# Chapter 1
# Incompressible Newtonian Fluid Mechanics

We present the properties of incompressible Newtonian viscous fluids and their modelling based on the Navier–Stokes equations. At constant ambient temperature, incompressible fluids are characterized by their invariable density. They are present in nature as well as in many technical applications. Incompressible flows offer very rich and very complex physical phenomena and therefore, their study is appealing and exciting.

Many books are devoted to incompressible fluid mechanics as well as to hydrodynamics. Without wishing to be exhaustive, we can cite the books of Batchelor [10], Guyon, Hulin, Petit and Mitescu [37], Lamb [47], Landau and Lifschitz [48], Meyer [57], Ockendon and Ockendon [64], Panton [71], Rieutord [79], Ryhming [83], Tritton [108], Truesdell and Rajagopal [111], and Yih [125]. The review article of synthesis by Serrin [90] is also a source of information and inspiration.

In this chapter, we will write the fundamental governing relations obtained from the principles of Continuum Mechanics. The reader who wishes to go through all the needed developments is referred to the monograph by Botsis and Deville [16] where full details are provided.

## 1.1 Introduction

Incompressible viscous fluids are part of our daily lives without our clear consciousness. The most obvious example is that of water, which accompanies our most ordinary actions: coffee or tea we drink, the baths, the wetting rain, etc. Then, we understand that water is ubiquitous, both in nature: oceans, rivers, lakes, waves as well as in technical applications: hydraulic turbines, forced ducts, boat design, canals, etc. Water, in the eyes of the fluid mechanicist, is a typical example of Newtonian viscous incompressible fluid. The Newtonian qualification will be explained in the sequel.

Some incompressible fluids have rheological behaviors different from that of water, such as, for example, blood, molten polymers, mud, agro-alimentary fluids, …. In the case of blood, it contains the formed elements: platelets, white and red cells. Their presence within the flow modifies the mechanical behavior of the fluid. As part of the microcirculation, blood is a non-Newtonian shear-thinning fluid with a viscosity decreasing when the shear stress increases.

For polymer melts, it is the long chains of molecules that affect the rheology which also depends on the concentration of the polymer in the solvent. Finally, agro-alimentary fluids have a range of features, such as stress thresholds, viscosity depending on the local shear rate, etc. All these fluids form the class of non-Newtonian fluids; some of them, for example polymers, are viscoelastic and exhibit memory effects such as creep and stress relaxation, which take into account the history of their deformation. However, we will not examine them in this monograph. The interested reader is referred to the book of Deville and Gatski [24] to discover the foundations of complex fluids and flows.

Fluid constitutive models and basic equations are derived from the mechanics of continuous media [5, 16]. We will use the Eulerian description which is the representation where the fluids are generally studied. We place ourselves in a fixed spatial position and observe the flow from this point.

We will use a system of rectangular Cartesian coordinates that is the set formed by point 0 taken as the origin and three orthonormal basis vectors $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ at this origin. Therefore the Cartesian coordinates of a point P in the system is given by the associated vector position of a fluid particle with respect to $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ such that $\boldsymbol{x} = (x_1, x_2, x_3)$ (this particle is understood at the macroscopic scale, but being small enough to carry out infinitesimal analysis). The position will also depend on time $t$. The velocity vector has three components that in this system, we note classically $\boldsymbol{v} = (v_1, v_2, v_3)$. In some cases, we will use cylindrical coordinates with $\boldsymbol{v} = (v_r, v_\theta, v_z)$ or spherical coordinates with $\boldsymbol{v} = (v_r, v_\varphi, v_\theta)$.

The flow behavior is characterized by a dimensionless number, the Reynolds number defined by the relation

$$Re = \frac{UL}{\nu} \, , \tag{1.1}$$

where $U$ and $L$ are a reference velocity and length of the problem at hand, respectively, while $\nu$ is the kinematic viscosity of the fluid expressed in SI (Système International in French; International System of Units) units, i.e. $m^2\,s^{-1}$. Let us recall that for water, $\nu_{water} = 10^{-6}\,m^2\,s^{-1}$. In this last case, one concludes that the $\nu$ presence in the denominator of Eq. (1.1) produces immediately a high $Re$ value. In reality, $Re$ is comprised between values close to zero for slow or creeping flows that are laminar to values of $10^6 \ldots 10^7$ for turbulent flows. For these high $Re$ values, we are in the case of developed turbulence whose space-time dynamics is very fluctuating. Between these two extreme values, flows are subject to instabilities and bifurcations that are the features of transition phenomena. Recently, chaos theory allowed to penetrate more deeply in the concept of weak turbulence where $Re$ presents moderate values going from a few hundreds to a few (dozens of) thousands.

**Fig. 1.1** Symmetric Taylor vortices. Courtesy of N. Borhani with permission
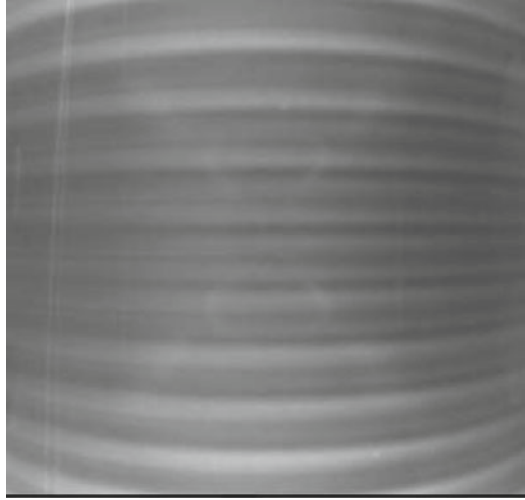


### 1.1.1 Circular Couette Flow

The so-called spectral transition is well illustrated by the circular Couette flow between two concentric vertical cylinders. Let us examine the particular case where the inner cylinder of radius $R_1$ is rotating with a constant angular velocity $\omega$ and the outer cylinder of radius $R_2$ is fixed.

The Reynolds number is now defined as

$$Re = \frac{\omega R_1 (R_2 - R_1)}{\nu} \, ,$$ (1.2)

where $(R_2 - R_1)$ is the gap dimension. For small $Re$ values, the steady-state laminar flow is described by the azimuthal velocity

$$v_\theta = Ar + B/r \, ,$$ (1.3)

with $A = -\omega R_1^2 / (R_2^2 - R_1^2)$ and $B = \omega R_1^2 R_2^2 / (R_2^2 - R_1^2)$.

If the angular velocity of the inner cylinder is increased, the flow goes through a first transition towards the Taylor vortices [103]. This discovery impacted deeply the study of fluid dynamics so much so that this flow is now named the Taylor–Couette flow. Figure 1.1 shows these vortices for an experimental set-up with $R_1 = 8$ cm and a gap $R_2 - R_1 = 0.5$ cm. The rotation velocity is such that the Reynolds number is $Re = 150$ and the Taylor vortices are axisymmetric and steady-state. In a meridian plane, they appear in counter rotating pairs. A fluid particle follows an helical path placed on the surface of a torus centered on the rotation axis. Increasing the inner cylinder rotation, a second transition occurs and the Taylor toruses are deformed in
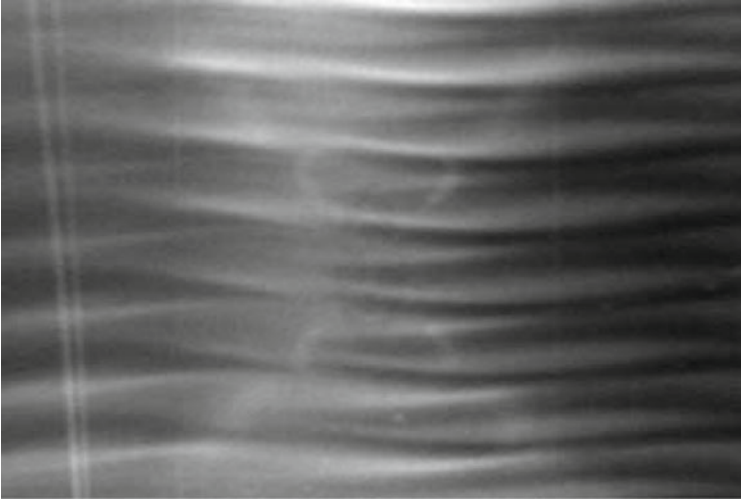
**Fig. 1.2** Wavy Taylor vortices. Courtesy of N. Borhani with permission

the azimuthal direction. Figure 1.2 displays the steady-state wavy Taylor vortices pattern for $Re = 200$.

Subsequent transitions modify the number of vortices pairs in the vertical direction and the wavenumber of the azimuthal deformations in direction $\theta$. At some value of the Reynolds number, the flow becomes unsteady. This physical phenomenon is a Hopf bifurcation giving in phase space a limit cycle with a clearly identified time period. Donald Coles [20] has shown experimentally that by increasing the inner cylinder rotation the route to turbulence is not the same as by decreasing the rotation velocity. Typically the flow presents a hysteresis.

### 1.1.2   Flow Around a Cylinder

The uniform, parallel flow upstream of a horizontal circular cylinder is of paramount importance in hydrodynamics. As we will note in the sequel, this flow can be transformed in the flow around an airfoil through the Joukowski transformation. The Reynolds number is defined by $U$, the uniform upstream velocity, $L = D$, the diameter of the cylinder, and $\nu$, the kinematic viscosity of the fluid. Figure 1.3[1] shows the flow at $Re \simeq 0$ for which the streamlines are symmetric with respect to the horizontal, vertical, and diagonal directions. The streamlines follow closely the shape of the obstacle.

