, the point is singular. Singularities are poles, essential singular points and branch points. Without rigorous definitions, consider the next examples to illustrate these concepts. The function $1/z^2$ has a pole of order two at the origin. Function $1/(z^2 + 1)$ has two simple poles $z = \pm i$. Function $e^{\frac{1}{z-a}}$ has an essential singular point in $z = a$. Function $z^{\frac{1}{n}}$ presents a branch point at the origin.

Let us now examine two cases in detail.

1. If z_0 is the center of a circle of radius R where the flow is regular, cf. Fig. 6.2, it is possible to develop w as a Taylor series with complex coefficients such that

$$w = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots , \quad | z - z_0 | < R \ . \tag{6.23}$$

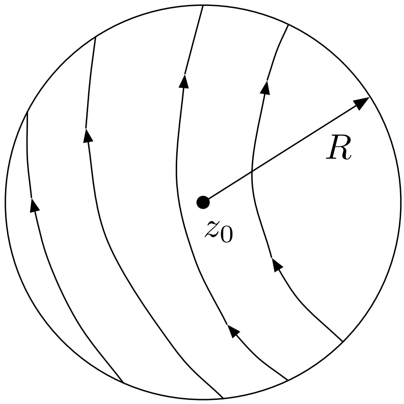


Fig. 6.2 Regular flow in a circle of radius R centered in z_0

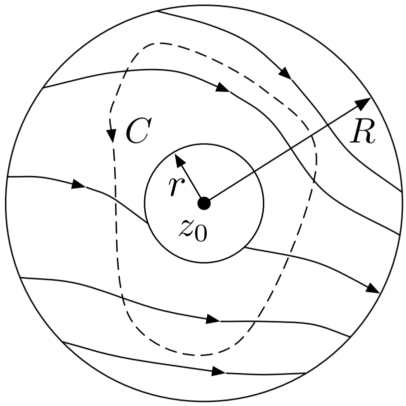


Fig. 6.3 Regular flow between two circles centered in z_0

2. If z_0 is the center of an annular domain comprised between two circles of respective radii r and R where the flow is regular, cf. Fig. 6.3, w is developed in Laurent series

$$w = \sum_{n=-\infty}^{n=\infty} a_n \,(z - z_0)^n, \quad r <\mid z - z_0 \mid< R \,. \tag{6.24}$$

The Laurent series includes two power series, one of ratio $z - z_0$ that converges for $\mid z - z_0 \mid< R$, the other one of ratio $1/(z - z_0)$ converging for $\frac{1}{|z-z_0|} < \frac{1}{r}$. Integrating the series (6.24), one obtains the complex potential

$$f(z) = \sum_{n=0}^{n=\infty} \frac{a_n}{n+1}(z - z_0)^{n+1} + a_{-1} \ln(z - z_0) + \sum_{n=-\infty}^{n=-2} \frac{a_n}{n+1}(z - z_0)^{n+1} + C \,. \tag{6.25}$$

In (6.25), the presence of the multiform complex function $\ln(z - z_0)$ influences the integration of the complex potential in relation (6.22). If the integration path is an arbitrary closed curve C that circles around z_0 only once, then the theorem of residues allows writing that the complex circulation $\mathbf{\Gamma}$ is

$$\mathbf{\Gamma}_C = \int_C df(z) = 2\pi i a_{-1} \,. \tag{6.26}$$

When the radius r is zero and the isolated singularity in z_0 is a pole, the calculus of residue utilises the relation

$$a_{-1} = \lim_{z \to z_0} (z - z_0) w \tag{6.27}$$

for a pole of order one and

$$a_{-1} = \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \frac{(z - z_0)^n w}{(n-1)!} \tag{6.28}$$

for a pole of order n. If R goes to infinity and the velocity at infinity is uniform, the complex potential (6.25) is such that $a_i = 0, i = 1, \ldots, \infty$ and the velocity at infinity is given by

$$w(\infty) = a_0 \,. \tag{6.29}$$

6.3 Elementary Complex Potential Flows

The irrotational flow is essentially a linear problem as the velocity potentials and the stream function satisfy each of them a Laplace equation. Thus in order to build up a somewhat complicated flow, we take full advantage of the superposition principle by combining simple complex potentials. This new potential will allow the calculation of the flow rate and the resulting circulation using the same superposition principle. For the pressure field, the Bernoulli equation will be our tool for analysis.

In the following figures, the equipotentials will be represented by dashed lines and the streamlines by solid lines with arrows pointing in the flow direction.

6.3.1 Parallel Homogeneous Flow

The complex potential

$$f(z) = Az, w = A = const \,, \tag{6.30}$$

is a homogeneous parallel flow. If A is real, the flow is parallel to direction x while if A is imaginary, it is in direction y. For complex A we have a superposition of both previous cases.

6.3.2 *Vortex and Source*

Let the potential be

$$f(z) = (A + iB)\ln z \tag{6.31}$$

with A and B real. In polar coordinates,

$$f(z) = (A + iB)(\ln r + i\theta) = (A\ln r - B\theta) + i(B\ln r + A\theta) \tag{6.32}$$

The complex velocity is

$$w = f'(z) = \frac{A + iB}{z} \; . \tag{6.33}$$

In polar coordinates, by (6.21),

$$\frac{\partial f}{\partial r} = v_r - i v_\theta = \frac{A + iB}{r} \; . \tag{6.34}$$

This velocity has a pole of order one and the residue is $(A + iB)$. The integration of the velocity involves a closed contour encircling once the origin. By (6.22), one has

$$\Gamma_0 + i Q_0 = 2\pi i(A + iB) \tag{6.35}$$

and thus

$$\Gamma_0 = -2\pi B, \quad Q_0 = 2\pi A \; . \tag{6.36}$$

Let us inspect the various cases corresponding to this potential.

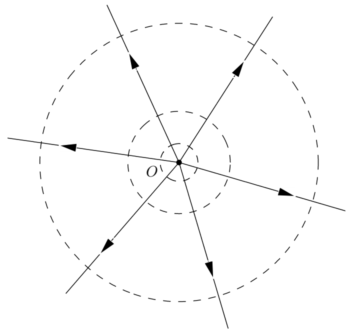
6.3.2.1 Sink or Source

$$A \neq 0, \quad B = 0 \; ,$$

$$\varphi = A\ln r, \qquad v_r = \frac{A}{r}, \quad \Gamma = 0 \; , \tag{6.37}$$

$$\psi = A\theta, \qquad v_\theta = 0, \quad Q = 2\pi A \; . \tag{6.38}$$

Fig. 6.4 Source flow, $f = A \ln z$



In Fig. 6.4, one observes that the equipotentials are circles centered at the origin and the streamlines are half-straight lines from the same point. The velocity is radial and inversely proportional to the distance to the origin. The flow is a source for $A > 0$ or a sink ($A < 0$) with a flow rate $2\pi A$.

6.3.2.2 Vortex

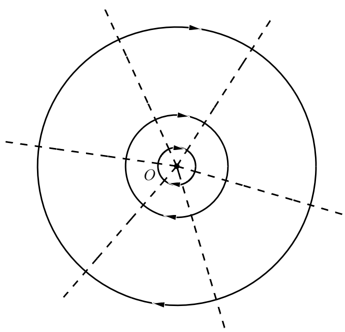
$$A = 0, \quad B \neq 0 \, ,$$

$$\varphi = -B\theta, \qquad v_r = 0, \quad \Gamma = -2\pi B \, , \tag{6.39}$$

$$\psi = B \ln r, \qquad v_\theta = -\frac{B}{r}, \quad Q = 0 \, . \tag{6.40}$$

Here (cf. Fig. 6.5), we may just swap the streamlines and the equipotentials of the previous case. The equipotentials are radial lines while the streamlines are circles centered at the origin. A fluid particle rotates around the origin with a velocity inversely proportional to the distance to the origin. If $B > 0$, the rotation is clockwise (on an analog watch). Note that the motion is irrotational except at the origin where

Fig. 6.5 Vortex flow, $f = iB \ln z$



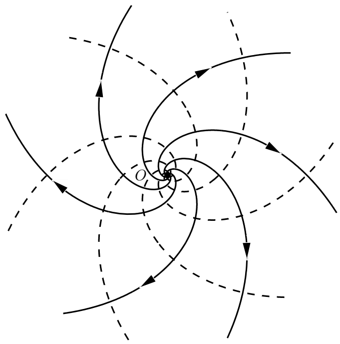


Fig. 6.6 Logarithmic spiral flow, $f = (A + iB)\ln z$

a vortex is concentrated in the singularity. This is the reason why Γ_0 is different from zero.

6.3.2.3 Spiral Flow

$$A \neq 0, \quad B \neq 0 . \tag{6.41}$$

The velocity potential and the stream function read

$$\varphi = A \ln r - B\theta, \quad \psi = A\theta + B \ln r . \tag{6.42}$$

One deduces

$$v_r = A/r, \quad \Gamma = -2\pi B, \quad v_\theta = -B/r, \quad Q = 2\pi A . \tag{6.43}$$

The fluid particles rotate around the origin while moving away if there is a source at the origin. The equipotentials and the streamlines are orthogonal nets of logarithmic spirals as can be seen in Fig. 6.6.

