Chapter 4
Vorticity and Vortex Kinematics

In Sect. 1.8, the incompressible Navier–Stokes equations were derived for a viscous Newtonian fluid in terms of primitive variables: velocity and pressure. The phenomena observed in fluid flows have been interpreted by an equilibrium between the inertial forces, the pressure gradient, the volume forces such as gravity, and the viscous forces. In this chapter, we take a different point of view based on the concept of vorticity.

The presence of vorticity in a flow is an indication of the importance of the viscous effects, given that they are generated by viscous stresses. However rotational flows are computed by solving Euler equation, which show the presence of vortices. For example, the flow over a delta wing may be obtained by solving Euler equation. Vorticity can also be generated by a baroclinic mechanism for compressible flow as in Rayleigh–Taylor instability.

Therefore, under certain assumptions, vorticity possesses the following properties:

(i) in the absence of viscosity, it is transported by the flow as an elementary material vector;
(ii) in the presence of viscosity, it diffuses into the surrounding fluid while being continually produced at the solid walls that delimit the flow.

Thus the vorticity produced on a solid wall introduces the notion of a boundary layer for which we are led to modify certain conclusions coming from the theory of irrotational perfect fluids. In turbulence, flow dynamics is mostly the result of the stretching or shortening of vortex filaments and their deformation.

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4.1 Kinematic Considerations

The velocity gradient tensor $\boldsymbol{L}$ can be decomposed into the sum of a symmetric strain rate tensor $\boldsymbol{d}$ and an antisymmetric rotation rate tensor $\dot{\boldsymbol{\omega}}$ according to Eq. (1.36). The tensor $\boldsymbol{d}$ is given by (1.33) and $\dot{\boldsymbol{\omega}}$ by (1.35). Recall that the dual vector $\dot{\boldsymbol{\Omega}}$, corresponding to the rotation rate tensor, is the rotation rate vector introduced by (1.37).

In fluid mechanics, we classically introduce the vorticity vector $\boldsymbol{\omega}$, defined as the curl of the velocity (1.40). To acquire familiarity with the concept of vorticity, we study the flow near a stagnation point at the origin. The velocity components are such that we have, with a constant C,

$$v_1 = Cx_1,\ v_2 = -Cx_2,\ v_3 = 0\ . \tag{4.1}$$

We easily calculate that for this flow $\boldsymbol{\omega} = \boldsymbol{0}$. A flow with zero vorticity is called ***irrotational***.

Now consider the plane Poiseuille flow in a channel of height h. If the coordinate system has its origin on the lower wall, the velocity profile, (3.19) with definition (3.24), is given by the relation

$$v_1 = 4v_{max}\frac{x_2}{h}(1 - \frac{x_2}{h})\ , \tag{4.2}$$

with v_{max} being the maximum velocity on the centerline of the channel at $x_2 = h/2$. The only component of the vorticity is ω_3. It is perpendicular to the plane of the flow and its value is

$$\omega_3 = \varepsilon_{321}\frac{\partial v_1}{\partial x_2} = -\frac{4v_{max}}{h}(1 - \frac{2x_2}{h})\ . \tag{4.3}$$

In this case, the absolute value of the vorticity attains a maximum at the two walls and goes to zero on the centerline of the channel.

From these examples we can conclude that the concept of vorticity has no relation to the curvature of the streamlines. In the first case, the streamlines are curved ($\psi = Cx_1x_2$), but the vorticity is zero; while in the second example, the streamlines are straight lines and there is finite vorticity.

The Stokes theorem or the curl theorem transforms the flux of the curl of a vector through a surface S into the line integral of that vector around the curvy boundary C of the surface. The theorem reads

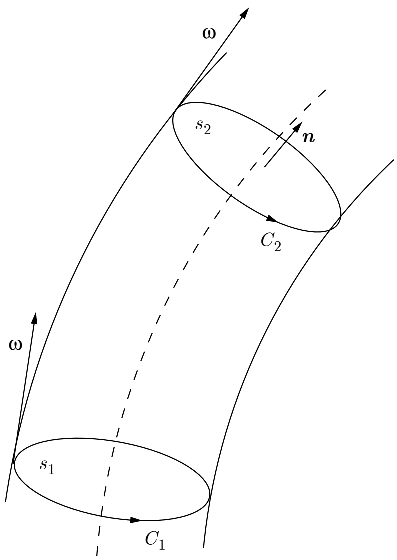


Fig. 4.1 Vortex tube

Theorem 4.1 (Stokes theorem)

$$\int_S \boldsymbol{\omega} \cdot \boldsymbol{n}\, dS = \oint_C \boldsymbol{v} \cdot \boldsymbol{\tau}\, dl\ . \tag{4.4}$$

From the definition of vorticity, (1.40), and the Stokes theorem (4.4), we obtain the identity

$$I(S) = \int_S \boldsymbol{\omega} \cdot \boldsymbol{n}\, dS = \int_S \mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{n}\, dS = \oint \boldsymbol{v} \cdot \boldsymbol{\tau}\, dl = \Gamma\ , \tag{4.5}$$

where $I(S)$ is the intensity of the vortex tube. The curvilinear integral in (4.5) defines the velocity circulation, Γ, along the closed curve C, of the unit tangent vector $\boldsymbol{\tau}$. It is thus equal to the vorticity vector flux through an arbitrary surface bounded by the curve. In the following, this property will permit us to systematically link the concept of circulation to an interpretation in terms of vorticity.

Recall that a vortex line (Fig. 4.1) is a line tangent at all its points to the vorticity vector, and that a vortex tube is a family of vortex lines circumscribed by a closed curve. The intensity of a vortex tube, for a surface S defined by a closed line enclosing the vortex tube, is the flux $I(S)$ of vorticity through the surface.

Theorem 4.2 (Helmholtz) (Vorticity properties) *Helmholtz main theorems about vorticity are as follows:*

- *the vorticity flux through a closed surface is always zero;*
- *the intensity of a vortex tube does not depend on the transverse section considered;*

- *a vortex tube can only end connected to itself or extend to infinity unless it is cut by a wall.*

The proof of these theorems can be found in Panton's book [71].

4.2 Dynamic Vorticity Equation

4.2.1 General Equation

The formulation of the equation that governs vorticity dynamics requires the establishment of certain preliminary relations.

