An Initial Study on Typical Hesitant (T,N)-Implication Functions

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Abstract. In the theory of Hesitant Fuzzy Sets (HFS), the membership degree of an element is characterized by a membership function which always returns a fuzzy set. This approach enables one to express, for example, the hesitance of several experts in the process of decision making based on multiple attributes and multiple criteria. In this work, we focus on the study of a class of implication functions for typical hesitant fuzzy sets (THFS). The novelty of our proposal lies on the fact that it is the first time that an admissible order is used to define operators on hesitant fuzzy setting. Thus, we introduce typical hesitant fuzzy negations, typical hesitant t-norms and typical hesitant implication functions considering an admissible order, which allows the comparison of typical hesitant fuzzy elements with different cardinalities.

Keywords: Hesitant Fuzzy Sets · Admissible orders on THFS · Typical Hesitant Implication Functions · (T,N)-implication functions

1 Introduction

In situations where there are conflicts among the several experts in the process of decision making based on multiple attributes and multiple criteria, it is common to use Hesitant Fuzzy Sets (HFS). In the HFS theory [25], one considers, as the membership degree of an element, a membership function which always returns a fuzzy set expressing this hesitance. Since its introduction in 2010, relevant research in decision making has used HFS theory, for example, the studies found

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in [12,28–30]. In particular, several weighted average and ordered weighted average (OWA)-like operators have been proposed to be used in decision making, as we can see in [5,29,34].

A frequent issue in the context of decision making is that it is not always possible to find a consensus between a group of experts. So, it seems more appropriate to consider a set of possible values taking into account everyone's opinion. For instance, in order to provide a membership degree for an element of the universe, HFS can be useful to express this membership degree through a set of typical hesitant fuzzy elements (THFE), which will consider each opinion given by everyone in the group of experts.

On the other hand, it is common sense the importance of fuzzy implication functions which have been widely investigated and applied in many fields, as for example in decision making [9,17,23] and clustering [27]. In order to have a better understanding of logical connectives, one must know their properties and main characteristics. There are many different ways to model implication-like operators. In [6,18,19], a class of implication functions named (T,N)-implications was investigated and in their definition it is used a t-norm and a fuzzy negation. In this context, this work presents the definition of Typical Hesitant Implication Functions, including the class of (T,N)-implications, and the correspondent analysis of their main properties. Besides, and important contribution of the present work is that an admissible order on a HFS is provided allowing the comparison of hesitant fuzzy elements with different cardinalities. This novelty corroborates with the meaning of hesitant implication functions, providing semantic interpretation for implications setting found in multi-valued fuzzy logics. Therefore, the main properties proposed in the literature for fuzzy implications were studied and extended to HFS, which we discuss in this paper and present the properties of what we call Typical Hesitant (T,N)-Implication Functions (THIF).

This work is organized as follows: some preliminary and necessary concepts are given in Sect. 2, which allow us to provide in Sect. 3 an admissible order for the HFS elements and also allow us to introduce some operators, such as the typical hesitant fuzzy negations and typical hesitant t-norms. Then, Sect. 4 presents typical hesitant implication functions and discusses their main properties, including an incipient study on typical hesitant (T,N)-implication functions. Finally, Sect. 5 concludes the study.

2 Preliminaries

We start with some basic concepts of aggregation functions on the unit interval $[0,1]$, and then we recall triangular norms, fuzzy negations and fuzzy implication functions, for more details refer to [1,3,7,8,14,15].

Definition 1. *A function* $A : [0,1]^n \rightarrow [0,1]$ *is an* n*-ary aggregation function (AF) if it verifies, respectively, the isotonicity and boundary conditions, as follows:*

(A1) If $x_i \leq y_i$ *for each* $i = 1, \ldots, n$*, then* $A(x_1, \ldots, x_n) \leq A(y_1, \ldots, y_n)$*;*
(A2) $A(0, \ldots, 0) = 0$ *and* $A(1, \ldots, 1) = 1$*.*

Definition 2. *A function* $A : \bigcup_{n=1}^{\infty}[0,1]^n \to [0,1]$ *is an extended aggregation function (EAF) if the following condition holds:*

(A3) For each natural number $n \geq 2$, $A \restriction [0,1]^n \colon [0,1]^n \to [0,1]$ *is an AF and* $A(x,\ldots,x) = x$, *for each* $x \in [0,1]$.

Definition 3. *A function* $T\colon [0,1]^2 \to [0,1]$ *is a t-norm if, for each* $x, y, z \in [0,1]$, *it satisfies:*

(T1) $T(x,y) = T(y,x)$ *(commutativity);*
(T2) $T(x, T(y,z)) = T(T(x,y), z)$ *(associativity);*
(T3) If $x \leq y$ *then* $T(x,z) \leq T(y,z)$ *(isotonicity);*
(T4) $T(x,1) = x$ *(neutrality of* 1*-element).*

Observe that each t-norm is a bivariate aggregation function.

Definition 4. *A function* $N\colon [0,1] \to [0,1]$ *is a fuzzy negation if*

(N1) $N(0) = 1$ *and* $N(1) = 0$*;*
(N2) If $x \leq y$ *then* $N(y) \leq N(x)$, *for all* $x, y \in [0,1]$.

A fuzzy negation N *is strict if it is continuous and* $N(x) < N(y)$ *when* $y < x$ *and additionally, it is strong if it is involutive, i.e.*

(N3) $N(N(x)) = x, \forall x \in [0,1]$.

