Isotone $\mathcal{L}$-Fuzzy Formal Concept Analysis and $\mathcal{L}$-Valued Fuzzy Measures and Integrals

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Abstract. The main idea of the paper is to generalize the concept of lattice valued fuzzy measures and integrals for data from complete residuated lattice where double negation law holds and then to show their relationship to isotone $\mathcal{L}$-fuzzy concept forming operators.

Keywords: Fuzzy measure · Fuzzy integral · Formal concept analysis

1 Introduction

Once I saw a very nice presentation about fuzzy sets and its applications and basic technics where very simple but nice toy example of fuzzy measure and fuzzy integral were presented. It was my almost first meeting with such an area where I found some small intuitive connection with fuzzy Formal concept analysis (FCA) [2,3,5,6,12,13] that is my main topic for years. After that experience my scientific curiosity led me to papers that seemed to me as a newcomer as most understandable [1,7,14–17] and also very useful was to read two papers that deals with both fuzzy integrals and fuzzy FCA [4,8].

Main topic of the paper is to first connect two notions, lattice valued fuzzy measures and integrals [17] and t-norm and t-conorm fuzzy measures and integrals [16] into one "complete residuated lattice"-valued fuzzy measures and integrals, where double negation law has to be preserved. Next step is to show their connection with isotone derivation (concept forming) operators of fuzzy formal context that are the most important part of FCA.

Second section is dedicated to basics of isotone FCA based on data from complete residuated lattice. Third section is about proposition of so called one-sided fuzzy concept-forming operators and the definition of new complete residuated lattice valued fuzzy measures and integrals. All needed and some new properties are proved.

2   Isotone $\mathcal{L}$-fuzzy Formal Concept Analysis

Definition 1. *An algebra* $\mathcal{L} = \langle L, \wedge, \vee, 0, 1, \otimes, \rightarrow \rangle$ *is said to be a* complete residuated lattice *if*

1. $\langle L, \wedge, \vee, 0, 1 \rangle$ *is a complete lattice where* 0 *and* 1 *are the bottom and top elements (resp.).*
2. $\langle L, \otimes, 1 \rangle$ *is a commutative monoid.*
3. $\langle \otimes, \rightarrow \rangle$ *is an adjoint pair, i.e.* $k \otimes m \leq n$ *if and only if* $k \leq m \rightarrow n$, *for all* $k, m, n \in L$, *where* $\leq$ *is the ordering generated by* $\wedge$ *and* $\vee$.

It will be important to have $\mathcal{L}$ with double neggation law, ie. $\neg\neg k = k$ for any $k \in L$, where $\neg k = k \rightarrow 0$.

Definition 2. *A* Girard monoid *is a residuated lattice* $\mathcal{L}$ *satisfying the* law of double negation, *namely, the equality* $x = (x \rightarrow 0) \rightarrow 0$ *holds for all* $x \in L$.

This notion represents one of the several flavours in which one can find residuated lattices, it was used by Girard in his development programme for linear logics, and a study of its structure in the particular case of the unit interval can be found in [9]. Other well-known enriched versions of residuated lattices include, for instance, Heyting algebras (satisfying $x \otimes y = x \wedge y$), BL-algebras (satisfying divisibility, i.e. $x \wedge y = x \otimes (x \rightarrow y)$, and the prelinearity, i.e. $(x \rightarrow y) \vee (y \rightarrow x) = 1$), or MV-algebras (BL-algebra satisfying the law of double negation).

Definition 3. *The operator* $\oplus \colon L \times L \rightarrow L$ *is defined by*

$$a \oplus b = \neg a \rightarrow b = (a \rightarrow 0) \rightarrow b.$$

Assuming that we are working on a Girard monoid, it is not difficult to check that $\oplus$ is commutative and associative. Furthermore, the De Morgan laws between $\otimes$ and $\oplus$, and also between $\vee$ and $\wedge$, and contraposition law also hold. Hereafter, we will assume that $\mathcal{L}$ is a Girard monoid.

Let X be any final set. The ordered set (complete lattice) of all $\mathcal{L}$-sets over X will be denoted by $\mathcal{L}^X$.