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[1] Figures 1.1–1.7 are taken from text [114]. Attempts to identify the copyright owner have not as yet succeeded, and he or she is invited to contact the publisher.
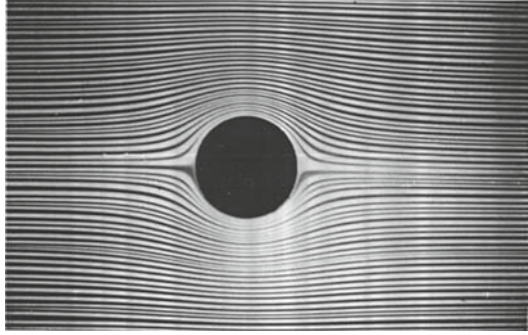
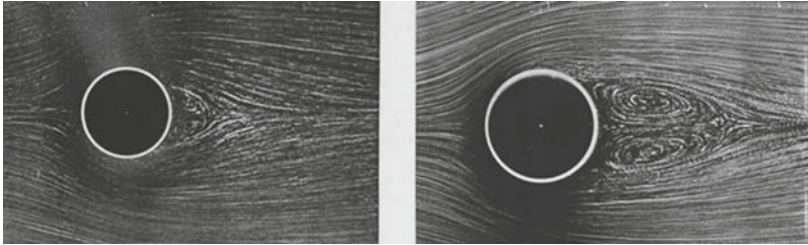**Fig. 1.3** Flow around a cylinder at $Re \simeq 0$



**Fig. 1.4** Flow around a cylinder for (left) $Re = 13.1$ and (right) 26

As $Re$ grows, for the values 13.1 and 26 shown in Fig. 1.4, it is seen that the flow is stationary and symmetric with respect to the horizontal axis. However, two counter-rotating recirculation zones appear behind the cylinder. The length of the recirculation zone increases linearly with $Re$ while the distance separating the centers of the vortices grows as $\sqrt{Re}$.

At $Re = 47.5$, the first critical Reynolds number is reached, at which point the physical phenomena become unstable. Results over the last decade have shown that this is truly not a fixed number. The associated Hopf bifurcation can be multiple in number. Please note the experimental observations of Homann in [89]. A von Kármán vortex street is produced behind the cylinder with vortices alternately shed above and below. A similar vortex street is shown in Fig. 1.5 for $Re = 140$, taken from [114]. The shed vortices are regularly produced at a frequency corresponding to a limit cycle in phase space: a Hopf bifurcation. This frequency, denoted $f$, leads to the definition of the Strouhal number, $St$

$$St = \frac{fD}{U} \ .\tag{1.4}$$

For values of $Re$ around one hundred, $St$ is 0.13.

Stability analyses are based on the Ginzburg–Landau equation [26] which determines the non-linear development of perturbations superimposed on an underlying
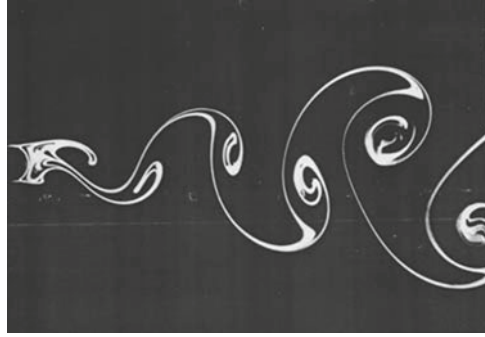
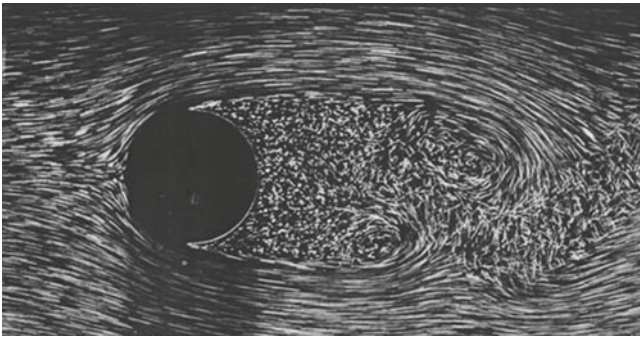**Fig. 1.5**  von Kármán vortex street for $Re = 140$



**Fig. 1.6**  von Kármán vortex street for $Re = 2000$

flow. This theory extends over a vast domain that this book cannot cover. We refer the reader to specialized texts, for example, [18, 27, 86]. If the Reynolds number is again increased, the flow passes through transitional regimes before finally attaining the turbulent state. An excellent synthesis of the dynamics of the wakes of circular cylinders is that of Williamson [121].

Figure 1.6 shows the flow pattern for weak turbulence. The boundary layer, where viscous effects are of the same order of magnitude as inertial effects, is laminar in front of the cylinder, develops around it, undergoes a separation, and produces a turbulent wake. It is still possible to observe two vortices resulting from the non-linear dynamics.

At $Re = 10^4$ as in Fig. 1.7, the flow has roughly the same form, with two identifiable vortices.
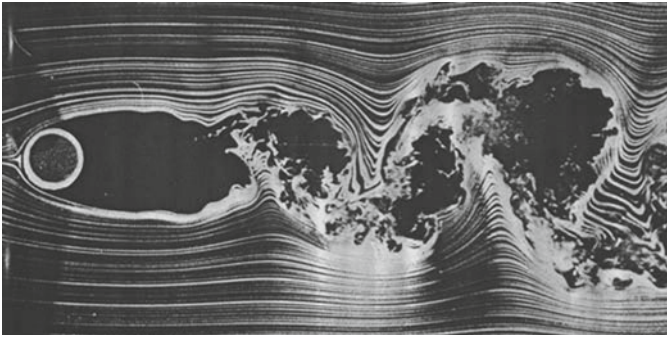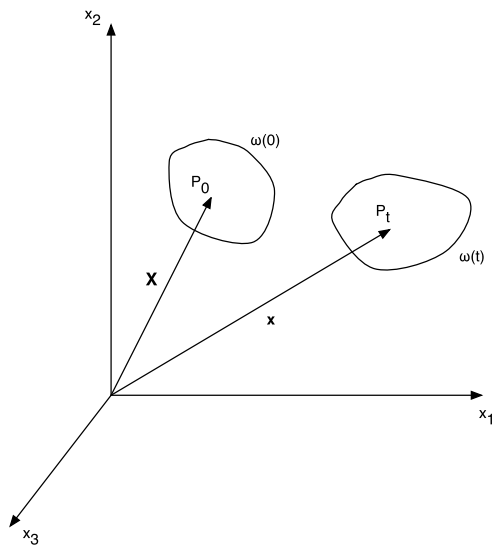
**Fig. 1.7** von Kármán vortex street for $Re = 10^4$

**Fig. 1.8** Representation of the fluid deformation configurations



## 1.2 Fluid Kinematics

Fluid flows in a three-dimensional Euclidean space. In order to describe its deformation, the concept of motion is needed. For the sake of simplicity, a system of Cartesian rectangular orthonormal coordinates is chosen. Let $X$ be the initial position vector of a fluid particle $P_0$ in a material volume $\omega(0)$ at time $= 0$ (cf. Fig. 1.8). At the present time $t \geq 0$, the position of this particle $P_t$ is located by the actual position vector $x$ in the volume $\omega(t)$. The particle motion is described by a vector function $\chi$ defined over time $t$ that depends on $X$:

$$x = \chi(X, t). \tag{1.5}$$

If the initial reference position ($t = 0$) coincides with the current position, the function $\chi$ must satisfy the condition

$$X = \chi(X, 0).$$ (1.6)

The motion $\chi$ is a bijection ensuring a one-to-one correspondence between the initial and current positions of the fluid particles. The existence of the function $\chi$ and its inverse $\chi^{-1}$

$$X = \chi^{-1}(x, t)$$ (1.7)

with

$$X = \chi^{-1}(X, 0)$$ (1.8)

guarantees the integrality and the unity of the fluid body as a whole.

### 1.2.1  Material and Spatial Descriptions

The material description, also called the Lagrangian description, of fluid mechanics signifies the study of physical phenomena under consideration by observing what happens to a fluid particle or in its neighborhood. Alternatively the spatial description, known as the Eulerian description, consists of observing the events occurring at a fixed point in space. Uppercase letters for the material representation and lowercase letters for the spatial representation will be used to distinguish them clearly without any ambiguity.

Hydrodynamics problems are most of the time expressed in spatial description as fluids undergo huge deformations (let us think about the river flowing from its source till the sea or the ocean). The spatial description will use $x, t$ as independent variables.

### 1.2.2  Velocity, Material Derivative and Acceleration

#### 1.2.2.1  Velocity

The velocity of a material particle at time $t$ is the derivative of the motion function in (1.5) with respect to time. By definition, in the material description, we have

$$V(X, t) = \frac{\partial \chi(X, t)}{\partial t}$$ (1.9)

$$V_i(X, t) = \frac{\partial \chi_i(X, t)}{\partial t}.$$ (1.10)

The vector $V(X, t)$ expresses the velocity at time $t$ of the particle that initially was at $X$. Note that (1.9) is obtained using (1.5), taking into account that $X$ is one of the independent variables.

