Chapter 2
Dimensional Analysis

2.1 Principles and General Concepts

For a deep and thorough presentation of dimensional analysis we refer the reader to the monograph by Barenblatt [9] that covers amply the noticeable properties produced by adequately chosen scalings. Fluid mechanics is very much based on experiments to ascertain the relevant physical phenomena, to watch them and to give them a quantitative approach. The goal of dimensional analysis consists in providing the similarity conditions between the observed phenomena in the physical reality or on a prototype and those yielded on a reduced model or a mock-up.

Let us start by noticing that in dimensional analysis, there is no basic or natural measurement unit for the physical variables. Several universal physical constants are available like the charge of the electron, Planck's constant, the speed of light, etc. But these constants are not characteristic of all physical phenomena. The electron charge is not a fundamental unit to measure the current intensity in an electrical engine. Likewise the flow velocity around an airfoil or the ocean wave velocity are not measured with the light speed as the basic unit. Therefore we may conclude that the measurement scales are arbitrary conventions that are playing no essential role in the physical processes. If we change the size of the length unit, all variables implying a length are rescaled in an appropriate manner.

When considering the dimensional aspect of a problem, it can always be simplified and highlight interesting informations. This is all the more true as dimensional analysis rests on the principle of dimensional invariance that states "the structure of the equations remains unchanged if one performs a modification in size of the system's units".

The aim of dimensional analysis consists in searching the relevant dimensionless numbers that characterize the fluid flow phenomenon. In some cases the values of these numbers offer the possibility of simplifying the mathematical model of the problem at hand. Another benefit of dimensional analysis is the collapse of all experimental data on a single curve that alleviates the burden of various sources of observations, like the drag curve versus the Reynolds number for the flow around

M. O. Deville, *An Introduction to the Mechanics of Incompressible Fluids*,
https://doi.org/10.1007/978-3-031-04683-4_2

a circular cylinder. Please see different vortex shedding for the same $Re = 53$ in the same wind tunnel with two different diameters circular cylinder experimented at different free stream speed in [89].

Suppose the physical problem depends on n variables $v_1, v_2, \ldots, v_n$. Each variable v_i has a dimension denoted by $[v_i]$ such that one writes

$$v_i = u\,[v_i]\ . \tag{2.1}$$

Here u expresses the size of the variable in the chosen system of units. For example if it is the length of a rope, we will write $l = 5[l]$. Defining the dimension of length as $[l] = L$, one has $l = 5L$. Referring to the International System of Units (SI) the dimension L is given in meters (m) and thus $l = 5[l] = 5\,\mathrm{m}$. Besides L, the dimensions M and T are introduced for mass and time, respectively.

The SI system is composed of the meter (m), kilogram (kg), second (s), ampere (A), kelvin (K), candela (cd) and mole (mol). These are the primary units. All other units are secondary and derived from the primary units. As an example of a secondary unit, the definition of the newton (N) reads "The newton is that force which gives to a mass of 1 kg an acceleration of 1 meter per second per second".

2.2 The Vaschy–Buckingham Theorem

From the viewpoint of dimensions, the force F in rational mechanics is defined by the product of the mass m and the acceleration $\ddot{x}$ and leads to the relation

$$[F] = MLT^{-2}\ , \tag{2.2}$$

where M, L, T are the fundamental dimensions of mass, length and time, respectively and $[F]$ is the dimension of F. We notice that in order to express the dimension of a variable, one writes a monomial of powers of the basic quantities. More generally, a physical problem will deal with a model that is the representation of intrinsic relationships between the various variables. We will have

$$f(v_1, v_2, \ldots, v_N) = 0\ , \tag{2.3}$$

or

$$v = g(v_1, v_2, \ldots, v_{N-1})\ . \tag{2.4}$$

As an example, let us consider the stationary flow of a viscous incompressible fluid in a Couette apparatus with the exterior cylinder of radius R_2 fixed and the interior cylinder of radius R_1 rotating with the angular velocity ω. The pressure p in every point of the flow with position $\boldsymbol{x}$ is a dependent variable that involves other quantities describing the problem. One writes

$$f(p, R_1, R_2, \omega, \rho, \mu, \boldsymbol{x}) = 0 \,. \tag{2.5}$$

The aim of dimensional analysis is to collect several variables to elaborate a reduced or dimensionless variable. Here we have the group

$$\Pi = \frac{p}{\rho(\omega R_1)^2} \,. \tag{2.6}$$

In this case, the dynamic pressure $\rho(\omega R_1)^2$ is considered as a natural measuring scale for the pressure p. Using this information, one obtains by (2.5)

$$\Pi = \frac{p}{\rho(\omega R_1)^2} = \frac{1}{\rho(\omega R_1)^2}\, g(R_1, R_2, \omega, \rho, \mu, \boldsymbol{x}) \,. \tag{2.7}$$

The relation (2.4) is independent of the chosen units' system as a result of the principle of dimensional invariance. This implies that the relation be homogeneous from the dimensional point of view. In other words, the dimension of each variable can be written as

$$[v_i] = P_1^{a_i} P_2^{b_i} P_3^{c_i} \ldots \,, \tag{2.8}$$

with P_i denoting a fundamental (primary) quantity (M, L, T).