Definition 4. $\mathcal{L}$-*Fuzzy* Formal Context *is a triple* $\langle B, A, \mathcal{L}, r \rangle$ *where* B *is the set of* objects, A *is the set of* attributes *and* $r : B \times A \rightarrow \mathcal{L}$ *is a binary relation between objects and attributes.*

Definition 5. *Let us define two pairs of so called* isotone derivation opera-tors *between ($\mathcal{L}$-fuzzy) powerset complete lattices over the sets of objects and attributes as follows:*

1. – $\nearrow \colon \mathcal{L}^B \longleftarrow \mathcal{L}^A$ *and* $\diagup \colon \mathcal{L}^A \longleftarrow \mathcal{L}^B$.
   – $\nearrow (f)(a) = \bigvee_{b \in B}(f(b) \otimes r(b, a))$ *for any* $f \in \mathcal{L}^B$.
   – $\diagup (g)(b) = \bigwedge_{a \in A}(r(b, a) \rightarrow g(a))$ *for any* $g \in \mathcal{L}^A$.
2. – $\searchsignal \colon \mathcal{L}^B \longleftarrow \mathcal{L}^A$ *and* $\searrow \colon \mathcal{L}^A \longleftarrow \mathcal{L}^B$.
   – $\searchsignal (f)(a) = \bigwedge_{b \in B}(r(b, a) \rightarrow f(b))$ *for any* $f \in \mathcal{L}^B$.

– $\searcher (g)(b) = \bigvee_{a \in A}(g(a) \otimes r(b, a))$ *for any* $g \in \mathcal{L}^A$.

For above defined operators holds $(\swarrow, \nearrow)$ and $(\searrow, \searcher)$ form an isotone Galois connections between $\mathcal{L}^B$ and $\mathcal{L}^A$, i.e. for any $f \in \mathcal{L}^B$ and $g \in \mathcal{L}^A$ holds

$$\nearrow (f) \leq g \quad \Leftrightarrow \quad f \leq \swarrow (g) \quad \text{and} \quad g \leq \searcher (f) \quad \Leftrightarrow \quad \searrow (g) \leq f.$$

As a consequence of previous facts the compositions

– $\swarrow\nearrow$ and $\searcher\searrow$ forms closure operators on $\mathcal{L}^B$ or $\mathcal{L}^A$ respectively.
– $\nearrow\swarrow$ and $\searrow\searcher$ forms interior operator on $\mathcal{L}^A$ or $\mathcal{L}^B$ respectively.

That means that

– all compositions are monotone
– $\swarrow\nearrow$ and $\searcher\searrow$ are inflationary (i.e. $\searcher\searrow (g) \geq g$ for any $g \in \mathcal{L}^A$)
– $\nearrow\swarrow$ and $\searrow\searcher$ are deflationary (i.e. $\searrow\searcher (f) \leq f$ for any $f \in \mathcal{L}^B$).
– all compositions are idempotent (i.e. $\swarrow\nearrow\swarrow\nearrow (f) = \swarrow\nearrow (f)$ for any $f \in \mathcal{L}^B$)

Pairs $(f, g) \in \mathcal{L}^B \times \mathcal{L}^A$ such that $\nearrow (f) = g$ and $\swarrow (g) = f$ is called $\mathcal{L}$-*fuzzy formal concept* of $\mathcal{L}$-context $\langle B, A, \mathcal{L}, r \rangle$ with respect to operators $(\nearrow, \swarrow)$. f is then called *extent* and g is called *intent*. All extents will be denoted by $\text{Ext}(B, A, \mathcal{L}, r, \nearrow, \swarrow)$ and all intents by $\text{Int}(B, A, \mathcal{L}, r, \nearrow, \swarrow)$. Similarly for the other pair of operators $(\searcher, \searrow)$. Sets of all extents and intents ordered by $\mathcal{L}$-fuzzy set inclusion form complete lattices.

3 $\mathcal{L}$-fuzzy Isotone Derivation Operators and $\mathcal{L}$-fuzzy Measures and Integrals

Krajči in [10,11] defined a modification of antitone $\mathcal{L}$-fuzzy derivation operators $(\uparrow, \downarrow)$, mappings between $\mathcal{L}^B$ and $\mathcal{L}^A$ into the case of mappings between 2^B and $\mathcal{L}^A$, due to better understanding or interpretation of possible results.