6.3.3 Complex Potential in Power of z

Let the complex potential be

$$f(z) = Az^n = Ar^n\, e^{in\theta} = Ar^n(\cos n\theta + i \sin n\theta) \tag{6.44}$$

with $A \in \mathrm{R}$ and $A > 0$. The velocity potential is

$$\varphi = Ar^n \cos n\theta \tag{6.45}$$

and the stream function

$$\psi = Ar^n \sin n\theta \ . \tag{6.46}$$

The complex velocity reads

$$w = f'(z) = nAz^{n-1} \ , \tag{6.47}$$

and presents a singularity at the origin for n non-positive integer or negative n. In polar coordinates, by (6.21),

$$\frac{\partial f}{\partial r} = v_r - i v_\theta = nAr^{n-1}(\cos n\theta + i \sin n\theta) \ . \tag{6.48}$$

In order to obtain a uniform velocity everywhere in the complex plane, n must be an integer. The streamlines $\psi = 0$ are radial lines going through the origin given by

$$\theta = \frac{k\pi}{n}, k = 0, 1, \dots \ . \tag{6.49}$$

For non-integer values of n, cuts in the complex plane are needed to make the velocity uniform. The stream lines $\psi = 0$ are again radial lines through the origin.

Let us examine successively the following cases: integer $n \geq 0$, non integer $n \geq 0$, integer $n < 0$.

6.3.3.1 Positive Integer *n*

1. $\mathbf{n = 0}$. The fluid is at rest, i.e. hydrostatic regime. One has

$$w = 0 \ . \tag{6.50}$$

2. $\mathbf{n = 1}$. This is the case of homogeneous and parallel flow described in Sect. 6.3.1.
3. $\mathbf{n = 2}$. The complex potential gives

$$f(z) = A(x^2 - y^2) + 2iAxy \ , \tag{6.51}$$

the velocity components of which are

$$u = 2Ax, \quad v = -2Ay \ . \tag{6.52}$$

The equipotentials are equilateral hyperbolas whose asymptotes are bisectors of the coordinate axes. The streamlines are also equilateral hyperbolas that are orthogonal to the previous ones and whose asymptotes are the coordinate axes, cf. Fig. 6.7. The origin of the axes is called the stagnation point.

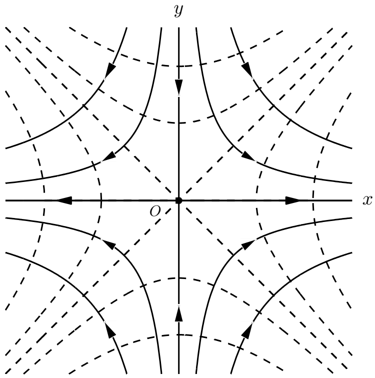


Fig. 6.7 Flow corresponding to $f = Az^2$

6.3.3.2 Non Integer $n \geq 1/2$

The use of fractional values for n is possible inasmuch one carries out cuts in the complex plane to get a uniform velocity in the domain at hand. These cuts are made most of the time along streamlines that are assimilated to obstacles. The best choice to obtain a very wide domain consists in utilizing cut lines going through the origin.

1. $\mathbf{n} = \frac{1}{2}$. The potential

$$f(z) = Ar^{1/2}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}) \tag{6.53}$$

produces the streamlines

$$\psi = Ar^{1/2}\sin\frac{\theta}{2}\ . \tag{6.54}$$

For values $\theta = 0$ or $\theta = 2\pi$, $\psi = 0$. One has the flow around a thin plate (cf. Fig. 6.8) that coincides with the semi-axis $x > 0$. One notices that the radial velocity component depends on $r^{-1/2}$ indicating that it goes to infinity at the origin.

2. $\mathbf{n} > \frac{1}{2}$ .The streamlines $\psi = 0$ are obtained for $\theta = \frac{k\pi}{n}$, $k = 0, 1, \ldots,$. Figure 6.9 shows that the flow is generated in the concave part of a dihedron $n > 1$ or around the convex part of a dihedron $n < 1$. The velocity at the tip of the convex dihedron ($n < 1$) goes to infinity, cf. Fig. 6.10 (left) while for a concave dihedron cf. Fig. 6.10 (right), the velocity at the origin goes to zero. Bernoulli's law that connects pressure and the square of the velocity shows that for a convex dihedron, we will have an infinite negative pressure at the origin and for the concave dihedron, pressure will be maximum at the vertex (Fig. 6.10).

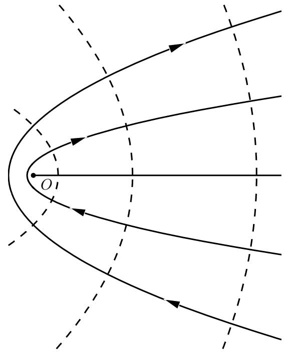


Fig. 6.8 Flow corresponding to $f = Az^{1/2}$

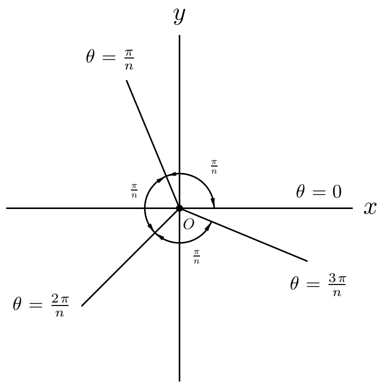


Fig. 6.9 Flow corresponding to $f = Az^n, n > 1/2$

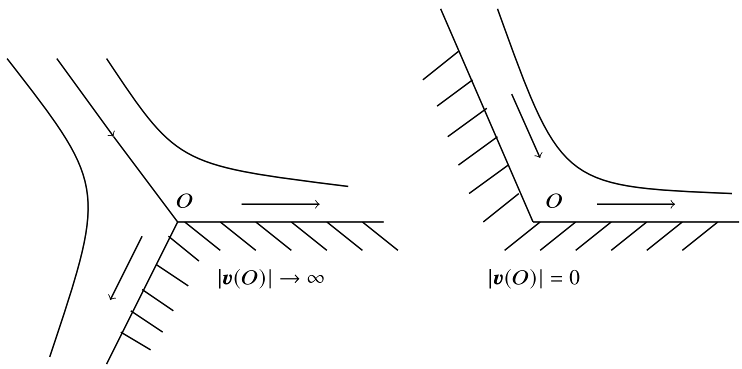


Fig. 6.10 Convex dihedron Concave dihedron

6.3.3.3 Dipole

Consider the case $n = -1$. The complex potential

$$f(z) = \frac{A}{z} = \frac{A}{r}(\cos\theta - i\sin\theta) \tag{6.55}$$

yields real and imaginary parts

$$\varphi = \frac{Ax}{x^2 + y^2} = \frac{A}{r}\cos\theta, \quad \psi = \frac{-Ay}{x^2 + y^2} = -\frac{A}{r}\sin\theta\,. \tag{6.56}$$

The equipotential lines are circles centered on Ox and tangent to Oy at the origin, while the streamlines are circles centered on Oy and tangent to Ox in O, cf. Fig. 6.11. The f function and its derivative present a pole at the origin. The velocity

$$w = -\frac{A}{z^2} \tag{6.57}$$

has a pole of order two at the origin. The relations (6.57) and (6.22) show that the residue a_{-1} at the origin is zero. Thus the flow rate and the circulation associated to a closed contour around the origin also vanish. In fact everything happens as if we had at the origin a source and a sink of the same flow rate. This case corresponds to a doublet or dipole.

The shape of the streamlines (Fig. 6.11) show that they become tighter at the origin. The velocity increases when one approaches to the origin and goes to infinity at that precise point. It is possible to demonstrate that the potential of a dipole is the limit of the potential of a couple source-sink located from either side of the origin on the x axis, when the distance between the singularities goes to zero, while their opposite flow rates go to infinity (cf. Fig. 6.12)

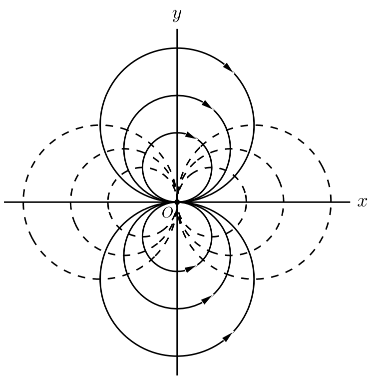


Fig. 6.11 Flow corresponding to $f = Az^{-1}$

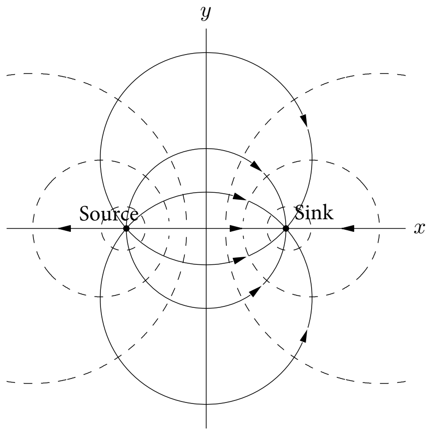


Fig. 6.12 Flow corresponding to a doublet

6.4 Flow Around a Circular Cylinder

For the sake of simplicity, the cylinder has a circular cross section of radius a centered at the origin.