Let us introduce the concept of dimensional matrix. This matrix is composed by the list of the exponents $a_i, b_i, c_i, \ldots$ of the fundamental quantities of each variable or parameter of the problem. It allows to control the linear independence of the variables in terms of the chosen primary quantities. For the Couette problem, the dimensional matrix is the following

	p	$\boldsymbol{x}$	R_1	R_2	μ	ρ	ω
M	1	0	0	0	1	1	0
L	−1	1	1	1	−1	−3	0
T	−2	0	0	0	−1	0	−1

Via the principle of dimensional homogeneity and since each dimension can be written as a monomial of powers, one can demonstrate the following theorem

Theorem 2.1 (Vaschy–Buckingham) *If a physical problem is described by N variables and parameters in r dimensions, i.e. in r independent variables, then it is possible to organise the original variables in dimensionless groups such that*

$$\Pi = v_1^{\alpha_1} v_2^{\alpha_2} \ldots v_N^{\alpha_N} \,. \tag{2.9}$$

The function (2.3) may be written in a simpler manner because it does contain only $M = N - r$ variables

$$\Phi(\Pi_1, \Pi_2, \ldots, \Pi_M) = 0 \,, \tag{2.10}$$

with r the rank of the dimensional matrix.

We will not prove the theorem and for this purpose the reader is referred to Barenblatt [9] or to Panton [71]. In the field of isothermal fluid mechanics, we will suppose that $r = 3$, which corresponds to the choice $P_1 = M$, $P_2 = L$, $P_3 = T$.

2.3 Application of Vaschy–Buckingham Theorem

2.3.1 Circular Couette Flow

Let us choose three independent primary variables R_1, ρ, ω, whose minor of order three in the previous dimensional matrix is different from zero. Let us examine the pressure p and let us construct the first group that is denoted by Π_1. We have with $\alpha_1 = \alpha, \alpha_2 = \beta, \alpha_3 = \gamma$, etc.,

$$\Pi_1 = p R_1^{\alpha} \rho^{\beta} \omega^{\gamma} \ , \tag{2.11}$$

or

$$[\Pi_1] = M^0 L^0 T^0 = M L^{-1} T^{-2} L^{\alpha} (M L^{-3})^{\beta} (T^{-1})^{\gamma} \ . \tag{2.12}$$

Equating the exponents of the fundamental quantities M, L, T of the two sides of Eq. (2.12), one has

$$\begin{aligned} 1 + \beta &= 0 \\ -1 + \alpha - 3\beta &= 0 \\ -2 - \gamma &= 0 \end{aligned} \tag{2.13}$$

Solving system (2.13) one obtains $\alpha = -2, \beta = -1, \gamma = -2$ and the group Π_1 is such that $\Pi_1 = p/(\rho\,\omega^2 R_1^2)$. Afterwards we compute $\Pi_2 = \boldsymbol{x}/R_1, \Pi_3 = R_2/R_1, \Pi_4 = \mu/(\rho\omega R_1^2)$. Therefore

$$\Pi_1 = \frac{p}{\rho(\omega R_1)^2} = \frac{1}{\rho(\omega R_1)^2} h\left(\frac{\boldsymbol{x}}{R_1}, \frac{R_2}{R_1}, \frac{\mu}{\rho\omega R_1^2}\right) . \tag{2.14}$$

The group Π_4 is the inverse of the Reynolds number that in this case reads

$$(\Pi_4)^{-1} = Re = \frac{\omega R_1^2}{\nu} \ , \tag{2.15}$$

where one finds ωR_1 the characteristic velocity, R_1 the geometrical reference length, $\nu = \mu/\rho$.

Choosing now R_1, μ, ω as basic variables, one generates the Π groups: $\Pi_1 = \frac{p}{\mu\omega}, \Pi_2 = \frac{\boldsymbol{x}}{R_1}, \Pi_3 = \frac{R_2}{R_1}, \Pi_4 = \frac{\omega R_1^2}{\nu}$.

We observe that there is no unique way to set up a problem in reduced form, since the rank of the dimensional matrix allows selecting the basic variables very differently.

2.3.2 *Flow in a Pump*

We consider the flow of an incompressible fluid with density ρ and dynamic viscosity μ in a pump whose rotor rotates at constant angular velocity ω. The respective pressures at suction and discharge sections of the pump are denoted by p_1 and p_2. A choice must be made between several homothetic pumps. According to this homothety, it suffices to fix a characteristic dimension D of the pump, inasmuch that the other dimensions are proportional to it, including the diameters of the input and output sections where the pressure are measured as p_1 and p_2. It is obvious that the flow rate Q is a function of the quantities already introduced ρ, D, μ, ω as well as the pressure increase created by the pump. We have the relationship

$$Q = f(p_2 - p_1, \rho, \mu, D, \omega) . \tag{2.16}$$

It is asked to write a dimensionless relation for the cases when the fluid is perfect (without viscosity) and a real viscous fluid.

