On Uncertain Discontinuous Functions and Quasi-equilibrium in Some Economic Models

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Abstract. In the paper is studied some properties of uncertain discontinuous mappings, the so-called w-discontinuous mappings. Based on them, the existence of a quasi-equilibrium for a new economic model is proved.

Keywords: Discontinuity · Fixed point theorem · Market equilibrium · Quasi-equilibrium

1 Introduction

One of the basic assumptions in a mathematical modelling of the standard economic model is the continuity of the excess demand function involved. There are reasons to maintain that the necessity of this assumption is caused by the methods provided by mathematics. First of all the fixed points theorems of Brouwer and Kakutani have to be mentioned, since both require the continuity of the maps. They are the main tools for establishing the existence of an equilibrium. However, the necessity of the assumption of continuity has also some economic motivation: in a neoclassical exchange economy due to the strict convexity and strict monotony of the preferences of all consumers the excess demand function is continuous (s. [2], Th.1.4.4).

The paper offers a possibility to substitute the continuity of the excess demand function by the w-discontinuity of this function and therefore to deal, in some extent, with unstable economies. We will examine the properties of w-discontinuous mappings and finally, under some additional conditions, we prove the existence of a generalized equilibrium. The concept w-discontinuity includes uncertainty about the deviation of a function from continuity.

The classical microeconomic models have their origins mainly in the work of L. Walras [18], (1954), a wider discussion of them is presented by K. J. Arrow and G. Debreu [3], (1954) and also by K.J. Arrow and F.H. Hahn [4], (1991). An extended description of the classical model can also be found in textbooks on

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microeconomics, for example, H. Varian [17], (1992), D.M. Kreps [14], (1990), W. Nicholson [15], (1992) or R.M. Starr [16], (2011). For a strictly functional-analytic approach we refer to the book of C.D. Aliprantis, D.J. Brown and O. Burkinshaw [2], (1990).

2 w-Discontinuous Mappings and Their Properties

A class of discontinuous mappings is defined as follows. Let (X, d) and (Y, ϱ) be two metric spaces and w a positive number.

Definition 1. *A mapping $f\colon X \to Y$ is said to be w-discontinuous at the point $x_0 \in X$ if for every $\varepsilon > 0$ there exists δ such that whenever $x \in X$ and $d(x_0, x) < \delta$ follows that $\varrho(f(x_0), f(x)) < \varepsilon + w$.*

The constant w may not be the best possible (smallest) one. Very often, especially in economic applications, there is known only a rough upper estimation for the "jump". Exactly the constant w includes uncertainty about the division of a function from continuity.

A mapping f is called *w-discontinuous in X* if it is w-discontinuous at all points of X.

The notion of w-discontinuous maps is not new. It is already found in [12] as the concept of *oscillation* or as *continuity defect* in [8]. The notion of w-discontinuity (former w-continuity) was introduced by the author in [5].

Example 1. The usual Dirichlet function on $\mathbf{R}$ and also the generalized Dirichlet function $f : \mathbf{R}^n \to \{a, b\}$, a, $b \in \mathbf{R}$, $a \neq b$, defined for all $x = (x_1, x_2, ..., x_n) \in \mathbf{R}^n$ by

$$f(x) = \begin{cases} a, \text{ if all components } x_i \in \mathbf{Q} \\ b, \text{ if there exists } i_0 \text{ such that } x_{i_0} \in \mathbf{R} \setminus \mathbf{Q} \end{cases}$$

are $|a - b|$-discontinuous (and for any $w \geq |a - b|$ also w-discontinuous) functions. ■

If X, Y, V are real normed vector spaces the following properties of w-discontinuous mappings are established (similar as for continuous mappings). For proofs see [7].

Proposition 1. *Let be $f_i : X \to Y$, $\alpha_i \in \mathbf{R}$, $i = 1, \ldots, k$ and $g = \alpha_1 f_1 + \cdots + \alpha_k f_k$. Suppose $w_i > 0$ and that f_i is w_i-discontinuous on the set X for each $i = 1, \ldots, k$. Then $g = \alpha_1 f_1 + \cdots + \alpha_k f_k$ is a $|\alpha_1| w_1 + \cdots + |\alpha_k| w_k$-discontinuous mapping.*

From the Definition 1, which makes sense also for $w = 0$, immediately follows that the 0-discontinuous mappings are exactly the continuous ones.