So now the isotone derivation operators will be modified into one sided form:

Let $X \subseteq B$ be an arbitrary classical subset. χ_X is a characteristic function of X. Let us define a new mappings $\overline{\nearrow} : 2^B \to \mathcal{L}^A$ and $\overline{\searrow} : 2^B \to \mathcal{L}^A$

$$\overline{\nearrow}(X)(a) = \nearrow (\chi_X)(a) = \bigvee_{b \in B} \chi_X(b) \otimes r(b, a)$$

$$= \bigvee_{b \in X} 1 \otimes r(b, a) \vee \bigvee_{b \in X^c} 0 \otimes r(b, a)$$

$$= \bigvee_{b \in X} 1 \otimes r(b, a) \vee 0 = \bigvee_{b \in X} r(b, a)$$

$$\overline{\diagdown}(X)(a) = \diagdown (\chi_X)(a) = \bigwedge_{b \in B} (r(b,a) \rightarrow \chi_X(b))$$

$$= \bigwedge_{b \in X} (r(b,a) \rightarrow 1) \wedge \bigwedge_{b \in X^c} (r(b,a) \rightarrow 0)$$

$$= 1 \wedge \bigwedge_{b \in X^c} (r(b,a) \rightarrow 0) = \bigwedge_{b \in X^c} \neg r(b,a)$$

Definition 6. *Let B be an arbitrary set and $\mathcal{L}$ be the lattice with bottom 0 and top element 1. The mapping $\mu : \mathcal{P}(B) \rightarrow \mathcal{L}$ is called $\mathcal{L}$-valued fuzzy measure iff μ is monotone, $\mu(\emptyset) = 0$ and $\mu(B) = 1$.*

Before the theorem about a new measure, let us define a notion of normal $\mathcal{L}$-set and normal $\mathcal{L}$-context.

Definition 7. *Any $\mathcal{L}$-set from $\mathcal{L}^X$ is called* normal *when its maximal membership value is equal to 1 (top of $\mathcal{L}$). Let $\langle B, A, \mathcal{L}, r \rangle$ be a $\mathcal{L}$-fuzzy formal context, such that its columns $r(-, a)$ are normal $\mathcal{L}$-fuzzy sets from $\mathcal{L}^B$. Such $\mathcal{L}$-context is also called* normal.

Theorem 1. *Let $\langle B, A, \mathcal{L}, r \rangle$ be a normal $\mathcal{L}$-fuzzy formal context. Let $a \in A$ be arbitrary attribute. The following upper and lower $\mathcal{L}$-valued mappings*

– $\underline{\mu}_a$ *where for any $X \subseteq B$ is $\underline{\mu}_a(X) = \overline{\nearrow}(X)(a)$*
– $\overline{\mu}_a$ *where for any $X \subseteq B$ is $\overline{\mu}_a(X) = \overline{\diagdown}(X)(a)$.*

are $\mathcal{L}$-valued fuzzy measures.

Proof. It is well known that operators $\nearrow$ and $\diagdown$ are monotone, hence also their one-sided fuzzy form will also be monotone. Moreover

$$\overline{\mu}_a(B) = \bigwedge_{b \in B^c} \neg r(b,a) = 1 \quad \text{and} \quad \underline{\mu}_a(\emptyset) = \bigvee_{b \in \emptyset} r(b,a) = 0$$

for any $a \in A$.

What can be little questionable are the following two facts. In general $\overline{\mu}_a(\emptyset)$ need not to be equal to 0 and $\underline{\mu}_a(B)$ need not to be equal to 1. This is why the precondition of normality is here, which says that there for any $a \in A$ there exists at least one $b \in B$ such that $r(b,a) = 1$. Hence

$$\overline{\mu}_a(\emptyset) = \bigwedge_{b \in \emptyset^c} \neg r(b,a) = \bigwedge_{b \in B} \neg r(b,a) = 0 \quad \text{and} \quad \underline{\mu}_a(B) = \bigvee_{b \in B} r(b,a) = 1$$

$\square$

Before the definition of integral let us first define some auxiliary notation. Let $f \in \mathcal{L}^B$ be an arbitrary $\mathcal{L}$-set over B. Then

- $f_\alpha = \{b \in B | f(b) \geq \alpha\}$ is well known α-cut of $\mathcal{L}$-set for some $\alpha \in \mathcal{L}$
- $f^\alpha = \{b \in B | f(b) \not\leq \alpha\}$

The following definition is a generalisation of the definition of lower and upper-lattice valued fuzzy integrals from [17].