6.4.1 Flow Without Circulation Around a Cylinder

Let us apply the superposition principle by combining a dipole with a homogeneous parallel flow. The complex potential reads

$$f(z) = U\left(z + \frac{a^2}{z}\right) . \tag{6.58}$$

This potential corresponds to the definition (6.25). Decomposing real and imaginary parts, the velocity potential is given by

$$\varphi = Ux\left(1 + \frac{a^2}{x^2 + y^2}\right) \tag{6.59}$$

and the streamline

$$\psi = Uy\left(1 - \frac{a^2}{x^2 + y^2}\right) . \tag{6.60}$$

The streamline $\psi = 0$ is composed of the circle of radius a centered at the origin $x^2 + y^2 = a^2$ and of the abscissa axis $y = 0$. The velocity is easily evaluated as

$$w = f'(z) = U\left(1 - \frac{a^2}{z^2}\right) = U[1 - \frac{a^2(x^2 - y^2 - 2ixy)}{(x^2 + y^2)^2}] . \tag{6.61}$$

Inspection of (6.61) shows that the velocity is zero at the two points of intersection of the abscissa axis with the circle in $x = \pm a$, $y = 0$; these are the stagnation points S_1 and S_2 of the flow. We note also that the velocity reaches its maximum norm equal to $2U$ at the intersection points of the circle with the ordinates axis in $x = 0$, $y = \pm a$.

To obtain the velocity on the cylinder, it is easier to work in polar coordinates. One has

$$f(z) = U\left(re^{i\theta} + \frac{a^2}{re^{i\theta}}\right) . \tag{6.62}$$

The equation of the circle is $z = ae^{i\theta}$. The components v_r and v_θ are obtained by (6.21)

$$v_r - iv_\theta = U\left(e^{i\theta} - \frac{a^2}{r^2e^{i\theta}}\right) . \tag{6.63}$$

On the cylinder, the velocity is such that

$$v_r = 0, \quad v_\theta = -2U\sin\theta \tag{6.64}$$

showing that it is tangent to the cylinder with a sine variation.

For the steady state flow with the effects of gravity neglected, Bernoulli's relation (4.41) gives

$$C = \frac{p}{\rho} + \frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2} , \tag{6.65}$$

We have

$$\frac{p_\infty}{\rho} + \frac{U^2}{2} = \frac{p}{\rho} + 2U^2\sin^2\theta . \tag{6.66}$$

Defining the pressure coefficient C_p by

$$C_p = \frac{p - p_\infty}{\rho\frac{U^2}{2}} , \tag{6.67}$$

one obtains

$$C_p = 1 - 4\sin^2\theta . \tag{6.68}$$

The pressure field is symmetric with respect to the axial plane orthogonal to the flow direction and therefore if we integrate the pressure on the cylinder to obtain the lift force F_y, it will be zero. The streamlines are exhibited in Fig. 6.13.

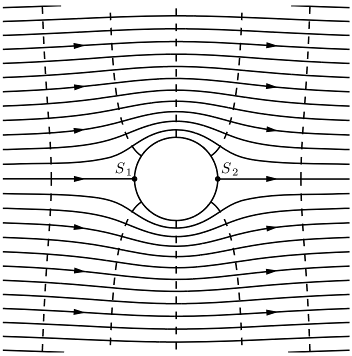


Fig. 6.13 Flow without circulation around a cylinder

6.4.2 Flow with Circulation Around a Cylinder

To produce circulation, it is necessary to introduce a logarithmic term in the potential

$$f(z) = U\left(z + \frac{a^2}{z}\right) - i\frac{\Gamma}{2\pi}\ln\frac{z}{a} \,. \tag{6.69}$$

This logarithmic term is modified in such a way that the streamline $\psi = 0$ contains the circle of radius a centered at the origin. The velocity is computed as

$$w = f'(z) = U\left(1 - \frac{a^2}{z^2}\right) - \frac{i\Gamma}{2\pi z} \,. \tag{6.70}$$

The velocity potential and the streamlines are easily obtained

$$\varphi = Ur\cos\theta\left(1 + \frac{a^2}{r^2}\right) + \Gamma\frac{\theta}{2\pi} \tag{6.71}$$

$$\psi = Ur\sin\theta\left(1 - \frac{a^2}{r^2}\right) - \frac{\Gamma}{2\pi}\ln\frac{r}{a} \,. \tag{6.72}$$

The velocity components are

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U\cos\theta\left(1 - \frac{a^2}{r^2}\right) \tag{6.73}$$

$$v_\theta = -\frac{\partial\psi}{\partial r} = -U\sin\theta\left(1 + \frac{a^2}{r^2}\right) + \frac{\Gamma}{2\pi r} \,. \tag{6.74}$$

The radial component vanishes on the circle. The calculation of the flow stagnation points is carried out via (6.70)

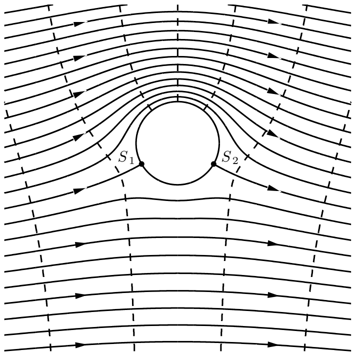


Fig. 6.14 Flow around a circular cylinder with $\Gamma = 4\pi U a \sin\theta$

$$z^2 - \frac{i\Gamma}{2\pi U} z - a^2 = 0 \; . \tag{6.75}$$

The solution reads

$$z = \frac{i\Gamma}{4\pi U} \pm \sqrt{a^2 - (\frac{\Gamma}{4\pi U})^2} \; . \tag{6.76}$$

Consider the case of a positive discriminant

$$a^2 - (\frac{\Gamma}{4\pi U})^2 > 0 \; . \tag{6.77}$$

Let us set $\sin\theta = \Gamma/(4\pi U a)$. The solution (6.76) becomes

$$z = a(\pm\cos\theta + i\sin\theta) \; . \tag{6.78}$$

If $\mid \Gamma \mid < 4\pi U a$, Eq. (6.78) shows that two stagnation points are located symmetrically with respect to the vertical axis and the streamlines are shown in Fig. 6.14. For $\Gamma = 4\pi U a$ with $\Gamma < 0$, $\sin\theta = -1$ and $\theta = 3\pi/2$, $z = -ia$ and the stagnation point is unique and situated at the bottom point of the cylinder. Figure 6.15 displays the streamlines of this particular case.

If $\mid \Gamma \mid > 4\pi U a$, then one finds

$$z = i\left(\frac{\Gamma}{4\pi U} \pm \sqrt{(\frac{\Gamma}{4\pi U})^2 - a^2}\right) \; . \tag{6.79}$$

The two stagnation points are conjugated with respect to the circle of radius a. One of these points is inside the circle and does not participate to the flow, cf. Fig. 6.16.

The streamline $\psi = 0$ separates the flow in two distinct regions. The fluid close to the cylinder stays locked up because of the vortex strength. The velocities in that region are high and consequently, pressure decreases. The pressure on the cylinder surface is obtained by the Bernoulli equation (4.41)

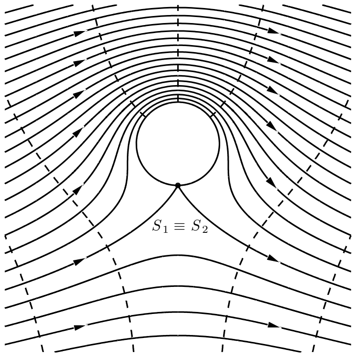


Fig. 6.15 Flow around a circular cylinder with $\Gamma = 4\pi U a$

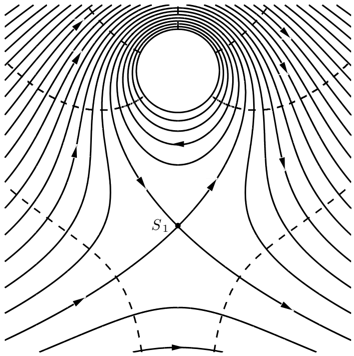


Fig. 6.16 Flow around a circular cylinder with $\mid \Gamma \mid > 4\pi U a$

$$\frac{p}{\rho} = \frac{p_\infty}{\rho} + \frac{1}{2}U^2 - \frac{1}{2}\left(4U^2 \sin^2\theta + \frac{\Gamma^2}{4\pi^2 a^2} - \frac{2\Gamma U}{\pi a}\sin\theta\right) . \tag{6.80}$$

The last term in the parenthesis generates the lift as we will verify in the sequel.

6.5 Blasius Theorem: Forces and Moment

Let C be a closed contour corresponding to the wall of a rigid obstacle in the steady irrotational flow of an inviscid fluid, cf. Fig. 6.17.