For the perfect fluid, the dimensional matrix is

	$p_2 - p_1$	Q	D	ρ	ω
M	1	0	0	1	0
L	−1	3	1	−3	0
T	−2	−1	0	0	−1

Choosing ρ, D, ω as basic variables, one obtains

$$\frac{Q}{D^3\omega} = f\left(\frac{p_2 - p_1}{\rho D^2 \omega^2}\right) . \tag{2.17}$$

This is nothing else than the characteristic curve of the pump, Fig. 2.1.

For the Newtonian viscous fluid, one has

$$\frac{Q}{D^3\omega} = f\left(\frac{p_2 - p_1}{\rho D^2 \omega^2}, Re\right) . \tag{2.18}$$

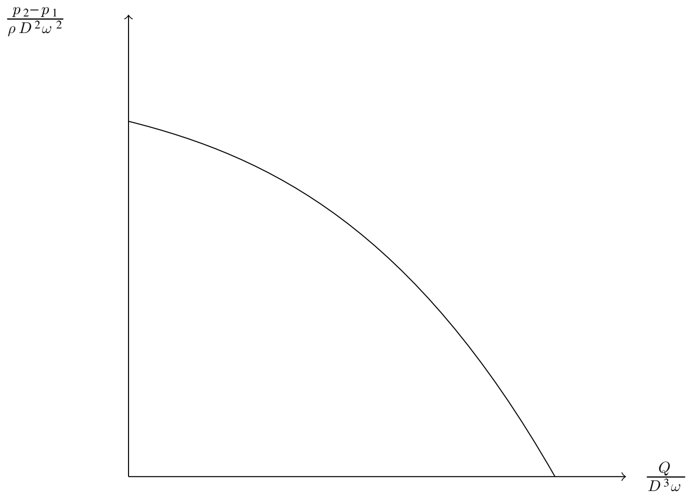

Fig. 2.1 Characteristic of a hydraulic pump

2.4 Dynamic Similarity

Two physical phenomena are said similar if the variables describing them can be matched. Let us suppose that the physical phenomenon of interest is described by the relation

$$\Pi_1 = f(\Pi_2, \ldots, \Pi_n) \ . \tag{2.19}$$

If two flows are such that the dimensionless groups $\Pi_2, \ldots, \Pi_n$ are given the same values, the dependent variables are also identical. If the first flow is the one of the reduced scale model and the second one of reality (the prototype), we have

$$(\Pi_i)_m = (\Pi_i)_p, \quad i = 1, \ldots, n \ . \tag{2.20}$$

This amounts to imposing the equality of the Reynolds numbers if we want to study the flow around an airfoil on a mock-up to draw operational conclusions for the real profile. However the geometrical similarity resting on the Π theorem is sometimes not feasible. Indeed the boundary conditions of an experimental set-up may also affect the similarity rules.

In certain cases, it is impossible to apply strictly the similarity laws because dimensional constants such as gravity step in. As an example, let us study the drag on a boat that is given by the force in direction x_1 of the main flow with velocity U_∞. Building up the dimensional matrix and generating the variables F_{x_1}, ρ, μ, U_∞, g, d, with d the boat length overall, one obtains

$$C_{x_1} = \frac{F_{x_1}}{\rho d^2 U_\infty^2} = C_{x_1}(Re, Fr) \;, \tag{2.21}$$

with the Froude number defined by the relationship

$$Fr = \left(\frac{U_\infty^2}{gd}\right) \;. \tag{2.22}$$

To fix the ideas, let us work with a ratio of scales between mock-up and prototype equal to $1/20$. The similarity rules impose the equality of the Reynolds and Froude numbers. The equality of the Froude numbers leads to

$$\left(\frac{U_\infty^2}{gd}\right)_m = \left(\frac{U_\infty^2}{gd}\right)_p \tag{2.23}$$

or dropping the ∞ subscript

$$\left(\frac{U_m}{U_p}\right)^2 = \frac{d_m}{d_p} = S_d \;, \tag{2.24}$$

with S_d the ratio of the dimensions scales (S). We deduce the ratio of the velocity scales S_U

$$S_U = \frac{U_m}{U_p} = S_d^{1/2} \;. \tag{2.25}$$

The equality of the Reynolds numbers produces the relation

$$\left(\frac{Ud}{\nu}\right)_m = \left(\frac{Ud}{\nu}\right)_p \;, \tag{2.26}$$

and consequently

$$S_\nu = \frac{\nu_m}{\nu_p} = \frac{U_m}{U_p}\frac{d_m}{d_p} = S_U S_d = S_d^{3/2} \;. \tag{2.27}$$

As $S_d = 1/20$, one obtains

$$S_\nu = (\frac{1}{20})^{3/2} = 0.011 \;. \tag{2.28}$$

This last equation means that the model should use a fluid of kinematic viscosity a hundred times less than that of water. Such a fluid is impossible to find. In this case, we will have to use approximate methods in which we have different Reynolds numbers for the model and the prototype, with the hope that the change in Reynolds number will have little effect on the drag measured with the model.

