Chapter 5
Stokes Flow

In this chapter, let us consider first the Stokes equation, valid for very slow flows that we conventionally call creeping flows or Stokes flows. These flows are dominated by viscous forces which are much larger than inertial forces. Examples come from technologies in such diverse domains as convection currents in high temperature glass melting furnaces, lubricants in bearings, and the flow of oils and mud (although the latter may have pronounced non-Newtonian behavior). In nature (another source of interesting cases), we find convection in terrestrial magma, the flow of lava, the swimming of fish, the propulsion of microorganisms, and the squirming of spermatozoon.

We assume that the Reynolds number $Re \ll 1$ and therefore the Navier–Stokes equations reduce to the Stokes equation. As the latter is linear, a complete analytic treatment is possible.

Taking the divergence of the Stokes equation (2.54) and taking into account the solenoidal character of the velocity field, we get

$$\Delta p = 0 \, . \tag{5.1}$$

The pressure is thus a harmonic function for a Stokes flow.

Taking the curl of the Stokes equation (2.54), we obtain

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nu \Delta \boldsymbol{\omega} \, , \tag{5.2}$$

where we have introduced the vorticity, $\boldsymbol{\omega} = \mathbf{curl}\, \boldsymbol{v}$ using Eq. (1.40). If the flow is stationary, the components of the vorticity are also harmonic functions.

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5.1 Plane Creeping Flows

Consider a plane flow for which we have

$$\boldsymbol{v} = (v_1(x_1, x_2, t),\ v_2(x_1, x_2, t),\ 0); \quad p = p(x_1, x_2, t) . \tag{5.3}$$

In such a two-dimensional problem, incompressibility (1.73) is automatically satisfied by the introduction of a ***stream function*** ψ so that

$$v_1 = \frac{\partial \psi}{\partial x_2}, \quad v_2 = -\frac{\partial \psi}{\partial x_1} . \tag{5.4}$$

As the vorticity reduces to a single component $\boldsymbol{\omega} = (0, 0, \omega)$, it follows that

$$\omega = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = -\Delta \psi , \tag{5.5}$$

and relation (5.2) becomes

$$\frac{\partial \Delta \psi}{\partial t} = \nu \Delta \Delta \psi . \tag{5.6}$$

For a stationary problem, we will have

$$\Delta \Delta \psi = 0 , \tag{5.7}$$

showing that in this case the stream function is a biharmonic function.

In polar coordinates (r, θ), the conservation of mass becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0 . \tag{5.8}$$

A stream function ψ also exists such that

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad v_\theta = -\frac{\partial \psi}{\partial r} . \tag{5.9}$$

5.1.1 Flow in a Corner

Let us consider the flow in a corner as presented in Fig. 5.1. The lower wall is fixed while the wall inclined at an angle θ_0 is in uniform translational motion at the constant velocity U in the direction x_1. Near the origin, the velocity gradients are large; nonetheless, we expect the viscous forces to dominate in the neighborhood of the origin. To formulate the problem in steady state, we choose a coordinate system with the origin at the intersection of the two walls, in motion with the inclined wall. In this case, the boundary conditions are written as

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = -U, \qquad \frac{\partial \psi}{\partial r} = 0 \quad \text{at} \quad \theta = 0 \tag{5.10}$$

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0, \qquad \frac{\partial \psi}{\partial r} = 0 \quad \text{at} \quad \theta = \theta_0 \; . \tag{5.11}$$

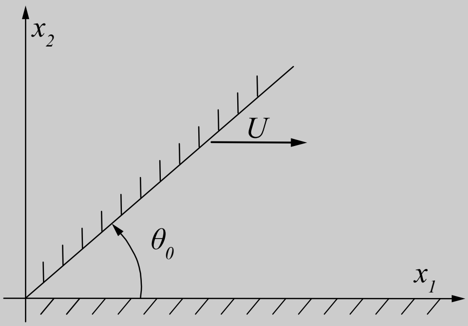


Fig. 5.1 Flow in a corner of angle θ_0. The coordinate system moves at the upper wall velocity U

The form of the boundary conditions suggests that we can write ψ in the following form:

$$\psi = r\, f(\theta) \; . \tag{5.12}$$

Substituting (5.12) in the biharmonic equation

$$\nabla^4 \psi = \left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}\right) = 0 \; , \tag{5.13}$$

we find the relation

$$\frac{1}{r^3}\left(\frac{d^4 f}{d\theta^4} + 2\frac{d^2 f}{d\theta^2} + f\right) = 0 \; , \tag{5.14}$$

for which the solution is

$$f(\theta) = A\sin\theta + B\cos\theta + C\theta\sin\theta + D\theta\cos\theta\ . \tag{5.15}$$

(Recall that if H is a harmonic function, θH is a biharmonic function). The imposition of the boundary conditions (5.10) and (5.11) allows us to evaluate the constants which are

$$A, B, C, D = \left(-\theta_0^2, 0, \theta_0 - \sin\theta_0\cos\theta_0, \sin^2\theta_0\right)\frac{U}{\theta_0^2 - \sin^2\theta_0}\ . \tag{5.16}$$

For the special case of a right angle, we have

$$\psi = \frac{rU}{(\frac{\pi}{2})^2 - 1}\left(-(\frac{\pi}{2})^2\sin\theta + \frac{\pi}{2}\theta\sin\theta + \theta\cos\theta\right)\ , \tag{5.17}$$

from which we can easily obtain the velocity components

$$v_r = \frac{U}{(\frac{\pi}{2})^2 - 1}\left((1 - \frac{\pi^2}{4})\cos\theta + \frac{\pi}{2}(\sin\theta + \theta\cos\theta) - \theta\cos\theta\right) \tag{5.18}$$

$$v_\theta = -\frac{U}{(\frac{\pi}{2})^2 - 1}\left(-(\frac{\pi}{2})^2\sin\theta + \frac{\pi}{2}\theta\sin\theta + \theta\cos\theta\right)\ . \tag{5.19}$$

In retrospect we can examine the correctness of the creeping flow assumption. We see that the acceleration components (A.11) and (A.12) evaluated with the preceding solution are proportional to U^2/r with a factor that depends on θ, that is, of the order of unity. As for the viscous effects, they are of the order of $\mu U/r^2$. Thus the creeping flow assumption is met when $\rho r U/\mu \ll 1$. This is the case in the region close to the origin such that $r \ll \nu U$. Further away, the solution will no longer be correct as the inertial forces rapidly become of the same order of magnitude as the viscous forces.