The wall coincides with a streamline. To evaluate the force exerted by the fluid on the wall, one integrates the density of the contact forces on the curve C. Using Cauchy theorem (1.53) and the constitutive relation for the perfect fluid (1.70), one obtains

$$\boldsymbol{F} = \int_C \boldsymbol{t}(\boldsymbol{n}) ds = \int_C -p\boldsymbol{n}\, ds . \tag{6.81}$$

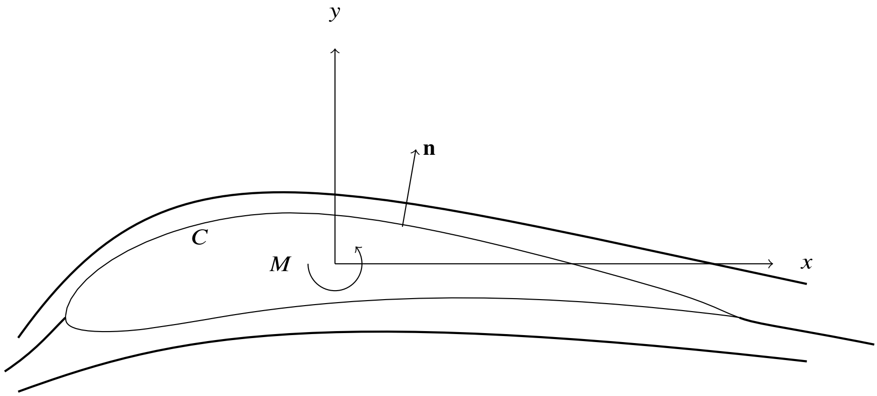


Fig. 6.17 Flow around an airfoil

In complex variables, the wall unit normal is given by

$$\boldsymbol{n} = n_x + i n_y = \frac{dy}{ds} - i\frac{dx}{ds} = -i\frac{dz}{ds}\ , \tag{6.82}$$

where s is a curvilinear coordinate along C.

Combining (6.81) and (6.82) we find

$$\boldsymbol{F} = i\int_C p\, dz\ . \tag{6.83}$$

The pressure field on the profile C is governed by Bernoulli equation

$$\frac{p}{\rho} + \frac{(\boldsymbol{v}\cdot\boldsymbol{v})_C}{2} = \frac{p_0}{\rho}\ , \tag{6.84}$$

with the last term evaluated at the upstream airfoil stagnation point. The term $(\boldsymbol{v}\cdot\boldsymbol{v})_C$ is given by

$$(\boldsymbol{v}\cdot\boldsymbol{v})_C = (u^2+v^2)_C = [(u-iv)(u+iv)]_C = (w\overline{w})_C = (\frac{df}{dz}\frac{d\overline{f}}{d\overline{z}})_C\ , \tag{6.85}$$

with the overline indicating complex conjugation. Therefore the Bernoulli equation is rewritten as

$$\frac{p}{\rho} + \frac{1}{2}\frac{df}{dz}\frac{d\overline{f}}{d\overline{z}} = \frac{p_0}{\rho}\ . \tag{6.86}$$

Consequently, the force vector $\boldsymbol{F}$ (6.83) becomes

$$\begin{aligned}\boldsymbol{F} = F_x + iF_y &= i\int_C (p_0 - \frac{\rho}{2}\frac{df}{dz}\frac{d\overline{f}}{d\overline{z}})dz \\ &= ip_0\int_C dz - \frac{i\rho}{2}\int_C \frac{d\overline{f}}{d\overline{z}}df\ . \end{aligned} \tag{6.87}$$

On the streamline C, $\psi = cst$, thus the differential $df = d\varphi + id\psi$ is real as $d\psi = 0$ and thus $df = d\overline{f}$. Moreover $d\overline{f}/d\overline{z} = df/dz$. By Cauchy theorem, the first term of the right hand side of (6.87) is zero. One obtains

$$\overline{\boldsymbol{F}} = \frac{i\rho}{2}\int_C (\frac{df}{dz})^2 dz = \frac{i\rho}{2}\int_C w^2 dz\ . \tag{6.88}$$

The moment exerted by the fluid on the obstacle with respect to the origin of the axes is given by

$$\begin{aligned} M &= \int_C -p\,\boldsymbol{n}\times\boldsymbol{x}\,ds = -\int_C p(n_x y - n_y x)ds = \int_C -p(x\frac{dx}{ds} + y\frac{dy}{ds})ds \\ &= -\int_C p\Re(zd\overline{z}) = \Re[\int_C -p_0(zd\overline{z}) + \frac{\rho}{2}\int_C z\frac{df}{dz}d\overline{f}]\,, \end{aligned} \tag{6.89}$$

where $\Re$ indicates the real part of the expression. Along a streamline, one has

$$M = \frac{\rho}{2}\Re\int_C z\overline{w}^2 dz = \frac{\rho}{2}\Re\int_C zw^2 dz\ , \tag{6.90}$$

as $(\overline{w}d\overline{z})_C = (w\,dz)_C$.

Let us apply these force and moment relations to the case of the cylinder in a flow with circulation. By (6.70) and the theorem of residues, one gets

$$\overline{\boldsymbol{F}} = i\rho\Gamma U, \quad F_x = 0, \quad F_y = -\rho\Gamma U\ . \tag{6.91}$$

We find a zero drag and a lift proportional to the circulation. As we assumed $\Gamma < 0$ and $U > 0$, the force F_y is positive and oriented in direction Oy.

Let us now examine the moment. From (6.70) and (6.90), we have

$$\begin{aligned} M &= \frac{\rho}{2}\Re\int_C \left(U(1 - \frac{a^2}{z^2}) - i\frac{\Gamma}{2\pi\,z}\right)^2 zdz \\ &= \frac{\rho}{2}\Re\left(2\pi\,i\,a^2\left[-2U^2 + \frac{\Gamma^2}{4\pi^2 a^2}\right]\right) = 0\ . \end{aligned} \tag{6.92}$$

Thus the couple exerted on the cylinder vanishes as the resulting M is purely imaginary.

6.6 The Method of Conformal Transformation

Let z and ζ be two complex planes, the first one with variable $z = x + iy$ and the second one with variable $\zeta = \xi + i\eta$. We consider a one-to-one correspondance, i.e. a bijective transformation between these two planes, given by

$$\zeta = \zeta(z) \,, \tag{6.93}$$

such that each point z in the z plane has an image in the ζ plane. Therefore a curve in z plane is equivalent to another curve in ζ plane in such a way that the second curve is an image or mapping of the first one.

6.6.1 *A Few Properties of the Conformal Transformation*

The mapping is conformal because it preserves or conserves the angles but allows stretching or shortening of lengths. In Fig. 6.18, two elementary vectors dz_1 and dz_2 of the z plane are mapped onto the corresponding counterpart vectors $d\zeta_1$ and $d\zeta_2$. We write

$$d\zeta_1 = \frac{d\zeta}{dz} dz_1, \quad d\zeta_2 = \frac{d\zeta}{dz} dz_2 \,. \tag{6.94}$$

As ζ is an analytic function, the value of $d\zeta/dz$ depends on the location of point z and is independent of the orientation. Therefore by (6.94) with the relations $z = re^{i\theta}$, $\zeta = se^{i\vartheta}$, one obtains

$$\arg d\zeta_1 - \arg d\zeta_2 = \vartheta_1 - \vartheta_2 = \arg dz_1 - \arg dz_2 = \theta_1 - \theta_2 \,. \tag{6.95}$$

The angle between the two vectors is conserved by the transformation. Moreover, it is easily shown that

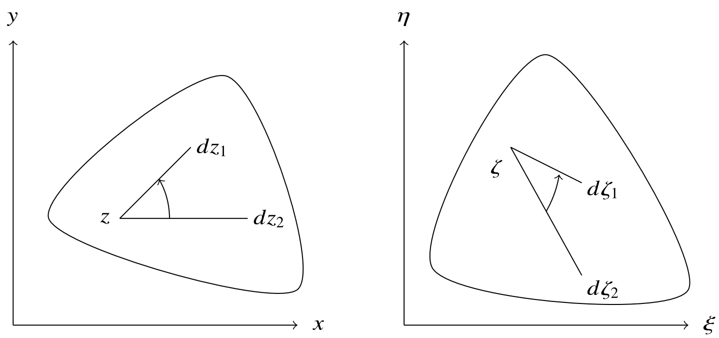


Fig. 6.18 z (left) and ζ (right) planes

$$|\frac{dz_2}{dz_1}| = |\frac{d\zeta_2}{d\zeta_1}| \ . \tag{6.96}$$

The ratio of the length of two infinitesimal vectors in a plane is equal to the ratio of the length of the corresponding vectors in the other plane.

6.6.2 Application to Potential Flows

In the z plane, the complex potential $f(z)$ transformed via (6.93) yields the complex potential G in the ζ plane. Indeed we have

$$f(z) = f(z(\zeta)) \equiv G(\zeta) \ . \tag{6.97}$$

The argument $z(\zeta)$ is the inverse function of the transformation (6.93). The analytical function $G(\zeta)$ defines a flow in the ζ plane. The streamlines and the equipotentials in the original plane z are transformed and present different shapes in the ζ plane. By analogy with the definition (6.7) one has

$$G(\zeta) = \Phi(\xi, \eta) + i\Psi(\xi, \eta) \ . \tag{6.98}$$

Equating f and G by (6.97), one obtains

$$\Re f(z) = \Re G(\zeta), \quad \Im f(z) = \Im G(\zeta) \ , \tag{6.99}$$

along homologous curves. The symbol $\Im$ denotes the imaginary part.

The complex velocity in the ζ plane is computed as

$$w_\zeta = G'(\zeta) = \frac{df}{dz}\frac{dz}{d\zeta} = \frac{w(z)}{\zeta'(z)} \ . \tag{6.100}$$

Let us note that both velocities w_ζ and w vanish at homologous points inasmuch $\zeta'(z) \neq 0$, i.e. when the mapping remains conformal.