2.5 Self-similarity

Consider the flow generated by the instantaneous motion of a wall in its own plane. As we will note in Sect. 3.3.1, the governing equation reads

$$\frac{\partial v_1}{\partial t} = \nu \frac{\partial^2 v_1}{\partial x_2^2} \ . \tag{2.29}$$

The boundary conditions are

$$t < 0, \quad v_1 = 0, \quad \forall x_1 \ , \tag{2.30}$$

$$t \geq 0, \quad v_1 = U, \text{ for } x_2 = 0, \tag{2.31}$$

$$v_1 = 0, \text{ for } x_2 \to \infty. \tag{2.32}$$

The dimensional matrix is as follows

	v_1	U	x_2	t	ν
M	0	0	0	0	0
L	1	1	1	0	2
T	−1	−1	0	1	−1

As the first line of the matrix is zero, its rank is equal to two. Let us choose as basic variables U and ν. One compute the next three dimensionless groups

$$\Pi_1 = \frac{v_1}{U}, \ \Pi_2 = \frac{x_2 U}{\nu} = \xi, \ \Pi_3 = \frac{tU^2}{\nu} = \zeta \ . \tag{2.33}$$

Therefore we write

$$\frac{v_1}{U} = f(\xi, \zeta) \ . \tag{2.34}$$

However, this relation is too general. We simplify it by noting that the Eqs. (2.29) and (2.31) are always satisfied if we apply a scaling factor α to x_2 and a factor α^2 to time t. As we desire to obtain self-similar solutions by enforcing the condition

$$v_1(t_1, \frac{x_2}{g(t_1)}) = v_1(t_2, \frac{x_2}{g(t_2)}) \ , \tag{2.35}$$

where the function g constitutes a scaling factor for the space, we replace the dimensionless group Π_2 by

$$\Pi_2^* = \frac{\Pi_2}{\sqrt{\Pi_3}} = \frac{x_2}{\sqrt{\nu t}} = \eta \ . \tag{2.36}$$

The relation (2.34) becomes

$$\frac{v_1}{U} = f^*(\eta, \zeta) \ . \tag{2.37}$$

Eq. (2.29) is now written

$$2\frac{\partial^2 f^*}{\partial \eta^2} + \eta\,\frac{\partial f^*}{\partial \eta} = 0\,, \tag{2.38}$$

while the conditions (2.30)–(2.32) are

$$\frac{v_1}{U} = 0 \text{ for } \eta \to \infty\,, \tag{2.39}$$

$$\frac{v_1}{U} = 1 \text{ for } \eta = 0\,. \tag{2.40}$$

Inspecting (2.38)–(2.40), one observes that the function f^* depends only on η and thus eventually

$$\frac{v_1}{U} = f^*(\eta)\,. \tag{2.41}$$

Thanks to (2.41), the partial derivatives problem is reduced to an ordinary differential equation. Relation (2.38) is a self-similar relation.

2.6 Dimensionless Form of the Navier–Stokes Equations

The dimensionless presentation of the Navier–Stokes equations for the incompressible fluid is essential for the understanding of the flow physics. Indeed, by this analysis, we may distinguish the dominating phenomenon and simplify the equations to be tackled that will be more amenable to closed-form solutions.

The dimensional matrix reads

	t	$\boldsymbol{x}$	$\boldsymbol{v}$	p	μ	ρ	L	U	$\boldsymbol{b}$
M	0	0	0	1	1	1	0	0	0
L	0	1	1	−1	−1	−3	1	1	1
T	1	0	−1	−2	−1	0	0	−1	−2

Choosing as basic variables L, ρ, U, with a minor equal to -1, we generate the dimensionless groups

$$\Pi_1 = \frac{tU}{L},\quad \Pi_2 = \frac{\boldsymbol{x}}{L},\quad \Pi_3 = \frac{\boldsymbol{v}}{U},\quad \Pi_4 = \frac{p}{\rho U^2},\quad \Pi_5 = \frac{\mu}{\rho U L},\quad \Pi_6 = \frac{\boldsymbol{b}L}{U^2}\,. \tag{2.42}$$

These Π groups allow the introduction of dimensionless variables and functions (denoted with primes) by the relations:

$$x_i = Lx_i'\,,\ t = \frac{L}{U}t'\,,\ v_i = Uv_i'\,,\ p' = \frac{p - p_0}{\rho U^2}\ ,b_i = U^2\frac{b_i'}{L}\,.$$