Theorem 2. *Let $\langle B, A, \mathcal{L}, r \rangle$ be the normal $\mathcal{L}$-fuzzy formal context and for each $a \in A$ there are corresponding $\underline{\mu}_a$ and $\overline{\mu}_a$ lower and upper $\mathcal{L}$-fuzzy measures. The following lower and upper $\mathcal{L}$-valued mappings defined for any $f \in \mathcal{L}^B$ and $X \subseteq B$ as follows*

$$\underline{\int}_X f d\underline{\mu}_a = \bigvee_{\alpha \in \mathcal{L}} \alpha \otimes \underline{\mu}_a(f_\alpha \cap X) \quad and \quad \overline{\int}_X f d\overline{\mu}_a = \bigwedge_{\alpha \in \mathcal{L}} \alpha \oplus \overline{\mu}_a(f^\alpha \cap X)$$

are lower and upper $\mathcal{L}$-valued fuzzy integrals.

Proof. In [14] where the definition of fuzzy integral says that it has to be idempotent homeomorphism. So we have to prove for above defined $\mathcal{L}$-valued mappings the following properties:

1. they are monotone
2. they map top $\mathcal{L}$-set from $\mathcal{L}^B$ to 1 and bottom to 0
3. for any $\gamma \in \mathcal{L}$ a constant $\mathcal{L}$-set $\overline{\gamma}$ defined as $\overline{\gamma}(b) = \gamma$ for any $b \in B$ holds:
   - $\underline{\int}_B \overline{\gamma} d\underline{\mu}_a = \gamma$
   - $\overline{\int}_B \overline{\gamma} d\overline{\mu}_a = \gamma$

1. The monotonicity follows directly from the definition of integrals.
2. From interior and closure operator properties of $\nearrow$ and $\swarrow$ from the Sect. 2 and the normality precondition the following holds.

$$\underline{\int}_B \chi_B d\underline{\mu}_a = \bigvee_{\alpha \in \mathcal{L}} \alpha \otimes \underline{\mu}_a(B) = \bigvee_{\alpha \in \mathcal{L}} \alpha \otimes 1 = 1 \otimes 1 = 1$$

$$\underline{\int}_B \chi_\emptyset d\underline{\mu}_a = \bigvee_{\alpha \in \mathcal{L}} \alpha \otimes \underline{\mu}_a(\emptyset) = \bigvee_{\alpha \in \mathcal{L}} \alpha \otimes 0 = 0$$

$$\overline{\int}_B \chi_B d\overline{\mu}_a(B) = \bigwedge_{\alpha \in LL} \alpha \oplus \overline{\mu}_a(B) = \bigwedge_{\alpha \in \mathcal{L}} \alpha \oplus 1 = 1$$

$$\overline{\int}_B \chi_B d\overline{\mu}_a(\emptyset) = \bigwedge_{\alpha \in LL} \alpha \oplus \overline{\mu}_a(\emptyset) = \bigwedge_{\alpha \in \mathcal{L}} \alpha \oplus 0 = 0 \oplus 0 = 0$$

3.

$$\int_{-B} \overline{\gamma} d\underline{\mu}_a = \bigvee_{\alpha \in \mathcal{L}} \alpha \otimes \underline{\mu}_a (B \cap \overline{\gamma}_\alpha)$$

$$= \bigvee_{\alpha \in \mathcal{L}; \alpha \leq \gamma} \alpha \otimes \underline{\mu}_a (B \cap \overline{\gamma}_\alpha) \vee \bigvee_{\alpha \in \mathcal{L}; \alpha \nleq \gamma} \alpha \otimes \underline{\mu}_a (B \cap \overline{\gamma}_\alpha)$$

$$= \bigvee_{\alpha \in \mathcal{L}; \alpha \leq \gamma} \alpha \otimes \underline{\mu}_a (B) \vee \bigvee_{\alpha \in \mathcal{L}; \alpha \nleq \gamma} \alpha \otimes \underline{\mu}_a (\emptyset)$$