The most common strong fuzzy negation is $N_S(x) = 1 - x$, also known as the standard or Zadeh negation. Each strong fuzzy negation is strict but the converse does not hold. For example, the negation $N(x) = 1 - \sqrt{x}$ is strict but it is not strong.

An important notion in our work is the concept of implication functions, in the sense of Fodor and Roubens, see [2,3,10,20]) for additional information.

Definition 5. *A fuzzy implication is a function* $I\colon [0,1]^2 \to [0,1]$ *such that, for every* $x, y, z \in [0,1]$*:*

(I1) If $x \leq y$ *then* $I(y,z) \leq I(x,z)$ *(first place antitonicity);*
(I2) If $y \leq z$ *then* $I(x,y) \leq I(x,z)$ *(second place isotonicity);*
(I3) $I(0,y) = 1$ *(left boundary);*
(I4) $I(x,1) = 1$ *(right boundary);*
(I5) $I(1,0) = 0$ *(corner condition).*

Finally, let us recall the notions of partial ordering. Let P be a non-empty set, we say that a partial order $\preccurlyeq$ on the set P is a binary relation on P which satisfies, respectively, the reflexivity, antisymmetry and transitivity properties:

(P1) $p \preccurlyeq p$, for each $p \in P$,
(P2) If $p \preccurlyeq q$ and $q \preccurlyeq p$, then $p = q$ for all $p, q \in P$,
(P3) If $p \preccurlyeq q$ and $q \preccurlyeq r$, then $p \preccurlyeq r$ for all $p, q, r \in P$.

Note that we say $a \prec b$ when (a,b) is in a relation $\preccurlyeq$ but $a \neq b$. A set P with a partial order $\preccurlyeq$ is referred to as a partially ordered set (poset) and denoted by $(P, \preccurlyeq)$. If any two elements a, b are comparable in a poset $(P, \preccurlyeq)$, i.e. either $a \preccurlyeq b$ or $b \preccurlyeq a$, then the partial order $\preccurlyeq$ is said to be a linear (or total) order (and then P is a chain).

2.1 Typical Hesitant Fuzzy Sets

Hesitant Fuzzy Sets (HFS) were introduced by Torra in [24] and Torra and Narukawa in [25]. In their work, the membership degree of an element that belongs to a set was represented by means of a subset of $[0,1]$. In the process of decision-making, HFS can be useful to handle situations where there is indecision among many possible values for the preferences over objects. Formally, let $\wp([0,1])$ be the power set of $[0,1]$. A HFS A defined over U, where U is a non-empty set, is given by:

$$A = \{(x, \mu_A(x)) | x \in U\}. \tag{1}$$

and $\mu_A\colon U \rightarrow \wp([0,1])$, where μ_A is the membership function. There is a particular case when $\mu_A(x)$ is finite and non-empty for each $x \in U$, and in this case we have Typical Hesitant Fuzzy Sets (THFS).

Definition 6. [5] *Let $\mathbb{H} = \{X \subseteq [0,1] | X$ is finite and $X \neq \emptyset\}$. A THFS A defined over U is given by Eq. (1), where $\mu_A\colon U \rightarrow \mathbb{H}$.*

Each $X \in \mathbb{H}$ is named Typical Hesitant Fuzzy Element (THFE) of $\mathbb{H}$ and the cardinality of X, i.e. the number of elements of X, is referred to as $\#X$. The i^{th} smallest element of a THFE X will be denoted by $X^{(i)}$.

Some examples of THFS are $X = \{0.1, 0.4, 0.7\}$ and $Y = \{0.1, 0.6, 0.9\}$ where $\#X = \#Y = 3$. In those examples, $X^{(1)} = 0.1$ and $Y^{(2)} = 0.6$.

Definition 7. *From every EAF A, and knowing that the least and the greatest elements are $\mathbf{0}_{\mathbb{H}} = \{0\}$ and $\mathbf{1}_{\mathbb{H}} = \{1\}$, respectively, we define the function $f_A\colon \mathbb{H} \rightarrow [0,1]$ as:*

$$f_A(X) = \begin{cases} 0, & \text{if } X = \mathbf{0}_{\mathbb{H}} \\ 1, & \text{if } X = \mathbf{1}_{\mathbb{H}} \\ k \cdot A(X^{(1)}, \ldots, X^{(\#X)}) + \frac{1-k}{2}, & \text{otherwise.} \end{cases}$$

where $0 < k < 1$.

For example, if A is the arithmetic average, $k = 0.8$ and $X = \{0.1, 0.2, 0.4, 0.9\}$ then $f_A(X) = 0.8 \cdot 0.4 + \frac{0.2}{2} = 0.42$.

In the literature, one can find many proposals of orders for THFE, such as the ones found in [5,13,26,31–33]. The unique consensus among all these orders is that all of them refine[1] the following order on $\mathbb{H}$:

$$X \preceq_{\mathbb{H}} Y \text{ iff } X = \mathbf{0}_{\mathbb{H}} \text{ or } Y = \mathbf{1}_{\mathbb{H}} \text{ or } (\#X = \#Y \text{ and } X^{(i)} \leq Y^{(i)}, \forall i = 1, \ldots, \#X) \tag{2}$$

considered in [5]. However, this is very restrictive, since for two THFE to become comparable, it is required that both have the same cardinality.

[1] A partial order $\leq_1$ on a set S refines another partial order $\leq_2$ on S if $(S, \leq_2) \subseteq (S, \leq_1)$, i.e. for each $x, y \in S$ such that $x \leq_2 y$ we have that $x \leq_1 y$.