The transformation is no longer conformal at the singular points $\zeta'(z) = 0$. A first case occurs when the z point is a stagnation point ($w = 0$) and the derivative ζ' is finite, then the velocity w_ζ remains finite. The second case corresponds to $\zeta'(z) = 0$ with the second derivative $\zeta''(z) \neq 0$. The homologous point ζ is a cusp (cf. Fig. 6.19) because by (6.93), we write

$$d\zeta = 0 + \frac{1}{2}\frac{d^2\zeta}{dz^2}dz^2 \ , \tag{6.101}$$

where the infinitesimal quantity $d\zeta$ is the dominating term in the development in series of the difference $\zeta(z + dz) - \zeta(z)$. This leads to the result that two infinitesimal

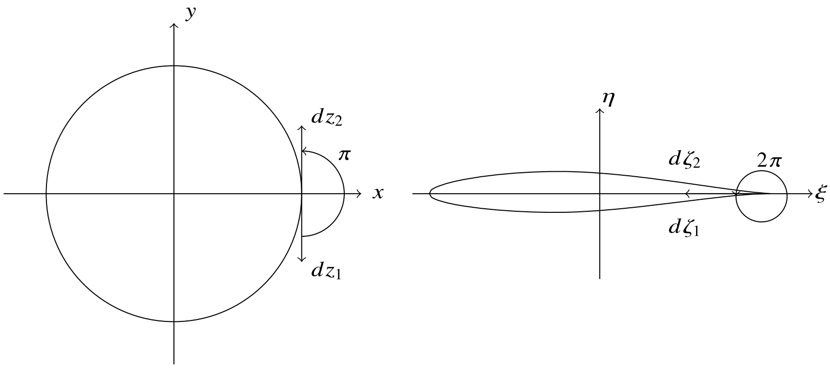


Fig. 6.19 Cusp in the ζ plane. The airfoil on the right is a NACA 64-1112 profile

segments dz_1 and dz_2 tangents to the circle in the z plane are transformed into two segments $d\zeta_1$ and $d\zeta_2$ such that

$$\frac{d\zeta_2}{d\zeta_1} = \left(\frac{dz_2}{dz_1}\right)^2 . \tag{6.102}$$

Therefore one has

$$\arg d\zeta_1 - \arg d\zeta_2 = 2(\arg dz_1 - \arg dz_2) . \tag{6.103}$$

As $\arg dz_1 - \arg dz_2 = \pi$, we obtain $\arg d\zeta_1 - \arg d\zeta_2 = 2\pi$ that corresponds indeed to a cusp.

The complex circulation remains unchanged in the transformation

$$(\Gamma + iQ)_\zeta = \int_{C_\zeta} G'(\zeta)d\zeta = \int_{C_z} \frac{w}{\zeta'(z)}\frac{d\zeta}{dz}dz = (\Gamma + iQ)_z . \tag{6.104}$$

6.7 Schwarz-Christoffel Transformation

The polygonal boundary of a domain located in the complex plane z with interior angles $\alpha, \beta, \gamma, \ldots$ can be transformed in the real axis ξ of the complex plane ζ by a conformal transformation. The transformed domain is the half plane $\eta > 0$ in Fig. 6.20. The transformation is given by a differential equation that is integrated for any polygonal shape. The equation defining the transformation $z = f(\zeta)$ is given by the relationship

$$\frac{dz}{d\zeta} = K(\zeta - a)^{\frac{\alpha}{\pi}-1}(\zeta - b)^{\frac{\beta}{\pi}-1}(\zeta - \gamma)^{\frac{\gamma}{\pi}-1} \ldots , \tag{6.105}$$

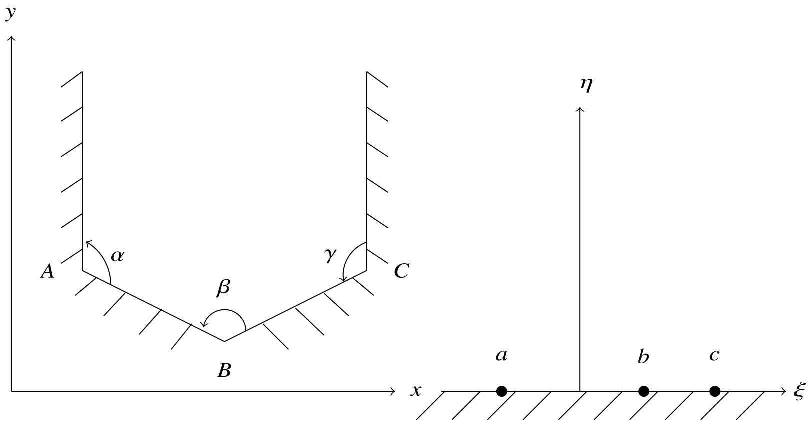


Fig. 6.20 Schwarz-Christoffel transformation of polygonal contour

where K is a constant and $a, b, c, \ldots$ are the real values of the complex variables corresponding to the polygon vertices in the z plane. If the polygon is closed and possesses n vertices, from Euclid's geometry, the quantities $\alpha, \beta, \gamma, \ldots$ are such that $\alpha + \beta + \gamma + \cdots = (n-2)\pi$. The function f can be expressed by an indefinite integral

$$z = z_0 + K \int (\zeta - a)^{\frac{\alpha}{\pi}-1}(\zeta - b)^{\frac{\beta}{\pi}-1}(\zeta - \gamma)^{\frac{\gamma}{\pi}-1} \ldots d\zeta \ . \tag{6.106}$$

The K and z_0 constants allow the determination of the position and the size of the domain.

It is sometimes convenient to place at infinity the point in the ζ plane corresponding to one of the vertices, e.g. $\zeta = a$. The factor $(\zeta - a)^{\frac{\alpha}{\pi}-1}$ is incorporated in the K constant generating a new constant $K' = (-a)^{\frac{\alpha}{\pi}-1}$ such that we can write

$$\frac{dz}{d\zeta} = K'\zeta - b)^{\frac{\beta}{\pi}-1}(\zeta - \gamma)^{\frac{\gamma}{\pi}-1} \ldots \ . \tag{6.107}$$

It is possible to show that two real values corresponding to the vertices may be chosen arbitrarily. This will be implemented in the following examples.

6.7.1 Mapping of a Semi-infinite Strip

Let us consider in Fig. 6.21 a semi-infinite strip A, B, C, D of height h. This strip may be assimilated to a rectangle with two vertices at infinity. The point A will be mapped on the point $\zeta = -\infty$, while B, C will be at $\zeta = b$, $\zeta = c$, respectively. We

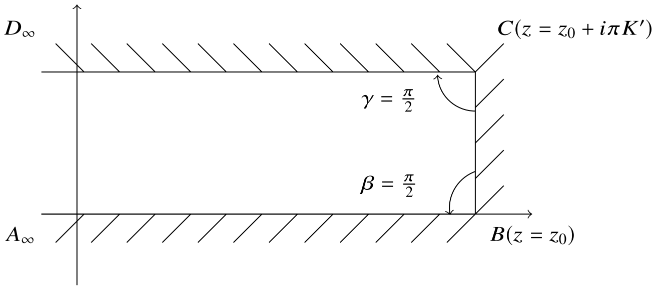


Fig. 6.21 Schwarz-Christoffel transformation of a semi-infinite strip

can conclude easily that D will be at $\zeta = \infty$. The interior angles that are relevant for the transformation are those at B and C, and then $\beta = \gamma = \pi/2$.

Eq. (6.107) is

$$\frac{dz}{d\zeta} = K'(\zeta - b)^{-\frac{1}{2}}(\zeta - c)^{-\frac{1}{2}} = \frac{K'}{\sqrt{(\zeta - b)(\zeta - c)}} = \frac{K'}{\sqrt{(\zeta - \frac{b+c}{2})^2 - (\frac{b-c}{2})^2}} . \tag{6.108}$$

Let us set

$$\delta = \frac{\zeta - \frac{b+c}{2}}{\left|\frac{b-c}{2}\right|} . \tag{6.109}$$

Therefore the Schwarz-Christoffel relation (6.108) becomes

$$\frac{dz}{d\delta} = \frac{K'}{\sqrt{\delta^2 - 1}} . \tag{6.110}$$

The integration leads to (cf. [1])

$$z = z_0 + K' \ln(\delta + \sqrt{\delta^2 - 1}) = z_0 + K' \cosh^{-1} \delta . \tag{6.111}$$

Consequently

$$\delta = \cosh\left(\frac{z - z_0}{K'}\right) . \tag{6.112}$$

Using 6.109) and (6.112), we write

$$\zeta = \frac{b+c}{2} + \left|\frac{b-c}{2}\right| \cosh\left(\frac{z - z_0}{K'}\right) . \tag{6.113}$$

If $b \geq c$ and $z = z_0$, (6.113) yields $\zeta = b$. With $z = z_0 + i\pi\, K'$, we obtain $\zeta = c$.

6.7.2 Mapping of a Plane Channel

The problem of plane channel may be considered as a polygon with two vertices located at infinity with zero angles as shown in Fig. 6.22.

The Schwarz-Christoffel formula (6.106) allows writing

$$\frac{dz}{d\zeta} = K(\zeta - a)^{0-1}(\zeta - b)^{0-1} . \tag{6.114}$$

Let us choose a at infinity and b any value. Equation (6.114) becomes

$$\frac{dz}{d\zeta} = K'(\zeta - b)^{0-1} , \tag{6.115}$$

which gives

$$\zeta = b + e^{(z-z_1)/K'} , \tag{6.116}$$

where z_1 is a complex constant corresponding to any point of the channel. Note that $\zeta = b$ for $z \to -\infty$ and $\zeta = b + 1$ when $z = z_1$.