It is interesting to compute the pressure from Eq. (A.22):

$$\frac{\partial p}{\partial\theta} = r\mu\left(\triangle v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r^2}\right) = r\mu\left(-\triangle\frac{\partial\psi}{\partial r} + \frac{2}{r^3}\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{r^2}\frac{\partial\psi}{\partial r}\right)\ , \tag{5.20}$$

which yields

$$p = \frac{2\mu}{r(\theta_0^2 - \sin^2\theta_0)}\left((\frac{1}{2}\sin 2\theta_0 - \theta_0)\sin\theta - \sin^2\theta_0\cos\theta\right)\ . \tag{5.21}$$

Observe that the pressure varying like r^{-1} becomes unbounded when we approach the corner. This dismal performance comes from the fact that at the corner the boundary conditions are not consistent with the real problem, which has always a tiny gap so that the forces remain finite.

5.2 Two-Dimensional Corner Moffatt Eddies

Plane Stokes flows occur in engineering or physical problems in the neighborhood of slots or cracks in a wall. This situation is modeled by analyzing the creeping flow close to the vertex of a sharp wedge with an aperture angle 2α formed by the walls $\theta = \pm\alpha$ as shown in Fig. 5.2.

The forcing mechanism generating the corner flow is "far" from the wedge vertex. For example, in the lid-driven square cavity problem, corner vortices are generated by the influence of the main primary vortex.

For the flow in a wedge, two geometrical configurations are possible: the flow is asymmetric with respect to the symmetry axis $\theta = 0$; the flow is symmetric in θ and the symmetry axis plays the role of a mirror.

Here we will concentrate on the asymmetric case shown in Fig. 5.2 with the assumption of 2D creeping flow described by the stream function. The governing equation is still (5.7) which in polar coordinates is given by

$$\Delta\Delta\psi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \psi \; . \tag{5.22}$$

We look for separable solutions of (5.22) expressed by

$$\psi = r^\lambda f(\theta) \; , \tag{5.23}$$

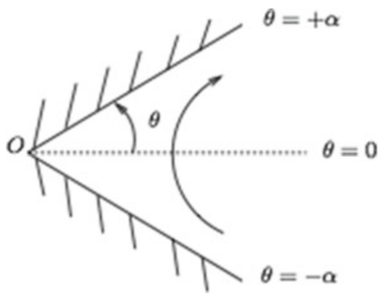


Fig. 5.2 Geometry of the corner flow

where λ is a real or complex number. Carrying through the algebra, the biharmonic equation yields

$$\Delta\Delta\psi = r^{\lambda-4}\left(\frac{d^2}{d\theta^2} + \lambda^2\right)\left(\frac{d^2}{d\theta^2} + (\lambda-2)^2\right) f = 0 \,. \tag{5.24}$$

For $\lambda \neq 0, 1, 2$, the solution of (5.24) is given by

$$\psi = r^{\lambda}(Ae^{i\lambda\theta} + Be^{i(\lambda-2)\theta}) \,. \tag{5.25}$$

Following Moffatt [60], the two-dimensional streamfunction is expanded in a series of basic solutions $\psi = r^{\lambda} f(\theta)$

$$\psi = \Re \sum_{n=1}^{\infty} A_n r^{\lambda_n} f_{\lambda_n}(\theta) \,, \tag{5.26}$$

where the A_n are complex numbers and the λ_n satisfy the condition

$$1 < \Re\lambda_1 < \Re\lambda_2 < \cdots \tag{5.27}$$

The first inequality imposes that the flow vanish at the origin, which is located at the corner. The remaining inequalities indicate that the first term in (5.26) dominates over the others and then

$$\psi \approx A_1 r^{\lambda_1} f_{\lambda_1}(\theta) = Ar^{\lambda} f_{\lambda}(\theta) \,. \tag{5.28}$$

As for the asymmetric solution $v_r(r, -\theta) = -v_r(r, \theta)$ and $v_\theta(r, -\theta) = v_\theta(r, \theta)$, the function f_λ has to be even in θ. The term in between parentheses in solution (5.25) is rewritten as

$$A \sin\lambda\theta + B\cos\lambda\theta + C\cos(\lambda-2)\theta + D\sin(\lambda-2)\theta \,, \tag{5.29}$$

and is such that the constants A and D vanish:

$$f_\lambda = B\cos\lambda\theta + C\cos(\lambda-2)\theta \,. \tag{5.30}$$

The no slip boundary conditions on the two walls $f_\lambda(\pm\alpha) = f'_\lambda(\pm\alpha) = 0$ yield the system

$$B\cos\lambda\alpha + C\cos(\lambda-2)\alpha = 0 \,, \tag{5.31}$$

$$B\lambda\sin\lambda\alpha + C(\lambda-2)\sin(\lambda-2)\alpha = 0 \,. \tag{5.32}$$

For a nonzero solution the determinant of this system must vanish, i.e.

$$\sin 2(\lambda - 1)\alpha + (\lambda - 1)\sin 2\alpha = 0\,. \tag{5.33}$$

5.2.1 Real Solutions for λ (α > 73.15°)

The nonlinear equation (5.33) gives real solutions for an angle $\alpha > 73.15°$. Figure 5.3 shows the real solutions obtained as the intersections of the sine function $\sin 2(\lambda - 1)\alpha$ in black and the straight line $-(\lambda - 1)\sin 2\alpha$ in green with respect to the variable $(\lambda - 1)\alpha$. The smallest value is the relevant one when we approach the corner $r \to 0$ as the solution goes like r^{λ}.