Our aim in the present work is to establish an admissible order to allow comparisons between THFE without this restriction. The idea of admissible order was presented in [11] for interval-valued fuzzy sets and after in [16] for interval-valued Atanassov's intuitionistic fuzzy sets. And in [21], the study of admissible total orders on hesitant fuzzy sets was included as a challenge. We acknowledge that some efforts have already been made in order to establish an admissible ordering for hesitant fuzzy sets, as seen in [26]. However, their proposal requires that both THFE must have the same cardinality.

In the next section, we present an admissible order in the typical hesitant fuzzy setting, which will allow us to introduce the notion of some typical hesitant connectives.

3 Admissible Orders for Typical Hesitant Fuzzy Elements

Take $\mathbb{H}^{(m)} = \{X \subseteq [0,1] | \#X = m\}$, we start by defining an admissible ordering for the typical hesitant fuzzy elements with cardinality m.

Definition 8. [26] *A total order $\leq_{\mathbb{H}^{(m)}}$ on $\mathbb{H}^{(m)}$ is said to be admissible if for all $X, Y \in \mathbb{H}^{(m)}$, we have that $X \leq_{\mathbb{H}^{(m)}} Y$ if and only if $X^{(i)} \leq Y^{(i)}$ for each $1 \leq i \leq m$.*

Example 1. (i) At first, take $\mathbb{H}^{(m)}$ for $m \geq 1$, and then consider the lexicographical order (with respect to the first variable) [11]. So, we have that $X \leq_{\mathbb{H}^{(m)}} Y$, if $X = Y$ or exists an i such that $X^{(i)}$ is strictly less than $Y^{(i)}$ and for all $j < i, X^{(j)} = Y^{(j)}$. For instance, $X = \{0.1, 0.4, 0.7\} \leq_{\mathbb{H}^{(3)}} \{0.1, 0.6, 0.9\} = Y$.

Definition 9. *A total order $\leq_{\mathbb{H}}$ on $\mathbb{H}$ is said to be admissible if, for all $X, Y \in \mathbb{H}$, we have that $X \leq_{\mathbb{H}} Y$ whenever $X \preceq_{\mathbb{H}} Y$.*

Observe that $(\mathbb{H}, \leq_{\mathbb{H}})$ is a bounded chain with the least and the greatest elements $\mathbf{0}_{\mathbb{H}} = \{0\}$ and $\mathbf{1}_{\mathbb{H}} = \{1\}$, respectively.

Remark 1. Note that for all admissible orders $\leq_{\mathbb{H}}$ on $\mathbb{H}$, their restriction to $\mathbb{H}^{(m)}$ is an admissible order on $\mathbb{H}^{(m)}$.

Next, we provide a method to generate admissible order for THFE based on an indexed family of admissible order $\leq_{\mathbb{H}^{(m)}}$, where $m \in \mathbb{N}^+$.

Theorem 1. *Let $(\leq_{\mathbb{H}^{(m)}})_{m \in \mathbb{N}^+}$ be family of indexed admissible orders and an EAF operator $A\colon \bigcup_{n=1}^{\infty} [0,1]^n \to [0,1]$. Then, the binary relation*

$$X \leq_{\mathbb{H}}^{A} Y \Leftrightarrow \begin{cases} f_A(X) < f_A(Y), \text{ or} \\ f_A(X) = f_A(Y) \text{ and } \#Y < \#X, \text{ or} \\ f_A(X) = f_A(Y) \text{ and } \#Y = \#X = m \text{ and } X \leq_{\mathbb{H}^{(m)}} Y \end{cases} \quad (3)$$

is an admissible order on $\mathbb{H}$.

Proof. It is straightforward the prove that the binary relation $\leq_{\mathbb{H}}^{A}$ is reflexive and antisymmetric. In addition, the relation $\leq_{\mathbb{H}}^{A}$ is also a transitive relation on $\mathbb{H}$, which is shown as follows:

(*i*) If $X \leq_{\mathbb{H}}^{A} Y$ and $Y \leq_{\mathbb{H}}^{A} Z$ for a given $X, Y, Z \in \mathbb{H}$, then $f_A(X) \leq f_A(Y) \leq f_A(Z)$. In case $f_A(X) < f_A(Y)$ and $f_A(Y) \leq f_A(Z)$ or $f_A(X) \leq f_A(Y)$ and $f_A(Y) < f_A(Z)$, it follows that $f_A(X) < f_A(Z)$. In case $f_A(X) = f_A(Y) = f_A(Z)$, we need to consider the four situations as described below:
Case 1:

$$\left.\begin{aligned} f_A(X) &= f_A(Y) \text{ and } \#Y < \#X \\ f_A(Y) &= f_A(Z) \text{ and } \#Z < \#Y \end{aligned}\right\} \Rightarrow f_A(X) = f_A(Z) \text{ and } \#Z < \#X;$$

Case 2:

$$\left.\begin{aligned} & f_A(X) = f_A(Y) \text{ and } \#Y<\#X \\ & f_A(Y) = f_A(Z) \text{ and } \#Z = \#Y = m \text{and } Y \leq_{\mathbb{H}(m)} Z \end{aligned}\right\} \Rightarrow f_A(X) = f_A(Z) \text{ and } \#Z<\#X;$$

Case 3:

$$\left.\begin{aligned} & f_A(X) = f_A(Y) \text{ and } \#X = \#Y = m \text{ and } X \leq_{\mathbb{H}(m)} Y \\ & f_A(Y) = f_A(Z) \text{ and } \#Z < \#Y \end{aligned}\right\} \Rightarrow f_A(X) = f_A(Z) \text{ and } \#Z<\#X;$$