The acceleration term $\boldsymbol{a}$ can be written as follows:

$$\boldsymbol{a} = \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{v} + \boldsymbol{grad}\left(\frac{\boldsymbol{v} \cdot \boldsymbol{v}}{2}\right) , \tag{4.6}$$

that can be proved by induction

$$\begin{aligned} a_i &= \frac{\partial v_i}{\partial t} + \varepsilon_{ijk}\omega_j v_k + \frac{\partial}{\partial x_i}\left(\frac{v_j v_j}{2}\right) , \\ &= \frac{\partial v_i}{\partial t} + \varepsilon_{ijk}\varepsilon_{jlm}\left(\frac{\partial v_m}{\partial x_l}\right) v_k + v_j \frac{\partial v_j}{\partial x_i} , \\ &= \frac{\partial v_i}{\partial t} + (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})\left(\frac{\partial v_m}{\partial x_l} v_k\right) + v_j \frac{\partial v_j}{\partial x_i} , \end{aligned}$$

or

$$a_i = \frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k} .$$

This last expression is none other than the definition of acceleration (1.23). Thus, relation

$$\mathbf{curl}\,\boldsymbol{a} = \frac{D\boldsymbol{\omega}}{Dt} - \boldsymbol{\omega} \cdot \boldsymbol{grad}\,\boldsymbol{v} \tag{4.7}$$

is an identity. As can be seen, applying the curl operator to relation (4.6) leads to

$$\mathbf{curl}\,\boldsymbol{a} = \frac{\partial}{\partial t}\mathbf{curl}\,\boldsymbol{v} + \mathbf{curl}(\boldsymbol{\omega} \times \boldsymbol{v}) + \mathbf{curl}\,\boldsymbol{grad}\left(\frac{\boldsymbol{v} \cdot \boldsymbol{v}}{2}\right)$$

or

$$\mathbf{curl}\,\boldsymbol{a} = \frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{curl}(\boldsymbol{\omega} \times \boldsymbol{v}) . \tag{4.8}$$

The term $\mathbf{curl}(\boldsymbol{\omega} \times \boldsymbol{v})$ can be developed as follows:

$$\mathbf{curl}(\boldsymbol{\omega} \times \boldsymbol{v}) = \boldsymbol{v} \cdot \boldsymbol{grad}\, \boldsymbol{\omega} - (\nabla \boldsymbol{v})\ \boldsymbol{\omega} + \boldsymbol{\omega} \operatorname{div} \boldsymbol{v} - \boldsymbol{v} \operatorname{div} \boldsymbol{\omega}\ . \tag{4.9}$$

The last term of (4.9) is zero as div $\mathbf{curl}\, \boldsymbol{v} = 0$. From (4.8) and (4.9), it follows that

$$\mathbf{curl}\, \boldsymbol{a} = \frac{D\boldsymbol{\omega}}{Dt} - (\nabla \boldsymbol{v})\ \boldsymbol{\omega} + \boldsymbol{\omega} \operatorname{div} \boldsymbol{v}\ .$$

From the mass conservation equation (1.50), we obtain the relation

$$\mathbf{curl}\, \boldsymbol{a} = \frac{D\boldsymbol{\omega}}{Dt} - (\nabla \boldsymbol{v})\ \boldsymbol{\omega}\ ,$$

which is equivalent to Eq. (4.7) that can be written in the form

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \boldsymbol{grad})\, \boldsymbol{v} + \mathbf{curl}\, \boldsymbol{a}\ . \tag{4.10}$$

From the conservation of momentum (1.58), we write

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \boldsymbol{grad})\, \boldsymbol{v} + \mathbf{curl}\, (\boldsymbol{b} + \frac{1}{\rho}\, \mathbf{div}\, \boldsymbol{\sigma})\ . \tag{4.11}$$

Using the constitutive equation (1.67) in (4.11), we have

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \boldsymbol{grad})\, \boldsymbol{v} + \mathbf{curl}\, \boldsymbol{b} - \frac{1}{\rho} \mathbf{curl}(\nabla p) + 2\nu\, \mathbf{curl}(\mathbf{div}\, \boldsymbol{d})\ . \tag{4.12}$$

If the body force is conservative, it can be derived from a potential χ, as is the case for gravity. Then we write

$$\boldsymbol{b} = -\nabla\, \chi\ . \tag{4.13}$$

Consequently, $\mathbf{curl}\, \boldsymbol{b} = \mathbf{0}$, and this term disappears from (4.12). We adopt this hypothesis for the rest of the discussion. Furthermore as for the scalar field p one has $\mathbf{curl}\ (\nabla\ p) = 0$, Eq. (4.12) is simplified as

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \boldsymbol{grad})\, \boldsymbol{v} + 2\nu\, \mathbf{curl}(\mathbf{div}\, \boldsymbol{d})\ . \tag{4.14}$$

The left-hand side of relation (4.14) contains the material derivative of the vorticity. On the right-hand side, we find two terms that describe the deformation (stretching and shrinking) and the curvature (bending-tilting) of the vortex lines and the viscous diffusion of the vorticity. We notice that in the incompressible case, the vorticity equation does not contain any pressure contribution unlike the compressible case where a baroclinicity term appears (cf. Botsis-Deville [16]).

4.2.2 Physical Interpretation of Vorticity Dynamics for the Incompressible Perfect Fluid

For an incompressible fluid ($\rho = constant$), that is inviscid ($\nu = 0$), Eq. (4.14) yields

$$\frac{D\boldsymbol{\omega}}{Dt} = (\nabla \boldsymbol{v})\ \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{v}\ . \tag{4.15}$$

The term

$$(\boldsymbol{\omega} \cdot \nabla)\boldsymbol{v}$$

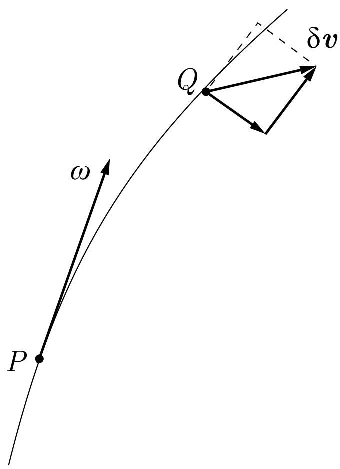
does not correspond to any term in the Navier–Stokes equations written with the primitive variables, velocity and pressure. Let us examine what that term means from the physical point of view.