The spatial description of velocity, written as $v$ according to our convention, is obtained by

$$v(x, t) = V(\chi^{-1}(x, t), t) = V(X, t). \tag{1.11}$$

The vector $v(x, t)$ expresses the velocity at an instant $t$ of the particle that, at that time, passes through the position $x$.

### 1.2.2.2 Material Derivative

Let us introduce the notion of material derivative. Let $\varphi$ be a scalar field. During a motion $\chi$, the material derivative of $\varphi(x, t)$, written $\dot{\varphi}$ or $D\varphi/Dt$, is the rate of change of $\varphi(x, t)$ with time (the derivative with respect to time) for a single particle in a fixed space position. In the material description, that is $\varphi(\chi(X, t), t) = \Phi(X, t)$, we simply have

$$\frac{D\varphi(x, t)}{Dt} = \dot{\varphi} = \left.\frac{\partial \Phi(X, t)}{\partial t}\right|_{X=\chi^{-1}(x,t)}. \tag{1.12}$$

The last equation shows that the material derivative is applied to the same particle. For that reason, some authors call it the particle derivative. Since we can write $\Phi(X, t) = \Phi(\chi^{-1}(x, t), t) = \varphi(x, t)$, we obtain

$$\frac{\partial \Phi(X, t)}{\partial t} = \frac{\partial \varphi}{\partial x_1}\frac{\partial \chi_1}{\partial t} + \frac{\partial \varphi}{\partial x_2}\frac{\partial \chi_2}{\partial t} + \frac{\partial \varphi}{\partial x_3}\frac{\partial \chi_3}{\partial t} + \left.\frac{\partial \varphi}{\partial t}\right|_{x=\chi(X,t)}. \tag{1.13}$$

Using the definition of the velocity (1.9), the preceding equation takes the following form:

$$\frac{\partial \Phi(X, t)}{\partial t} = \left.\frac{\partial \varphi}{\partial t}\right|_{x=\chi(X,t)} + V_i(X, t)\left.\frac{\partial \varphi}{\partial x_i}\right|_{x=\chi(X,t)}, \quad i = 1, 2, 3, \tag{1.14}$$

where the Einstein convention of summation on repeated indices holds. The notation $\partial \varphi/\partial x_i$ designates the Cartesian components of the gradient of the scalar field $\varphi$, namely $\nabla \varphi$, which is a vector field.

Since the goal is to express the rightmost term of (1.14) in spatial coordinates, we make the substitution $X = \chi^{-1}(x, t)$ in the last equation which gives

$$\left.\frac{\partial \Phi(X, t)}{\partial t}\right|_{X=\chi^{-1}(x,t)} = \frac{\partial \varphi}{\partial t} + v_i(x, t)\frac{\partial \varphi}{\partial x_i}, \tag{1.15}$$

where we used

$$V_i(X, t)\Big|_{X=\chi^{-1}(x,t)} = v_i(x, t) . \tag{1.16}$$

Now we can define the following derivative:

$$\dot{\varphi}(x, t) = \frac{D\varphi(x, t)}{Dt} \equiv \frac{\partial \Phi(X, t)}{\partial t}\Big|_{X=\chi^{-1}(x,t)} , \tag{1.17}$$

where, from (1.15),

$$\frac{D\varphi(x, t)}{Dt} = \frac{\partial \varphi(x, t)}{\partial t} + v(x, t) \cdot \nabla\varphi(x, t) , \tag{1.18}$$

$$= \frac{\partial \varphi(x, t)}{\partial t} + v_j \frac{\partial \varphi(x, t)}{\partial x_j} . \tag{1.19}$$

The dot in Eq. (1.18) represents the scalar product of two vectors. The derivative $D\varphi(x, t)/Dt$ is called the **material derivative** and represents the rate of change of the function $\varphi$ following the same particle whose velocity is $v(x, t)$. Alternatively, this derivative can be considered as giving the change of $\varphi$ over time, as seen by an observer moving with the particle that is at $x$.

For a vector field $w$, we have a similar formula for its material derivatives:

$$\frac{Dw}{Dt} = \dot{w} = \frac{\partial W(X, t)}{\partial t}\Big|_{X=\chi^{-1}(x,t)} \tag{1.20}$$

$$\frac{Dw_i}{Dt} = \dot{w}_i = \frac{\partial W_i(X, t)}{\partial t}\Big|_{X=\chi^{-1}(x,t)}$$

$$\dot{w} = \frac{\partial w(x, t)}{\partial t} + \left(\nabla w(x, t)\right)\frac{\partial \chi(X, t)}{\partial t}\Big|_{X=\chi^{-1}(x,t)} \tag{1.21}$$

$$\dot{w}_i = \frac{\partial w_i(x, t)}{\partial t} + \frac{\partial w_i(x, t)}{\partial x_j} v_j .$$

### 1.2.2.3   Acceleration

The acceleration $A$ of a material particle at time $t$ is the derivative of its velocity $V$ with respect to time, that is, the material derivative of $V$. In the material description, we have

$$A(X, t) = \frac{\partial V(X, t)}{\partial t} = \frac{\partial^2 \chi(X, t)}{\partial t^2} \tag{1.22}$$

$$A_i = \dot{V}_i = \frac{\partial^2 \chi_i(X, t)}{\partial t^2} ,$$

and in the spatial description, we have

$$a = \dot{v} = \frac{\partial v(x, t)}{\partial t} + \big(\nabla v(x, t)\big) v(x, t) \tag{1.23}$$

$$a_i = \dot{v}_i = \frac{\partial v_i(x, t)}{\partial t} + \frac{\partial v_i(x, t)}{\partial x_j} v_j(x, t).$$

The notation $\nabla v$ represents the velocity gradient, a second-order tensor that has the components $\partial v_i / \partial x_j$. The first term on the right-hand side of (1.23) can be considered as the acceleration due to the time dependence of the velocity at a fixed point in space. The second term can be interpreted as the contribution to the acceleration of the material particle due to the heterogeneity of the velocity field. These terms are sometimes called the local and convective (or advective) parts, respectively, of the acceleration. The advection corresponds to the transport of the velocity field by itself.

### 1.2.3 Jacobian

The Jacobian of the transformation (1.5), i.e.

$$J = \det\left(\frac{\partial \chi_i}{\partial X_j}\right) \tag{1.24}$$

represents the dilatation of an infinitesimal volume during the motion. For the inverse function $\chi^{-1}$ to be differentiable, we have the condition

$$0 < J < \infty. \tag{1.25}$$

An elegant relationship due to L. Euler reads

$$\dot{J} = J \operatorname{div} v. \tag{1.26}$$

### 1.2.4 Reynolds Transport Theorem

**Theorem 1.1** (Reynolds theorem) *If the current time t value of the integral I (t) of property $f(x, t)$ over a fluid volume $\omega(t)$ is defined by the equation*

$$I(t) = \int_{\omega(t)} f(x, t)\, dV, \tag{1.27}$$

*the time derivative of I is given by*

$$\mathrm{d}I(t) = \int_{\omega(t)} \left[ \frac{Df}{Dt} + f\, div\, \boldsymbol{v} \right] dV \; , \tag{1.28}$$

where $\frac{D}{Dt}$ is the material derivative defined by (1.18) such that

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + v_j \frac{\partial f}{\partial x_j} = \frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} f \; . \tag{1.29}$$

We can consider that Reynolds theorem is somehow the generalization to a fluid material of the one-dimensional Leibnitz' integral whose integration limits are also function of the integration variable. In (1.27), the function $f$ may be a scalar, vector or tensor function.

## 1.3  Velocity Gradient and Associated Tensors

In numerous problems of fluid mechanics, an interesting kinematic quantity is not the change in the shape of the material volume, but the rate at which this change is produced.

Let $\mathcal{V}$ be the neighborhood of the point P with coordinates $x_i$, and Q an arbitrary point belonging to $\mathcal{V}$ with coordinates $x_i + dx_i$. The spatial velocity of Q is given by

$$v_i(x_j + dx_j, t) = v_i(x_j, t) + \frac{\partial v_i(x_j, t)}{\partial x_j} dx_j + \cdots \; . \tag{1.30}$$