Corollary 1. *Suppose that $f, g : X \to Y$, f is w'-discontinuous and g is w''-discontinuous. Then $f + g$ and $f - g$ are $w' + w''$-discontinuous mappings. In particular, if one of the mappings (f or g) is continuous, then $f \pm g$ are w'-discontinuous (or w''-discontinuous).*

Corollary 2. *If $f\colon X \to Y$ is w- discontinuous and c is a constant then $c \cdot f$ is a $|c|w$- discontinuous mapping.*

Proposition 2. *Let $f\colon \mathrm{dom} f \to \mathbf{R}$ and $g\colon \mathrm{dom}\, g \to \mathbf{R}$ be w'-, w''-discontinuous functions, respectively. Then the functions $f \wedge g$ and $f \vee g$ are $w' + w''$-discontinuous on $\mathrm{dom} f \cap \mathrm{dom}\, g$.*

Corollary 3. *If f is w- discontinuous and g is continuous then $f \vee g$ is w-discontinuous.*

In order to consider the product of mappings we need the notation of the product in a normed space.

Definition 2 ([13]). *Let X, Y, Z be real normed vector spaces. A mapping $\pi\colon X \times Y \to Z$ is called a product if it satisfies the following conditions: for all $a, b \in X,\ u, v \in Y$ and $\lambda \in \mathbf{R}$ one has*

1. $\pi((a + b, v)) = \pi((a, v)) + \pi((b, v))$
2. $\pi((a, u + v)) = \pi((a, u)) + \pi((a, v))$
3. $\pi((\lambda a, u)) = \lambda\pi((a, u)) = \pi((a, \lambda u))$
4. $\|\pi((a, u))\|_Z \leq \|a\|_X \|u\|_Y$.

A simple example is given by $X = Y = \mathbf{R}^n,\ Z = \mathbf{R}$ and $\pi((x, y)) = \langle x, y\rangle$ – the scalar product in $\mathbf{R}^n$, i.e., $\langle x, y\rangle = \sum\limits_{i=1}^{n} x_i\, y_i$.

Let V, X, Y, Z be real normed vector spaces and let $\pi\colon X \times Y \to Z$ be a product. The product of the mappings $f\colon \mathrm{dom} f \subseteq V \to X$ and $g\colon \mathrm{dom}\, g \subseteq V \to Y$ is understood pointwise, i.e.,

$$(f \cdot g)(v) = \pi\big(f(v), g(v)\big), \quad \forall v \in \mathrm{dom} f \cap \mathrm{dom}\, g,$$

where $\mathrm{dom} f$, $\mathrm{dom}\, g \subseteq V$.

Proposition 3. *Suppose that $f\colon \mathrm{dom} f \to X$ is w'-discontinuous and $g\colon \mathrm{dom}\, g \to Y$ is w''-discontinuous on $\mathrm{dom} f \cap \mathrm{dom}\, g$. Then $f \cdot g$ is a $(w'w'' + w'\|g(x_0)\|_Y + w''\|f(x_0)\|_X)$-discontinuous mapping at every point $x_0 \in \mathrm{dom} f \cap \mathrm{dom}\, g$.*

Corollary 4. *If $f : V \to X$ is w-discontinuous and $g : V \to Y$ is continuous then $f \cdot g$ is a $\|g(x_0)\|_Y\, w$- discontinuous mapping at every point $x_0 \in V$.*

For the division we reconcile with simplified situation, where (X, d) is again a metric space.

Proposition 4. *Let the function $f\colon X \to \mathbf{R}$ be w-discontinuous at the point x_0 and $f(x_0) \neq 0$. If there exists a neighborhood U of x_0 and a number $\alpha_0 > 0$ such that $|f(x)| \geq \alpha_0$ for all $x \in U$ then the function $\frac{1}{f}$ is $\frac{w}{\alpha_0 |f(x_0)|}$-discontinuous at x_0.*

As a special case we get

Corollary 5. *If $f : X \to [1, +\infty[$ is w-discontinuous then $\frac{1}{f}$ is a $\frac{w}{f(x_0)}$- discontinuous mapping for every point $x_0 \in X$*

If the domain of definition for a continuous mapping is compact, then its range is also compact and, in particular, bounded. The boundedness of the range is guaranteed for w-discontinuous mappings as well, however, compactness may not hold.

Example 2. Define $f\colon [0; 2] \to [0; 2]$ as

$$f(x) = \begin{cases} 1, \text{ if } x \in \{0,\ 2\} \\ x, \text{ if } \ x \in]0, 2[. \end{cases}$$

The function f is 1-discontinuous and its range $]0, 2[$ is bounded, but not compact. ■

Theorem 1. *Suppose that $A \subset X$ is compact and let $f\colon A \to X$ be w-discontinuous. Then $f(A)$ is bounded.*

The following essential result is proved by O. Zaytsev in [19] and can be considered as a generalization of the Bohl-Brouwer-Schauder fixed point theorem for w-discontinuous mappings.