In Fig. 4.2, consider two neighboring points Pr and Q on a vortex line. The points P and Q also define a material line of length $dx = \| \boldsymbol{dx} \|$, and we can show the equality

$$\frac{D(dx_i)}{Dt} = dv_i = \frac{\partial v_i}{\partial x_j} dx_j \ \text{ or } \ \frac{D(\boldsymbol{dx})}{Dt} = \boldsymbol{dx} \cdot \boldsymbol{grad}\ \boldsymbol{v}\ . \tag{4.16}$$

This last equation simultaneously expresses the changes in length and direction of a material line element. Comparison of (4.15) and (4.16) shows that the vorticity vector $\boldsymbol{\omega}$ plays a role analogous to that of the vector $\boldsymbol{dx}$. Thus, $\boldsymbol{\omega}$ behaves *as if* it were a material line element instantaneously coinciding with a portion of the vortex line. Let $\delta \boldsymbol{v}$ be the relative velocity of the fluid at Q with respect to P. In relation (4.15), we can make the substitution

Fig. 4.2 Portion of a vortex line



$$(\nabla \boldsymbol{v})\ \boldsymbol{\omega} = \parallel \boldsymbol{\omega} \parallel \lim_{PQ \to 0} \frac{\delta \boldsymbol{v}}{PQ}\ .$$

One part of the change of $\boldsymbol{\omega}$ measured by (4.15) comes from the rigid body rotation of the material line element (from the component of $\delta \boldsymbol{v}$ normal to $\boldsymbol{\omega}$), and the other part is generated by the shrinking or stretching of the elementary line (from the component $\delta \boldsymbol{v}$ parallel to $\boldsymbol{\omega}$). Finally, Eq. (4.15) can be interpreted as follows: the vorticity is transported by the fluid particles, while being oriented and deformed *as if* it were an elementary material vector.

4.2.3 The Vorticity Number

Truesdell [110] proposed to measure the vorticity in a flow by means of a dimensionless invariant named the vorticity number. This number evaluates the relative strengths of the rotation and stretching and is defined by the relation

$$\mathcal{W} = \sqrt{\frac{\dot{\boldsymbol{\omega}} : \dot{\boldsymbol{\omega}}}{\boldsymbol{d} : \boldsymbol{d}}}\ , \tag{4.17}$$

where $\dot{\boldsymbol{\omega}}$ and $\boldsymbol{d}$ are the vorticity tensor (1.35) and rate of deformation tensor (1.33), respectively. The symbol : defines the scalar product of two tensors. For example, $\boldsymbol{d} : \boldsymbol{d} = d_{ij} d_{ij}$.

The two limit cases are the irrotational motion where $\dot{\boldsymbol{\omega}} = 0$ and $\mathcal{W} = 0$, and the rigid body motion where $\boldsymbol{d} = 0$ with $\dot{\boldsymbol{\omega}} \neq 0$ and thus $\mathcal{W} = \infty$. All flows with non zero vorticity will be measured through the vorticity number the range of which is in between 0 and infinity.

As an example, let us consider a generalized Poiseuille flow with the following velocity profile

$$v_1 = v_2 = 0, \quad v_3 = v_3(x_1, x_2)\ . \tag{4.18}$$

It is easily computed that $\mathcal{W} = 1$.

4.3 Vorticity Equation for a Viscous Newtonian Fluid

We assume now that the viscosity μ is invariable. With (1.68), we write

$$\mathbf{div}\, \boldsymbol{d} = \frac{1}{2} \left(\boldsymbol{grad}\, (\mathrm{div}\, \boldsymbol{v}) + \Delta\, \boldsymbol{v} \right) \tag{4.19}$$

or using the identity for an arbitrary vector $\boldsymbol{a}$

$$\Delta \boldsymbol{a} = \nabla \cdot \nabla \boldsymbol{a} = \nabla(\nabla \cdot \boldsymbol{a}) - \mathbf{curl\ curl}\ \boldsymbol{a} \tag{4.20}$$

the relation (4.19) becomes

$$\mathbf{div}\,\boldsymbol{d} = \boldsymbol{grad}\,(\mathrm{div}\,\boldsymbol{v}) - \frac{1}{2}\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v}\ . \tag{4.21}$$

Taking the curl of (4.21) leads to

$$\mathbf{curl}(\mathbf{div}\,\boldsymbol{d}) = -\frac{1}{2}\mathbf{curl}(\,\mathbf{curl}\,\boldsymbol{\omega})\ . \tag{4.22}$$

The vorticity dynamics equation is obtained by combining (4.14) and (4.22):

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \boldsymbol{grad})\,\boldsymbol{v} - \nu\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\omega}\ . \tag{4.23}$$

Equation (4.23) may be rewritten with the help of (4.20) as

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \boldsymbol{grad})\,\boldsymbol{v} + \nu\,\Delta\boldsymbol{\omega}\ . \tag{4.24}$$

Special Case for Two-Dimensional Flow

For an incompressible two-dimensional flow, Eq. (4.23) becomes, with the notation $\omega_3 = \omega$,

$$\frac{D\omega}{Dt} = \nu\,\Delta\omega\ \ , \tag{4.25}$$

because in this special case the term $(\boldsymbol{\omega} \cdot \boldsymbol{grad}\ \boldsymbol{v})$ is zero since $\boldsymbol{\omega}$ is orthogonal to the flow plane and thus to $\boldsymbol{grad}\ \boldsymbol{v}$. We note that Eq. (4.25) is analogous to that for heat conduction, with the kinematic viscosity replacing the thermal diffusivity. We also notice that Eq. (4.25) is satisfied for $\omega = 0$, that is, for an irrotational flow. However, that solution is inadequate. To understand why, we reason by analogy with the heat equation, which also allows an identically zero solution. We know from the study of heat flow, that any non-uniform distribution of temperature at the wall or non-zero heat flux will generate a variable temperature field in the material. Thus the analogy leads us to conclude that, in the case of a viscous fluid, the vorticity that is generated at the walls will diffuse out by shear and then be carried away by the flow. The creation of vorticity at the wall is the result of the shear stress on the wall. To obtain the value of vorticity at the wall, we can resort to the classical method of Green's functions [101].