5.2.2 Complex Solutions for λ (α < 73.15°)

Let us write $\lambda = p + 1 + iq$. The azimuthal velocity component is

$$v_\theta(r, \theta) = -\frac{\partial\psi}{\partial r} = \Re\left(-\lambda r^{\lambda-1} f(\theta)\right)\,. \tag{5.34}$$

On the symmetry axis of the corner, $\theta = 0$, and therefore

$$v_\theta(r, 0) = \Re\left(-\lambda r^{\lambda-1} f(0)\right) = \Re\left(r^{\lambda-1} C\right)\,, \tag{5.35}$$

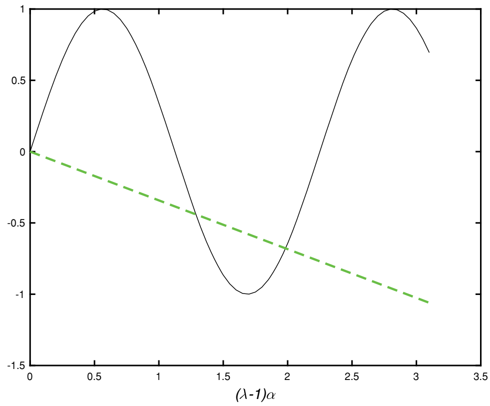


Fig. 5.3 Real solutions for $\alpha = 80°$

where $C = |C|e^{i\beta} \equiv -\lambda f(0)$. Equation (5.35) yields

$$v_\theta(r, 0) = \Re\left(r^p |C| e^{iq\ln r} e^{i\beta}\right) = r^p |C| \cos(q \ln r + \beta)\,. \tag{5.36}$$

When $r \to 0$, $\ln r \to -\infty$ and the velocity $v_\theta(r, 0)$ changes sign infinitely often. This behavior means that a string of counter-rotating vortices is present in the corner. The center of the nth corner eddy denoted by r_n is the distance of this center to the origin. It is given by the relation $v_\theta(r, 0) = 0$ leading to

$$q \ln r_n + \beta = -(2n+1)\frac{\pi}{2},\ n = 0, 1, 2, \ldots, \quad \text{or} \quad r_n = e^{-(2n+1)\frac{\pi}{2q}} e^{-\frac{\beta}{q}}\,. \tag{5.37}$$

A simple calculation yields

$$\frac{r_n}{r_{n+1}} = \frac{r_n - r_{n+1}}{r_{n+1} - r_{n+2}} = e^{\frac{\pi}{q}}\,, \tag{5.38}$$

which shows that the sizes of the vortices fall off in geometrical progression with a common ratio $e^{\pi/q}$, depending on the aperture angle of the corner. If we now inspect the velocity maxima, we find $v_{\theta,max} = r^p|C|$ at points $r_{n+\frac{1}{2}} = e^{-n\frac{\pi}{2q}} e^{-\frac{\beta}{q}}$. The maximum velocity will be called the intensity of the eddy. The corner vortices have their intensities falling off in geometrical progression with the common ratio

$$\left|\frac{v_n}{v_{n+1}}\right| = \left|\frac{r_{n+\frac{1}{2}}}{r_{n+\frac{3}{2}}}\right|^p = e^{\pi p/q}\,, \tag{5.39}$$

which also depends on α.

The numerical solution of Eq. (5.33) for the angle α is given in Table 5.1.

We observe that λ decreases when α increases. Furthermore, the imaginary part $\Im\lambda$ goes to zero when α reaches the value 73.15°, meaning that λ then becomes real.

Table 5.1 Main eigenvalue λ with respect to the corner half angle

α	λ
2°	$61.34043791 + i\,32.2266675$
10°	$13.0794799 + i\,6.3843883$
20°	$7.0578309 + i\,3.0953659$
30°	$5.0593290 + i\,1.9520499$
40°	$4.0674345 + i\,1.3395862$
50°	$3.4792155 + i\,0.9303733$
60°	$3.0941391 + i\,0.6045850$
70°	$2.8268686 + i\,0.2616953$
73.155°	$2.7634862 + i\,0$

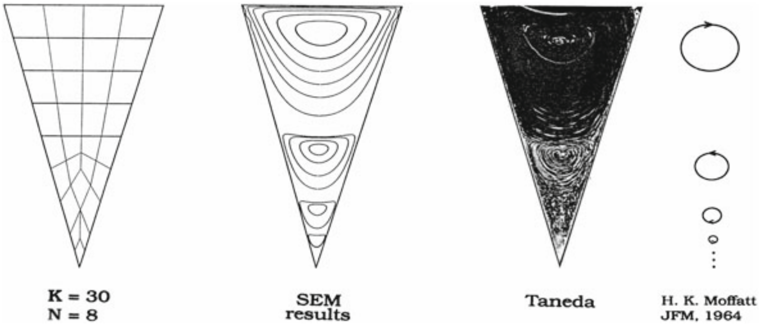


Fig. 5.4 Corner flow in a wedge of aperture $\alpha = 28.5°$. (Courtesy of E. Rønquist [81]. Taneda picture [102] (c) (1979) The Physics Society of Japan, is reprinted with permission)

Figure 5.4 shows a spectral element solution computed by Einar Rønquist [81] for the Stokes flow in a wedge. As the top lid moves at unit velocity, a series of Moffat corner eddies is generated. These eddies are stacked from top to bottom in an infinite cascade to the tip. The wedge shown has an aperture angle of 28.5°. The asymptotic ratio of successive eddy intensities is 405. With the discretization comprising 30 elements of polynomial degree 8 shown in the figure, one obtains four eddies. The ratio of the strength of two successive eddies is from top to bottom 386, 406, and 411. We observe also that the computed results are very close to the experimental data provided by Taneda [102] and Van Dyke [114].

5.3 Stokes Eigenmodes

Most state-of-the-art numerical methods dealing with the Navier–Stokes equation rely on an implicit treatment of the Stokes operator and an explicit scheme for the nonlinearity (rejecting it as a source term). Hence it is essential to understand the structure of the Stokes operator. Furthermore if we can write the eigenmodes in closed form, or if we can compute them accurately, then we are able to use those modes as the basis for the approximation of the Stokes equation, see for example Batcho-Karniadakis [12].

The Stokes eigenproblem is defined by setting $\frac{\partial \boldsymbol{v}}{\partial t} = \lambda \boldsymbol{v}$ in Eq. (2.54) and assuming that $\boldsymbol{b} = \boldsymbol{0}$. The eigenvalue λ provides the growth or decay rate of the velocity field. Now the eigensystem becomes

$$\lambda \boldsymbol{v} - \nu \Delta \boldsymbol{v} + \nabla p = \boldsymbol{0} \quad \text{in} \quad \Omega\,, \tag{5.40}$$

$$\operatorname{div} \boldsymbol{v} = 0 \quad \text{in} \quad \Omega\,, \tag{5.41}$$

$$\boldsymbol{v} = \boldsymbol{0} \quad \text{on} \quad \partial\Omega\,, \tag{5.42}$$

where p is normalized by ρ, Ω the problem domain and $\partial\Omega$ its boundary.