Theorem 2. *Let K be a nonempty, compact and convex subset in a normed vector space X. For every w-discontinuous mapping $f : K \to K$ $(w > 0)$ there exists a point $x^* \in K$ such that $\| x^* - f(x^*) \| \leq w$.*

3 Market Equilibrium of the Standard Economic Model

We give the description of a simple economic model $\mathcal{E}$ considered by Arrow and Hahn in [4].

Let there be n ($n \in \mathbf{N}$) different goods (commodities) on the market: services and wares, and a finite number of economic agents: households and firms, where each household can be considered as a firm, and, vice versa, each firm can be considered as a household.

Let x_{hi} be the quantity of good i which is needed to the household h. If $x_{hi} < 0$ then $|x_{hi}|$ denotes the quantity of good i which is supplied by the household h. If $x_{hi} \geq 0$ then x_{hi} is the (real) demand of good i by h, including the zero demand. The summation over all households will be indicated by $x_i = \sum\limits_h x_{hi}$ – the total demand of good i, $i = 1, \ldots, n$.

The quantity of good i that is supplied by the firm f will be denoted by y_{fi}. Again, if $y_{fi} < 0$ then $|y_{fi}|$ is the demand (input) of good i by f. If $y_{fi} \geq 0$ then y_{fi} is the supplied quantity (output) of i by f, where the zero supply again is included. The summation over all firms will be indicated by $y_i = \sum\limits_f y_{fi}$ – the supply of good i, $i = 1, \ldots, n$.

The initially available amount (or resources) of good i in all households will be denoted by $\overline{x_i}$. Note that $\overline{x_i}$ must be non-negative.

A market equilibrium, which is one of the most important characteristics of any economy (see f. e. [1,2,4,9,11,16]), describes the economic situation that the total demand of each good in the economy is satisfied by its total supply. This fact is obviously expressed by saying that the difference between the total demand of each good and its total supply is less than or equal to zero. The total supply of good i is understood as the sum of the supply of the good i and the quantity of i which is already available, i.e. the total supply of the good i equals to $y_i + \overline{x_i}$. The excess demand of good i is then defined as $x_i - y_i - \overline{x_i}$, $i = 1, ..., n$.

If economic agents at the market are faced with a system of prices, i.e. with a price vector $p = (p_1, \ldots, p_n)$, where p_i is the price of one unit of good i, then the quantities x_{hi}, y_{fi} and also $x_i, y_i, \overline{x_i}$ depend on p. Now we denote the excess demand of the good i by $z_i(p)$, i.e.

$$z_i(p) = x_i(p) - \Big(y_i(p) + \overline{x_i}(p)\Big).$$

If prices are involved then an equilibrium price (a price system at which an equilibrium is reached) clears the markets.

Further on we frequently make use of the natural order in $\mathbf{R}^n$ introduced by the positive cone

$$\mathbf{R}^n_+ = \{x = (x_1, \ldots, n) \in \mathbf{R}^n \mid x_i \geq 0,\ i = 1, \ldots, n\},$$

i.e. for two vectors $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$ we write $x \leq y$ iff $x_i \leq y_i$ for all $i = 1, \ldots, n$, we write $x < y$ iff $x \leq y$ and $x_{i_0} < y_{i_0}$ for at least one index i_0. The norm we will use in the space $\mathbf{R}^n$ is defined as

$$\|x\| = \sum_{i=1}^{n} |x_i|,\ x = (x_1, ..., x_n) \in \mathbf{R}^n.$$

This norm is equivalent to the euclidean norm which is introduced by means of the scalar product $\langle x, y\rangle = \sum\limits_{i=1}^{n} x_i\, y_i$. Note that in economic publications the scalar product of two vectors $x, y \in \mathbf{R}^n$ is usually written as $x\, y$.

For the standard economic model the following four assumptions have to be met (see [4]).

Assumption 1. Let $p = (p_1, ..., p_n)$ be an n-dimensional price vector with the prices p_i for one unit of good i as components, $i = 1, 2, ..., n$. For any p let the excess demand for i be characterized by a unique number $z_i(p)$ and so the unique vector $z(p) = (z_1(p), \ldots, z_n(p))$ - the excess demand function with excess demand functions for i as components $(i = 1, 2, ..., n)$ - is well defined.

Assumption 2. $z(p) = z(\lambda p), \quad \forall p > \mathbf{0}$ and $\lambda > 0$.

The Assumption 2 asserts that z is a homogeneous vector-function of degree zero. Economically this means that the value of the excess demand function does not depend on the price system if the latter is changed for all the goods simultaneously by the same portion.