The tensor $\boldsymbol{L}$ whose components are

$$L_{ij} = \frac{\partial v_i}{\partial x_j} = \left( \boldsymbol{\nabla} \boldsymbol{v} \right)_{ij} \tag{1.31}$$

is the ***velocity gradient*** that already appeared in Eq. (1.23). The symmetric part of $\boldsymbol{L}$, that is,

$$d_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \tag{1.32}$$

$$\boldsymbol{d} = \frac{1}{2} \left( \boldsymbol{\nabla} \boldsymbol{v} + (\boldsymbol{\nabla} \boldsymbol{v})^T \right) \tag{1.33}$$

is called the ***rate of deformation tensor***, and the antisymmetric part of $\boldsymbol{L}$, that is,

$$\dot{\omega}_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right) \tag{1.34}$$

$$\dot{\boldsymbol{\omega}} = \frac{1}{2} \left( \boldsymbol{\nabla} \boldsymbol{v} - (\boldsymbol{\nabla} \boldsymbol{v})^T \right) \tag{1.35}$$

is called the ***rotation rate tensor*** or ***vorticity tensor***. The upper index $T$ in (1.33) and (1.35) denotes the transpose. Thus we can write

$$L = d + \dot{\omega} . \tag{1.36}$$

As an antisymmetric tensor has only three independent components, they can form the dual or axial vector $\dot{\Omega}_i$ associated with the rotation rate tensor, that is,

$$\dot{\Omega}_i = \frac{1}{2}\varepsilon_{ikj}\dot{\omega}_{jk} = -\frac{1}{2}\varepsilon_{ijk}\dot{\omega}_{jk} = \frac{1}{2}\varepsilon_{ijk}\dot{\omega}_{kj} \quad , \tag{1.37}$$

is called the ***rotation rate vector***. The permutation symbol $\varepsilon_{ijk}$ is a third order tensor defined as follows

$$\varepsilon_{ijk} = \begin{cases} 1 \text{ if } ijk \text{ is an even permutation of 123} \\ -1 \text{ if } ijk \text{ is an odd permutation of 123} \\ 0 \text{ all other cases,} \end{cases} \tag{1.38}$$

or as

$$\varepsilon_{ijk} = \frac{1}{2}(i - j)(j - k)(k - i) . \tag{1.39}$$

Note that in fluid mechanics, one typically introduces the vorticity vector $\boldsymbol{\omega}$ with the definition as the curl of the velocity. Then

$$\boldsymbol{\omega} = \mathbf{curl}\, v = \nabla \times v \tag{1.40}$$

or

$$\omega_i = \varepsilon_{ijk}\frac{\partial v_k}{\partial x_j} . \tag{1.41}$$

And we easily deduce that

$$\boldsymbol{\omega} = 2\dot{\boldsymbol{\Omega}}. \tag{1.42}$$

To better understand the vorticity vector, consider the decomposition of a local motion of a fluid. Let P be a point at position $x$ and P′ a neighboring point as shown in Fig. 1.9.

The vector position of P′ relative to P is $dx$. After an infinitesimal lapse of time, P and P′ occupy new positions. P moves with the local velocity $v$ and P′ with the velocity $v + dv$. We consider P to be the principal fluid particle and, subtracting its translational velocity, we describe the motion of P′ as observed from this principal particle. This reasoning is valid only when the distance $dx$ is very small. We can decompose the motion of P and P′ into three distinct parts: a translation, a rigid body rotation, and a strain. The translational motion is given by the velocity $v$ of P. All the other motions, taken together, are given by $dv$, the velocity of P′ with respect to P. We then have

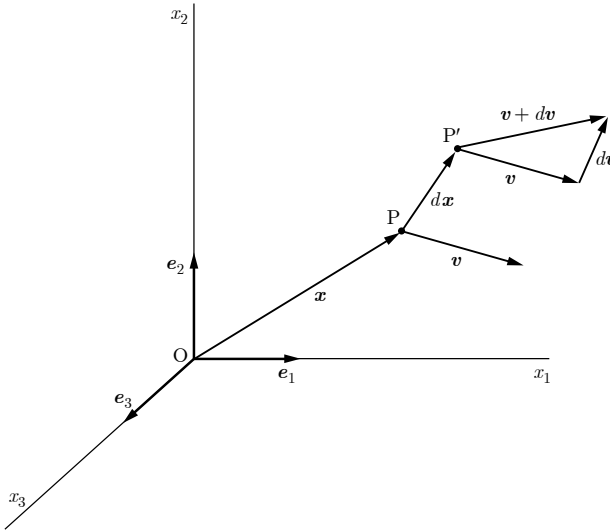$$dv = \frac{\partial v}{\partial x}dx = L\, dx . \tag{1.43}$$

**Fig. 1.9**  Relative motion of two fluid particles

By (1.36), the strain motions (stretching, shortening, ...) of P′ with respect to P are described by $\boldsymbol{d}$. Consequently the rotational motion of P′ with respect to P is taken into account by $\dot{\boldsymbol{\omega}}$. We can write

$$dv^{(r)} = \dot{\boldsymbol{\omega}}\,dx\;, \tag{1.44}$$

where the superscript $r$ refers to rotation.

The rigid body rotational motion of P′ with respect to P must have the form of the equation $\boldsymbol{v} = \boldsymbol{\Omega} \times \boldsymbol{x}$, where $\boldsymbol{\Omega}$ is the rate of angular rotation (a vector). By (1.37) and (1.42), we have $-\dot{\omega}_{ij} = \frac{1}{2}\varepsilon_{ijk}\omega_k = \dot{\omega}_{ji}$. Thus the rotational component of motion is given by

$$dv_j^{(r)} = \dot{\omega}_{ji}\,dx_i = \frac{1}{2}\varepsilon_{ijk}\omega_k\,dx_i$$
$$= \frac{1}{2}\varepsilon_{jki}\,(\omega_k)\,dx_i\;. \tag{1.45}$$

This last equation is of the form $d\boldsymbol{v} = \boldsymbol{\Omega} \times d\boldsymbol{x}$. The vorticity vector $\boldsymbol{\omega}$ corresponds to an angular velocity such that the vorticity $\boldsymbol{\omega}$ is equal to $2\boldsymbol{\Omega}$, that is two times the vector rate of rigid body rotation of P′ with respect to P.

Note that in the case of rotation of a rigid body, one can show that $\boldsymbol{d} = 0$ and $\boldsymbol{L} = \dot{\boldsymbol{\omega}}$. The rotation rate tensor is thus entirely determined by the instantaneous rotation of the body.

## 1.4 Mass Conservation

The principle of mass conservation reads as follows: *For the same material volume $\omega(t)$, the mass M(t) remains constant in time.*

One has:

$$\mathrm{d}M(t) = 0 \,. \tag{1.46}$$

In order to write the local form of the principle of mass conservation, the Reynolds transport theorem is needed. Indeed, the mass $M(t)$ can be expressed by the definition

$$M(t) = \int_{\omega(t)} \rho \, dV \,, \tag{1.47}$$

where $\rho$ is the mass density of the fluid. It has for SI[2] units kg/m$^3$ or dimensions ML$^{-3}$ where M is the mass and L a length. For water, at standard sea level temperature, $\rho$ is 1,000 kg/m$^3$. Fluid mechanicists name incorrectly $\rho$ density. Strictly speaking, density is defined as the ratio of the volumic mass of the fluid (or the material) at hand to that of water. Therefore water density (specific gravity) is equal to 1.

As $\rho$ can in general depend on position and time, the Eqs. (1.28) and (1.47) yield in local form:

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \boldsymbol{v} = 0 \,, \tag{1.48}$$

or

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \, \boldsymbol{v}) = 0 \,. \tag{1.49}$$

A fluid is considered as incompressible when the density $\rho(\boldsymbol{x}, t)$ is constant and $D\rho(\boldsymbol{x}, t)/Dt = 0$. It also follows from (1.49) that the ***incompressibility condition*** can be expressed by the following relationship:

$$\operatorname{div} \boldsymbol{v} = \frac{\partial v_i}{\partial x_i} = 0 \,. \tag{1.50}$$

Note that the velocity field that satisfies (1.50) is solenoidal. As $\operatorname{div} \boldsymbol{v} = 0$, it follows from (1.26) that $\dot{J} = 0$, so $J$ remains constant over time. Since $J(\boldsymbol{X}, 0) = 1$, the motion of an incompressible material takes place with constant volume and is often called isochoric.

The incompressibility condition (1.50) can be satisfied by the vector potential $\boldsymbol{\Psi}$ such that

$$\boldsymbol{v} = \operatorname{\mathbf{curl}} \boldsymbol{\Psi} \,, \tag{1.51}$$

as this verifies the vector identity $\operatorname{div} \operatorname{\mathbf{curl}} = 0$.

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[2] International System of Units, Système International d'Unités, designated SI in all languages.

## 1.5  Equation of Motion

In a fluid, the applied forces on the volume $\omega(t)$ are of two types

1. volume or body forces (acting at a distance) by unit volume such as gravity $\boldsymbol{g}$ or electromagnetic forces, including the Lorentz force, defined by $\rho \boldsymbol{b}(\boldsymbol{x}, t)$
2. contact forces by unit surface $\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n})$.

In the contact forces, appears $\boldsymbol{n}$ the unit normal vector to the surface $\partial \omega(t)$ where the force is exerted.

The principle of conservation of momentum which is the generalization of Newton's law, reads: *The rate of change of the momentum of an arbitrary volume of a fluid at time t is equal to the sum of the forces applied to $\omega(t)$ at that instant.*

The principle states

$$\frac{d}{dt} \int_{\omega(t)} \rho(\boldsymbol{x}, t) \boldsymbol{v}(\boldsymbol{x}, t) \, dV = \int_{\omega(t)} \rho(\boldsymbol{x}, t) \boldsymbol{b}(\boldsymbol{x}, t) \, dV + \int_{\partial \omega(t)} \boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n}) \, dS \,.$$

$$(1.52)$$

**Theorem 1.2** (Cauchy theorem) *The stress vector $\boldsymbol{t}$ is linked to the outward unit normal $\boldsymbol{n}$ of the closed material surface $\partial \omega(t)$ via the stress tensor $\boldsymbol{\sigma}$.*

$$\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n}) = \boldsymbol{\sigma}(\boldsymbol{x}, t) \, \boldsymbol{n} \quad or \quad t_i(\boldsymbol{x}, t, \boldsymbol{n}) = \sigma_{ij}(\boldsymbol{x}, t) n_j \,. \qquad (1.53)$$

The symbol $\sigma_{ij}$ is the stress component in direction $i$ of the Cartesian rectangular coordinate system acting on a surface element with a unit normal oriented in the direction of the basis vector $\boldsymbol{e}_j$. Taking mass conservation into account, the left hand side of (1.52) becomes

$$\frac{d}{dt} \int_{\omega(t)} \rho \, \boldsymbol{v} \, dV = \int_{\omega(t)} \rho \, \frac{D\boldsymbol{v}}{Dt} \, dV \,. \qquad (1.54)$$

Let us introduce the Gauss theorem that will be useful for many developments.