We rewrite Eqs. (1.73) and (1.74) with dimensionless quantities

$$\frac{\partial v_i'}{\partial t'} + v_k' \frac{\partial v_i'}{\partial x_k'} = -\frac{\partial p'}{\partial x_i'} + \frac{\mu}{UL\rho} \frac{\partial^2 v_i'}{\partial {x_j'}^2} + b_i' \,, \tag{2.43}$$

$$\frac{\partial v_j'}{\partial x_j'} = 0 \,. \tag{2.44}$$

In Eq. (2.43) appears the dimensionless Reynolds number,

$$Re = \frac{\rho U L}{\mu} = \frac{UL}{\nu} \,.$$

The symbol ν represents the kinematic viscosity defined by the relation

$$\nu = \frac{\mu}{\rho} \,. \tag{2.45}$$

It is expressed in $m^2 s^{-1}$. Its value for water at ambiant temperature is $\nu_{water} = 1.138\,10^{-6}\,m^2\,s^{-1}$. The Reynolds number expresses the relative importance of the inertia forces with respect to the viscous forces. It takes values close to 0 for creeping flows dominated by viscous effects up to values of the order of $10^6 \ldots 10^8$ where inertia is the main driving force. In this last case, the flow is turbulent. An example of creeping flow is that of thermal convection in the earth's mantle or the convective currents in a bath of molten glass. The turbulent flows are widespread in nature or in technological applications: the water flow on a boat hull, the aerodynamics design of a car, meteorology, etc.

The Reynolds number can still be interpreted as the ratio of the characteristic time of viscous fluid flows. If we introduce the inertial time $t_{inert} = L/U$ and the viscous time $t_{vis} = L^2/\nu$, the Reynolds number becomes

$$Re = \frac{t_{vis}}{t_{inert}} \,. \tag{2.46}$$

We note that for high values of the Reynolds number, the time scale significant for the fluid inertia is much shorter than the time scale for the action of the viscous effects. This situation explains the stiff character of the numerical integration of the Navier–Stokes equations at high values of the Reynolds number, given the disparity of inertial and viscous time scales. Indeed the numerical integration will march in time with a time step imposed by the inertial dynamical behavior over time ranges long enough to take the viscous effects into account.

The Navier–Stokes equations in dimensionless form read:

$$\frac{D\boldsymbol{v}'}{Dt'} = -\nabla\, p' + \frac{1}{Re}\Delta\boldsymbol{v}' + \boldsymbol{b}'. \tag{2.47}$$

$$\nabla \cdot \boldsymbol{v}' = 0\ , \tag{2.48}$$

If we fix the coordinates x_i and time t and we let the Reynolds number go to infinity, $Re \to \infty$, the system (2.47)–(2.48) leads to the Euler equation for perfect fluids. Note that if $Re \to 0$, we face an inconsistency for (2.47).

When the body force is gravity, with $\boldsymbol{b}' = \boldsymbol{g}' = \boldsymbol{g}/g$ and $g = \parallel \boldsymbol{g} \parallel$ is the gravity acceleration, Eq. (2.47) becomes

$$\frac{D\boldsymbol{v}'}{Dt'} = -\nabla\, p' + \frac{1}{Re}\Delta\boldsymbol{v}' + \frac{1}{Fr}\boldsymbol{g}'\ . \tag{2.49}$$

Here we have the Froude number (2.22)

$$Fr = \frac{U^2}{Lg}\ .$$

This number compares inertia forces to gravity forces.

The limit form of the Navier–Stokes equations is obtained by (2.49) when $Re \to \infty$,

$$\frac{D\boldsymbol{v}'}{Dt'} = -\nabla\, p' + \frac{1}{Fr}\boldsymbol{g}'\ . \tag{2.50}$$

These are the Euler equations. If we come back to dimensional variables, we have

$$\rho\frac{D\boldsymbol{v}}{Dt} = -\nabla\, p + \rho\boldsymbol{g}\ . \tag{2.51}$$

Let us consider again Eq. (1.74) and normalize the reduced form of time and pressure by the viscous effects (this is the viewpoint adopted by rheologists):

$$t' = \frac{\nu t}{L^2} \quad \text{and} \quad p' = \frac{(p - p_0)}{(\frac{\mu U}{L})}\ ,$$

the dimensionless form of the Navier–Stokes equations for an incompressible fluid reads

$$\frac{\partial v_i'}{\partial t'} + Re\left(v_k'\frac{\partial v_i'}{\partial x_k'}\right) = -\frac{\partial p'}{\partial x_i'} + \Delta v_i' + \frac{Re}{Fr}g_i'\ . \tag{2.52}$$

Equations (2.49) and (2.52) are different because the time normalization is made on the one hand by the time linked to advection (inertial term) t_{inert}, and on the other hand by the characteristic time of molecular diffusion t_{vis}.