From the Assumption 2 follows that prices can be normalized (see [4], p.20 or [9], p.10). If for some price p one has $z(p) = 0$ then $z(\lambda p) = 0$ for all prices of the ray $\{\lambda p \colon \lambda > 0\}$. Therefore, further on we consider only prices from the $n-1$-dimensional simplex of R^n

$$\Delta_n = \{p = (p_1, p_2, ..., p_n) \mid p_i \geq 0 \ \text{ and } \ \sum_{i=1}^{n} p_i = 1\}.$$

We rule out the situations when all the prices are zero or some of them are negative. Note that Δ_n is a compact and convex set in the space $\mathbf{R}^n$ equipped with one of its (equivalent) norms.

Assumption 3 or Walras' Law. $p\,z(p) = 0, \quad \forall p \in \Delta_n.$

Walras' Law can be regarded as an attempt to have a model sufficiently truly reflecting rationally motivated activities of economic agents. According to Walras' Law all the firms and all the households both spend their financial resources completely [9].

Assumption 4. The excess demand function z is continuous on its domain of definition Δ_n.

It means that a small change of a price system will imply only a small change in the excess demand. As a consequence from continuity of z, the standard model can be used only for the description of economies with continuous excess demand functions. Sometimes they are called stable economies.

In economies such prices are important at which the excess demand for each good is nonpositive, i.e. the total supply of each good satisfies at least its total demand.

Definition 3. *A price $p^* \in \Delta_n$ is called an* **equilibrium** *(price) if $z(p^*) \leq 0$.*

If p^* is an equilibrium price then $\sum\limits_{i=1}^{n} z_i(p^*) \leq 0$.

For the standard model of an economy with a finite number of goods and agents such prices always exist as is proved in the following theorem.

Theorem 3 ([4]). *If an economy $\mathcal{E}$ with a finite number of goods and agents satisfies the Assumptions 1–4, then there exists an equilibrium in $\mathcal{E}$.*

4 Economic Models with Discontinuous Excess Demand Functions

If z is the excess demand function for a neoclassical exchange economy, then z is continuous on the set

$$S = \{p \in \Delta_n \mid p_i > 0, i = 1, 2, ..., n\}$$

(see [2], Th.1.4.4 and Th.1.4.6). A neoclassical exchange economy (see [2]) is characterized by a finite set of agents, where each agent i has a non-zero initial endowment ω_i and his preference relation $\succeq_i$ is continuous (a preference relation $\succeq$ is continuous if, given a two sequences $(x^n)_{n=1}^{\infty}$, $(y^n)_{n=1}^{\infty}$ with $\lim_{n\to\infty} x^n = x$, $\lim_{n\to\infty} y^n = y$ and $x^n \succeq y^n$, $n = 1, 2...$, then $x \succeq y$), strictly monotone and strictly convex (on $\mathbf{R}_+^n$) or else his preference relation $\succeq_i$ is continuous, strictly monotone and strictly convex on interior of $\mathbf{R}_+^n$, and everything in the interior is preferred to anything on the boundary and the total endowment $\omega = \sum_i \omega_i$ is strictly positive. If the preference relation $\succeq_i$ is continuous, strictly monotone and strictly convex then the corresponding utility function and the excess demand function are continuous on the set S. We will consider the situation with a discontinuous excess demand function. It is clear that in this case the properties of the preference relations differ from them in the neoclassical exchange economy.

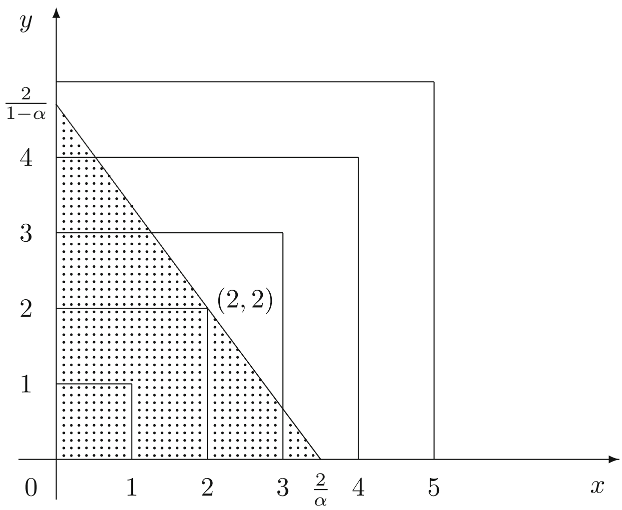

Fig. 1. The indifference curves of utility function $u(x, y) = \max\{x, y\}$ for the values 1, 2, 3, 4 and 5.