A point $z_1 + d$ with real d is transformed in a real $\zeta = b + e^{d/K'}$. The real half-line ζ with $\zeta > b$ corresponds to the channel boundary going through z_1. This was expected as the goal of the Schwarz-Christoffel transformation consists in building up the correspondance of the domain boundary with the real axis of the ζ plane. We also desire that the other plane channel wall corresponds to another portion of the real axis $\xi = 0$. Let H denote the channel height and d be a positive real value. We write

$$z = z_1 + iH + d \quad \Rightarrow \quad \zeta = b + e^{iH/K'} e^{d/K'} . \tag{6.117}$$

Therefore ζ will be real if $e^{iH/K'}$ is also real, i.e. if $\sin(H/K') = 0$. We deduce that we must have the condition $H/K' = n\pi$ with n, an integer. If we require that the second channel wall corresponds to the real half-line ζ with $\zeta < b$, one must choose an odd n and $K' = H/\pi$.

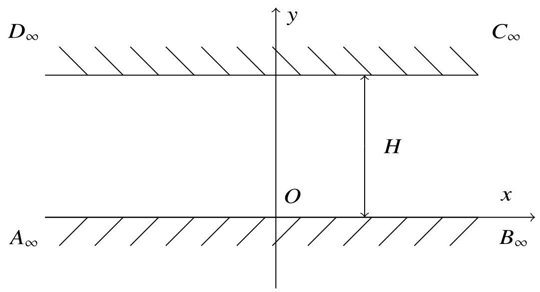


Fig. 6.22 Schwarz-Christoffel transformation of a plane channel

A parallel line to the channel walls is defined as

$$z = z_1 + i\alpha H + d \quad \text{with} \quad 0 \le \alpha \le 1 \, , \tag{6.118}$$

where α is a constant and d a variable parameter. The corresponding transformation reads

$$\zeta = b + e^{\pi d/H} e^{i\pi\alpha} \, . \tag{6.119}$$

This corresponds to a half-line that begins at point $\zeta = b$ and making an angle α with the positive real axis. Moreover, as $0 \le \alpha \le 1$, this half-line is located in the upper half-plane $\eta > 0$.

A straight line orthogonal to the channel walls is defined by the relationship (6.118), except that now α is varying and d is constant. Its transformation (6.119) is a semicircle centered in $\zeta = b$ with the radius $e^{\pi d/H}$.

6.7.3 *Schwarz-Christoffel Transformation of a Converging Channel*

In order to tackle the flow in a converging channel as shown in Fig. 6.23, the problem is solved with the help of the Schwarz-Christoffel transformation. We notice the following considerations:

- The first polygon angle is located at the vertices [2] and [3] coinciding at infinity. Therefore $\alpha_2 = \alpha_3 = 0$. The corresponding point in the ζ plane may be placed at the origin without loss of generality $\xi_2 = \xi_3 = 0$.
- For point [4], we have $\alpha_4 = 3\pi/4$ and $\xi_4 = a^4$.
- For point [5], we have $\alpha_5 = 5\pi/4$ and $\xi_5 = 1$.
- Point [1] (resp. [6]) is sent to $\xi = -\infty$ (resp. $\xi = \infty$) with zero angles in [1] and [6] (Fig. 6.23).

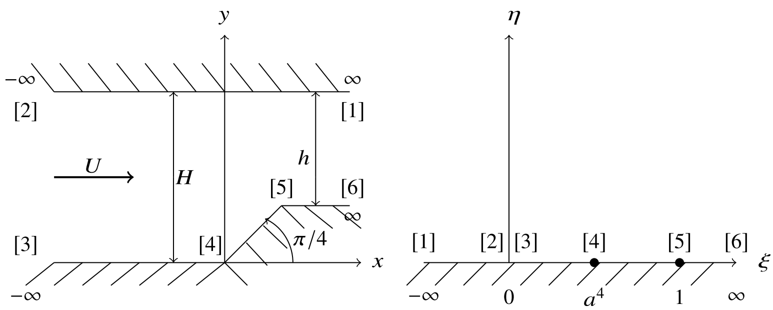


Fig. 6.23 Flow through a converging channel by the Schwarz-Christoffel method

With the help of (6.107), we write

$$\frac{dz}{d\zeta} = K'(\zeta - 0)^{0-1}(\zeta - a^4)^{3/4-1}(\zeta - 1)^{5/4-1} = \frac{K'}{\zeta}\left(\frac{\zeta - 1}{\zeta - a^4}\right)^{1/4} . \quad (6.120)$$

In the z plane, the entry flow is uniform with the velocity U. The volume flow rate Q is UH. In the ζ plane, we look for a complex potential with this flow rate generated between points [2] and [3] that coincide in $\xi = 0$. This is achieved by a source located at the origin with the complex potential

$$f(\zeta) = \frac{Q}{2\pi} \ln \zeta = \frac{UH}{\pi} \ln \zeta . \quad (6.121)$$

The last equality is due to the fact that $Q = 2UH$ as the source flows also in the half-plane $\eta < 0$. The complex velocity in the ζ plane is given by

$$w_\zeta = \frac{df}{d\zeta} = \frac{UH}{\pi\zeta} . \quad (6.122)$$

At the corresponding points in the z plane, the velocity is

$$w_z(z) = \frac{df}{d\zeta}\frac{d\zeta}{dz} = \frac{UH}{\pi K'}\left(\frac{\zeta - a^4}{\zeta - 1}\right)^{1/4} . \quad (6.123)$$

Let us determine the K' value in order that the velocity at [6] in the z plane be equal to the velocity at point [6] in the ζ plane. Due to the flow rate conservation, one has

$$w_z([6]) = \frac{UH}{h} \quad \Rightarrow \quad \frac{UH}{\pi K'}\left(\frac{\xi_6 - a^4}{\xi_6 - 1}\right)^{1/4} = \frac{UH}{h} . \quad (6.124)$$

But $\xi_6 = \infty$. Taking the limit of (6.124), one gets

$$\frac{UH}{\pi K'} = \frac{UH}{h} \quad \rightarrow \quad K' = \frac{h}{\pi} . \quad (6.125)$$

We must now compute the abscissa of point [4] in the ζ plane such that $\zeta_4 = \xi_4$. The z velocity in [4] is U. Using (6.123), we have

$$U = \frac{UH}{h}\left(\frac{0 - a^4}{0 - 1}\right)^{1/4} \quad \Rightarrow \quad a = \frac{h}{H} \quad \Rightarrow \quad \xi_4 = a^4 = \left(\frac{h}{H}\right)^4 . \quad (6.126)$$

With (6.120) and (6.125), we write

$$dz = \frac{h}{\pi\zeta}\left(\frac{\zeta - 1}{\zeta - a^4}\right)^{1/4} d\zeta \quad (6.127)$$

In order to ease the integration of (6.127), we introduce the change of variable

$$t^4 = \frac{\zeta - a^4}{\zeta - 1} \ . \tag{6.128}$$

Inverting (6.128) we have

$$\zeta = \frac{t^4 - a^4}{t^4 - 1} \ . \tag{6.129}$$

We can compute $d\zeta/\zeta$ and with this intermediate result, Eq. (6.127) becomes using factorization

$$dz = \frac{2h}{\pi}\left(\frac{1/2a}{t-a} - \frac{1/2a}{t+a} - \frac{1}{2(t-1)} + \frac{1}{2(t+1)} + \frac{1}{t^2+a^2} - \frac{1}{t^2+1}\right) dt \ . \tag{6.130}$$

Integrating we get

$$\begin{aligned} z = z_0 + \frac{h}{\pi}\bigg(& -\frac{1}{a}\ln(t+a) + \frac{1}{a}\ln(t-a) + \ln(t+1) - \ln(t-1) \\ & + \frac{2}{a}\arctan(\frac{t}{a}) - 2\arctan t \bigg) \ . \end{aligned} \tag{6.131}$$

The integration constant z_0 is obtained by imposing that the point $\zeta_4 = a^4 \leftrightarrow t = 0$ correspond to the origin in the z plane. This leads to the following expression (as $\ln(-1) = (2k+1)i\pi$)

$$z_0 = i(H + h) \ . \tag{6.132}$$

6.8 Joukowski Transformation

The Joukowski transformation is defined by

$$\zeta = z + \frac{a^2}{z} \ , \tag{6.133}$$

with a a real number. For large values of z, one has $\zeta \simeq z$ and the homologous flows given by (6.133) are identical at infinity. Let us consider the transformation of the region outside a circle of radius R in the z plane given by

$$z = R\, e^{i\theta} \ . \tag{6.134}$$

With (6.133) and (6.134), one obtains

$$\zeta = \xi + i\eta = \left(R + \frac{a^2}{R}\right)\cos\theta + i\left(R - \frac{a^2}{R}\right)\sin\theta \ . \tag{6.135}$$

Eliminating θ one gets

$$\frac{\xi^2}{r^2} + \frac{\eta^2}{s^2} = 1 \ , \tag{6.136}$$

$$r = R + \frac{a^2}{R}, \quad s = R - \frac{a^2}{R} \ . \tag{6.137}$$

The counterpart of the circle of radius R is an ellipse of semi-axes r and s. The two foci[1] of the ellipse are located on the ξ axis in $\xi = \pm 2a$. If the circle of ζ plane is such that $R = a$, it is transformed in the segment $-2a \leq \xi \leq 2a$ of the ξ axis. From Eq. (6.135) we have

$$\xi = a\, e^{i\theta} + a\, e^{-i\theta} = 2a\cos\theta \ . \tag{6.138}$$

For $\theta = 0, \pi/2, \pi, 3\pi/2, 2\pi$, one has $\xi = 2a, 0, -2a, 0, 2a$. Therefore when a point goes around the circle $R = a$ in the z plane, the homologous point in the ζ plane sweeps first the upper part of the segment and afterwards the bottom part. This means that in the ζ plane we have a flat plate of span $4a$.