If we now let $Re \to 0$, Eq. (2.52) is simplified and yields:

$$\frac{\partial \boldsymbol{v}'}{\partial t'} = -\nabla\, p' + \Delta \boldsymbol{v}'. \tag{2.53}$$

These are the linear Stokes equations. In dimensional variables, they read

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} = -\nabla\, p + \mu\, \Delta \boldsymbol{v}. \tag{2.54}$$

2.7 Dimensional Analysis of the Compressible Navier–Stokes Equations

The set of the Navier–Stokes equations for the compressible fluid reads

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \boldsymbol{v} = 0\,, \tag{2.55}$$

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\nabla\, p + \nabla\,(\lambda\, tr\boldsymbol{d}) + \operatorname{div}\,(\boldsymbol{2}\mu\, \boldsymbol{d}) + \rho \boldsymbol{b}\ \ , \tag{2.56}$$

$$\rho c_p \frac{DT}{Dt} = \frac{Dp}{Dt} + \lambda(\mathrm{tr}\,\boldsymbol{d})^2 + 2\mu\, \boldsymbol{d} : \boldsymbol{d} + \mathrm{div}(\mathrm{k}\nabla \mathrm{T}) + \mathrm{r}\,, \tag{2.57}$$

$$p = \rho RT\ . \tag{2.58}$$

We will simplify these relations with the assumptions $r = 0$ and λ, μ, k constants. In addition, we use Stokes' hypothesis

$$3\lambda + 2\mu = 0\,. \tag{2.59}$$

The Stokes relation has been established based on reasoning from the kinetic theory of gases. Although this hypothesis is valid for monatomic gases, it is not valid for polyatomic gases. Nonetheless it is widely used in aerodynamics applications. From the monograph by Langlois and Deville [49] we quote "Neither the theoretical foundation nor the experimental verification of the Stokes relation is especially convincing. Also, Truesdell [109] remarked on p. 229 that "The Stokes relation implies the anomalous result that a spherical mass of fluid may perform symmetrical oscillations in perpetuity, without frictional loss". Stokes himself never took the relation very seriously, and it is now generally conceded to be invalid, except for monatomic gases, with the hard-to-obtain experimental data leniently interpreted."

Table 2.1 Dimensional matrix of the compressible Navier–Stokes variables

	t	$\boldsymbol{x}$	$\boldsymbol{v}$	p	μ	ρ	L	U	$\boldsymbol{b}$	c_p	k	T
M	0	0	0	1	1	1	0	0	0	0	1	0
L	0	1	1	−1	−1	−3	1	1	1	2	1	0
T	1	0	−1	−2	−1	0	0	−1	−2	−2	−3	0
Θ	0	0	0	0	0	0	0	0	0	−1	−1	1

Equations (2.56) and (2.57) become

$$\rho \frac{Dv_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j} + \frac{\mu}{3} \frac{\partial}{\partial x_i}(d_{kk}) + \rho b_i \tag{2.60}$$

$$\rho c_p \frac{DT}{Dt} = \frac{Dp}{Dt} + k \frac{\partial^2 T}{\partial x_j \partial x_j} - \frac{2}{3}\mu (d_{kk})^2 + 2\mu\, d_{ij}\, d_{ij}\,. \tag{2.61}$$

The fundamental variables to build the dimensional matrix are as usual M, L, T plus the thermodynamic temperature Θ. Recall that the SI units for c_p and k are J kg^{-1} K^{-1} and W m^{-1} k^{-1} respectively, where J is kg m^2 s^{-2}. The dimensional matrix reads (Table 2.1).

Choosing as primary variables ρ, U, L, T, the associated minor is different from zero. The dimensional matrix has rank 4 and we are left with eight dimensionless groups

$$\Pi_1 = \frac{tU}{L}, \quad \Pi_2 = \frac{\boldsymbol{x}}{L}, \quad \Pi_3 = \frac{\boldsymbol{v}}{U}, \quad \Pi_4 = \frac{p}{\rho U^2} \tag{2.62}$$

$$\Pi_5 = \frac{\mu}{\rho U L}, \quad \Pi_6 = \frac{\boldsymbol{b}L}{U^2}, \quad \Pi_7 = \frac{c_p T}{U^2}, \quad \Pi_8 = \frac{kT}{\rho L U^3}\,. \tag{2.63}$$

In Π_5 we recognize the inverse of the Reynolds number and Π_6 is the inverse of the Froude number. The combination $\Pi_5 \Pi_7/\Pi_8$ yields $Pr = \mu c_p/k$ which is the definition of the Prandtl number.

The careful aerodynamicist is still puzzled by the absence of the Mach number. In fact this number is hidden in Π_4. Using (1.98) we have

$$\frac{p}{\rho U^2} = \frac{RT}{U^2} = \frac{a^2}{\gamma U^2} = \frac{1}{\gamma M^2} \tag{2.64}$$

with the definition of the Mach number

$$M = \frac{U}{a}\,. \tag{2.65}$$

Let us now investigate the set of dimensionless Navier–Stokes equations for the ideal gas with constant heat capacity. Denote the reference values of length, speed, pressure, density, and temperature that characterize the flow under consideration by L, U, p_0, ρ_0, and T_0. The variables p_0, ρ_0, and T_0 designate a thermodynamic reference state. The time scale is L/U and the scale for inertial forces is U^2/L. Now we introduce non-dimensional variables and functions (denoted with primes) with relations

$$x_i = Lx'_i \qquad t = \frac{L}{U}t' \qquad v_i = Uv'_i \qquad p = p_0 p'$$
$$\rho = \rho_0 \rho' \qquad T = T_0 T' \qquad b_i = U^2 \frac{b'_i}{L} .$$