5.3.1 Periodic Stokes Eigenmodes

Let us consider the fully periodic solutions of the transient Stokes problem Eqs. (2.54) in the open square domain $\Omega =]-1, +1[^2$ with the Fourier approximation

$$\begin{aligned} \boldsymbol{v}(\boldsymbol{x}, t) &= \sum_{\|\boldsymbol{k}\|<\infty} \widehat{\boldsymbol{v}}(\boldsymbol{k}) e^{i(\boldsymbol{k}\cdot\boldsymbol{x}+\omega t)}, \quad p = \sum_{\|\boldsymbol{k}\|<\infty} \widehat{p}(\boldsymbol{k}) e^{i(\boldsymbol{k}\cdot\boldsymbol{x}+\omega t)}, \\ \boldsymbol{b}(\boldsymbol{x}, t) &= \sum_{\|\boldsymbol{k}\|<\infty} \widehat{\boldsymbol{b}}(\boldsymbol{k}) e^{i(\boldsymbol{k}\cdot\boldsymbol{x}+\omega t)}, \end{aligned} \tag{5.43}$$

where $\widehat{\boldsymbol{v}}$, $\widehat{p}$, $\widehat{\boldsymbol{b}}$ are the complex Fourier coefficients, $\boldsymbol{k}$ the wavevector and ω a complex frequency. The notation $\|\boldsymbol{k}\| < \infty$ is defined to mean $-\infty < k_i < +\infty$ for $i = 1, 2$.

Let us denote $\boldsymbol{e_k}$ the unit vector in the direction of $\boldsymbol{k}$. Then $\boldsymbol{k} = k\,\boldsymbol{e_k}$. The solution is easily obtained

$$\widehat{p} = -i\frac{\boldsymbol{e_k}\cdot\widehat{\boldsymbol{b}}}{k}, \quad (i\omega + \nu k^2)\widehat{\boldsymbol{v}} = \left(\widehat{\boldsymbol{b}} - \boldsymbol{e_k}(\boldsymbol{e_k}\cdot\widehat{\boldsymbol{b}}\right)\ . \tag{5.44}$$

The resulting periodic eigenmode corresponding to $\lambda = i\,\omega$ is

$$\widehat{p} = 0, \quad (\lambda + \nu k^2)\widehat{\boldsymbol{v}} = \boldsymbol{0}, \quad \boldsymbol{k}\cdot\widehat{\boldsymbol{v}} = 0\ . \tag{5.45}$$

The periodic Stokes modes are constant pressure modes driven only by diffusion as $\lambda = -\nu k^2$. The incompressibility constraint div $\boldsymbol{v} = 0$ does not influence the space configuration, except that the wavevector $\boldsymbol{k}$ must be orthogonal to the velocity. Geometrically speaking the velocity is contained in a plane perpendicular to $\boldsymbol{k}$, while the pressure is aligned with the wavevector.

5.3.2 Channel Flow Stokes Eigenmodes

The problem is based on the plane channel flow between horizontal plates as treated in [69]. The flow is assumed periodic in the x_2 direction while it is confined by rigid walls in $x_1 = \pm 1$. We seek a solution of the two-dimensional Stokes equation in the form

$$\boldsymbol{v}(\boldsymbol{x}, t) = \left(u(x_1)e^{ikx_2+\lambda t}, v(x_1)e^{ikx_2+\lambda t}\right), \quad p = p(x_1)e^{ikx_2+\lambda t}\ , \tag{5.46}$$

where k is a chosen wavenumber and u, v are complex functions. The Stokes equations satisfied by u, v, p are

$$\begin{aligned}\lambda u &= -\frac{dp}{dx_1} + \nu(\frac{d^2u}{dx_1^2} - k^2u) \ , \\ \lambda v &= -ikp + \nu(\frac{d^2v}{dx_1^2} - k^2v) \ , \\ \frac{du}{dx_1} &+ ikv = 0 \ ,\end{aligned} \tag{5.47}$$

for $-1 \leq x_1 \leq +1$. The boundary conditions are

$$\boldsymbol{v}(\pm 1, x_2, t) = \mathbf{0} \ , \tag{5.48}$$

for no-slip walls. The elimination of v and p in Eqs. (5.47) yields

$$\lambda \left(D^2 - k^2\right) u = \nu \left(D^2 - k^2\right)^2 u \ , \tag{5.49}$$

with $u(\pm 1) = Du(\pm 1) = 0$, where $D = d/dx_1$. The solutions of Eq. (5.49) are either symmetric in x_1

$$u(x_1) = \cos\mu \cosh kx_1 - \cosh k \cos\mu x_1 \ , \tag{5.50}$$

or antisymmetric

$$u(x_1) = \sin\mu \sinh kx_1 - \sinh k \sin\mu x_1 \ . \tag{5.51}$$

The eigenvalues are

$$\lambda = -\nu(\mu^2 + k^2) \ , \tag{5.52}$$

satisfying the relations

$$k \tanh k = -\mu \tan\mu \ , \tag{5.53}$$

for (5.50) and

$$k \coth k = \mu \cot\mu \ , \tag{5.54}$$

for (5.51). For $k = 1$ and $k = 10$, the first symmetric eigenmode decays with $\lambda/\nu = -9.3137$ and $\lambda/\nu = -103.0394$, respectively, while, for $k = 1$ and $k = 10$, the antisymmetric eigenmode decays with $\lambda/\nu = -20.5706$ and $\lambda/\nu = -112.0836$. All the eigenvalues λ are real and negative, indicating strong damping by the viscous forces.

This problem is an excellent benchmark to test the accuracy of numerical methods treating the velocity-pressure formulation of the Navier–Stokes equations.