4.4 Circulation Equation

In the context of the hypotheses introduced in the previous section, we prove that for a material curve $C(t)$, along which the circulation of the velocity vector is $\Gamma(t)$, we can write the following:

$$\frac{d\Gamma}{dt} = -\oint_{C(t)} \nu(\textbf{curl curl}\, \boldsymbol{v}) \cdot d\boldsymbol{x} \ . \tag{4.26}$$

This relation expresses the fact that the variation of the circulation along the material curve is due to the viscosity which dampens the motion.

To obtain (4.26), we must first prove that for a material curve $C(t)$, we have the following identity:

$$\frac{d\Gamma}{dt} = \oint_{C(t)} \boldsymbol{a} \cdot d\boldsymbol{x} \ . \tag{4.27}$$

For that purpose, we can write the equation

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\oint_{C(t)} v_i dx_i = \frac{d}{dt}\oint_{C_0} V_i \frac{\partial x_i}{\partial X_j} dX_j \ ,$$

in which C_0 denotes the material curve $C(t)$ at the instant $t = t_0$ and X_i are the associated Lagrangian coordinates. Denoting by A_i and V_i the Lagrangian representations of acceleration (1.22) and velocity (1.9), we have:

$$\begin{aligned}\frac{d}{dt}\oint_{C_0} V_i \frac{\partial x_i}{\partial X_j} dX_j &= \oint_{C_0}\left(A_i \frac{\partial x_i}{\partial X_j} + V_i \frac{\partial V_i}{\partial X_j}\right) dX_j \\ &= \oint_{C(t)} a_i dx_i + \oint_{C_0} \frac{\partial}{\partial X_j}\left(\frac{V_i V_i}{2}\right) dX_j \ .\end{aligned}$$

The last term of the right-hand side of this equality is zero on a closed curve.

With relation (1.67), which we use in the motion equation (4.21), taking into account the vector identity (4.20) and Eq. (1.74), we can write

$$\boldsymbol{a} = -\boldsymbol{grad}\, \chi - \frac{1}{\rho}\boldsymbol{grad}\, p + 2\nu\, \boldsymbol{grad}\,(\operatorname{div} \boldsymbol{v}) - \nu\, \textbf{curl curl}\, \boldsymbol{v} \ .$$

By the conservation of mass, it leads to

$$\boldsymbol{a} = -\boldsymbol{grad}\left(\frac{p}{\rho} + \chi\right) - \nu\, \textbf{curl curl}\, \boldsymbol{v} \ . \tag{4.28}$$

Inserting (4.28) in (4.27), we obtain (4.26).

4.5 Vorticity Equation for a Perfect Fluid

For an incompressible, perfect ($\nu = 0$) fluid, the vorticity dynamics equation (4.24) becomes

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \boldsymbol{grad}\, \boldsymbol{v} . \tag{4.29}$$

In the two-dimensional case, $\boldsymbol{\omega}$ is orthogonal to $\boldsymbol{grad}\, \boldsymbol{v}$ and this relation reduces to

$$\frac{D\omega}{Dt} = 0 . \tag{4.30}$$

From Eq. (4.29), we deduce that, for a perfect incompressible fluid, if the flow is irrotational at an instant, it remains so. In particular, an initially uniform flow will remain irrotational afterwards.

In the case of a perfect fluid, Eq. (4.26) yields Kelvin's theorem (cf. [71], Sect. 13.10)

Theorem 4.3 (Kelvin theorem) *The circulation of the velocity along a closed material line does not change, for an incompressible perfect fluid*

$$\frac{d\Gamma}{dt} = 0 . \tag{4.31}$$

4.6 Bernoulli's Equation

Bernoulli's equation is obtained from the Navier–Stokes equation (1.74) written for perfect fluids ($\mu = 0$). Assume that the volume forces can be derived from a potential (4.13), then

$$\frac{D\boldsymbol{v}}{Dt} = -\frac{1}{\rho}\boldsymbol{\nabla}\, p - \boldsymbol{\nabla}\chi . \tag{4.32}$$

Using the vector identity

$$\boldsymbol{v} \cdot \boldsymbol{\nabla}\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{v} + \boldsymbol{\nabla}\left(\frac{\boldsymbol{v} \cdot \boldsymbol{v}}{2}\right) , \tag{4.33}$$

in the material derivative of the velocity, we obtain

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{\omega} \times \boldsymbol{v} - \frac{1}{\rho}\boldsymbol{\nabla}\, p - \boldsymbol{\nabla}\left(\frac{v^2}{2} + \chi\right) . \tag{4.34}$$

We also assume that the flow is irrotational, $\boldsymbol{\omega} = \boldsymbol{0}$. This assumption is strong, because real fluids produce rotational flows, such as those produced, for example, by the viscous effects near a wall. Thus, Eq. (4.34) can now be written as

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\frac{1}{\rho}\nabla\, p - \nabla\left(\frac{v^2}{2} + \chi\right) . \tag{4.35}$$

Since the flow is irrotational, the velocity field can be derived from a potential, Φ, such that

$$\boldsymbol{v} = \nabla\Phi\ . \tag{4.36}$$

Applying the divergence to (4.36) shows that the potential satisfies a Laplace equation

$$\Delta\Phi = 0\ . \tag{4.37}$$

The Euler equation (4.35) yields

$$\nabla\left(\frac{\partial\Phi}{\partial t} + \frac{v^2}{2} + \chi\right) = -\frac{1}{\rho}\nabla\, p. \tag{4.38}$$

As the left-hand side of (4.38) corresponds to the gradient of a scalar function, the same must be the case for the right-hand side. Consequently, Eq. (4.38) becomes with the assumption $\rho = \text{cnst}$

$$\nabla\left(\frac{\partial\Phi}{\partial t} + \frac{v^2}{2} + \chi + \frac{p}{\rho}\right) = 0\ . \tag{4.39}$$

We integrate this equation to obtain the general form of *Bernoulli's equation:*

$$\frac{\partial\Phi}{\partial t} + \frac{p}{\rho} + \frac{v^2}{2} + \chi = C(t)\ . \tag{4.40}$$

If the flow is stationary, then (4.40) yields the steady state form of Bernoulli's equation

$$\frac{p}{\rho} + \frac{v^2}{2} + \chi = C, \tag{4.41}$$

which, as is suggested by the second term, is an integral of the energy. Therefore, Bernoulli's equation is a first integral of the Euler equation for the case of a stationary, irrotational, perfect fluid.