For example, consider the preference relation on $\mathbf{R}_+^2$ that is represented by the utility function $u(x, y) = \max\{x, y\}$ and an initial endowment $\omega = (2, 2)$. The utility function is continuous, but it is not strictly monotone (for example, $(2, 2) > (2, 1)$ but $u(2, 2) = 2 = u(2, 1)$) and it is not strictly concave, it is convex. The indifference curves for the values 1, 2, 3, 4 and 5 are illustrated in Fig. 1. Let $p = (\alpha, 1-\alpha)$ be a fixed price vector for some $0 < \alpha < 1$. We maximize the utility function u subject to the budget constraint $\alpha x+(1-\alpha)y = 2\alpha+2(1-\alpha) = 2$. This line goes through the point $(2, 2)$ and intersects the axis in the points $(0, \frac{2}{1-\alpha})$ and $(\frac{2}{\alpha}, 0)$. From Fig. 1 we see that the maximal vector of u over budget set (the dotted region in Fig. 1) is the point $(0, \frac{2}{1-\alpha})$ if $\alpha > \frac{1}{2}$ and $(\frac{2}{\alpha}, 0)$ if $\alpha < \frac{1}{2}$, respectively. If $\alpha = \frac{1}{2}$ then $\frac{2}{1-\alpha} = \frac{2}{\alpha}$ and therefore we have two maximizing vectors. The demand function in this case is

$$d(p) = d(\alpha, 1-\alpha) = \begin{cases} (0, \frac{2}{1-\alpha}), & \alpha > \frac{1}{2}, \\ \{(0,4),(4,0)\}, & \alpha = \frac{1}{2}, \\ (\frac{2}{\alpha}, 0), & \alpha < \frac{1}{2}. \end{cases}$$

In the point $(\frac{1}{2}, \frac{1}{2})$ the demand multifunction is discontinuous.

In [1] it is proved that in a neoclassical exchange economy the condition $p_n \to p \in \partial S$ with $(p_n)_{n \in \mathbf{N}} \subset S$ implies $\lim_{n\to\infty} ||z(p_n)|| = \infty$. It is also not our case (see Theorem 1). In [1] it is shown that a utility function $u\colon X \to \mathbf{R}$ (X - topological space) representing a continuous preference relation is not necessarily continuous. If we start with an arbitrary chosen discontinuous utility function then we have no mathematical tools for finding the corresponding demand function (in the classical situation an agent maximizes the utility function with respect to the budget constraint and uses the Lagrange multiplier method for finding demand function). We note that there exist preference relations which cannot be represented by a real-valued function, for example, the lexicographic preference ordering of $\mathbf{R}^2$ (by definition $(a,b) \succeq (c,d)$ if (1) $a > c$ or (2) $a = c$ and $b > d$) (see [10], notes to chapt.4).

The above situation inspires one to consider models without explicitly given preference relations. In which cases is the excess demand function discontinuous? Consider some good i and a fixed price system p. In the case that this good is, e.g. an aeroplane or a power station, its demand $x_i(p)$ is naturally an integer. A function like $x_i(p) = \left[\frac{30000}{1+\alpha}\right]$, where $[x]$ denotes the integer part of x, provides an example. Obviously, if the good is a piece-good (table, shoes, flower and other) then the demand for this good is an integer. Similarly, the supply of piece-goods is an integer. Therefore the demand and supply functions for piece-goods are discontinuous and consequently the excess demand function too.

We will analyse some model of an economy with w-discontinuous excess demand functions.

For the economies under consideration we keep the two first assumptions from the standard model and change the two last as follows.

Assumption 4'. The excess demand function z is w-discontinuous on its domain of definition Δ_n.

The w-discontinuity of the excess demand function makes our model available to describe some properties of an unstable economy as well.

It is quite natural that for every price vector $p \in \Delta_n$ there exist at least one good i with the price $p_i > 0$ and such that the demand for them is satisfied, i.e. $z_i(p) \leq 0$.

If for some economy $\mathcal{E}$ with the excess demand vectors $z(p)$, $p \in \Delta_n$ there holds the Walras' Law, i.e. $p\,z(p) = 0$ for any $p \in \Delta_n$, then for each $p \in \Delta_n$ the inequality

$$\gamma_p = \sum_{i:z_i(p)\le 0} p_i > 0$$

is satisfied. (We write further "$z_i(p) \le 0$" instead of "$i : z_i(p) \le 0$" and in similar cases.) Indeed, if for some $p = (p_1, \ldots, p_n) \in \Delta_n$ there would be $\sum\limits_{z_i(p)\le 0} p_i = 0$, then

$$\sum_{z_i(p)\le 0} p_i + \sum_{z_i(p)>0} p_i = \sum_{i=1}^{n} p_i = 1$$

would imply the existence of an index i_0 such that $p_{i_0} > 0$ and $z_{i_0}(p) > 0$, which hold then, because of $\sum\limits_{i=1}^{n} p_i = 1$, for some i_0 there must be $p_{i_0} > 0$ and $z_{i_0}(p) > 0$, which yields $p\,z(p) = \sum\limits_{i=1}^{n} p_i z_i(p) \ge p_{i_0} z_{i_0}(p) > 0$, a contradiction to Walras' Law.