$$= \gamma \otimes \underline{\mu}_a (B) \vee \bigvee_{\alpha \in \mathcal{L}; \alpha \nleq \gamma} \alpha \otimes 0$$

$$= (\gamma \otimes 1) \vee 0 = \gamma$$

$$\overline{\int_B} \overline{\gamma} d\overline{\mu}_a = \bigwedge_{\alpha \in \mathcal{L}} \alpha \oplus \overline{\mu}_a (B \cap \overline{\gamma}^\alpha)$$

$$= \bigwedge_{\alpha \in \mathcal{L}; \alpha \leq \gamma} \alpha \oplus \overline{\mu}_a (B \cap \overline{\gamma}^\alpha) \wedge \bigwedge_{\alpha \in \mathcal{L}; \alpha \nleq \gamma} \alpha \oplus \overline{\mu}_a (B \cap \overline{\gamma}^\alpha)$$

$$= \bigwedge_{\alpha \in \mathcal{L}; \alpha \leq \gamma} \alpha \oplus \overline{\mu}_a (\emptyset) \wedge \bigwedge_{\alpha \in \mathcal{L}; \alpha \nleq \gamma} \alpha \oplus \overline{\mu}_a (B)$$

$$= \gamma \oplus \overline{\mu}_a (\emptyset) \wedge \bigwedge_{\alpha \in \mathcal{L}; \alpha \nleq \gamma} \alpha \oplus 1$$

$$= \gamma \wedge 1 = \gamma$$

$\square$

The following theorem will show a relationship between new integrals and concept forming operators.

Theorem 3. *Let $\langle B, A, \mathcal{L}, r \rangle$ be a $\mathcal{L}$-context and $\{\underline{\mu}_a | a \in A\}$ and $\{\overline{\mu}_a | a \in A\}$ be its collections of measures. Then for corresponding integrals holds:*

$$\int_{-B} f d\underline{\mu}_a = \nearrow (f)(a) \quad and \quad \overline{\int_B} f d\overline{\mu}_a = \searrow (f)(a).$$

Proof. Let f be an arbitrary from $\mathcal{L}^B$.

$$\int_{-B} f d\underline{\mu}_a = \bigvee_{\alpha \in \mathcal{L}} \alpha \otimes \underline{\mu}_a (B \cap f_\alpha)$$

$$= \bigvee_{\alpha \in \mathcal{L}} \alpha \otimes \bigvee_{b \in f_\alpha} r(b, a)$$

$$= \bigvee_{\alpha \in \mathcal{L}} \alpha \otimes \bigvee_{b \in B; f(b) \geq \alpha} r(b, a)$$

$$= \bigvee_{\alpha \in \mathcal{L}} \bigvee_{b \in B; f(b) \geq \alpha} \alpha \otimes r(b, a)$$

$$= \bigvee_{b \in B} \bigvee_{\alpha \in \mathcal{L}; f(b) \geq \alpha} \alpha \otimes r(b, a)$$

$$= \bigvee_{b \in B} f(b) \otimes r(b, a) = \nearrow (f)(a)$$

$$\overline{\int}_B f d\overline{\mu}_a = \bigwedge_{\alpha \in \mathcal{L}} \alpha \oplus \overline{\mu}_a(B \cap f^\alpha)$$

$$= \bigwedge_{\alpha \in \mathcal{L}} \neg\alpha \rightarrow \bigwedge_{b \in (f^\alpha)^c} \neg r(b, a)$$

$$= \bigwedge_{\alpha \in \mathcal{L}} \bigwedge_{b \in B; f(b) \leq \alpha} \neg\alpha \rightarrow \neg r(b, a)$$

$$= \bigwedge_{\alpha \in \mathcal{L}} \bigwedge_{b \in B; f(b) \leq \alpha} r(b, a) \rightarrow \alpha$$

$$= \bigwedge_{b \in B} \bigwedge_{\alpha \in \mathcal{L}; f(b) \leq \alpha} r(b, a) \rightarrow \alpha$$

$$= \bigwedge_{b \in B} r(b, a) \rightarrow f(b) = \nwarrow (f)(a)$$

Theorem 4. *Let* $\langle B, A, \mathcal{L}, r \rangle$ *be a* $\mathcal{L}$*-context and* $\{\underline{\mu}_a | a \in A\}$ *and* $\{\overline{\mu}_a | a \in A\}$ *be its collections of measures. Then for corresponding integrals holds:*

$$\underline{\int}_X f d\underline{\mu}_a = \nearrow (f \cap \chi_X)(a) \quad and \quad \overline{\int}_X f d\overline{\mu}_a = \nwarrow (f \cap \chi_X)(a)$$