4.7 Vorticity Production on a Solid Wall

The presence of a solid wall in a flow generates vorticity. An important fact to report is the direct link between the viscous wall shear stress and the produced vorticity. It is proposed in this section to establish certain properties of the effort exercised by a viscous fluid on a fixed wall. To this end, we can show that for a viscous

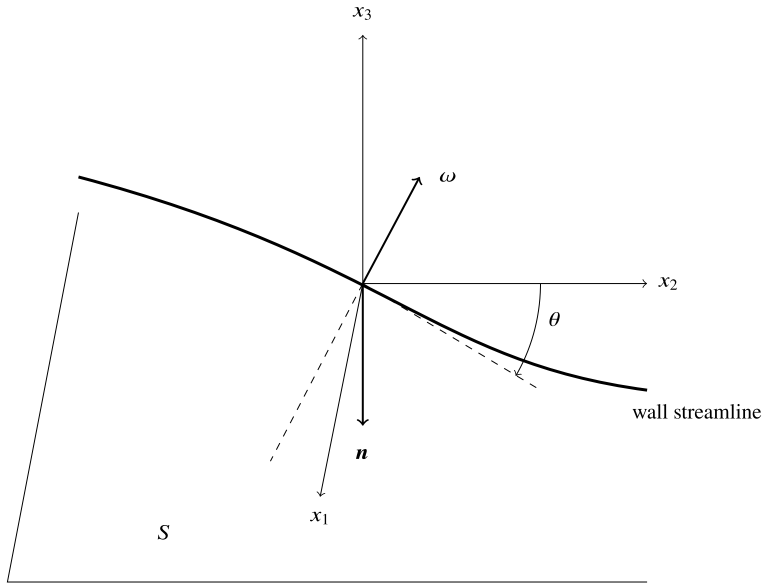


Fig. 4.3 Vorticity generated on a plane wall

incompressible fluid, considered in a point of the fixed wall, we have the following properties:

- the normal component of the contact force is the pressure;
- the tangential component of the contact force is equal to the product of the dynamic viscosity by the vorticity vector $\boldsymbol{\omega}$ rotated by 90° in the plane tangent to the wall, in the positive direction around the normal to this plane.

Let S be a plane wall, $\boldsymbol{n}$ the unit normal vector to S pointing to the outside of the fluid. The system of Cartesian rectangular axes is chosen such that the axes x_1, x_2 are located in S and the normal $\boldsymbol{n}$ is oriented in the negative direction of the axis x_3, cf. Fig. 4.3.

The continuity equation expressed at the origin of the axes reads

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0 \,. \tag{4.42}$$

As the components v_1, v_2 vanish on the wall, we obtain

$$\frac{\partial v_1}{\partial x_1} = \frac{\partial v_2}{\partial x_2} = 0 \,. \tag{4.43}$$

The relations (4.42) and (4.43) imply consequently

$$\frac{\partial v_3}{\partial x_3} = 0 \,. \tag{4.44}$$

Taking these relations into account, we develop the velocity components in Taylor series with respect to the normal direction to the wall. The dominating terms are

$$v_1 = 0 + \frac{\partial v_1}{\partial x_3}\,|_0\, x_3 + \cdots \,, \tag{4.45}$$

$$v_2 = 0 + \frac{\partial v_2}{\partial x_3}\,|_0\, x_3 + \cdots \tag{4.46}$$

$$v_3 = 0 + 0 + \frac{\partial^2 v_3}{\partial x_3^2}\,|_0\, \frac{x_3^2}{2} + \cdots \,. \tag{4.47}$$

It is then possible to evaluate the slope of the wall streamline (w.s.), i.e. the streamline obtained when the distance to the wall goes to zero

$$\frac{dx_2}{dx_1}\,|_{w.s.} = \tan\theta = \lim_{x_3 \to 0} \frac{v_2}{v_1} = \frac{\frac{\partial v_2}{\partial x_3}|_0}{\frac{\partial v_1}{\partial x_3}\,|_0} \,. \tag{4.48}$$

In this relation θ is the angle at the origin of the axes between the wall streamline and the x_2 axis. This particular streamline is indeed in the wall as the angle that it makes with the planes (x_3, x_1) and (x_3, x_2), that is the limits of the respective ratios v_3/v_1 and v_3/v_2, vanish altogether.

Let us consider now the vorticity lines. On the wall at the origin of the axes, the vorticity components are

$$\begin{aligned}
\omega_1\,|_0 &= \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} = -\frac{\partial v_2}{\partial x_3}|_0 \,, \\
\omega_2\,|_0 &= \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} = \frac{\partial v_1}{\partial x_3}\,|_0 \,, \\
\omega_3\,|_0 &= \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = 0 \,.
\end{aligned} \tag{4.49}$$

The normal component of the wall vorticity is zero. It results that the vorticity vector is entirely located in the plane of the wall, where it is generated. We can also show that the wall vorticity lines (v.l.) are always orthogonal to the wall streamlines. Indeed, we have the relationship

$$\frac{dx_2}{dx_1}\,|_{v.l.} = \frac{\omega_2}{\omega_1} = \frac{\frac{\partial v_1}{\partial x_3}\,|_0}{-\frac{\partial v_2}{\partial x_3}\,|_0} = -\frac{1}{\frac{dx_2}{dx_1}|_{w.s.}} \tag{4.50}$$

When moving away from the wall, vorticity lines and streamlines do not necessarily remain orthogonal, especially in a three-dimensional flow.