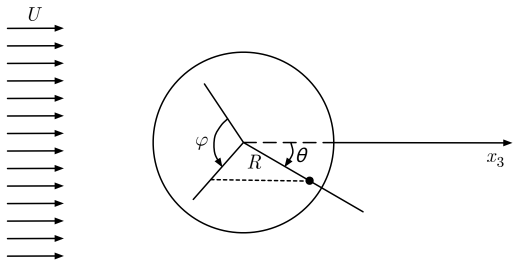


Fig. 5.5 Flow around a sphere

5.4 Parallel Flow Around a Sphere

A sphere of radius R is in a viscous steady state flow for which the velocity at infinity upstream is U. We assume a creeping flow such that we can have a solution of the Stokes equation (2.54). We place the Cartesian coordinate system such that the axis x_3 is oriented in the direction of the flow incident on the sphere (Fig. 5.5). The boundary conditions expressed in spherical coordinates (see Fig. B.1) are

$$\boldsymbol{v} = \mathbf{0} \quad \text{at} \quad r = R, \tag{5.55}$$

$$\boldsymbol{v} = U\boldsymbol{e}_3 \quad \text{at} \quad r = \infty\,. \tag{5.56}$$

The problem thus posed is symmetric about the axis Ox_3, and with respect to the longitude of the sphere. Consequently, $\partial(\bullet)/\partial\varphi \equiv 0$. Also then, $v_\varphi = 0$. Thus the mass conservation equation (B.20) reduces to

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\, v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta) = 0\,. \tag{5.57}$$

We deduce that a stream function ψ exists such that

$$v_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \quad v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\,. \tag{5.58}$$

Given the two-dimensional character of the flow, the vorticity will have a single component in the direction of the vector $\boldsymbol{e}_\varphi$ that we denote ω. We write (cf. (B.5))

$$\omega(r,\theta) = -\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right]\,. \tag{5.59}$$

In the case of the Stokes equation, vorticity is a harmonic function. We have (recall that the Laplacian of a vector is not equal to the Laplacian of its components, cf. (B.7))

$$
\begin{aligned}
\Delta\omega - \frac{\omega}{r^2 \sin^2\theta} &= \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial \omega}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\omega}{\partial\theta}\right) - \frac{\omega}{\sin^2\theta}\right) \\
&= \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\omega) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\omega\sin\theta)\right) \qquad (5.60) \\
&= 0\,. \qquad (5.61)
\end{aligned}
$$

The combination of relations (5.59)–(5.61) gives the following biharmonic equation:

$$
\left(\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right)^2 \psi = 0\,. \qquad (5.62)
$$

The boundary conditions (5.55) and (5.56), expressed in terms of the stream function, become

$$
\begin{aligned}
&\frac{\partial\psi}{\partial\theta} = \frac{\partial\psi}{\partial r} = 0, \quad \text{at} \quad r = R, \\
&v_r = U\cos\theta, \quad \frac{\partial\psi}{\partial\theta} = Ur^2\sin\theta\cos\theta, \quad \text{at} \quad r = \infty \\
&v_\theta = -U\sin\theta, \quad \frac{\partial\psi}{\partial r} = Ur\sin^2\theta. \qquad (5.63)
\end{aligned}
$$

The condition at infinity can be easily integrated. It follows that

$$
\psi_\infty = \frac{1}{2}Ur^2\sin^2\theta\,. \qquad (5.64)
$$

The form of this expression for ψ suggests that the stream function can be written in the general form

$$
\psi = \sin^2\theta\, f(r)\,. \qquad (5.65)
$$

Introducing (5.65) into (5.62), we find

$$
\frac{d^4 f}{dr^4} - \frac{4}{r^2}\frac{d^2 f}{dr^2} + \frac{8}{r^3}\frac{df}{dr} - \frac{8}{r^4}f = 0\,. \qquad (5.66)
$$

Seeking a solution as a power series in r^n, we obtain the characteristic polynomial

$$
(n-2)(n-1)\left(n^2 - 3n - 4\right) = 0\,,
$$

whose roots are $n = -1, 1, 2$, and 4. The function $f(r)$ is thus

$$
f = \frac{C_{-1}}{r} + C_1 r + C_2 r^2 + C_4 r^4\,. \qquad (5.67)
$$

The imposition of the boundary condition at infinity, (5.64), requires $C_4 = 0$ and $C_2 = \frac{1}{2}U$, while at the sphere, with $v_r = v_\theta = 0$, we can determine $C_{-1} = (1/4)U R^3$, $C_1 = -(3/4)U R$. The stream function is then

$$\psi = \frac{UR^2}{2}\sin^2\theta\left(\frac{R}{2r} - \frac{3}{2}\frac{r}{R} + (\frac{r}{R})^2\right). \tag{5.68}$$

We can easily deduce the velocities from (5.58)

$$v_r = U\cos\theta\left[\frac{1}{2}(\frac{R}{r})^3 - \frac{3}{2}\frac{R}{r} + 1\right] \tag{5.69}$$

$$v_\theta = U\sin\theta\left[\frac{1}{4}(\frac{R}{r})^3 + \frac{3}{4}\frac{R}{r} - 1\right]. \tag{5.70}$$

The Cartesian velocity components are linked to the spherical components by the transformation

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}\begin{pmatrix} v_r \\ v_\theta \\ v_\varphi \end{pmatrix}. \tag{5.71}$$

Using (5.71), Eqs. (5.69) and (5.70) yield

$$\begin{aligned}
v_1 &= -\frac{3}{4}\frac{URx_1x_3}{r^3}\left(1 - \frac{R^2}{r^2}\right), \\
v_2 &= -\frac{3}{4}\frac{URx_2x_3}{r^3}\left(1 - \frac{R^2}{r^2}\right), \\
v_3 &= U\left[1 - \frac{3}{4}\frac{Rx_3^2}{r^3}\left(1 - \frac{R^2}{r^2}\right) - \frac{1}{4}\frac{R}{r}\left(3 + \frac{R^2}{r^2}\right)\right],
\end{aligned} \tag{5.72}$$

as we have

$$\begin{aligned}
x_1 &= r\sin\theta\cos\varphi \\
x_2 &= r\sin\theta\sin\varphi \\
x_3 &= r\cos\theta\,.
\end{aligned}$$