Case 4:

$$\left.\begin{aligned} & f_A(X) = f_A(Y) \text{ and } \#X = \#Y = m \text{ and } X \leq_{\mathbb{H}(m)} Y \\ & f_A(Y) = f_A(Z) \text{ and } \#Y = \#Z = m \text{ and } Y \leq_{\mathbb{H}(m)} Z \end{aligned}\right\} \Rightarrow \begin{aligned} & f_A(X) = f_A(Z) \text{ and } \#X = \#Z = m \\ & \text{and } X \leq_{\mathbb{H}(m)} Z. \end{aligned}$$

For any of the above cases, $X \leq_{\mathbb{H}}^{A} Z$ and, therefore, the $\leq_{\mathbb{H}(m)}$-transitivity holds.

(*ii*) Besides, we also have to prove that either $X \leq_{\mathbb{H}}^{A} Y$ or $Y \leq_{\mathbb{H}}^{A} X$. There are three possible situations: (1) $f_A(X) < f_A(Y)$ and therefore $X \leq_{\mathbb{H}}^{A} Y$. (2) $f_A(Y) < f_A(X)$ and therefore $Y \leq_{\mathbb{H}}^{A} X$. (3) $f_A(X) = f_A(Y)$ and so, we also have three cases:

(3a) $\#X < \#Y$, so $Y \leq_{\mathbb{H}}^{A} X$.
(3b) $\#Y < \#X$, so $X \leq_{\mathbb{H}}^{A} Y$.
(3c) $\#X = \#Y = m$, so since $\leq_{\mathbb{H}(m)}$ is admissible, then $X \leq_{\mathbb{H}(m)} Y$ or $Y \leq_{\mathbb{H}(m)} X$.

Hence, $X \leq_{\mathbb{H}}^{A} Y$ or $Y \leq_{\mathbb{H}}^{A} X$.

(*iii*) Finally, let $X, Y \in \mathbb{H}$ and suppose $X \prec_{\mathbb{H}} Y$, then by Eq. (2), we have three possibilities: (1) $X = \mathbf{0}_{\mathbb{H}}$, and in this case, $f_A(X) = 0$ and $f_A(Y) > \frac{k}{2}$ and so, $f_A(X) < f_A(Y)$, i.e. $X <_{\mathbb{H}}^{A} Y$. (2) $Y = \mathbf{1}_{\mathbb{H}}$, which is analogous to (1). At last, (3) ($\#X = \#Y = m$ and $X^{(i)} \leq Y^{(i)}, \forall i = 1, \ldots, m$) then, because A is an EAF, we have $A(X^{(1)}, \ldots, X^{(m)}) \leq A(Y^{(1)}, \ldots, Y^{(m)})$ and therefore, $f_A(X) \leq f_A(Y)$. Hence, $X \leq_{\mathbb{H}(m)} Y$. Therefore, Theorem 1 holds.

Example 2. Considering the THFS $X = \{0.1, 0.4, 0.7\}, Y = \{0.1, 0.6, 0.9\}$ and $Z = \{0, 1\}$ the following EAF operators:

(1) $A_1(x_1,\ldots,x_n) = \sum_{i=1}^{n} \frac{x_i}{n}$;
(2) $A_2(x_1,\ldots,x_n) = \max\{x_i\}_{1\leq i\leq n}$;
(3) $A_3(x_1,\ldots,x_n) = \sqrt[n]{\prod_{i=1}^{n} x_i}$.

Thus, one can easily observe the following relations:

i. $X \leq_{\mathbb{H}}^{A_1} Z \leq_{\mathbb{H}}^{A_1} Y$.
ii. $X \leq_{\mathbb{H}}^{A_2} Y \leq_{\mathbb{H}}^{A_2} Z$.
iii. $Z \leq_{\mathbb{H}}^{A_3} X \leq_{\mathbb{H}}^{A_3} Y$.

In the sequence, some operators are given regarding admissible ordering on $\mathbb{H}$.

3.1 Typical Hesitant Fuzzy Negations

In [4,22], different definitions of Typical Hesitant Fuzzy Negations (THFN) were provided, both using partial orders. Now we introduce the concept of $\langle\mathbb{H},\leq_{\mathbb{H}}\rangle$-negations, which consider an admissible order $\leq_{\mathbb{H}}$.

Definition 10. *Let $\mathcal{N}:\mathbb{H}\to\mathbb{H}$ be a function. $\mathcal{N}$ is said to be a THFN with respect to an admissible order $\leq_{\mathbb{H}}$, $\langle\mathbb{H},\leq_{\mathbb{H}}\rangle$-negation in short, if the following conditions hold:*

($\mathcal{N}1$) *$\mathcal{N}(\mathbf{0}_{\mathbb{H}}) = \mathbf{1}_{\mathbb{H}}$ and $\mathcal{N}(\mathbf{1}_{\mathbb{H}}) = \mathbf{0}_{\mathbb{H}}$.*
($\mathcal{N}2$) *If $X \leq_{\mathbb{H}} Y$ then $\mathcal{N}(Y) \leq_{\mathbb{H}} \mathcal{N}(X)$.*

Additionally, we state that the $\langle\mathbb{H},\leq_{\mathbb{H}}\rangle$-negation $\mathcal{N}$ is strong if it is involutive, i.e. if for each $X\in\mathbb{H}$, it satisfies a third property, namely:

($\mathcal{N}3$) $\mathcal{N}(\mathcal{N}(X)) = X$.