Let us rotate the system of axes around the x_3 axis in such a way that the angle θ be zero. By (4.48), one has

$$\frac{\partial v_2}{\partial x_3}\ |_0 = 0\ . \tag{4.51}$$

The viscous component of the contact force exerted on the wall by the fluid will be denoted $\boldsymbol{t}^v$. We evaluate its value by the relations (4.43), (4.44), (4.51) and the Cauchy theorem (1.53)

$$t_1^v = n_3\sigma_{13} = -\mu\,\frac{\partial v_1}{\partial x_3} = -\mu\,\omega_2\ . \tag{4.52}$$

The wall contact viscous force is thus tangential and directly proportional to the wall vorticity. It is possible to generalize (4.52) for an arbitrary system of axes for a wall that is not a plane, cf. Berker [14]. One obtains the relation

$$\boldsymbol{t} = p\,\boldsymbol{n} + \mu(\boldsymbol{n} \times \boldsymbol{\omega})\ . \tag{4.53}$$

In the same way that the temperature of a wall gives no indication of the amount of energy leaving it, the wall vorticity does not give information on the vorticity intensity which goes in or out the flow. For the sake of comparison, the theory of heat flow establishes that the heat flux through a plane of normal $\boldsymbol{n}$ is given by the product $\boldsymbol{n} \cdot \boldsymbol{q}$ where $\boldsymbol{q}$ denotes the heat flux vector. By analogy, we define the diffusive vorticity flux for a viscous incompressible fluid by the equation

$$\begin{aligned} \boldsymbol{\zeta} &= -\,(\nabla\boldsymbol{\omega})\,\boldsymbol{n} \\ \zeta_i &= -n_j\,\frac{\partial \omega_i}{\partial x_j}\ . \end{aligned} \tag{4.54}$$

The wall value of ζ_i is computed from Eq. (4.6) and the momentum equation (1.74) with no body force term

$$\frac{\partial \boldsymbol{v}}{\partial t} + \nabla(\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2} + \frac{p}{\rho}) = -\boldsymbol{\omega} \times \boldsymbol{v} - \nu\,\mathbf{curl}\,\boldsymbol{\omega}\ . \tag{4.55}$$

On a fixed wall the velocity $\boldsymbol{v}$ is zero and (4.55) yields

$$\nabla p = -\mu\,\mathbf{curl}\,\boldsymbol{\omega}\ . \tag{4.56}$$

By (4.49), (4.54) and (4.56), one obtains

$$\frac{\partial p}{\partial x_1} = \mu\frac{\partial \omega_2}{\partial x_3} = \mu\,\zeta_2, \tag{4.57}$$

$$\frac{\partial p}{\partial x_2} = -\mu\frac{\partial \omega_1}{\partial x_3} = -\mu\,\zeta_1. \tag{4.58}$$

We notice that it is necessary to have a pressure gradient along the wall to maintain vorticity production inside the fluid.

The third component of the diffusive vorticity flux is computed by the property $\nabla \cdot \boldsymbol{\omega} = 0$. One has

$$\zeta_3 = -n_3 \frac{\partial \omega_3}{\partial x_3} = -(\frac{\partial \omega_1}{\partial x_1} + \frac{\partial \omega_2}{\partial x_2}) \, . \tag{4.59}$$

The necessary pressure gradient for the wall vorticity generation is produced at the flow start-up as the term $\rho \, \partial \boldsymbol{v}/\partial t$ is the only one during the first instants that can be compensated by $-\nabla \, p$, because the viscous part of the Navier–Stokes operator comes into play on longer time scales, especially when the Reynolds number is large, cf. Morton [61].

4.8 Flow Behind a Grid

Kovasznay [45] examines the steady state two-dimensional exact solution of the Navier–Stokes equation for the laminar flow behind a periodic array of cylinders or rods. The velocity field is assumed to be such that $v_1 = U + u_1$, $v_2 = u_2$, where U is the mean velocity in the x_1 direction. The vorticity equation (4.25) yields

$$\frac{\partial \omega_3}{\partial t} + (U + u_1)\frac{\partial \omega_3}{\partial x_1} + u_2 \frac{\partial \omega_3}{\partial x_2} = \nu \Delta \omega_3 \, . \tag{4.60}$$

Denoting the grid spacing by δ, we define the Reynolds number as $Re = \delta U/\nu$. The dimensionless vorticity becomes $\omega = \omega_3 \, \delta/U$. The other dimensionless variables are $x = x_1/\delta$, $y = x_2/\delta$, $\tau = tU/\delta$, $1 + u = v_1/U$, $v = v_2/U$. The governing equation (4.60) becomes

$$\frac{\partial \omega}{\partial \tau} + (1 + u)\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \frac{1}{Re}\Delta \omega \, . \tag{4.61}$$

As steady state solutions are sought, the term $\partial \omega/\partial \tau$ vanishes. We are left with

$$\Delta \omega - Re\frac{\partial \omega}{\partial x} - Re\left(u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y}\right) = 0 \, . \tag{4.62}$$

To build up the analytical solution, the trick consists in finding an expression that cancels the nonlinear term. The streamfunction is introduced to satisfy the continuity equation

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \, , \tag{4.63}$$

and therefore the vorticity is

$$\omega = -\Delta \psi \, . \tag{4.64}$$

Taking the y periodicity into account, the streamfunction is set up such that

$$\psi = f(x) \sin 2\pi y \ . \tag{4.65}$$

With (4.65), the nonlinear term of (4.62) gives

$$f' f'' - f f''' = 0 \ . \tag{4.66}$$

Assuming that none of the derivatives vanish, we write

$$\frac{f'''}{f''} = \frac{f'}{f} \ . \tag{4.67}$$

Integrating (4.67) we obtain

$$f'' = k^2 f \ , \tag{4.68}$$

where k is a real or complex arbitrary constant. A further integration yields

$$f = C e^{kx} \ . \tag{4.69}$$

With the stream function

$$\psi = C e^{kx} \sin 2\pi y \tag{4.70}$$

canceling the nonlinear term in (4.62), we have to seek a solution of the equation

$$\Delta\omega - Re \frac{\partial \omega}{\partial x} = 0 \ . \tag{4.71}$$

Setting

$$\omega = g(x) \sin 2\pi y \ , \tag{4.72}$$

we have

$$g'' - Re\, g' - 4\pi^2 g = 0 \ , \tag{4.73}$$

the solution of which is

$$g(x) = A e^{\lambda_1 x} + B e^{\lambda_2 x} \ , \tag{4.74}$$

where

$$\lambda_{1,2} = \frac{Re}{2} \pm \sqrt{\frac{Re^2}{4} + 4\pi^2} \ . \tag{4.75}$$

Combining (4.72) and (4.74), the vorticity is

$$\omega = \left(A e^{\lambda_1 x} + B e^{\lambda_2 x} \right) \sin 2\pi y \ , \tag{4.76}$$