The singular points of the transformation are the zeroes of the relation

$$\frac{d\zeta}{dz} = 1 - \frac{a^2}{z^2} \ . \tag{6.139}$$

These are the points $z = \pm a$ and thus $\zeta = \pm 2a$. The transformed circle generates cusps at those points.

6.8.1 Flow over a Flat Plate

Let us consider the flow over a circular cylinder with circulation that is inclined by an angle α with respect to the x axis. This is taken into account by the transformation $z = z_0 e^{i\alpha}$ where z_0 corresponds to the case of a horizontal incoming flow. With reference to relation (6.69) corresponding to z_0, the complex potential reads

$$f(z) = U\left(ze^{-i\alpha} + \frac{a^2}{z}e^{i\alpha}\right) - i\frac{\Gamma}{2\pi}\ln\left(\frac{z}{a}e^{-i\alpha}\right) \ . \tag{6.140}$$

Taking the derivative of (6.140) with respect to z and taking (6.139) into account, one produces the velocity relation on the plane $\zeta = a\, e^{i\theta}$

[1] The distance of the focus c to the ellipse center is such that $c = \sqrt{a^2 - b^2}$, where a and b are the semi-axes and the ellipse equation is $(x/a)^2 + (y/b)^2 = 1$.

$$w_\zeta = \frac{U}{1 - e^{-2i\theta}} \left(e^{i\alpha} - e^{i(\alpha - 2\theta)} - i\frac{\Gamma}{2\pi a U} e^{-i\theta} \right) . \tag{6.141}$$

With the identity

$$\frac{1}{1 - e^{-2i\theta}} = \frac{1 - e^{2i\theta}}{4\sin^2\theta} \tag{6.142}$$

we obtain

$$w_\zeta|_{plate} = \frac{U}{2\sin^2\theta} \left(\cos\alpha - \cos(\alpha - 2\theta) - \frac{\Gamma}{2\pi a U} \sin\theta \right) . \tag{6.143}$$

One notes that for the particular values $\alpha = \Gamma = 0$, one gets $w_\zeta|_{plate} = U$, corresponding to the uniform and parallel flow not perturbed by the presence of the plate.

For $\Gamma = 0$, $w_\zeta = 0$ when

$$\cos\alpha = \cos(\alpha - 2\theta) \quad \text{or} \quad \theta_1 = \alpha, \quad \theta_2 = \alpha + \pi . \tag{6.144}$$

Figure 6.24 shows the streamlines for $\alpha = 25°$, $\Gamma = 0$. We notice the stagnation points S_1 and S_2 on the pressure and suction sides. The velocities at $x = \pm 2a$ are infinite. Using the identity

$$\cos\alpha - \cos(\alpha - 2\theta) = 2\sin^2\theta\cos\alpha - 2\sin\alpha\sin\theta\cos\theta$$

the velocity is

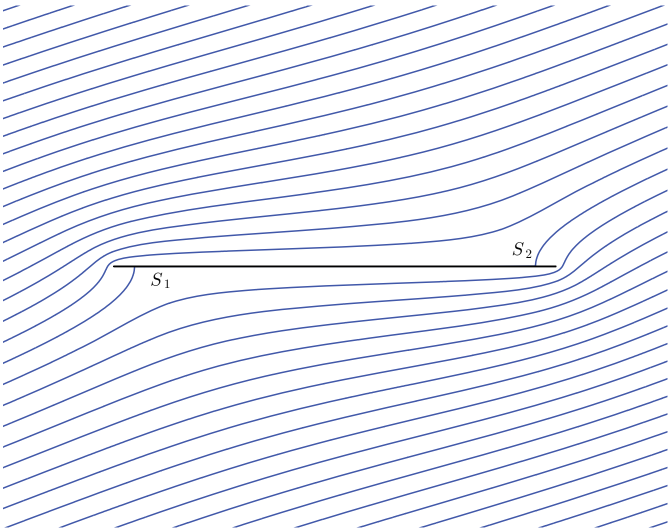


Fig. 6.24 Flow without circulation, inclined by an angle $\alpha = 25°$ around a flat plate

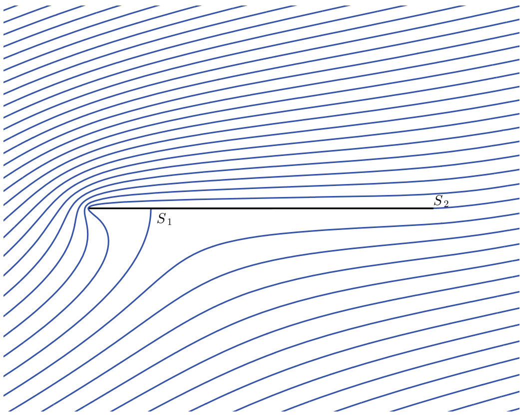


Fig. 6.25 Flow inclined by an angle $\alpha = 25°$ around a flat plate with $|\Gamma| = 4\pi a U \sin\alpha$

$$w_\zeta|_{plate} = U\cos\alpha - \frac{U}{2\sin\theta}\left(\frac{\Gamma}{2\pi a U} + 2\sin\alpha\cos\theta\right) . \tag{6.145}$$

In order to avoid an infinite velocity at the trailing edge that corresponds to $\xi = 2a$ and $\theta = 0$, we choose a Γ value that generates a finite velocity at that point. One has

$$|\Gamma| = 4\pi a U \sin\alpha . \tag{6.146}$$

Figure 6.25 presents the streamlines for this particular value of Γ, corresponding to a finite velocity at the trailing edge S_2.

6.8.2 Joukowski Profiles

With the help of Joukowski transformation, it is possible to generate a series of various profiles; among them, the ellipse and the flat plate constitute particular cases. This series called Joukowski profiles is obtained by locating the center of the circle of the transformation in a position different from the origin of the axes.

In Fig. 6.26, the circle is shifted in the negative direction of x axis. This procedure generates a thick symmetric profile such that the leading edge presents an elliptic shape and the trailing edge corresponds to the cusp of the flat plate case.

If the center of the circle is now located on the η axis, as it is shown in Fig. 6.27, one obtains an arc of a thin circle. This type of off-set generates a cambered profile.

Finally, Fig. 6.28 corresponds to the more general situation where the center of the circle is located inside the interior of the complex plane. We wish that the trailing edge be no longer a cusp, but a dihedral with a small aperture angle. This dihedral will be tied up with the intersection of the cylinder with the ξ axis. The center of the circle is off-set at point $z = z_{off}$. The transformation is given by the relationship

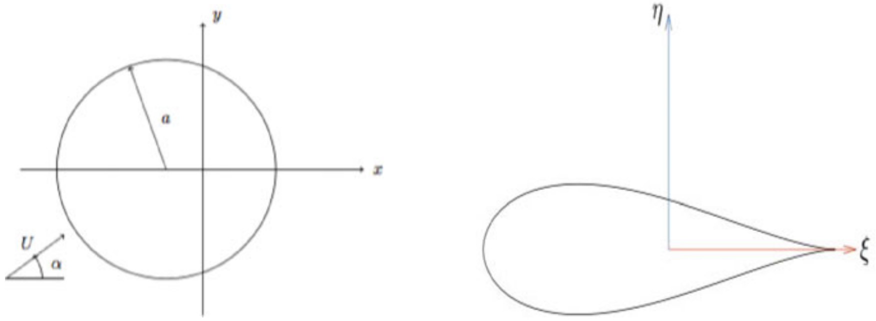


Fig. 6.26 Symmetric Joukowski profile

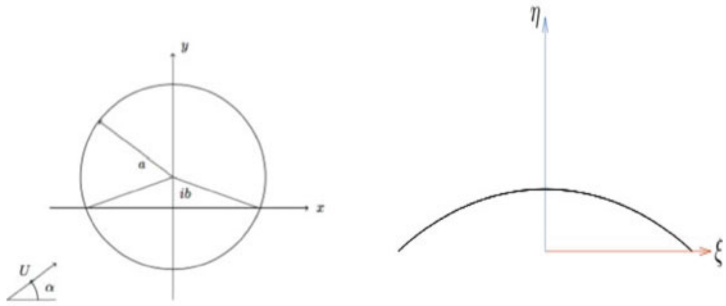


Fig. 6.27 Cambered Joukowski profile

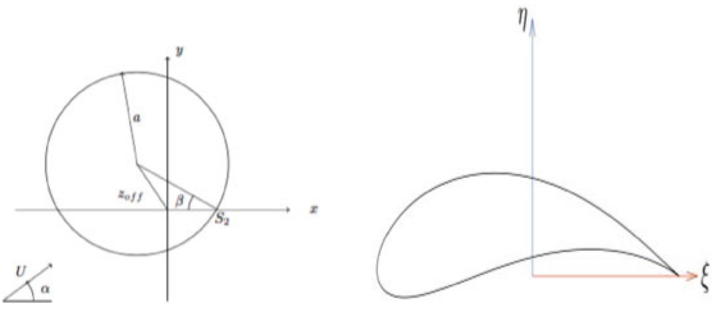


Fig. 6.28 General Joukowski profile

$$z = z_0 e^{i\alpha} + z_{off} \, . \tag{6.147}$$

The complex potential of the flow is then

$$f(z) = U\left(z - z_{off}\right)e^{-i\alpha} + \frac{Ua^2}{z - z_{off}}e^{i\alpha} - i\frac{\Gamma}{2\pi}\ln\left(\frac{z - z_{off}}{a}e^{-i\alpha}\right) \, . \tag{6.148}$$