The vorticity field is written (cf. (5.59))

$$\omega = -\frac{3}{2}UR\,(\frac{\sin\theta}{r^2}). \tag{5.73}$$

The calculation of the pressure field can easily be accomplished by taking into account the vector identity (4.20) which leads to the Stokes equation

$$\nabla p = -\mu \, \mathbf{curl}\, \boldsymbol{\omega} \, . \tag{5.74}$$

With (B.5), we generate the system of equations

$$\frac{\partial p}{\partial r} = -\frac{\mu}{r \sin\theta} \frac{\partial}{\partial \theta} (\omega \, \sin\theta) = 3\mu \, U \, R \frac{\cos\theta}{r^3} \, , \tag{5.75}$$

$$\frac{1}{r} \frac{\partial p}{\partial \theta} = \frac{\mu}{r} \frac{\partial (r\omega)}{\partial r} = \frac{3\mu}{2} U \, R \frac{\sin\theta}{r^3} \, . \tag{5.76}$$

Integration of (5.75) yields

$$p = -\frac{3\mu}{2} U \, R \frac{\cos\theta}{r^2} + q(\theta) \, .$$

Inserting this result in (5.76), we have

$$\frac{3\mu}{2} U \, R \frac{\sin\theta}{r^3} + \frac{q'(\theta)}{r} = \frac{3\mu}{2} U \, R \frac{\sin\theta}{r^3} \, .$$

The pressure field is then finally given by

$$p = -\frac{3\mu}{2} U \, R \frac{\cos\theta}{r^2} + p_0 \, , \tag{5.77}$$

with p_0 a constant reference pressure.

The uniform velocity flow around a sphere will generate pressure and shear forces. To calculate the pressure force in direction Ox_3, we integrate the elementary forces over the surface of the sphere

$$dF_{3,p} = -(\frac{3\mu}{2} U \, \frac{\cos\theta}{R} + p_0) \cos\theta \, (2\pi R^2 \sin\theta) \, d\theta \, . \tag{5.78}$$

The factor 2π comes from the symmetry of the problem which allows us to take into account the longitudinal part of the integral. Integrating from $\theta = 0$ to $\theta = \pi$, we obtain

$$F_{3,p} = -2\pi \, \mu \, U \, R \, . \tag{5.79}$$

Friction drag is obtained by integration over the sphere of the shear stress that acts on it, that is, $\sigma_{r\theta}$ (cf. (B.19)) which is $-3\mu U \sin\theta/(2R)$ for $r = R$. This leads to

$$F_{3,\sigma} = -\int_{\theta=0}^{\theta=\pi} (\sigma_{r\theta} \mid_{r=R} \sin\theta) \, (2\pi R^2 \sin\theta) \, d\theta = -4\pi \, \mu \, U \, R \, . \tag{5.80}$$

Total drag, $F_3 = F_{3,p} + F_{3,\sigma}$, known as ***Stokes drag***, is the sum of the pressure force and the friction force

$$F_3 = -6\pi \, \mu \, U \, R \, . \tag{5.81}$$

If we define the drag coefficient by

$$C_D = \frac{|F_3|}{\frac{1}{2}\rho U^2 \pi R^2}, \tag{5.82}$$

we obtain

$$C_D = \frac{24}{Re}, \tag{5.83}$$

where $Re = 2UR/\nu$. Note that the pressure drag represents one third of the total drag. Relation (5.83) is verified by experiments when $Re < 1$ which is valid in the neighborhood of the sphere. When we move away, the importance of the inertial terms grows and the Stokes solution diverges from the exact solution. Note that the solution that we have obtained is not applicable to the case of a set of spherical particles, as the presence of a spherical obstacle in the flow has impact relatively far away since the velocity profiles decrease as $1/r$.

The solution for uniform flow around a fixed sphere can be transposed to the case of translation at uniform velocity U of a sphere of radius R in a fluid at rest at infinity. In this case, the coordinate system is still attached to the sphere and thus in translation at uniform velocity. This modifies the sign of U to become $-U$ for the pressure and the vorticity. As for the velocity in the fluid, this is relative to the coordinate system, which leads to the following modifications: for the velocity and the stream function, U becomes $-U$ and the uniform velocity field must also be subtracted from the corresponding relations.

5.4.1 Oseen's Improvement

The Stokes solution was improved by Oseen [70] who proposed the solution of the Navier–Stokes equations (1.74) as a sum of uniform velocity field and a perturbation such that

$$\boldsymbol{v} = U\,\boldsymbol{e}_3 + \boldsymbol{v}' \,. \tag{5.84}$$

In the case of the flow around a fixed sphere, the velocity $\boldsymbol{v}'$ then takes into account the perturbation caused by the sphere in a flow uniform at infinity. With (5.84), the stationary inertial term takes the form

$$\rho \frac{D\boldsymbol{v}}{Dt} = \rho \left(v'_j \frac{\partial v'_i}{\partial x_j} + U \frac{\partial v'_i}{\partial x_3} \right) . \tag{5.85}$$

Oseen's assumption amounts to neglecting the first term with respect to the second term on the right-hand side of Eq. (5.85). We obtain a linearized Navier–Stokes equation

$$\rho\, U \frac{\partial \boldsymbol{v}'}{\partial x_3} = -\nabla p + \mu\, \Delta \boldsymbol{v}' + \rho \boldsymbol{b} \,. \tag{5.86}$$

The drag coefficient obtained with Oseen's solution is

$$C_D = \frac{24}{Re}(1 + \frac{3}{16}Re) \; . \tag{5.87}$$

Experimental results show that (5.87) is approximately valid for $Re < 5$. Using matched asymptotic expansions [49], the corrected coefficient becomes

$$C_D = \frac{24}{Re}(1 + \frac{3}{16}Re - \frac{19}{1280}Re^2 + O(Re^3)) \; . \tag{5.88}$$

Another approximation for the drag coefficient is given as

$$C_D = \frac{24}{Re}[1 + \frac{3}{16}Re + \frac{9}{160}Re^2 \ln Re + O(Re^2)] \; . \tag{5.89}$$

For a quantitative outline of the analysis leading to (5.89), we refer the reader to Proudman and Pearson [75].

A more accurate relation for the drag is proposed by Ockendon and Ockendon [64]:

$$C_D = \frac{24}{Re}[1 + \frac{3}{16}Re + \frac{9}{160}Re^2 \ln Re + \frac{1}{160}(9\gamma + 15\ln 2 - \frac{323}{40})Re^2 + \cdots] \; ,$$

where $\gamma = 0.5772156649$ is Euler's constant.