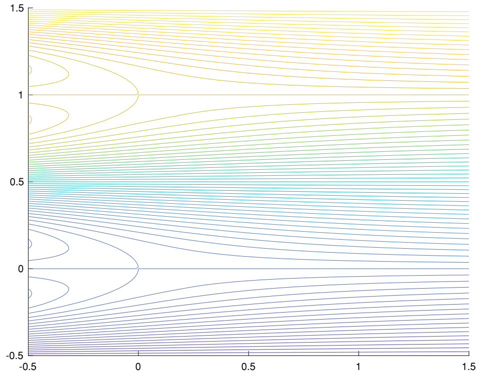


Fig. 4.4 Streamlines of the Kovasznay flow for $Re = 40$

while Eqs. (4.64) and (4.70) give

$$\omega = C(4\pi^2 - k^2)e^{kx} \sin 2\pi y \ . \tag{4.77}$$

Comparison of (4.76) and (4.77) shows that two solutions are possible

$$k = \lambda_1, \quad A = -Re\lambda_1 C, \quad B = 0, \tag{4.78}$$
$$k = \lambda_2, \quad A = 0, \quad B = -Re\lambda_2 C \ . \tag{4.79}$$

The constant C is obtained by fixing the stagnation point at $x = 0$ and therefore $C = -1/2\pi$. The corresponding streamfunction is

$$\psi = y - \frac{1}{2\pi} e^{\lambda_2 x} \sin 2\pi y \ . \tag{4.80}$$

With $Re = 40$ and the corresponding λ_2, the streamlines are shown in Fig. 4.4, with pairs of eddies generated behind the cylinders. The flow recovers uniformity downstream through the exponential term of the solution.

As the Kovasznay flow incorporates the nonlinear term, it is a good benchmark to test the numerical accuracy and space convergence of computational methods integrating the Navier–Stokes equation.

4.9 Taylor–Green Vortex

Taylor and Green [104] proposed an idealized model of a three-dimensional vortex field in order to test the dynamics of turbulence.

Orszag [66, 68] modified this model by shifting the origin of axis x_3 by a factor of $\frac{1}{2}\pi$ to obtain a two-dimensional initial velocity field that was simpler to handle by spectral Fourier methods in a cubic box of period 2π. The velocity is given as

$$
\begin{aligned}
v_1 &= \cos x_1 \sin x_2 \cos x_3 \\
v_2 &= -\sin x_1 \cos x_2 \cos x_3 \\
v_3 &= 0\,.
\end{aligned}
\tag{4.81}
$$

The corresponding vorticity components are

$$
\begin{aligned}
\omega_1 &= -\sin x_1 \cos x_2 \sin x_3 \\
\omega_2 &= -\cos x_1 \sin x_2 \sin x_3 \\
\omega_3 &= -2\cos x_1 \cos x_2 \cos x_3\,.
\end{aligned}
\tag{4.82}
$$

The initial streamlines are planar curves given by $\cos x_1 \cos x_2 = const$ in planes $x_3 = const$. Nonetheless, the velocity that will develop at later times is fully three-dimensional. The initial vortex lines are the curves

$$
\sin x_1/\sin x_2 = \text{const},\ \sin^2 x_1 \cos x_3 = \text{const}, \tag{4.83}
$$

so they are twisted and induce a velocity field able to stretch them. The Taylor–Green vortex is perhaps the simplest example of self-induced vortex stretching by a three-dimensional velocity field.

Orszag did not give any detail to obtain (4.83). Therefore, to compute the vorticity lines, we will rely on a paper by Nore et al. [63], especially the appendix "Taylor–Green Clebsch potentials".

The Clebsch potentials allow to decompose the velocity field as follows

$$
\boldsymbol{v} = \nabla\varphi + \lambda\nabla\mu\,, \tag{4.84}
$$

i.e. in a potential part (first term of the r.h.s.) and a rotational part (second term of r.h.s.). The rotational part is chosen to be a complex-lamellar field, that is a flow where the velocity field is orthogonal to its own **curl**. Call it $\boldsymbol{v}^{(\omega)}$. The complex-lamellar field may be considered as a potential if it is divided by an integration factor λ such that

$$
\boldsymbol{v}^{(\omega)} = \lambda\nabla\mu\,. \tag{4.85}
$$

The potential $\varphi(\boldsymbol{x})$, $\lambda(\boldsymbol{x})$, $\mu(\mathbf{x})$ are named Clebsch variables. Note that the decomposition (4.84) is not unique.

Taking into account the identity $\nabla\times(\lambda\nabla\mu) = \lambda(\nabla\times\nabla\mu) + \nabla\lambda\times\nabla\mu$, the vorticity of the field (4.84) yields

$$
\boldsymbol{\omega} = \mathbf{curl}\,\boldsymbol{v}^{(\omega)} = \nabla\lambda\times\nabla\mu\,. \tag{4.86}
$$

Geometrically speaking, the relation (4.86) shows that the surfaces of constant λ and μ are material vorticity surfaces. As λ and μ both contain vortex lines, their intersection describes the vortex lines. The potentials λ and μ must be invariants under vorticity flow dynamics.

Looking for a general invariant $\mathcal{I}$, one must satisfy the transport equation

$$\boldsymbol{\omega} \cdot \nabla \mathcal{I} = 0 \quad \text{or} \quad \omega_j \frac{\partial \mathcal{I}}{\partial x_j} = 0 \,. \tag{4.87}$$

Carrying through the algebra and dividing by $\cos x_1 \cos x_2 \sin x_3$, one writes

$$\tan x_1 \frac{\partial \mathcal{I}}{\partial x_1} + \tan x_2 \frac{\partial \mathcal{I}}{\partial x_2} + 2\frac{\frac{\partial \mathcal{I}}{\partial x_3}}{\tan x_3} = 0 \,. \tag{4.88}$$

This equation is solved by separation of variables. We divide (4.88) by $\mathcal{I} = u(x_1)v(x_2)w(x_3)$ to obtain the relation

$$\tan x_1 \frac{\partial[\ln u(x_1)]}{\partial x_1} + \tan x_2 \frac{\partial[\ln v(x_2)]}{\partial x_2} + 2\frac{\partial[\ln w(x_3)]/\partial x_3}{\tan x_3} = 0 \,. \tag{4.89}$$

Each term in (4.89) must be equal to a constant c_1, c_2, c_3 such that

$$c_1 + c_2 + 2c_3 = 0 \,. \tag{4.90}$$

The first two terms of (4.89) are of the form

$$\tan x_1 \frac{\partial[\ln u(x_1)]}{\partial x_1} = c_1 \,, \tag{4.91}$$

with a general solution given by $u(x_1) = \text{const}\,(\sin x_1)^{c_1}$. The last term yields the equation

$$\frac{\partial[\ln w(x_3)]/\partial x_3}{\tan x_3} = c_3 \,, \tag{4.92}$$

with the solution given by $w(x_3) = \text{const}\,(\cos x_3)^{-c_3}$.