We reformulate Eqs. (2.55), (2.56), (2.60), and (2.61) with non-dimensional values, including constant characteristic values μ_0 and k_0 estimated at the temperature T_0, as well as c_p, γ, and R:

$$\frac{\partial \rho'}{\partial t'} + v'_j \frac{\partial \rho'}{\partial x'_j} + \rho' \frac{\partial v'_j}{\partial x'_j} = 0 \tag{2.66}$$

$$\frac{\partial v'_i}{\partial t'} + v'_k \frac{\partial v'_i}{\partial x'_k} = -\frac{p_0}{\rho_0 U^2} \frac{1}{\rho'} \frac{\partial p'}{\partial x'_i}$$
$$+\frac{\mu_0}{UL\rho_0} \frac{1}{\rho'} \left(\frac{\partial^2 v'_i}{\partial x'^2_j} + \frac{1}{3} \frac{\partial}{\partial x'_i}(d'_{kk}) \right) + b'_i \tag{2.67}$$

$$\rho' \frac{DT'}{Dt'} = \frac{Dp'}{Dt'} + \frac{k_0}{\mu_0 c_p} \frac{\mu_0}{\rho_0 UL} \frac{\partial^2 T'}{\partial x'^2_j}$$
$$-\frac{\mu_0}{\rho_0 UL} \frac{U^2}{c_p T_0} \left(\frac{2}{3} (d'_{kk})^2 - \frac{1}{2} \left(\frac{\partial v'_i}{\partial x'_j} + \frac{\partial v'_j}{\partial x'_i} \right)^2 \right) \tag{2.68}$$

$$p' = \rho' T' , \tag{2.69}$$

if we set $p_0 = \rho_0 R T_0$.

In relations (2.66)–(2.68) three non-dimensional numbers appear:

- the ***Reynolds number***

$$Re = \rho_0 \frac{UL}{\mu_0} = \frac{UL}{\nu_0} ;$$

- the ***Prandtl number***

$$Pr = \frac{c_p \mu_0}{k_0} = \frac{\nu_0}{\Lambda} ;$$

- the ***Mach number***

$$M = \frac{U}{a_0} ,$$

which appear also in the group

$$\frac{p_0}{\rho_0 U^2} = \frac{RT_0}{U^2} = \frac{a_0^2}{\gamma U^2} = \frac{1}{\gamma M^2}\,.$$

The denominator of the Mach number a_0 is the characteristic speed of sound (1.98). The coefficient Λ defined by relation

$$\Lambda = \frac{k_0}{\rho_0 c_p} \tag{2.70}$$

appearing in the Prandtl number is called the ***thermal diffusivity***. The product of the Reynolds and Prandtl numbers yields the Péclet number

$$Pe = \frac{UL}{\Lambda}\,, \tag{2.71}$$

which is for the heat transfer equation, the counterpart of the Reynolds number for the Navier–Stokes equation.

The Reynolds number expresses the relative importance of the inertial forces with respect to the viscous forces. It takes values from zero up to several million. For $Re = 0$, the Navier–Stokes equations reduce to the Stokes equation. They govern the dynamics of slow or creeping laminar flows. For $Re \geq 10^6$, the flow is turbulent. The Prandtl number estimates the relative importance of the viscous and thermal diffusion phenomena ($Pr = 0.71$ for room temperature air).

The Mach number characterizes the compressibility effects. Its value is $M = 0$ for incompressible fluids. It is between zero and one, $0 < M < 1$ for subsonic flows and $M > 1$ for supersonic flows.

The Navier–Stokes equations take the non-dimensional form

$$\frac{D\rho'}{Dt'} + \rho' \,\mathrm{div}\boldsymbol{v}' = \mathbf{0} \tag{2.72}$$

$$\rho' \frac{D\boldsymbol{v}'}{Dt'} = -\frac{1}{\gamma M^2}\,\nabla p' + \frac{1}{Re}\left(\nabla^2 \boldsymbol{v}' + \frac{1}{3}\,\nabla\,(\mathrm{div}\boldsymbol{v}')\right) + \rho' \boldsymbol{b}' \tag{2.73}$$

$$\rho' \frac{DT'}{Dt'} = \frac{Dp'}{Dt'} + \frac{1}{Pr\,Re}\,\nabla^2 T' - (\gamma - 1)\frac{M^2}{Re}\left(\frac{2}{3}\left(\mathrm{div}\boldsymbol{v}'\right)^{\mathbf{2}} - \frac{\mathbf{1}}{\mathbf{2}}\left(\frac{\partial \mathbf{v}'_{\mathbf{i}}}{\partial \mathbf{x}_{\mathbf{j}}} + \frac{\partial \mathbf{v}'_{\mathbf{j}}}{\partial \mathbf{x}_{\mathbf{i}}}\right)^{\mathbf{2}}\right) \tag{2.74}$$

$$p' = \rho' T'\,. \tag{2.75}$$

If we fix the coordinates x_i, time t, and all the parameters M, Pr, γ, and take $Re \to \infty$, then the system (2.72)–(2.75) leads to the Euler system of equations for perfect (inviscid) fluids. Taking the limit where the Mach number goes to zero, with all the other parameters fixed, should lead to the Navier–Stokes equations for an incompressible fluid.