**Theorem 1.3** (Gauss theorem) *Gauss theorem, also known as the divergence theorem, transforms the volume integral of the divergence of a continuous media property into a surface integral*

$$\int_{\omega(t)} \boldsymbol{div} \, \boldsymbol{L} \, dV = \int_{\partial \omega(t)} \boldsymbol{L} \, \boldsymbol{n} \, dS \,. \qquad (1.55)$$

With Eq. (1.53) and the divergence theorem for a second-order tensor $\boldsymbol{L}$ the contact force term yields

$$\int_{\partial \omega(t)} \sigma_{ij} \, n_j \, dS = \int_{\omega(t)} \frac{\partial \sigma_{ij}}{\partial x_j} \, dV \quad or \quad \int_{\partial \omega(t)} \boldsymbol{\sigma} \boldsymbol{n} \, dS = \int_{\omega(t)} \boldsymbol{div} \, \boldsymbol{\sigma} \, dV. \quad (1.56)$$

Equation (1.52) now reads

$$\int_{\omega(t)} \left[ \rho \frac{D\boldsymbol{v}}{Dt} - \boldsymbol{div}\,\boldsymbol{\sigma} - \rho\boldsymbol{b} \right] dV = 0 \,. \tag{1.57}$$

Invoking the localization theorem

**Theorem 1.4**  *For an arbitrary material volume, the integral equation is identically satisfied when the integrand vanishes,*

Eq. (1.57) gives

$$\rho \frac{D\boldsymbol{v}}{Dt} = \boldsymbol{div}\,\boldsymbol{\sigma} + \rho\boldsymbol{b} \,. \tag{1.58}$$

In absence of volume torque, the principle of conservation of angular momentum implies the symmetry of the Cauchy stress tensor

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T \,. \tag{1.59}$$

This reduces the number of unknown stress components from 9 to 6.

## 1.6  Equation of Energy

The principle of conservation of energy is expressed as:

*For the same material volume $\omega(t)$, the time derivative of the total energy is equal to the sum of the power of the volume and contact forces and the rate of heat received by the material.*

The total energy is the sum of the kinetic energy defined by

$$\int_{\omega(t)} \frac{1}{2} \rho\,\boldsymbol{v} \cdot \boldsymbol{v}\,dV \tag{1.60}$$

and the internal energy

$$\int_{\omega(t)} \rho\,u\,dV, \tag{1.61}$$

where $u$ is the internal energy density per unit mass. One gets

$$\frac{d}{dt} \int_{\omega(t)} \left[ \frac{1}{2}(\boldsymbol{v} \cdot \boldsymbol{v}) + u \right] \rho\,dV = \int_{\partial\omega(t)} \boldsymbol{t} \cdot \boldsymbol{v}\,dS + \int_{\omega(t)} \rho\boldsymbol{b} \cdot \boldsymbol{v}\,dV$$

$$+ \int_{\partial\omega(t)} (-\boldsymbol{q} \cdot \boldsymbol{n})dS + \int_{\omega(t)} r\,dV \,. \tag{1.62}$$

In Eq. (1.62), $\boldsymbol{q}$ represents the heat flux vector across the surface $\partial\omega(t)$, while $r$ designates the heat produced or received per unit time and volume. For example, $r$

could be the heat produced or consumed by a chemical reaction in the material or heating by the Joule effect.

By the application of the conservation of mass (1.48), of Cauchy principle (1.53) and divergence and localization theorems to the relation (1.62), we obtain:

$$\rho \frac{D}{Dt}\left(\frac{1}{2}(\boldsymbol{v}\cdot\boldsymbol{v}) + u\right) = \text{div}\,(\boldsymbol{\sigma}\boldsymbol{v}) + \rho\boldsymbol{b}\cdot\boldsymbol{v} - \text{div}\,\boldsymbol{q} + r \ . \qquad (1.63)$$

The first term on the right hand side of Eq. (1.63) can be written as

$$\text{div}\,(\boldsymbol{\sigma}\boldsymbol{v}) = \text{div}\,(\boldsymbol{\sigma})\cdot\boldsymbol{v} + \boldsymbol{\sigma} : \boldsymbol{\nabla}\boldsymbol{v} \ . \qquad (1.64)$$

Subtracting from (1.63) the equation of motion (1.58) multiplied by $\boldsymbol{v}$, we have:

$$\rho \frac{Du}{Dt} = \boldsymbol{\sigma} : \boldsymbol{\nabla}\boldsymbol{v} - \text{div}\,\boldsymbol{q} + r \ . \qquad (1.65)$$

The product $\boldsymbol{\sigma} : \boldsymbol{\nabla}\boldsymbol{v}$ is the scalar product of two tensors such that in indexed form, one has

$$(\boldsymbol{\sigma} : \boldsymbol{\nabla}\boldsymbol{v}) = \sigma_{ij}\frac{\partial v_i}{\partial x_j} \ . \qquad (1.66)$$

## 1.7  Constitutive and State Equations

The elaboration of the constitutive equation for a viscous Newtonian fluid is based on the following assumptions:

i. For a fluid at rest, the viscous fluid behaves as a perfect (without viscosity) or inviscid fluid.

ii. For a moving fluid, the stress tensor depends through a linear relation on the strain rate tensor or the rate of deformation tensor. This hypothesis is characteristic of the Newtonian behavior of the fluid. The necessity of a quantitative relation between the internal forces in a moving fluid and the kinematical quantities describing the motion is due to Newton. Langlois and Deville [49] extract from his famous book *Philosophiae Naturalis Principia Mathematica* the following excerpt.

*Resistentiam quae oritur ex defectu lubricitatis partium fluidi, caeteris paribus, proportionatem esse velocitati, qua partes fluidi separantur ab invicem*[3].

iii. The fluid is isotropic.

These hypotheses allow writing the general constitutive equation of a viscous incompressible fluid in the form

$$\boldsymbol{\sigma} = -p\,\boldsymbol{I} + 2\,\mu\,\boldsymbol{d} \ . \qquad (1.67)$$

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[3] The resistance arising from imperfect slipping between fluid particles to be proportional to the velocity with which the particles are moving apart, other things being equal.

The symbol $p$ represents the pressure. The strain rate tensor $\boldsymbol{d}$, which is the symmetric part of the velocity gradient tensor $\nabla \boldsymbol{v}$, has for components:

$$d_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) . \tag{1.68}$$

The factor $\mu$ is the dynamic viscosity coefficient of dimension $ML^{-1}T^{-1}$ expressed in Pa.s. The tensor $\boldsymbol{I}$ is the unit tensor; using indices, it corresponds to the Kronecker symbol $\delta_{ij}$ defined by the relation

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases} \tag{1.69}$$

In an incompressible fluid, pressure is the scalar field resulting from the imposition of the continuity constraint. For the incompressible viscous fluid, the continuity Eq. (1.50) is equivalent to $tr(\boldsymbol{d}) = 0$, with $tr$ being the trace operator of a tensor defined as the sum of its diagonal elements. The incompressible perfect fluid is obtained for a vanishing viscosity in (1.67):

$$\boldsymbol{\sigma} = -p\,\boldsymbol{I}. \tag{1.70}$$

In this last case, the stress tensor is spherical, because only its diagonal elements are non zero.

The incompressible fluid is a particular case from the thermodynamics point of view, as $\rho$ is supposed to be a constant. The specific heat capacity is therefore unique and will be denoted by $c$. Furthermore we assume that the relation between the internal energy $u$ and $c$ is

$$\frac{du}{dT} = c(T) .$$

This relation can be integrated if the heat capacity is considered as a constant, and with an appropriate choice of the integration constants, we get

$$u = c\,T. \tag{1.71}$$

A last constitutive equation is still to be considered to close the system of equations to be solved, namely the heat flux. For the vast majority of Newtonian fluids, experimental data show that there exists a linear relation between the heat flux and the temperature gradient. This is given by Fourier's law

$$\boldsymbol{q} = -k(T)\,\boldsymbol{grad}\,T , \tag{1.72}$$

$k$ being the coefficient of thermal conductivity, which is, most often, only a function of temperature. Its SI units are W m$^{-1}$ K$^{-1}$. For air considered as an ideal gas, $k = 0.0262$ W m$^{-1}$K$^{-1}$.

## 1.8  Incompressible Navier–Stokes Equations

We suppose that the flow is isothermal with $T = cnst$. The principle of conservation of energy is satisfied. Inserting the constitutive equation (1.67) in the Eq. (1.58), we generate the incompressible Navier–Stokes equations or more precisely the Navier–Stokes equations for the incompressible fluid

$$\text{div } \boldsymbol{v} = 0 \tag{1.73}$$

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\boldsymbol{\nabla} p + \mu \triangle \boldsymbol{v} + \rho \boldsymbol{b} \,. \tag{1.74}$$

Equation (1.74) is a non-linear second-order partial differential equation. It states that acceleration is produced by the actions of the pressure gradient, the viscous forces, and the body forces. Note that the presence of the non-linearity in the acceleration term, namely the advective acceleration, renders the search for closed form or analytical solutions extremely dreadful. This partial differential equation requires initial and boundary conditions to be integrated and solved.