Example 3. Consider an admissible order $\leq_{\mathbb{H}}$ on the EAF A_1, A_2 and A_3, given in Example 2. Now take the function $\mathcal{N}_S:\mathbb{H}\to\mathbb{H}$, defined as follows:

$$\mathcal{N}_S(X) = \{1-x|x\in X\}$$

It is easy to see that $\mathcal{N}_S$ is a $\langle\mathbb{H},\leq_{\mathbb{H}}\rangle$-negation for $\leq_{\mathbb{H}}^{A_1}$ and $\leq_{\mathbb{H}}^{A_3}$, but not for $\leq_{\mathbb{H}}^{A_2}$.

Remark 2. $\mathcal{N}_S$ is a trivial example of a strong THFN with respect to the admissible orders $\leq_{\mathbb{H}}^{A_1}$ and $\leq_{\mathbb{H}}^{A_3}$.

3.2 Typical Hesitant Triangular Norms

The extension of the notion of t-norms for typical hesitant fuzzy elements was presented in [5], taking into account the partial order proposed in that paper. The following definition generalizes this notion by considering admissible orders on $\mathbb{H}$.

Definition 11. *Let $\mathcal{T}\colon \mathbb{H}^2 \to \mathbb{H}$ and let $\leq_{\mathbb{H}}$ be an admissible order on $\mathbb{H}$. $\mathcal{T}$ is a typical hesitant triangular norm with respect to $\leq_{\mathbb{H}}$, or $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-t-norm in short, if*

($\mathcal{T}1$) *It is commutative: $\mathcal{T}(X,Y) = \mathcal{T}(Y,X)$;*
($\mathcal{T}2$) *It is associative: $\mathcal{T}(X,\mathcal{T}(Y,Z)) = \mathcal{T}(\mathcal{T}(X,Y),Z)$;*
($\mathcal{T}3$) *It is monotonic, i.e., if $X \leq_{\mathbb{H}} Y$ then $\mathcal{T}(X,Z) \leq_{\mathbb{H}} \mathcal{T}(Y,Z)$; and*
($\mathcal{T}4$) *$\mathbf{1}_{\mathbb{H}}$ is the neutral element: $\mathcal{T}(X,\mathbf{1}_{\mathbb{H}}) = X$.*

Remark 3. Observe that each $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-t-norm also verifies the following property:

($\mathcal{T}5$) $\mathcal{T}(X,\mathbf{0}_{\mathbb{H}}) = \mathbf{0}_{\mathbb{H}}$, $\forall X \in \mathbb{H}$.

In fact, $\mathcal{T}(X,\mathbf{0}_{\mathbb{H}}) \leq_{\mathbb{H}} \mathcal{T}(\mathbf{1}_{\mathbb{H}},\mathbf{0}_{\mathbb{H}}) = \mathbf{0}_{\mathbb{H}}$, for all $X \in \mathbb{H}$.

Other additional property is reported below:

($\mathcal{T}6$) $\mathcal{T}(X,\mathcal{N}(X)) = \mathbf{0}_{\mathbb{H}}$, $\forall X \in \mathbb{H}$.

Example 4. Consider an admissible order $\leq_{\mathbb{H}}$ on the EAF A_1, A_2 and A_3, given in Example 2. Now take the functions $\mathcal{T}_P, \mathcal{T}_M, \mathcal{T}_L\colon \mathbb{H}^2 \to \mathbb{H}$, defined as follows:

i. $\mathcal{T}_P(X,Y) = \{x \cdot y \mid x \in X, y \in Y\}$
ii. $\mathcal{T}_M(X,Y) = \{\min\{x,y\} \mid x \in X, y \in Y\}$
iii. $\mathcal{T}_L(X,Y) = \{\max\{x+y-1,0\} \mid x \in X, y \in Y\}$

It is possible to prove that $\mathcal{T}_P$, $\mathcal{T}_M$ and $\mathcal{T}_L$ are $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-t-norms for $\leq_{\mathbb{H}}^{A_1}$, $\leq_{\mathbb{H}}^{A_2}$ and $\leq_{\mathbb{H}}^{A_3}$.

4 Typical Hesitant Implication Functions

Here we introduce the notion of $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-typical hesitant implication functions, $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-THIF in short, considering an admissible order $\leq_{\mathbb{H}}$, discussing their main properties.

The typical hesitant fuzzy approach for a fuzzy implication is conceived as an extension of axioms in Definition 5.

Definition 12. *Let $\mathcal{I} : \mathbb{H}^2 \to \mathbb{H}$ and let $\leq_{\mathbb{H}}$ be an admissible order. $\mathcal{I}$ is a typical hesitant fuzzy implication function with respect to $\leq_{\mathbb{H}}$, $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-THIF in short, if for each $X, Y, Z \in \mathbb{H}$, the following properties are verified:*

($\mathcal{I}1$) If $X \leq_{\mathbb{H}} Y$ then $\mathcal{I}(Y,Z) \leq_{\mathbb{H}} \mathcal{I}(X,Z)$ (first place antitonicity);
($\mathcal{I}2$) If $Y \leq_{\mathbb{H}} Z$ then $\mathcal{I}(X,Y) \leq_{\mathbb{H}} \mathcal{I}(X,Z)$ (second place isotonicity);
($\mathcal{I}3$) $\mathcal{I}(\mathbf{0}_{\mathbb{H}},\mathbf{0}_{\mathbb{H}}) = \mathbf{1}_{\mathbb{H}}$ (corner condition 1);
($\mathcal{I}4$) $\mathcal{I}(\mathbf{1}_{\mathbb{H}},\mathbf{1}_{\mathbb{H}}) = \mathbf{1}_{\mathbb{H}}$ (corner condition 2); and
($\mathcal{I}5$) $\mathcal{I}(\mathbf{1}_{\mathbb{H}},\mathbf{0}_{\mathbb{H}}) = \mathbf{0}_{\mathbb{H}}$ (corner condition 3).