The S_2 point corresponds to the intersection of the circle with ξ axis. Referring to Fig. 6.28, where the line connecting S_2 to the center of the circle makes the angle β, one gets by (6.147)

$$z_{S_2} = ae^{i\beta} + z_{off} \ . \tag{6.149}$$

Imposing a zero velocity at the trailing edge point S_2, we obtain

$$w(S_2) = U\left(e^{-i\alpha} - e^{+i(\alpha+2\beta)}\right) - \frac{i\Gamma}{2\pi a}e^{i\beta} = 0 \ . \tag{6.150}$$

We are then able to compute the circulation Γ

$$|\Gamma| = 4\pi a U \sin(\alpha+\beta) \ . \tag{6.151}$$

The lift on the profile is therefore

$$F_y = 4\pi\rho U^2 \sin(\alpha+\beta) \ . \tag{6.152}$$

Imposing this value for the circulation constitutes the Kutta condition which yields a finite velocity at the trailing edge on the pressure and suction sides of the profile. In the steady state plane case, there is no downstream vortex sheet. On the contrary, in the unsteady flow or downstream a three-dimensional profile, such a vortex sheet is present. From the physical point of view, this will generate a wake and some instabilities.

Consequently, the pressure coefficient is given by the relation

$$C_p = 1 - \frac{u^2+v^2}{U^2} = 1 - \frac{w\overline{w}}{U^2} \ . \tag{6.153}$$

Figure 6.29 shows the flow with an incidence angle of five degrees. The circle of the transformation is in $z_{off} = 0.1 + i\,0.14$ with radius $a = 1.16$. One notices that the pressure side (intrados) is subjected to high pressure and the suction side (extrados)

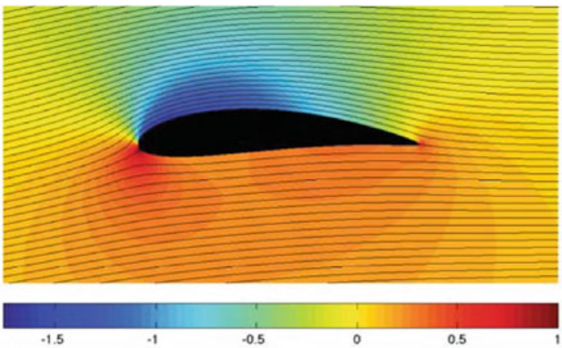


Fig. 6.29 Streamlines and isocontours of the pressure coefficient on a Joukowski airfoil for an incidence angle $\alpha = 5°$

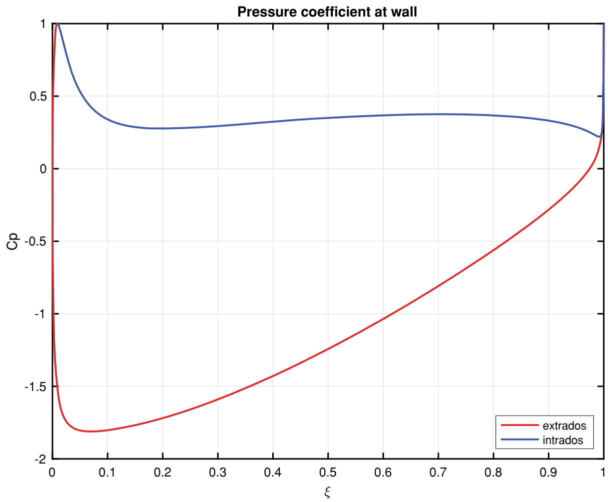


Fig. 6.30 Pressure coefficient on a Joukowski airfoil for an incidence angle $\alpha = 5°$

is low pressured. This phenomenon induces the lift of the airfoil. The next Fig. 6.30 shows the pressure variation between pressure and suction sides allowing computing the center of pressure which is the average location of the pressure variation. Through the center of pressure acts the aerodynamic force that is lift for an inviscid fluid and lift and drag in the viscous case.

Exercises

6.1 Let the complex potential of the flow be given by the expression

$$f(z) = m \ln\left(z - \frac{1}{z}\right), \quad m > 0\,. \tag{6.154}$$

- Find the locations of the sources and sinks.
- Compute the velocity potential φ and the streamfunction ψ and show that with these expressions it is possible to examine the flow in a positive half-disk $r < 1$.
- Evaluate the flow rate crossing the line between the points $z_1 = \frac{1}{2} + \frac{i}{2}$ and $z_2 = \frac{1}{2}$.

6.2 The complex potential of the flow reads

$$f(z) = (1+i)\ln(z^2-1) + (2-3i)\ln(z^2+4) + \frac{1}{z}\,. \tag{6.155}$$

- Find the positions of the singular points.
- Compute the flow rate across the circle $C : x^2 + y^2 = 9$ and the circulation around C.

6.3 The complex potential of the flow is given as

$$z = \cosh f \ . \tag{6.156}$$

- Compute the streamlines ψ and the equipotentials φ.
- Evaluate the velocity along the segment $[-1, 1]$ of the real axis.

Reminder

$$\cosh \alpha = \frac{e^{\alpha} + e^{-\alpha}}{2}$$
$$\sinh \alpha = \frac{e^{\alpha} - e^{-\alpha}}{2}$$
$$\cosh^2 \alpha - \sinh^2 \alpha = 1 \ .$$

6.4 Flow in front of a circular obstacle

Let us consider in the upper part ($\eta \geqslant 0$) of the ζ complex plane, the flow generated by a source with flow rate Q, at a distance $a < 1$ from a plane wall, and a sink with the same flow rate at a unit distance. The complex potential in the ζ plane is given by

$$g(\zeta) = \frac{Q}{2\pi} \ln \frac{\zeta^2 - a^2}{\zeta^2 - 1} \ . \tag{6.157}$$

- Justify the relation for the complex potential $g(\zeta)$ using the concept of hydrodynamic images analogous to that of electrical images in electrostatics.
- Express the complex velocity w_ζ in the ζ complex plane.

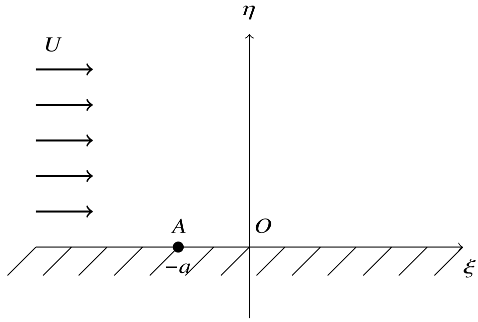
Consider now the conformal transformation between the complex ζ plane and the physical z plane :

$$z = g(\zeta) = \frac{1 + \zeta}{1 - \zeta} \ , \tag{6.158}$$

where $z = x + iy$ represents the point in the physical plane. This transformation is applied to the upper half plane $\eta \geq 0$.

- Evaluate the image in the z plane of the line $\xi = 0$ by the conformal transformation g.

Fig. 6.31 Flow over a step



- Obtain the images in the z plane of the sources and sinks in the ζ plane. For the sake of simplicity, set $b = (1 + a)/(1 - a)$.
- Calculate the image in the z plane of the half-plane $\eta \geq 0$.
- Express the complex velocity $w(z)$ in the physical plane as a function of w_ζ. Deduce the analytical expression of $w(z)$.
- Show that the complex potential in the physical plane reads

$$f(z) = \frac{Q}{2\pi}\left[\ln\left(1 - \frac{b}{z}\right) + \ln\left(z - \frac{1}{b}\right)\right]. \tag{6.159}$$

6.5 Flow over a forward facing step

Let us consider the conformal mapping $\zeta = f(z)$ of the upper half plane $\eta \geq 0$ towards the physical plane z defined by

$$\frac{dz}{d\zeta} = \left(\frac{\zeta}{\zeta + a}\right)^{1/2}, \tag{6.160}$$

where a is a positive real number. We denote by A the corresponding point of the complex plane such that $\zeta = \xi = -a$ as is exhibited in Fig. 6.31. Moreover, it is assumed that the origin O of the ζ plane corresponds to the origin of the physical plane.

- Evaluate the image of the half line $\xi > 0$ and $\eta = 0$ by the conformal mapping. To this end, use arguments based on angles.
- As in the previous item, evaluate the image of the segment AO and of the half line $\xi < -a$. Deduce the flow domain in the physical plane.
- The flow in the ζ plane is uniform and represented by the complex potential $g(\zeta) = U\zeta$. Express the complex velocity $w(z)$ in the physical plane as a function of ξ for the domain boundary $\eta = 0$. Show that the image of point A in the z plane is a stagnation point. Compute in the transformed plane the velocity at the origin (Fig. 6.31).