Proof. Let f be an arbitrary from $\mathcal{L}^B$ and $X \subseteq B$.

$$\underline{\int}_X f d\underline{\mu}_a = \bigvee_{\alpha \in \mathcal{L}} \alpha \otimes \underline{\mu}_a(f_\alpha \cap X)$$

$$= \bigvee_{\alpha \in \mathcal{L}} \alpha \otimes \bigvee_{b \in f_\alpha \cap X} r(b, a)$$

$$= \bigvee_{\alpha \in \mathcal{L}} \bigvee_{b \in f_\alpha \cap X} \alpha \otimes r(b, a)$$

$$= \bigvee_{\alpha \in \mathcal{L}} \bigvee_{b \in X, f(b) \geq \alpha} \alpha \otimes r(b, a)$$

$$= \bigvee_{b \in X} \bigvee_{\alpha \in \mathcal{L}, \alpha \leq f(b)} \alpha \otimes r(b, a)$$

$$= \bigvee_{b \in X} f(b) \otimes r(b, a)$$

$$= \bigvee_{b \in B} (f(b) \wedge \chi_X(b)) \otimes r(b, a)$$

$$= \nearrow (f \cap \chi_X)(a)$$

$$\overline{\int_X} f d\overline{\mu}_a = \bigwedge_{\alpha \in \mathcal{L}} \alpha \oplus \overline{\mu}_a(f^\alpha \cap X)$$

$$= \bigwedge_{\alpha \in \mathcal{L}} \alpha \oplus \bigwedge_{b \in (f^\alpha \cap X)^c} \neg r(b, a)$$

$$= \bigwedge_{\alpha \in \mathcal{L}} \alpha \oplus \bigwedge_{b \in (f^\alpha)^c \cup X^c} \neg r(b, a)$$

$$= \left(\bigwedge_{\alpha \in \mathcal{L}} \alpha \oplus \bigwedge_{b \in (f^\alpha)^c} \neg r(b, a) \right) \wedge \left(\bigwedge_{\alpha \in \mathcal{L}} \alpha \oplus \bigwedge_{b \in (X)^c} \neg r(b, a) \right)$$

$$= \left(\bigwedge_{\alpha \in \mathcal{L}} \neg \alpha \rightarrow \bigwedge_{f(b) \leq \alpha} \neg r(b, a) \right) \wedge \left(\bigwedge_{\alpha \in \mathcal{L}} \neg \alpha \rightarrow \bigwedge_{b \in (X)^c} \neg r(b, a) \right)$$

$$= \left(\bigwedge_{\alpha \in \mathcal{L}} \bigwedge_{f(b) \leq \alpha} r(b, a) \rightarrow \alpha \right) \wedge \left(\bigwedge_{\alpha \in \mathcal{L}} \bigwedge_{b \in (X)^c} r(b, a) \rightarrow \alpha \right)$$

$$= \left(\bigwedge_{b \in B} \bigwedge_{\alpha \in \mathcal{L}; \alpha \geq f(b)} r(b, a) \rightarrow \alpha \right) \wedge \left(\bigwedge_{b \in (X)^c} \bigwedge_{\alpha \in \mathcal{L}} r(b, a) \rightarrow \alpha \right)$$

$$= \left(\bigwedge_{b \in B} r(b, a) \rightarrow f(b) \right) \wedge \left(\left(\bigwedge_{b \in (X)^c} r(b, a) \rightarrow 0 \right) \wedge \left(\bigwedge_{b \in X} r(b, a) \rightarrow 1 \right) \right)$$

$$= \searrow (f)(a) \wedge \left(\bigwedge_{b \in B} (r(b, a) \rightarrow \chi_X(b)) \right)$$

$$= \searrow (f)(a) \wedge \searrow (\chi_X)(a) = \searrow (f \cap \chi_X)(a)$$

$\square$

Theorem 5. *Let $X \subseteq B$ and $f \in \mathcal{L}^B$ be arbitrary. Then*

$$\overline{\int_X} f d\mu_a = \neg \underline{\int_{X^c}} \neg f d\mu_a$$

Proof.