Table 1. Typical Hesitant Implication Functions

$\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-THIF	Restrictions
$\mathcal{I}_{FD}(X,Y) = \begin{cases} \mathbf{1}_{\mathbb{H}}, & \text{if } X \leq_{\mathbb{H}} Y, \\ \max(\mathcal{N}_S(X), Y), & \text{otherwise} \end{cases}$	$\mathcal{N}_S$ is a $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-negation
$\mathcal{I}_{GD}(X,Y) = \begin{cases} \mathbf{1}_{\mathbb{H}}, & \text{if } X \leq_{\mathbb{H}} Y, \\ Y, & \text{otherwise} \end{cases}$	–
$\mathcal{I}_{WB}(X,Y) = \begin{cases} \mathbf{1}_{\mathbb{H}}, & \text{if } X \leq_{\mathbb{H}} \mathbf{1}_{\mathbb{H}}, \\ Y, & \text{if } X = \mathbf{1}_{\mathbb{H}}. \end{cases}$	–
$\mathcal{I}_{GR}(X,Y) = \begin{cases} \mathbf{1}_{\mathbb{H}}, & \text{if } X \leq_{\mathbb{H}} Y, \\ \mathbf{0}_{\mathbb{H}}, & \text{otherwise} \end{cases}$	–

Proposition 1. *If $\mathcal{I}$ is an $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-THIF then it also satisfies the following properties:*

($\mathcal{I}$6a) $\mathcal{I}(\mathbf{0}_{\mathbb{H}}, Y) = \mathbf{1}_{\mathbb{H}}$ *(left boundary);*
($\mathcal{I}$6b) $\mathcal{I}(X, \mathbf{1}_{\mathbb{H}}) = \mathbf{1}_{\mathbb{H}}$ *(left and right boundary).*

Proof. Straightforward.

There are other properties that some $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-THIF can verify as the listed ones presented in the following.

($\mathcal{I}$7) $\mathcal{I}(\mathbf{1}_{\mathbb{H}}, X) = X$ (left neutrality property);
($\mathcal{I}$8) $\mathcal{I}(X, X) = \mathbf{1}_{\mathbb{H}}$ (identity principle);
($\mathcal{I}$9) $\mathcal{I}(X, \mathcal{I}(Y, Z)) = \mathcal{I}(Y, \mathcal{I}(X, Z))$ (exchange principle);
($\mathcal{I}$10) $\mathcal{I}(X, \mathcal{N}(Y)) = \mathcal{I}(Y, \mathcal{N}(X))$, if $\mathcal{N}$ is a strong $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-negation (right contraposition or contrapositive symmetry w.r.t. $\mathcal{N}$);
($\mathcal{I}$11) $\mathcal{I}(X, Y) = \mathcal{I}(\mathcal{N}(Y), \mathcal{N}(X))$, if $\mathcal{N}$ is a strong $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-negation (law of contraposition w.r.t. $\mathcal{N}$).

See in Table 1 examples illustrating the extension of important $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-THIF, namely: Fodor ($\mathcal{I}_{FD}$), Gödel ($\mathcal{I}_{GD}$), Weber ($\mathcal{I}_{WB}$) and Gaines-Rescher ($\mathcal{I}_{GR}$), with respect to the admissible $\leq_{\mathbb{H}}$-order.

4.1 Obtaining $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-THIF from $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-t-norms and $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-negations

Inspired in [6,18,19], which introduced a family of implication functions constructed from fuzzy negations and a triangular norm, in the following proposition we present a method to construct a $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-THIF from a $\mathbb{H}$-t-norm and a $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-negation.

Theorem 2. *Let $\mathcal{T}$ be a $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-t-norm and let $\mathcal{N}$ be a $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-negation. The function $\mathcal{I}_{\mathcal{T}}^{\mathcal{N}} : \mathbb{H}^2 \rightarrow \mathbb{H}$ defined by*

$$\mathcal{I}_{\mathcal{T}}^{\mathcal{N}}(X, Y) = \mathcal{N}(\mathcal{T}(X, \mathcal{N}(Y))) \tag{4}$$

is a typical hesitant implication function, denoted as $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-implication.