However, examination of the system (2.72)–(2.75) shows that this is not so, and that the term $-(1/\gamma M^2)\nabla p$ becomes dominant. This behavior is due to the choice of the non-dimensional pressure $p' = p/p_0$, which was made by considering pressure to be a thermodynamic variable. The momentum equation reveals that pressure is also a dynamic variable. It is more natural to choose

$$p^* = \frac{p - p_0}{\rho_0 U^2}$$

for the non-dimensional pressure.

In this case, Eq. (2.73) becomes

$$\rho' \frac{D\boldsymbol{v}'}{Dt'} = -\nabla p^* + \frac{1}{Re}\left(\nabla^2 \boldsymbol{v}' + \frac{1}{3}\nabla\left(\mathrm{div}\boldsymbol{v}'\right)\right) + \rho'\boldsymbol{b}' . \tag{2.76}$$

The limiting case of Eqs. (2.66), (2.76), (2.68), and (2.69) when the Mach number goes to zero, yields the relations

$$\frac{D\rho'}{Dt'} + \rho'\mathrm{div}\boldsymbol{v}' = \boldsymbol{0} \tag{2.77}$$

$$\rho' \frac{D\boldsymbol{v}'}{Dt'} = -\nabla p^* + \frac{1}{Re}\left(\nabla^2 \boldsymbol{v}' + \frac{1}{3}\nabla(\mathrm{div}\boldsymbol{v}')\right) + \rho'\boldsymbol{b}' \tag{2.78}$$

$$\rho' \frac{DT'}{Dt'} = \frac{1}{Pr\,Re}\nabla^2 T' \tag{2.79}$$

$$\rho' T' = 1\ , \tag{2.80}$$

valid for an ***incompressible fluid***, but which still may experience thermal expansion.

To obtain (2.79), we calculate

$$\frac{Dp'}{Dt'} = \frac{1}{p_0}\frac{Dp}{Dt'} = \frac{\rho_0 U^2}{p_0}\frac{Dp^*}{Dt'} = \frac{U^2}{RT_0}\frac{Dp^*}{Dt'} = \gamma M^2 \frac{Dp^*}{Dt'} .$$

Equation (2.80) comes from the following evaluation:

$$p' = \rho' T' = \frac{p}{p_0} = 1 + p^* \frac{U^2}{RT_0} = 1 + \gamma M^2 p^* .$$

If, in addition, we assume that at the domain wall $T' = 1$, then Eqs. (2.79) and (2.80) as well as the boundary conditions on T' are satisfied by

$$\rho' = 1 \tag{2.81}$$

$$T' = 1\,. \tag{2.82}$$

Consequently, in this case, Eqs. (2.77) and (2.78) reduce to the equations of an isothermal, incompressible flow.

In this section we may notice the different nature of pressure for compressible versus incompressible fluids. In the compressible case pressure is a thermodynamic variable that is computed through the equation of state as soon as we know ρ from mass conservation and T from energy governing equation. For the incompressible fluid there is no equation of state and the pressure scalar field is the variable that ensures a divergence free velocity field. In finite element theory for incompressible flows, pressure is the Lagrange variable associated with the constraint div $\boldsymbol{v} = 0$ in the weak formulation of the Navier–Stokes equation. A possible way to compute the pressure consists in generating a Poisson pressure equation by taking the divergence of the momentum equation. The difficulty is then to set up the correct boundary condition for the normal component of the pressure gradient at the wall. The reader is referred to the paper by Orszag et al. [69] that proposes several methods to solve this difficulty. A possible solution of the pressure Poisson equation can be obtained with Neumann boundary condition from wall-normal component of Navier–Stokes equation along with no-slip boundary condition. This is routinely done in aerodynamics to calculate lift and drag.

Exercises

2.1 Write the dimensionless Navier–Stokes equations in the framework of the Boussinesq approximation. To obtain a tractable heat equation we neglect in (1.112) the power dissipation and the volume heat production.

We defined the coefficient of thermal diffusivity expressed in $\mathrm{m^2\,s^{-1}}$ by the relation (2.70). It is possible to obtain three different sets of dimensionless equations according to the choice of the reference velocity: $U = \nu/L$; $U = \Lambda/L$; $U = (g\alpha(T - T_0)L)^{1/2}$. This produces the following dimensionless numbers : Prandtl, Rayleigh, Péclet, Grashof, denoted Pr, Ra, Pe, Gr, respectively, such that

$$Pr = \frac{\nu}{\Lambda},\ Ra = \frac{\alpha g(T - T_0)L^3}{\nu\Lambda},\ Pe = \frac{UL}{\Lambda},\ Gr = \frac{\alpha g(T - T_0)L^3}{\nu^2}\ . \tag{2.83}$$

2.2 Write the velocity profile of Exercise 1.4 in dimensionless form. Which dimensionless number is involved in the solution?