Our next assumption requires the existence of a uniform lower bound for the sums $\sum\limits_{z_i(p)\le 0} p_i$, for all $p \in \Delta_n$.

Assumption 3'. $\gamma = \inf\limits_{p\in\Delta_n} \gamma_p > 0$.

It seems to be clear that it would be hard to find out why an equilibrium exists in our model. But it will be possible if we can estimate the unsatisfied aggregate demand. This leads to the concept of quasi- or k-equilibrium.

Definition 4. *Let k be a positive real. A price vector $p^* \in \Delta_n$ is called a* **k-equilibrium** *if it satisfies the condition*

$$\sum_{z_i(p^*)>0} z_i(p^*) \le k.$$

The constant $k \in \mathbf{R}_+$ as a numerical value of the maximally possible unsatisfied demand for a given price $p^* \in \Delta_n$ characterizes to what state the economy differs from the market equilibrium (Definition 3).

We can prove now the following

Theorem 4. *Let $\mathcal{E}$ be an economy with n goods that satisfies the Assumptions 1, 2 and the Assumption 3' with some number $\gamma > 0$. Put*

$$w_+ = w_+(n,\gamma) = \frac{1}{2n}\left(-(n+1) + \sqrt{(n+1)^2 + 8n\gamma}\right).$$

If now the Assumption 4' is satisfied with $w \in [0, w_+)$, then the economy $\mathcal{E}$ possesses a k-equilibrium for each $k \ge \frac{nw^2+(n+1)w}{2\gamma - nw^2-(n+1)w}$.

Proof. For $p \in \Delta_n$ define $z_i^+(p) = \max\{0, z_i(p)\}$, $i = 1, ..., n$, $z^+(p) = (z_1^+(p), \ldots, z_n^+(p))$,

$$\nu(p) = \langle p + z^+(p), e\rangle = 1 + \sum_{z_i(p)>0} z_i(p) \quad \text{and} \quad t_i(p) = \frac{p_i + z_i^+(p)}{\nu(p)},\ i = 1, ..., n,$$

where $e = (1, \ldots, 1)$ denotes the vector of $\mathbf{R}^n$ with all components equal to 1. Note that $\|e\| = n$.

Define now a map $T\colon \Delta_n \to \Delta_n$ by $T(p) = \frac{p+z^+(p)}{\langle p+z^+(p), e\rangle}$, then $T(p) = (t_1(p), ..., t_n(p))$. Since $0 \le t_i(p) \le 1$ for each i and

$$\sum_{i=1}^n t_i(p) = \frac{\sum_{i=1}^n (p_i + z_i^+(p))}{\nu(p)} = \frac{1 + \sum_{z_i(p)>0} z_i(p)}{\nu(p)} = 1$$

one has $T(p) : \Delta_n \to \Delta_n$.

Now the particular maps which the map T consists of, possess the following properties. The identity map id on Δ_n is continuous, by Assumption 4' the map $z\colon \Delta_n \to \mathbf{R}^n$ is w-discontinuous and by Corollary 3 so is z^+. By Corollary 1 the map $id + z^+$ is w-discontinuous, what by Corollary 4 implies the $w\|e\|$-discontinuity, i.e. the nw-discontinuity of $\nu(p) = \langle p + z^+(p), e\rangle$. Since $\nu\colon \Delta_n \to [1, \infty)$ the function $\frac{1}{\nu}$ is $\frac{nw}{\nu(p)}$-discontinuous as a consequence of Corollary 5. Finally, based on Proposition 3, the map $T(p) = (p + z^+(p))\,\frac{1}{\nu(p)}$ is w_0-discontinuous at a every point $p \in \Delta_n$, where

$$w_0 = w_0(p) = \frac{nw^2}{\nu(p)} + \frac{w}{\nu(p)} + \frac{nw\|p + z^+(p)\|}{\nu(p)} = \frac{nw^2 + w}{\nu(p)} + nw \le nw^2 + (n+1)w \tag{1}$$

and so, the map T is also $nw^2 + (n+1)w$-discontinuous on the set Δ_n.