Proof. We have to prove that $\mathcal{I}_{\mathcal{T}}^{\mathcal{N}}$ satisfies the five properties of Definition 12.
($\mathcal{I}1$) If $X \leq_{\mathbb{H}} Y$ then by the monotonicity of $\mathcal{T}$, we have $\mathcal{T}(X,\mathcal{N}(Z)) \leq_{\mathbb{H}} \mathcal{T}(Y,\mathcal{N}(Z))$, and by ($\mathcal{N}1$), $\mathcal{N}(\mathcal{T}(Y,\mathcal{N}(Z))) \leq_{\mathbb{H}} \mathcal{N}(\mathcal{T}(X,\mathcal{N}(Z)))$. So, $\mathcal{I}_{\mathcal{T}}^{\mathcal{N}}(Y,Z) \leq_{\mathbb{H}} \mathcal{I}_{\mathcal{T}}^{\mathcal{N}}(X,Z)$.
($\mathcal{I}2$) If $Y \leq_{\mathbb{H}} Z$ then, by ($\mathcal{N}1$), $\mathcal{N}(Z) \leq_{\mathbb{H}} \mathcal{N}(Y)$. Therefore, by the monotonicity of $\mathcal{T}$, we have $\mathcal{T}(X,\mathcal{N}(Z)) \leq_{\mathbb{H}} \mathcal{T}(X,\mathcal{N}(Y))$ and $\mathcal{N}(\mathcal{T}(X,\mathcal{N}(Y))) \leq_{\mathbb{H}} \mathcal{N}(\mathcal{T}(X,\mathcal{N}(Z)))$. Thus, $\mathcal{I}_{\mathcal{T}}^{\mathcal{N}}(X,Y) \leq_{\mathbb{H}} \mathcal{I}_{\mathcal{T}}^{\mathcal{N}}(X,Z)$.
($\mathcal{I}3$) $\mathcal{I}_{\mathcal{T}}^{\mathcal{N}}(\mathbf{0}_{\mathbb{H}},\mathbf{0}_{\mathbb{H}}) \overset{Eq.(4)}{=} \mathcal{N}(\mathcal{T}(\mathbf{0}_{\mathbb{H}},\mathcal{N}(\mathbf{0}_{\mathbb{H}}))) \overset{(\mathcal{N}1)}{=} \mathcal{N}(\mathcal{T}(\mathbf{0}_{\mathbb{H}},\mathbf{1}_{\mathbb{H}})) \overset{(\mathcal{T}4)}{=} \mathcal{N}(\mathbf{0}_{\mathbb{H}}) \overset{(\mathcal{N}1)}{=} \mathbf{1}_{\mathbb{H}}$.
($\mathcal{I}4$) $\mathcal{I}_{\mathcal{T}}^{\mathcal{N}}(\mathbf{1}_{\mathbb{H}},\mathbf{1}_{\mathbb{H}}) \overset{Eq.(4)}{=} \mathcal{N}(\mathcal{T}(\mathbf{1}_{\mathbb{H}},\mathcal{N}(\mathbf{1}_{\mathbb{H}}))) \overset{(\mathcal{N}1)}{=} \mathcal{N}(\mathcal{T}(\mathbf{1}_{\mathbb{H}},\mathbf{0}_{\mathbb{H}})) \overset{(\mathcal{T}4)}{=} \mathcal{N}(\mathbf{0}_{\mathbb{H}}) \overset{(\mathcal{N}1)}{=} \mathbf{1}_{\mathbb{H}}$.
($\mathcal{I}5$) $\mathcal{I}_{\mathcal{T}}^{\mathcal{N}}(\mathbf{1}_{\mathbb{H}},\mathbf{0}_{\mathbb{H}}) \overset{Eq.(4)}{=} \mathcal{N}(\mathcal{T}(\mathbf{1}_{\mathbb{H}},\mathcal{N}(\mathbf{0}_{\mathbb{H}}))) \overset{(\mathcal{N}1)}{=} \mathcal{N}(\mathcal{T}(\mathbf{1}_{\mathbb{H}},\mathbf{1}_{\mathbb{H}})) \overset{(\mathcal{T}4)}{=} \mathcal{N}(\mathbf{1}_{\mathbb{H}}) \overset{(\mathcal{N}1)}{=} \mathbf{0}_{\mathbb{H}}$.
Therefore, Theorem 2 is verified.

Definition 13. *Let $\mathcal{T}$ be a $\langle\mathbb{H},\leq_{\mathbb{H}}\rangle$-t-norm and let $\mathcal{N}$ be a $\langle\mathbb{H},\leq_{\mathbb{H}}\rangle$-negation. The function $\mathcal{I}_{\mathcal{T}}^{\mathcal{N}}$ defined by Eq. (4) is called a typical hesitant (T,N)-implication function.*

Now, it is shown that a $\langle\mathbb{H},\leq_{\mathbb{H}}\rangle$-t-norm can be constructed from a $\langle\mathbb{H},\leq_{\mathbb{H}}\rangle$-THIF.

Proposition 2. [6] *Let $\mathcal{N}$ be a strong $\langle\mathbb{H},\leq_{\mathbb{H}}\rangle$-negation and let $\mathcal{T}$ be a $\langle\mathbb{H},\leq_{\mathbb{H}}\rangle$-t-norm. Then, for each $X,Y \in \mathbb{H}$,*

$$\mathcal{T}(X,Y) = \mathcal{N}(\mathcal{I}_{\mathcal{T}}^{\mathcal{N}}(X,\mathcal{N}(Y))).$$

Proof. Straightforward.

Proposition 3. *Let $\mathcal{I}_{\mathcal{T}}^{\mathcal{N}}$ be a typical hesitant (T,N)-implication function, let $\mathcal{T}$ be a $\langle\mathbb{H},\leq_{\mathbb{H}}\rangle$-t-norm and let $\mathcal{N}$ be a strong $\langle\mathbb{H},\leq_{\mathbb{H}}\rangle$-negation, then:*

(i) $\mathcal{I}_{\mathcal{T}}^{\mathcal{N}}$ satisfies the left neutrality property ($\mathcal{I}7$);
(ii) $\mathcal{I}_{\mathcal{T}}^{\mathcal{N}}$ satisfies the exchange principle ($\mathcal{I}9$);
(iii) $\mathcal{I}_{\mathcal{T}}^{\mathcal{N}}$ satisfies the law of right contraposition w.r.t. $\mathcal{N}$ ($\mathcal{I}10$);
(iv) $\mathcal{I}_{\mathcal{T}}^{\mathcal{N}}$ satisfies the law of contraposition w.r.t. $\mathcal{N}$ ($\mathcal{I}11$).