$$\overline{\int}_X f d\mu_a = \searrow (f \cap \chi_X)(a)$$

$$= \bigwedge_{b \in B} (r(b,a) \rightarrow (f \cap \chi_X)(b)) = \bigwedge_{b \in B} (\neg (f \cap \chi_X)(b) \rightarrow \neg r(b,a))$$

$$= \bigwedge_{b \in B} (\neg (f \cap \chi_X)(b) \rightarrow (r(b,a) \rightarrow 0))$$

$$= \bigwedge_{b \in B} ((\neg (f \cap \chi_X)(b) \otimes r(b,a)) \rightarrow 0)$$

$$= (\bigvee_{b \in B} (\neg (f \cap \chi_X)(b) \otimes r(b,a)) \rightarrow 0)$$

$$= \neg (\bigvee_{b \in B} (\neg (f \cap \chi_X)(b) \otimes r(b,a)))$$

$$= \neg \nearrow (\neg (f \cap \chi_X))(a) = \neg \nearrow (\neg f \cap \chi_{X^c})(a)$$

$$= \neg \underline{\int}_{X^c} \neg f d\mu_a$$

4 New Measures and Integrals Defined on Concept Lattices

And at the end we can also define a "new" measures and integrals for $\mathcal{L}$-context $\langle B, A, \mathcal{L}, r \rangle$ as follows:

- $\underline{\mu} : \mathcal{P}(B) \rightarrow \mathrm{Int}(B, A, \mathcal{L}, r, \nearrow, \swarrow)$ defined as $\underline{\mu}(X)(a) = \underline{\mu}_a(X)$
- $\overline{\mu} : \mathcal{P}(B) \rightarrow \mathrm{Int}(B, A, \mathcal{L}, r, \searrow, \searrow)$ defined as $\overline{\mu}(X)(a) = \overline{\mu}_a(X)$
- $\underline{\int}_X f d\underline{\mu} : \mathcal{L}^B \rightarrow \mathrm{Int}(B, A, \mathcal{L}, r, \nearrow, \swarrow)$ defined as $\underline{\int}_X f d\underline{\mu}(a) = \underline{\int}_X f d\underline{\mu}_a$
- $\overline{\int}_X f d\overline{\mu} : \mathcal{L}^B \rightarrow \mathrm{Int}(B, A, \mathcal{L}, r, \nearrow, \swarrow)$ defined as $\overline{\int}_X f d\overline{\mu}(a) = \underline{\int}_X f d\underline{\mu}_a$

In such a case the measures and integrals are mappings to closure and interior systems that are complete lattices

- $\mathrm{Int}(B, A, \mathcal{L}, r, \nearrow, \swarrow)$ with top $\nearrow (\chi_B)$ and bottom $\chi_\emptyset$
- $\mathrm{Int}(B, A, \mathcal{L}, r, \searrow, \searrow)$ with top χ_B and bottom $\searrow (\chi_\emptyset)$

Where all "inconveniences" with normality are hence solved.

Proposition 1. *Let $\underline{\mu}^B$ and $\overline{\mu}^B$ be $\mathcal{L}$-measures corresponded to $\mathcal{L}$-context $\langle B, A, \mathcal{L}, r \rangle$. Let $\underline{\mu}^A$ and $\overline{\mu}^A$ be $\mathcal{L}$-measures corresponded to $\mathcal{L}$-context $\langle A, B, \mathcal{L}, r^t \rangle$ where r^t is the transposition of r. Then*

$$\underline{\int}_B f d\underline{\mu}^B \leq g \quad \Leftrightarrow \quad f \leq \overline{\int}_A g d\overline{\mu}^A$$

and

$$\underline{\int}_A g d\underline{\mu}^A \leq f \quad \Leftrightarrow \quad g \leq \overline{\int}_B f d\overline{\mu}^B$$

Proof. From previous facts.

5 Conclusion

New lattice valued fuzzy measures and integrals are proposed by using data and operations from Girard monoid, i.e. complete residuated lattice with double negation law that are built from fuzzy formal context. Main result is to show the relationship between proposed measures and integrals and isotone concept-forming operators.

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