Two independent solutions of (4.90) are $c_1 = 1, c_2 = 0, c_3 = -1/2$ and $c_1 = 0, c_2 = 1, c_3 = -1/2$. The Clebsch potentials are

$$\lambda = \sin(x_1)\sqrt{\cos(x_3)} \tag{4.93}$$

$$\mu = \sin(x_2)\sqrt{\cos(x_3)} \,, \tag{4.94}$$

that give the vorticity lines (4.83).

Unlike the three-dimensional case that has no closed form solution, the two-dimensional Taylor–Green vortex is amenable to an analytical solution

$$\begin{aligned} v_1 &= e^{-2\nu t} \sin x_1 \cos x_2 \,, \\ v_2 &= -e^{-2\nu t} \cos x_1 \sin x_2 \,, \end{aligned} \tag{4.95}$$

$$p = -\frac{1}{4} e^{-4\nu t} \left(\cos(2x_1) + \cos(2x_2)\right) \,. \tag{4.96}$$

This solution is used in numerical fluid mechanics to check the accuracy and stability of Navier–Stokes solvers.

Exercises

4.1 Compute the vorticity in the circular Couette flow and verify the Stokes theorem (4.5).

4.2 By applying Bernoulli's theorem for perfect fluids (4.41), show that the velocity of a jet exiting an orifice in a wall at a distance h from the free surface of the fluid in a container is

$$v = \sqrt{2gh} \; . \tag{4.97}$$

4.3 Hill's Vortex

Hill's vortex [38] in an incompressible fluid represents the limit case where the vorticity is distributed in the volume inside a sphere of radius R. Outside the sphere the flow is considered irrotational. For example this situation concerns the constant velocity fall of a water drop in oil. The other limit case corresponds to filament vorticity distribution.

The problem in spherical coordinates has the $\boldsymbol{\omega}$ vorticity components

$$\omega_\varphi = \frac{\omega r \sin\theta}{R}, \quad \omega_r = \omega_\theta = 0, \quad \forall r \leq R \tag{4.98}$$

$$\boldsymbol{\omega} = \mathbf{0} \quad \forall r > R \; , \tag{4.99}$$

with C a constant.

We assume the problem is axisymmetric with $v_\varphi = 0$.

- Evaluate the velocity components $v_r = f(r)\cos\theta$ and $v_\theta = g(r)\sin\theta$ inside the vortex.
- Obtain with the velocity potential the velocity components outside the spherical vortex.
- Compute the streamfunction.

4.4 Simplified Vortex Study of the Draining of a Container

The physical situation concerns the draining of a container whose free surface is located at the altitude $z = 0$ in a cylindrical coordinate system, with the z axis oriented downwards. The fluid flow is axisymmetric, irrotational, incompressible with radial and axial velocity components given respectively by

$$v_r = -\frac{ar}{2}, \quad v_z = az \; , \tag{4.100}$$

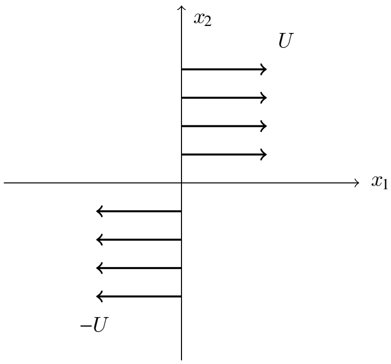


Fig. 4.5 Initial velocity distribution of the vortex sheet

with a a positive constant. We suppose that the flow is perturbed by the introduction of a low amplitude vortex (draining vortex) of vorticity $\boldsymbol{\omega} = \omega_z(r)\boldsymbol{e}_z$.

- Verify that the unperturbed flow is incompressible.
- Show that the unperturbed flow is irrotational.
- Show that ω_z verifies the steady state equation

$$\frac{a}{2}\omega_z r^2 + \nu r \frac{d\omega_z}{dr} = C \ , \tag{4.101}$$

  with C a constant.
- Show that the constant C in (4.101) vanishes by inspection of the ω_z behavior on short and long distances.
- Compute the solution for $\omega_z(r)$.
- Highlight the existence of a characteristic distance δ which will be expressed with the problem data.
- What is the physical meaning of δ?

4.5 Diffusion of a vortex sheet

At the initial time $t = 0$, a vortex sheet coinciding with the plane $x_2 = 0$ of the Cartesian coordinate system is given. This vortex sheet is a singularity that we will ignore. On each side of the sheet, the velocity distribution is uniform and such that (cf. Fig. 4.5)

$$\begin{aligned} v_1(x_2, 0) &= \quad U \quad \forall x_2 > 0 \\ v_1(x_2, 0) &= -U \quad \forall x_2 < 0 \ . \end{aligned} \tag{4.102}$$

The fluid is incompressible. The body forces are neglected. The pressure is uniform. The flow is one-dimensional in the plane Ox_1x_2.

Compute the time evolution of the vorticity resulting from the effects of viscous diffusion. To this end, follow the next steps.

- Show that the dynamic vorticity equation reduces to

$$\frac{\partial \omega_3}{\partial t} = \nu \frac{\partial^2 \omega_3}{\partial x_2^2} \, . \tag{4.103}$$

- Introduce the dimensionless variables $\eta = \frac{x_2}{\sqrt{\nu t}}$, $\upsilon = \frac{U^2 t}{\nu}$. Assuming that the vorticity may be written as $\omega_3(x_2, t) = f(\eta) g(\upsilon)$, show that ω_3 depends only on η.
- Solve the f equation with the hypotheses that $f(0)$ and $f'(0)$ have finite non zero values.
- Describe qualitatively the time evolution of ω_3.