Since Δ_n is a convex and compact subset in the normed vector space $\mathbf{R}^n$ and $T(p) : \Delta_n \to \Delta_n$ we conclude by means of Theorem 2 that there exists a vector $p^* \in \Delta_n$ satisfying the inequality

$$\|T(p^*) - p^*\| \le nw^2 + (n+1)w.$$

Using the norm in $\mathbf{R}^n$ this yields

$$\begin{aligned} \|T(p^*) - p^*\| &= \left\| \tfrac{p^* + z^+(p^*)}{\nu(p^*)} - p^* \right\| = \sum_{i=1}^n \left| \tfrac{p_i^* + z_i^+(p^*)}{\nu(p^*)} - p_i^* \right| \\ &= \sum_{i=1}^n \left| \frac{p_i^* + z_i^+(p^*) - p_i^* - p_i^* \sum_{z_i(p^*)>0} z_i(p^*)}{\nu(p^*)} \right| \le nw^2 + (n+1)w. \end{aligned}$$

Since $1 + \sum_{z_i(p^*)>0} z_i(p^*) > 0$ one has

$$\sum_{i=1}^n \left| z_i^+(p^*) - p_i^* \sum_{z_i(p^*)>0} z_i(p^*) \right| \le \big(nw^2 + (n+1)w\big)\ \nu(p^*). \tag{2}$$

The left side of inequality (2) can be splitted into two sums

$$\sum_{z_i(p^*)\le 0}\left|z_i^+(p^*) - p_i^* \sum_{z_i(p^*)>0} z_i(p^*)\right| + \sum_{z_i(p^*)>0}\left|z_i(p^*) - p_i^* \sum_{z_i(p^*)>0} z_i(p^*)\right|$$
$$= \sum_{z_i(p^*)\le 0} p_i^* \sum_{z_i(p^*)>0} z_i(p^*) + \sum_{z_i(p^*)>0}\left|z_i(p^*) - p_i^* \sum_{z_i(p^*)>0} z_i(p^*)\right|. \tag{3}$$

Using the triangle inequality we get the estimation

$$\left|\sum_{z_i(p^*)>0}\left(z_i(p^*) - p_i^* \sum_{z_i(p^*)>0} z_i(p^*)\right)\right| \le \sum_{z_i(p^*)>0}\left|z_i(p^*) - p_i^* \sum_{z_i(p^*)>0} z_i(p^*)\right| \tag{4}$$

and further the left hand side of (4) calculates as

$$\left|\sum_{z_i(p^*)>0}\left(z_i(p^*) - p_i^* \sum_{z_i(p^*)>0} z_i(p^*)\right)\right| = \left|\sum_{z_i(p^*)>0} z_i(p^*)\left(1 - \sum_{z_i(p^*)>0} p_i^*\right)\right|$$
$$= \sum_{z_i(p^*)>0} z_i(p^*)\left(1 - \sum_{z_i(p^*)>0} p_i^*\right) = \sum_{z_i(p^*)>0} z_i(p^*) \sum_{z_i(p^*)\le 0} p_i^*. \tag{5}$$

By means of the equalities (3), (5) and the inequalities (2), (4) we obtain now

$$2\sum_{z_i(p^*)>0} z_i(p^*) \sum_{z_i(p^*)\le 0} p_i^* \le \sum_{z_i(p^*)>0} z_i(p^*) \sum_{z_i(p^*)\le 0} p_i^*$$
$$+ \sum_{z_i(p^*)>0}\left|z_i(p^*) - p_i^* \sum_{z_i(p^*)>0} z_i(p^*)\right| \le \left(nw^2 + (n+1)w\right)\, \nu(p^*).$$

It follows by means of the Assumption 3'

$$2\gamma \sum_{z_i(p^*)>0} z_i(p^*) \le 2\sum_{z_i(p^*)>0} z_i(p^*) \sum_{z_i(p^*)\le 0} p_i^* \le \left(nw^2 + (n+1)w\right)\, \nu(p^*).$$

Since $\nu(p^*) = 1 + \sum_{z_i(p^*)>0} z_i(p^*)$ the last inequality yields

$$\sum_{z_i(p^*)>0} z_i(p^*) \le \frac{nw^2 + (n+1)w}{2\gamma - nw^2 - (n+1)w}, \quad \text{i.e.} \quad \sum_{z_i(p^*)>0} z_i(p^*) \le k,$$

where k satisfies $k \ge \frac{nw^2+(n+1)w}{2\gamma - nw^2-(n+1)w}$.