5.5 Parallel Flow Around a Cylinder

After the detailed study of the flow around a sphere, we would like to investigate the case of a uniform steady state parallel flow impinging a fixed circular cylinder. We will unfold the same steps as in the previous section in order to compare both theoretical developments and draw conclusions about the impact of the geometrical configurations on the analytical results.

A cylinder of radius R is placed in a uniform parallel flow for which the velocity at infinite upstream is U. We assume again a creeping flow described by the Stokes equation (2.54). We place the Cartesian coordinate system such that the axis x_3 is oriented in the direction of the flow incident on the cylinder (Fig. 5.6). The boundary conditions expressed in cylindrical or more precisely in polar coordinates are

$$\boldsymbol{v} = \mathbf{0} \quad \text{at} \quad r = R, \tag{5.90}$$

$$\boldsymbol{v} = U\boldsymbol{e}_3 \quad \text{at} \quad r = \infty \; . \tag{5.91}$$

The flow is two-dimensional and therefore $v_z = 0$. The velocity field is such that $\boldsymbol{v} = (v_r(r,\theta), v_\theta(r,\theta), 0)$. The mass conservation equation (A.2) reduces to

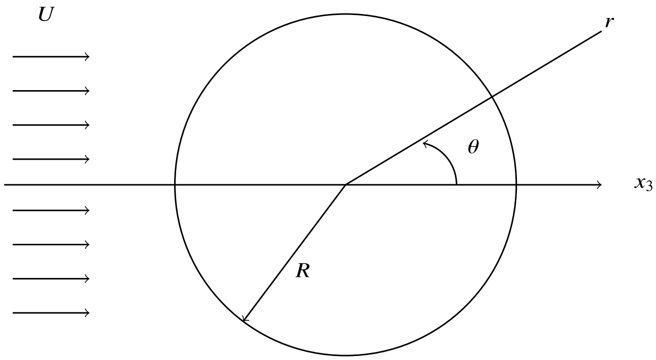


Fig. 5.6 Flow around a circular cylinder

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0\,. \tag{5.92}$$

We deduce that a stream function ψ exists such that

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad v_\theta = -\frac{\partial \psi}{\partial r}\,. \tag{5.93}$$

Given the two-dimensional character of the flow, the vorticity will have a single component in the direction orthogonal to the plane that we denote ω. We can then write (see Eq. (A.6))

$$\omega(r,\theta) = -\frac{1}{r}\left(\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right) \tag{5.94}$$

$$= -\left(\frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}\right)\,. \tag{5.95}$$

As vorticity is a harmonic function, its Laplacian is given by

$$\Delta\omega = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \omega}{\partial \theta^2} = \frac{1}{r}\frac{\partial \omega}{\partial r} + \frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \omega}{\partial \theta^2}\,. \tag{5.96}$$

The combination of relations (5.95)–(5.96) gives the following biharmonic equation:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \psi = 0\,. \tag{5.97}$$

The boundary conditions (5.90) and (5.91), expressed in terms of the stream function, become

$$\frac{\partial \psi}{\partial \theta} = \frac{\partial \psi}{\partial r} = 0, \quad \text{at} \quad r = R, \tag{5.98}$$

$$v_r = U \cos \theta, \quad \frac{\partial \psi}{\partial \theta} = U r \cos \theta, \quad \text{at} \quad r = \infty$$
$$v_\theta = -U \sin \theta, \quad \frac{\partial \psi}{\partial r} = U \sin \theta \,. \tag{5.99}$$

The condition at infinity suggests that the stream function can be written in the general form

$$\psi = f(r) \sin \theta \,. \tag{5.100}$$

Introducing (5.100) into (5.97), we find

$$\frac{d^4 f}{dr^4} + \frac{2}{r}\frac{d^3 f}{dr^3} - \frac{3}{r^2}\frac{d^2 f}{dr^2} + \frac{3}{r^3}\frac{df}{dr} - \frac{3}{r^4} f = 0 \,. \tag{5.101}$$

Seeking a solution as a power series in r^n, we obtain the characteristic polynomial

$$(n-1)^2 \left(n^2 - 2n - 3\right) = 0 \,,$$

whose roots are $n = -1, 1, 3$. Note that the root $n = 1$ is a double root. So far, the function $f(r)$ is

$$f = \frac{C_{-1}}{r} + C_1 r + C_3 r^3 \,. \tag{5.102}$$

We need a fourth term to complete the expression of f. Michell [58] provided the full solution of the biharmonic equation in polar coordinates from which we extract the expression $r \log r$. Hence, eventually, we have

$$f = \frac{C_{-1}}{r} + C_1 r + C_3 r^3 + C_4 r \log r \,. \tag{5.103}$$

The r^3 term is rejected as it violates the regularity conditions at infinity. Using the boundary conditions on the cylinder (5.98), we obtain

$$\psi = C_1 \sin \theta \left(\frac{1}{\overline{r}} - \overline{r} + 2\overline{r} \log \overline{r} \right) \,, \tag{5.104}$$

where $\overline{r} = r/R$. We note that it is impossible to satisfy the condition at infinity due to the presence of the logarithm.

Let us now satisfy the condition at infinity by setting $C_4 = 0$ and imposing $f(R) = 0$, one gets

$$f(\overline{r}) = C_1 (\overline{r} - \frac{1}{\overline{r}}) \,. \tag{5.105}$$

However it is impossible to satisfy $df/dr = 0$ at $r = R$.

This is Stokes paradox that shows there is no creeping flow around a circular cylinder unlike the flow around the sphere. To solve the paradox we need to resort to Oseen developments and matched asymptotic expansions for inner and outer expressions. We refer the reader to [46, 47].