## 1.9  Boundary and Initial Conditions

### 1.9.1  No Slip Wall

A viscous fluid in contact with a rigid wall will adhere to the wall due to the effects of viscosity. The no-slip condition can therefore be written as

$$\boldsymbol{v}_{fluid} = \boldsymbol{v}_{wall} \,. \tag{1.75}$$

### 1.9.2  Interface

An interface occurs when two immiscible fluids are in direct contact. For a two-dimensional flow, the interface is a planar curvy line, while in the three-dimensional case, the interface is a surface. Examples are provided by coating flows where multilayer materials are produced in the polymeric industrial applications.

Consider Fig. 1.10. The interface condition expresses the mechanical equilibrium of the contact forces applied by fluids I and II on either side of the interface. We have

$$\boldsymbol{t}_{\mathrm{I}} + \boldsymbol{t}_{\mathrm{II}} = \boldsymbol{0} \,, \tag{1.76}$$

with definition (1.53) for $\boldsymbol{t}$. Taking the equality $\boldsymbol{n}_{\mathrm{I}} = -\boldsymbol{n}_{\mathrm{II}}$ into account, one obtains
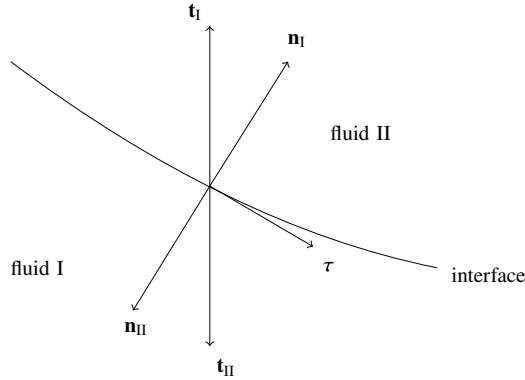
**Fig. 1.10**  Interface description

$$\sigma_{\mathrm{I}} \boldsymbol{n}_{\mathrm{I}} = \sigma_{\mathrm{II}} \boldsymbol{n}_{\mathrm{I}} . \tag{1.77}$$

The interface conditions are obtained by projecting Eq. (1.77) onto the normal $\boldsymbol{n}_{\mathrm{I}}$ and the tangent vector $\boldsymbol{\tau}$ giving

$$\sigma_{\mathrm{I}} \boldsymbol{n}_{\mathrm{I}} \cdot \boldsymbol{n}_{\mathrm{I}} = \sigma_{\mathrm{II}} \boldsymbol{n}_{\mathrm{I}} \cdot \boldsymbol{n}_{\mathrm{I}} , \tag{1.78}$$
$$\sigma_{\mathrm{I}} \boldsymbol{n}_{\mathrm{I}} \cdot \boldsymbol{\tau} = \sigma_{\mathrm{II}} \boldsymbol{n}_{\mathrm{I}} \cdot \boldsymbol{\tau} . \tag{1.79}$$

### *1.9.3   Laminar Free Surface*

An interesting particular case of an interface is that of a free surface (cf. Fig. 1.11) where a laminar viscous fluid is in contact with an inviscid fluid: e.g. air, assumed to be at rest.   Taking $\boldsymbol{n}$, the outgoing normal vector from the viscous fluid, as a reference, we have here $\sigma_{\mathrm{II}} = -p_{invis} \boldsymbol{I}$, where $p_{invis}$ is the pressure in the perfect (inviscid) fluid. The fluid index I is omitted for simplicity. Conditions (1.78) and (1.79) are now written as

$$\sigma \boldsymbol{n} \cdot \boldsymbol{n} = -p_{invis}(\boldsymbol{n} \cdot \boldsymbol{n}) = -p_{invis} , \tag{1.80}$$
$$\sigma \boldsymbol{n} \cdot \boldsymbol{\tau} = \boldsymbol{0} . \tag{1.81}$$

The latter relation is consistent with the fact that an inviscid fluid is incapable to sustain a shear force.

Free surface conditions imply that we know the form of the surface for their application. However, the form of the surface is itself part of the solution of the problem at hand. We find that free surface problems constitute one of the major difficulties in fluid mechanics as they are intrinsically non-linear. In some cases,
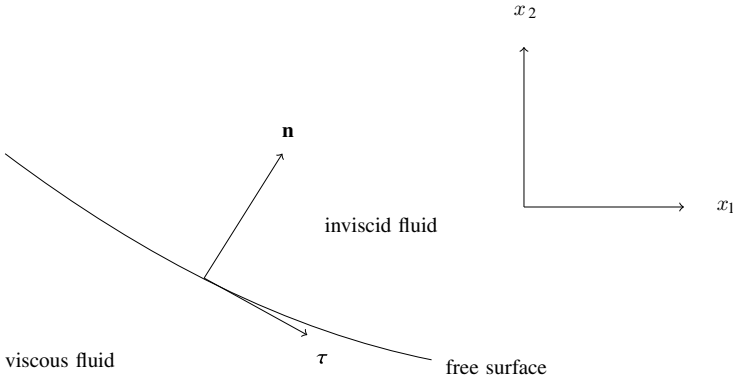
**Fig. 1.11** Free surface

closed form solutions are sought in Lagrangian formulation. The initial geometrical domain is prescribed and the solution method tracks the trajectories of fluid particles. This procedure is carried out in the breaking of a dam problem as described in Chap. 12 of the monograph by Stoker [100].

For certain fluids, condition (1.80) needs to be extended to take into account surface tension. Then we have

$$\boldsymbol{\sigma} \boldsymbol{n} \cdot \boldsymbol{n} = -p_{invis} + \gamma \left( \frac{1}{R_1} + \frac{1}{R_2} \right) , \tag{1.82}$$

in which $R_1$ and $R_2$ are the principal radii of curvature of the surface, and $\gamma$ is the viscous-inviscid surface tension coefficient expressed in N/m. The newton N is the force that gives to a mass of 1 kg an acceleration of 1 m/s. For example, for the water-air interface, we have $\gamma = 0.072$ N/m. The quantity inside the parentheses multiplying the surface tension coefficient in Eq. (1.82) is equal to $2\Xi$, where $\Xi$ is the average curvature of the surface. Its sign depends on the concavity $(-)$ or the convexity $(+)$ of the surface.

In practice, we generally limit the study to a part of the space occupied by the fluid. In this case, it is necessary to add the conditions on the entry section, where the velocity vector is typically prescribed, and the exit surface, where contact forces are usually imposed. The latter are most often taken to be zero, which corresponds to a situation where the fluid is allowed to exit at its own speed.

For the case of a transient problem, the initial conditions are the velocities, which are often zero at the start.

### 1.9.4  Perfect Fluid

As viscosity plays no role here, the fluid can slip along a wall. The adherence condition no longer applies. We impose that the normal component of the fluid velocity be zero with respect to the wall with which it is in contact. The slip condition is written as

$$\boldsymbol{v}_{fluid} \cdot \boldsymbol{n} = \boldsymbol{v}_{wall} \cdot \boldsymbol{n} \ . \tag{1.83}$$

Similarly, we impose the value of the normal component of the fluid velocity for the entry section and the pressure on the exit section. For transient flows, we proceed as for viscous flows.

Finally, in aerodynamics (external flows, for example, the flow around a wing profile or an airfoil), we very often find conditions to impose on an immaterial boundary (which may be at infinity). The typical example is that of a finite obstacle (like a slender body) placed in an unconfined flow. In this case we impose the condition that the flow is uniform at infinity.

## 1.10  Thermodynamics Considerations and Incompressibility

The speed of sound in ambient air at temperature 293 K is of the order of $340 \, \mathrm{ms}^{-1}$, whereas in water, it is $1500 \, \mathrm{ms}^{-1}$. By definition, the speed of sound is the variation of pressure with respect to density at fixed entropy $s$

$$a^2 = \left. \frac{\partial p}{\partial \rho} \right|_s . \tag{1.84}$$

As $a^2$ is obtained by the limit

$$a^2 = \lim_{\Delta \rho \to 0} \frac{\Delta p}{\Delta \rho} , \tag{1.85}$$

for an incompressible fluid, $\Delta \rho = 0$ and then the speed of sound is infinite. Therefore the question is raised: "How do we define the concept of an incompressible fluid?".

### 1.10.1  Compressible Fluid and Compressible Navier–Stokes Equations

To examine this matter, let us consider the Navier–Stokes equations for the compressible fluid in detail. The mass conservation equation is given by (1.48). To produce the

Navier–Stokes equations, we need the constitutive equation for compressible viscous
fluids

$$\boldsymbol{\sigma} = -p\,\boldsymbol{I} + \lambda(\rho, T)\,tr(\boldsymbol{d})\boldsymbol{I} + 2\,\mu(\rho, T)\,\boldsymbol{d} \ , \tag{1.86}$$

where $\lambda$ is the volume viscosity. The coefficients $\lambda$ and $\mu$ depend on the density $\rho$
and the temperature $T$. In the compressible case, the pressure is a thermodynamic
variable such that $p = p(\rho, T)$. If we assume that the fluid internal energy does not
depend on density, but only on temperature, we are in the framework of the ideal
gas that satisfies the Boyle–Mariotte law "at constant temperature, the product of the
pressure $p$ and the volume $V$ is constant". This law allows us to write the pressure
state equation

$$p = \rho RT \ , \tag{1.87}$$

that is a scalar constitutive equation for the pressure. $R$ is the ideal gas constant
expressed in J kg$^{-1}$K$^{-1}$. For ambient air, $R = 287$ J kg$^{-1}$K$^{-1}$.