In order to have the number $2\gamma - nw^2 - (n+1)w$ positive the value of w must belong to the interval $[0,\, w_+)$, where w_+ is the positive root of the equation $w^2 + \frac{n+1}{n}w - \frac{2\gamma}{n} = 0$. $\square$

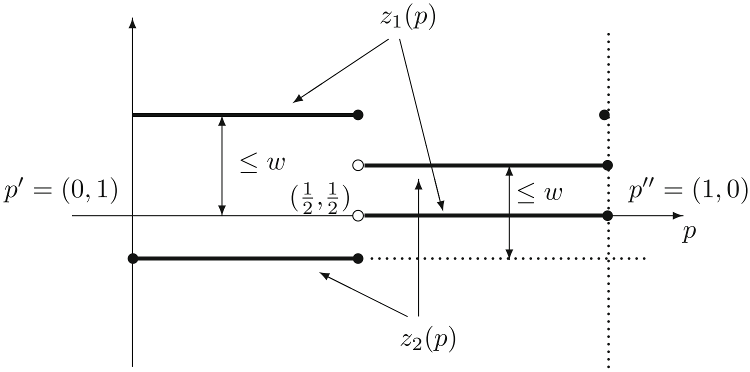

Fig. 2. No classical equilibrium, but k-equilibrium exists.

5 Conclusions

We make some remarks.

1. Let n and $\gamma > 0$ be fixed. Then $w_+ = w_+(n, \gamma)$ is defined as indicated in the theorem. For $w \in [0, w_+)$ put
$$k_0(n, w) = \frac{nw^2 + (n+1)w}{2\gamma - nw^2 - (n+1)w}.$$
The number $k_0(n, w)$ is non-negative as was shown above. Note that a sharper estimation (our estimation is based on the rough inequality $\nu(p) \geq 1$) in (1) would yield a smaller value of $k_0(n, w)$ and, therefore, would give a better result. In view of Theorem 2, however, an estimation has be obtained independently on p.
2. In Fig. 2 for $n = 2$ there is shown a situation without a classical equilibrium. It is clear that there is no $p \in \Delta_2$ which satisfies the inequality $z(p) = (z_1(p), z_2(p)) \leq \mathbf{0}$. The Assumptions 1, 2, 4' are obviously fulfilled. The Assumption 3' also holds. Indeed, represent $p = (p_1, p_2) \in \Delta_2$ as
$$p = (1-t)p' + tp'', \qquad t \in [0, 1],$$
then $t \in [0, \frac{1}{2}]$ implies $z_1(p) > 0$, $z_2(p) < 0$ and so $\gamma_p = p_2$ and $t \in (\frac{1}{2}, 1]$ implies $z_1(p) = 0$, $z_2(p) > 0$ and so $\gamma_p = p_1$. In both cases we get $\gamma_p \geq \frac{1}{2}$ which shows that the Assumption 3' holds with $\gamma = \frac{1}{2}$. Theorem 4 guarantees the existence of a k-equilibrium for $k \geq \frac{2w^2+3w}{1-2w^2-3w}$ if $w < -\frac{3}{4} + \frac{\sqrt{17}}{4}$. Note that Walras' Law is not satisfied.
3. The number $w_+(n, \gamma)$ is positive for each n and fixed $\gamma > 0$. If one takes $w = 0$ then $k_0(n, \gamma) = 0$ and with $k = 0$ there is obtained the classical case. Observe that in this case it is not necessary to use the Walras' Law for establishing a classical equilibrium.

4. Note that in the classical situation it is impossible to carry out any quantitative analysis. On the contrary, the inequalities from Theorem 4

$$w < w_+(n,\gamma) \quad \text{and} \quad k \geq k_0(n,w)$$

give a chance to analyse the behaviour of an economy for different numerical values of the parameters n, w, γ included in our model. From

$$0 \leq w_+(n,\gamma) = \frac{-(n+1) + \sqrt{(n+1)^2 + 8n\gamma}}{2n}$$
$$< \frac{-(n+1) + (n+1) + \sqrt{8n\gamma}}{2n} = \sqrt{\frac{2\gamma}{n}}$$

it follows that $\lim\limits_{n\to\infty} w_+(n,\gamma) = +0$. Since $k_0(n,0) = 0$, the positive number k can be chosen arbitrary small. This shows that the larger the number of goods the better the chance for a classical equilibrium.
5. It is reasonable to put $k_0(n, w_+(n,\gamma)) = +\infty$. If for fixed n and γ the value w is sufficiently close to $w_+(n,\gamma)$, then k is very large. In such a case the existence of an k-equilibrium seems to be of low economic meaning.
6. The results of this paper have been developed in a collaboration with prof. M. R. Weber from the Dresden University of Technology [7].
7. Other application of w-discontinuous mappings is to find a of quasi-equilibrium in economic models that the author has developed in [6] in a collaboration with a student D. Rika.

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