5.6 Three-Dimensional Stokes Solution

In this section we present a three-dimensional solution for the steady Stokes equations

$$
\begin{aligned}
-\mu\Delta \boldsymbol{v} + \nabla p &= \boldsymbol{0} \quad \text{in} \quad \Omega\,, && (5.106)\\
\operatorname{div}\, \boldsymbol{v} &= 0 \quad \text{in} \quad \Omega\,. && (5.107)
\end{aligned}
$$

This procedure is due to [106] and is based on harmonic solutions of the Laplace equation, assuming that separable solutions are relevant. The method works as follows. Suppose that $\boldsymbol{A}$ and B are vector and scalar fields satisfying Laplace's equations

$$
A_{i,jj} = 0, \quad B_{,jj} = 0\,. \tag{5.108}
$$

Then the velocity v_i and pressure p are given by the relationships

$$
\begin{aligned}
v_i &= \frac{\partial}{\partial x_i}\big(r_j A_j + B\big) - 2A_i\,, && (5.109)\\
\frac{p}{\mu} &= 2\,A_{j,j}\,, && (5.110)
\end{aligned}
$$

where r_j are the components of the position vector. The proofs given in [106] are based on theoretical developments coming from elasticity theory, more precisely on the Papkovich–Neuber type of solution, cf. Chap. 7 in Botsis-Deville [16]. For the sake of simplicity, we will skip them. However let us examine how the methodology of solving both Laplace equations (5.108) and then combining the two harmonic solutions through (5.109)–(5.110) yields the Stokes solution. In Cartesian coordinates with $r_j = x_j$, Eqs. (5.109)–(5.110) give

$$
\begin{aligned}
v_i &= \frac{\partial}{\partial x_i}(x_j A_j + B) - 2A_i = x_j A_{j,i} - A_i + B_{,i}\,, && (5.111)\\
\frac{p}{\mu} &= 2A_{j,j}\,. && (5.112)
\end{aligned}
$$

The incompressibility constraint is ensured

$$
v_{i,i} = A_{i,i} + x_j A_{j,ii} - A_{i,i} + B_{,ii} = 0\,. \tag{5.113}
$$

We next employ the equilibrium equation (5.106) and use (5.108) to eliminate some terms

$$
\begin{aligned}
v_{i,kk} &= \frac{\partial}{\partial x_k}\left(A_{k,i} + x_j A_{j,ik} - A_{i,k} + B_{i,k}\right) \\
&= A_{k,ik} + A_{k,ik} + x_j A_{j,ikk} - A_{i,kk} + B_{i,kk} = 2A_{k,ik} \\
&= \frac{1}{\mu}\frac{\partial p}{\partial x_i} \, .
\end{aligned} \tag{5.114}
$$

Let us apply the previous solution technique to the Stokes flow of a sphere of radius R moving at constant speed U in an infinite fluid, as reported in Sect. 5.4. We refer the problem to a Cartesian coordinate system with origin at the center of the sphere and with positive x_3-axis in the flow direction. The harmonic solutions are

$$
A_1 = A_2 = 0, \quad A_3 = -U + \frac{3UR}{4r}, \qquad B = -\frac{U x_3 R^2}{4r^3} \, . \tag{5.115}
$$

Using (5.115) in (5.111) we easily obtain the Eqs. (5.72).

Exercises

5.1 Hele-Shaw Flow

Let us consider the creeping flow between two fixed parallel plates, cf. Fig. 5.7, of a very viscous fluid in a layer of thickness $2h$. Inside the gap is placed an obstacle of cylindrical shape with its generators orthogonal to the plates, and of characteristic length L. The geometrical aspect ratio defined as

$$
\varepsilon = \frac{h}{L} \, , \tag{5.116}
$$

is such that $\varepsilon \ll 1$.

- With the incompressibility constraint (1.73) evaluate the order of magnitude of the velocity components.
- Evaluate the order of magnitude of the second order partial derivatives of v_1 and v_2.
- Show that the pressure is constant across the gap.
- Show that the velocity field may be written as

$$
v_i(x_1, x_2, x_3) = v_i(x_1, x_2, 0) f(x_3) \, , \tag{5.117}
$$

  with the origin of the axes located at mid-gap.
- Propose a solution for $f(x_3)$. Deduce the momentum equations for v_1, v_2.
- Show that

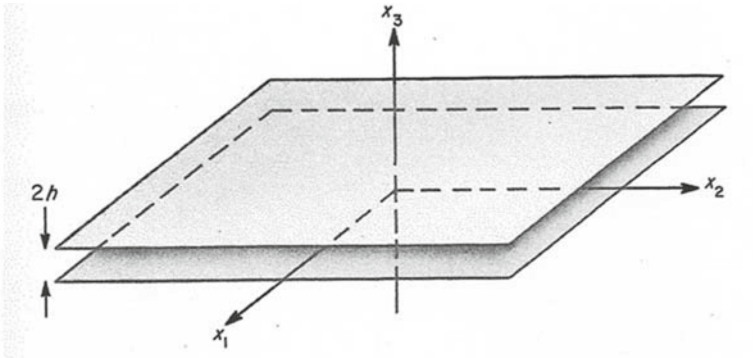


Fig. 5.7 Geometry for flow between parallel plates

$$v_i(x_1, x_2, 0) = -\frac{h^2}{2\mu}\frac{\partial p}{\partial x_i}\,. \tag{5.118}$$

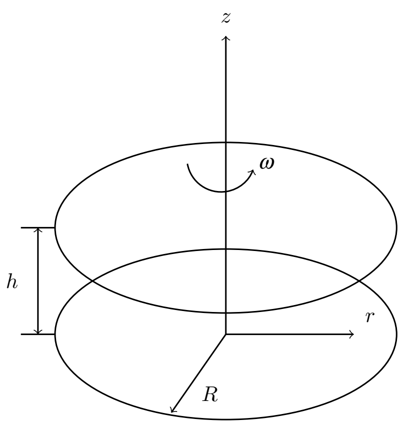
- Compute the velocity potential.

5.2 Flow between parallel discs

In the parallel discs viscometer the fluid to be tested is contained in the cylindrical region of radius R between two discs. The lower plate is fixed and lies in the $z = 0$ plane, while the upper disc in the $z = h$ plane rotates at the angular velocity ω, cf. Fig. 5.8. This figure represents schematically such a viscosimeter. It is supposed that the angular velocity of the upper plate is small and that for small values of the applied torque, the velocities in a cylindrical coordinate system are such that

$$\boldsymbol{v} = (v_r, v_\theta, v_z) = (0, r\, f(z), 0) \tag{5.119}$$

Fig. 5.8 Parallel discs and the associated coordinates



- Compute the velocity field
- As the angular velocity is small, it is assumed that $\partial p/\partial r = 0$. Deduce the hydrostatic pressure field.
- Evaluate the shear stress in the azimuthal direction. If the moment M to rotate the upper disc at velocity ω is known, generate the relation giving the viscosity μ as a function of M, h, ω, R.