Introducing the constitutive equation (1.86) in the momentum equation (1.58), we
obtain the Navier–Stokes equations for the compressible fluid

$$\rho\,\frac{D\boldsymbol{v}}{Dt} = -\boldsymbol{\nabla}\,p + \boldsymbol{\nabla}\,(\lambda\,tr\boldsymbol{d}) + \mathrm{div}\,(2\mu\,\boldsymbol{d}) + \rho\boldsymbol{b} \ . \tag{1.88}$$

The treatment of the energy equation (1.65) requires a few more concepts. Assuming
that the fluid is an ideal gas, its internal energy $u$ and its mass enthalpy $h$ are defined
as

$$h = u + \frac{p}{\rho} \ . \tag{1.89}$$

The variables $h$ and $u$ depend only on the temperature. The specific heat capacities
at constant volume and pressure are such that

$$du = c_v(T)dT \ , \quad dh = c_p(T)dT \ , \tag{1.90}$$

where $c_v$ and $c_p$ are the specific heat capacities at constant volume and constant
pressure, respectively. Taking into account the state equations (1.87) and (1.89), one
gets

$$dh = du + R\,dT \ , \tag{1.91}$$

from which we obtain

$$R = c_p(T) - c_v(T) \ . \tag{1.92}$$

For air considered as an ideal gas, $c_p = 1006$ J kg$^{-1}$ K$^{-1}$. If we assume that the
heat capacities are constant in the considered temperature range, we can write, within
an additive constant,

$$u = c_v \, T \; , \tag{1.93}$$

$$h = c_p \, T \; , \tag{1.94}$$

$$s = c_v \log p - c_p \log \rho \; , \tag{1.95}$$

where $s$ is the entropy. If the flow is isentropic (same entropy), we have

$$\frac{p}{\rho^\gamma} = \text{cnst} \; , \tag{1.96}$$

with the definition of the heat capacity ratio:

$$\gamma = \frac{c_p}{c_v} \; . \tag{1.97}$$

In the special case of an ideal gas, (1.84) takes the form, with the use of (1.96)

$$a^2 = \gamma \frac{p}{\rho} = \gamma \, R T \; . \tag{1.98}$$

With the help of (1.93), the energy equation (1.65) becomes

$$\rho \, c_v \, \frac{DT}{Dt} = \boldsymbol{\sigma} : \nabla \boldsymbol{v} - \operatorname{div} \boldsymbol{q} + r \; . \tag{1.99}$$

The expression $\boldsymbol{\sigma} : \nabla \boldsymbol{v}$ may be written successively

$$\sigma_{ij} \frac{\partial v_i}{\partial x_j} = \sigma_{ij} \, d_{ij} = -p \, \delta_{ij} \, d_{ij} + \lambda \, d_{kk} \, d_{ij} \, \delta_{ij} + 2\mu (d_{ij})^2$$

$$= -p \, d_{ii} + \lambda (d_{ii})^2 + 2\mu (d_{ij})^2$$

or

$$\boldsymbol{\sigma} : \boldsymbol{d} = -p \operatorname{tr} \boldsymbol{d} + \lambda (\operatorname{tr} \boldsymbol{d})^2 + 2\mu (\boldsymbol{d} : \boldsymbol{d}) \; . \tag{1.100}$$

To obtain the left hand side of Eq. (1.100), we have used the property stating that the trace of the inner product of a symmetric tensor $\boldsymbol{\sigma}^T$ with the antisymmetric part of the tensor $\nabla \boldsymbol{v}$, which is the rotation rate tensor, vanishes. From the mass conservation equation (1.48), we have

$$\operatorname{tr} \boldsymbol{d} = -\frac{1}{\rho} \frac{D\rho}{Dt} \; .$$

Therefore Eq. (1.99) yields

$$\rho c_v \frac{DT}{Dt} = \frac{p}{\rho} \frac{D\rho}{Dt} + \lambda \, (\operatorname{tr} \boldsymbol{d})^2 + 2\mu \, \boldsymbol{d} : \boldsymbol{d} - \operatorname{div} \boldsymbol{q} + \mathrm{r} . \tag{1.101}$$

Using the state equation (1.87), (1.101) is transformed as

$$\rho c_v \frac{DT}{Dt} = \frac{Dp}{Dt} - \rho R \frac{DT}{Dt} + \lambda (\text{tr}\, \boldsymbol{d})^2 + 2\mu\, \boldsymbol{d} : \boldsymbol{d} - \text{div}\boldsymbol{q} + \text{r}\,. \qquad (1.102)$$

Let us now modify the energy equation (1.102) by taking the Fourier law (1.72) and the relation (1.92) into account

$$\rho c_p \frac{DT}{Dt} = \frac{Dp}{Dt} + \lambda (\text{tr}\, \boldsymbol{d})^2 + 2\mu\, \boldsymbol{d} : \boldsymbol{d} + \text{div}(\text{k}\boldsymbol{\nabla}\text{T}) + \text{r}\,. \qquad (1.103)$$

The compressible Navier–Stokes equations comprise the mass conservation equation (1.48), the momentum equations (1.88), the energy equation (1.103) and the state equation (1.87). To investigate the limit case of the compressible set of equations to the incompressible formulation, we need to write the dimensionless form of the Navier–Stokes equations and let the Mach number go to zero. This analysis is deferred till the next chapter.

### 1.10.2  Incompressibility

From the mass conservation equation (1.48), we deduce

$$\text{div}\, \boldsymbol{v} = -\frac{1}{\rho}\frac{D\rho}{Dt}\,, \qquad (1.104)$$

from which one concludes that incompressibility is obtained when the density does not vary with time along a trajectory, i.e.

$$\frac{d\rho}{dt} = 0\,. \qquad (1.105)$$

Let us consider a compressible fluid whose pressure depends on the density and the temperature or whose density is a function of the pressure and the temperature. It is possible to expand the previous relationship

$$\frac{d\rho}{dt} = \left(\frac{\partial\rho}{\partial p}\right)_T \frac{dp}{dt} + \left(\frac{\partial\rho}{\partial T}\right)_p \frac{dT}{dt}\,, \qquad (1.106)$$

giving via (1.104)

$$\text{div}\, \boldsymbol{v} + \frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_T \frac{dp}{dt} + \frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p \frac{dT}{dt} = 0\,. \qquad (1.107)$$

The coefficient of isothermal compressibility $\chi_T$ is defined by

$$\chi_T = \frac{1}{\rho} \left( \frac{\partial \rho}{\partial p} \right)_T . \tag{1.108}$$

For water, one has $\chi_T = 5.10^{-10}\,\text{Pa}^{-1}$. For air, $\chi_T = 10^{-5}\,\text{Pa}^{-1}$. The other coefficient is the isobaric expansion

$$\alpha = \frac{1}{\rho} \left( \frac{\partial \rho}{\partial T} \right)_p . \tag{1.109}$$

The compressibility coefficient for liquids is very small. Batchelor [10] notices that water density increases by $\frac{1}{2}\%$ when the pressure augments from 1 to 100 bars, at constant ambient temperature (1 bar $= 10^5$ Pa). This large strength to compression of liquids is one of its main characteristics and from the viewpoint of fluid dynamics, it is a reason why we can consider them as incompressible with a great precision.

### 1.10.3  Boussinesq Approximation for Weakly Dilatable Fluids

Let us examine the non isothermal flow of a weakly compressible fluid. We are able to consider thermal effects for an incompressible fluid, like e.g. in the case of natural convection. This is achieved by Boussinesq approximation [17] that assumes a constant density in the full set of equations except in the body force term, where temperature differences in the fluid induce an Archimedes' thrust generated by density differences. This modeling is used in materials science, for example to study the convection currents in molten metal (Czochralski process for silicium production), im molten glass (floated glass process), etc. It is also widely used in geophysical flows and in meteorology.

Introducing the inner product

$$\boldsymbol{d} : \boldsymbol{d} = d_{ij} d_{ij}, \tag{1.110}$$

equation (1.99) becomes

$$\rho c \frac{DT}{Dt} = 2\mu\, \boldsymbol{d} : \boldsymbol{d} - \operatorname{div} \boldsymbol{q} + r . \tag{1.111}$$

Using the state equation (1.72), Eq. (1.111) is transformed

$$\rho c \frac{DT}{Dt} = 2\mu\, \boldsymbol{d} : \boldsymbol{d} + \operatorname{div} (k\, \boldsymbol{\nabla} T) + r . \tag{1.112}$$

We notice that the first term in the right hand side is the power dissipated by the viscous effects that in most cases, are negligible. However for very viscous fluids like molten glass or the terrestrial magma, these effects must be taken care of. The

temperature equation (1.112) is simplified for constant $k$. We have

$$\rho c \frac{DT}{Dt} = 2\mu\,\boldsymbol{d} : \boldsymbol{d} + k\,\triangle\,T + r\ . \tag{1.113}$$

In the Boussinesq approximation the body force is gravity and hence $\boldsymbol{b} = \boldsymbol{g}$. Furthermore for the weakly dilatable fluid, the state equation for density is linearized

$$\rho = \rho_0(1 - \alpha(T - T_0)), \tag{1.114}$$

with $\alpha$ the coefficient of volumic expansion expressed in $K^{-1}$. The quantity $\rho_0$ is the density at the reference temperature $T_0$. One observes that when the fluid gets hotter, it is lighter as $\rho$ decreases; when it is cooling, it is heavier as $\rho$ increases. Eventually, the Boussinesq approximation supposes that the fluid is incompressible with all its material characteristics constant, except in the gravity term. The set of Boussinesq equations reads

$$\text{div}\,\boldsymbol{v} = 0, \tag{1.115}$$

$$\rho_0 \frac{D\boldsymbol{v}}{Dt} = -\boldsymbol{\nabla} p\,+\,\mu\triangle\,\boldsymbol{v}\,+\,\rho_0((1 - \alpha(T - T_0))\boldsymbol{g}, \tag{1.116}$$

$$\rho_0 c \frac{DT}{Dt} = 2\mu\,\boldsymbol{d} : \boldsymbol{d} + k\,\triangle\,T + r. \tag{1.117}$$

In this system of equations, the velocity-temperature coupling is obtained by the Archimedes' thrust in the body force term, also called the buoyancy term.

It is customary to modify the pressure term in (1.116) with the following expression

$$\tilde{p} = p + \rho_0\boldsymbol{g}\cdot\boldsymbol{x}\ , \tag{1.118}$$

where the hydrostatic pressure is incorporated. The corresponding Navier–Stokes equation is

$$\rho_0 \frac{D\boldsymbol{v}}{Dt} = -\boldsymbol{\nabla}\tilde{p}\,+\,\mu\triangle\,\boldsymbol{v} - \rho_0\alpha(T - T_0)\boldsymbol{g}\ . \tag{1.119}$$

## 1.11  The Method of Control Volume

Instead of tackling the fluid mechanics partial differential equations, as will be the usual approach in this monograph, a large number of fluid problems can be solved using the integral formulation of mass and momentum conservation laws expressed on control volumes. The control volume is denoted $V$ with surface $S$.

The mass conservation law (1.50) yields

$$\int_V \text{div}\,\boldsymbol{v}\,dV = \int_S \boldsymbol{v}\cdot\boldsymbol{n}\,dS\ . \tag{1.120}$$

The momentum conservation equation (1.52) reads

$$\frac{d}{dt} \int_V \rho(\boldsymbol{x}, t) \boldsymbol{v}(\boldsymbol{x}, t) \, dV = \int_V \rho(\boldsymbol{x}, t) \boldsymbol{b}(\boldsymbol{x}, t) \, dV + \int_S \boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n}) \, dS. \quad (1.121)$$

For an incompressible fluid we have

$$\frac{d}{dt} \int_V \rho(\boldsymbol{x}, t) \boldsymbol{v}(\boldsymbol{x}, t) \, dV = \rho \int_V \frac{D\boldsymbol{v}}{Dt} \, dV. \quad (1.122)$$

The material derivative (1.23) can be recast in the following form

$$\frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_j} \, v_j = \frac{\partial v_i}{\partial t} + \frac{\partial v_i v_j}{\partial x_j}. \quad (1.123)$$

The second term in the right-hand side of (1.123) is the divergence of the tensor $\boldsymbol{V} = \boldsymbol{v} \otimes \boldsymbol{v}$. Using the divergence theorem one obtains

$$\int_V \boldsymbol{div}\,\boldsymbol{V} \, dV = \int_S \boldsymbol{V}\boldsymbol{n} \, dS = \int_S (\boldsymbol{v} \cdot \boldsymbol{n}) \boldsymbol{v} \, dS. \quad (1.124)$$

Finally, the relevant equation for the control volume method is

$$\rho \int_V \frac{\partial \boldsymbol{v}}{\partial t} dV + \rho \int_S (\boldsymbol{v} \cdot \boldsymbol{n}) \boldsymbol{v} \, dS = \int_V \rho \boldsymbol{b} dV + \int_S \boldsymbol{t} dS. \quad (1.125)$$

## Exercises

**1.1** Evaluate by the method of control volume the shearing force $\boldsymbol{F}$ exerted by the viscous fluid on the plate of length $L$ shown in Fig. 1.12 using the velocity profiles given in the inflow and outflow sections. It is assumed that

- the inflow velocity is uniform of value $U$ and the outflow profile is linear over the height $h$
- the pressure is uniform in the fluid
- the body force is neglected.

The control volume $ABCD$ in dashed lines has height $H$.

**1.2** Show that the velocity field $v_i = A x_i / r^3$, where $x_i x_i = r^2$ and $A$ is an arbitrary constant, satisfies the conservation of mass equation for an incompressible fluid.
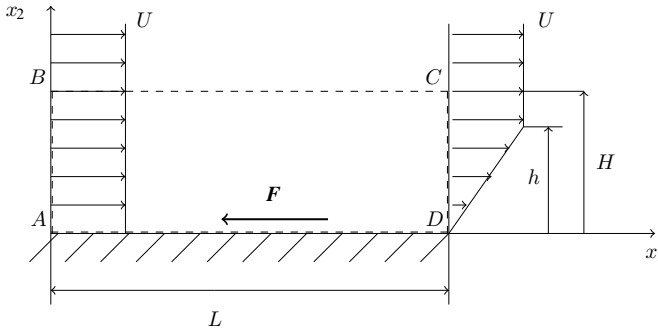
**1.3** Show that the flow given by the velocity field

**Fig. 1.12** Simplified flow on a flat plate

$$v_r = \frac{(1 - r^2)\cos\theta}{r^2}$$

$$v_\theta = \frac{(1 + r^2)\sin\theta}{r^2}$$

$$v_z = 0$$

satisfies the incompressibility equation.

## 1.4  Natural Convection Between Two Differentially Heated Vertical Parallel Walls

Let us consider the steady state, two-dimensional slow flow of a viscous incompressible fluid subjected to a variable temperature field. The fluid flows between two infinite parallel walls at different temperatures, cf. Fig. 1.13, such that $T_2 > T_1$. The heat transfer is by conduction only.

The problem is solved with the Boussinesq approximation. The viscous dissipation and the volume heat production are neglected.

Obtain the simplified temperature and momentum equations and compute the temperature and velocity fields.

## 1.5  Free Jet

A viscous fluid in laminar flow exits from the cylindrical pipe of a die as it is shown in Fig. 1.14. In the circular pipe the fluid velocity is that of a Poiseuille flow

$$v_z = v_{max}\left(1 - \frac{r^2}{R_0^2}\right). \tag{1.126}$$

This relation will be demonstrated in Sect. 3.2.2. The maximum velocity $v_{max}$ is reached on the pipe symmetry axis.

The problem is solved in two stages.

- After extrusion, the velocity profile is homogeneous. This means that it is assumed there is no diffusion with the exterior air at the pipe exit and no mixing. Compute
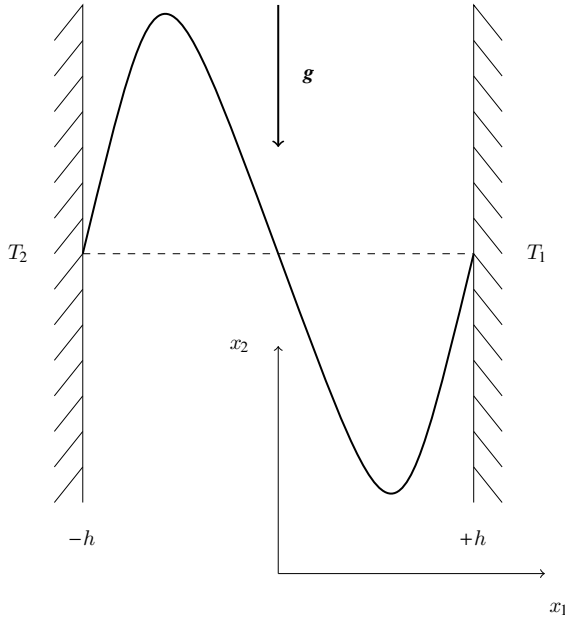
**Fig. 1.13** Natural convection in an infinite plane channel. The solid line is the velocity profile
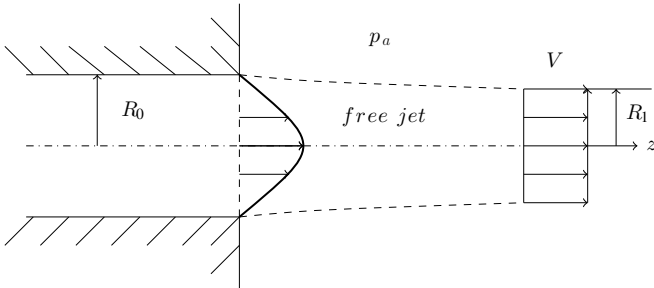


**Fig. 1.14** Free jet

the velocity ratio $V/v_{max}$ where $V$ is the flat velocity profile after extrusion. The control volume method is needed in this step.

• Evaluate the contraction coefficient

$$\beta = \frac{R_1^2}{R_0^2} \, , \tag{1.127}$$

where $R_1$ is the jet radius after extrusion.