Proof. ($\mathcal{I}7$) Bearing in mind that $\mathcal{T}$ is a $\langle\mathbb{H},\leq_{\mathbb{H}}\rangle$-t-norm, then for any $X \in \mathbb{H}$, we have: $\mathcal{I}_{\mathcal{T}}^{\mathcal{N}}(\mathbf{1}_{\mathbb{H}},X) \overset{Eq.(4)}{=} \mathcal{N}(\mathcal{T}(\mathbf{1}_{\mathbb{H}},\mathcal{N}(X))) \overset{(\mathcal{T}1)/(\mathcal{T}4)}{=} \mathcal{N}(\mathcal{N}(X)) \overset{(\mathcal{N}3)}{=} X$.

($\mathcal{I}9$) Once $\mathcal{N}$ is a strong $\langle\mathbb{H},\leq_{\mathbb{H}}\rangle$-negation and $\mathcal{T}$ is a $\langle\mathbb{H},\leq_{\mathbb{H}}\rangle$-t-norm, we have

$$\begin{aligned}
\mathcal{I}_{\mathcal{T}}^{\mathcal{N}}(X,\mathcal{I}_{\mathcal{T}}^{\mathcal{N}}(Y,Z)) &\overset{Eq.(4)}{=} \mathcal{N}(\mathcal{T}(X,\mathcal{N}(\mathcal{N}(\mathcal{T}(Y,\mathcal{N}(Z))))))\\
&\overset{(\mathcal{N}3)}{=} \mathcal{N}(\mathcal{T}(X,\mathcal{T}(Y,\mathcal{N}(Z))))\\
&\overset{(\mathcal{T}1)}{=} \mathcal{N}(\mathcal{T}(X,\mathcal{T}(\mathcal{N}(Z),Y)))\\
&\overset{(\mathcal{T}2)}{=} \mathcal{N}(\mathcal{T}(\mathcal{T}(X,\mathcal{N}(Z)),Y))\\
&\overset{(\mathcal{T}1)}{=} \mathcal{N}(\mathcal{T}(Y,\mathcal{T}(X,\mathcal{N}(Z))))\\
&\overset{(\mathcal{N}3)}{=} \mathcal{N}(\mathcal{T}(Y,\mathcal{N}(\mathcal{N}(\mathcal{T}(X,\mathcal{N}(Z)))))) \overset{Eq.(4)}{=} \mathcal{I}_{\mathcal{T}}^{\mathcal{N}}(Y,\mathcal{I}_{\mathcal{T}}^{\mathcal{N}}(X,Z)).
\end{aligned}$$

($\mathcal{I}$10) Due to the commutativity property of $\mathcal{T}$ and since $\mathcal{N}$ is a strong $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-negation, it follows that

$$\begin{aligned} \mathcal{I}_{\mathcal{T}}^{\mathcal{N}}(X, \mathcal{N}(Y)) &\overset{Eq.\ (4)}{=} \mathcal{N}(\mathcal{T}(X, \mathcal{N}(\mathcal{N}(Y)))) \\ &\overset{(\mathcal{N}3)}{=} \mathcal{N}(\mathcal{T}(X, Y)) \\ &\overset{(\mathcal{T}1)}{=} \mathcal{N}(T(Y, X)) \\ &\overset{(\mathcal{N}3)}{=} \mathcal{N}(T(Y, \mathcal{N}(\mathcal{N}(X)))) \overset{Eq.\ (4)}{=} \mathcal{I}_{\mathcal{T}}^{\mathcal{N}}(Y, \mathcal{N}(X)). \end{aligned}$$

($\mathcal{I}$11) Analogously, from the commutativity of $\mathcal{T}$ and as $\mathcal{N}$ is a strong $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-negation, we have the next results

$$\begin{aligned} \mathcal{I}_{\mathcal{T}}^{\mathcal{N}}(\mathcal{N}(Y), \mathcal{N}(X)) &\overset{Eq.\ (4)}{=} \mathcal{N}(\mathcal{T}(\mathcal{N}(Y), \mathcal{N}(\mathcal{N}(X)))) \\ &\overset{(\mathcal{N}3)}{=} \mathcal{N}(\mathcal{T}(\mathcal{N}(Y), X)) \\ &\overset{(\mathcal{T}1)}{=} \mathcal{N}(\mathcal{T}(X, \mathcal{N}(Y))) \overset{Eq.\ (4)}{=} \mathcal{I}_{\mathcal{T}}^{\mathcal{N}}(X, Y). \end{aligned}$$

Proposition 4. *Let $\mathcal{I}_{\mathcal{T}}^{\mathcal{N}}$ be a typical hesitant (T,N)-implication and $\mathcal{T}$ be a $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-t-norm satisfying $\mathcal{T}6$. Then $\mathcal{I}_{\mathcal{T}}^{\mathcal{N}}$ satisfies the identity principle ($\mathcal{I}$8).*

Proof. Suppose that $\mathcal{I}_{\mathcal{T}}^{\mathcal{N}}$ is a typical hesitant (T,N)-implication, $\mathcal{T}$ is a $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-t-norm such that $\mathcal{T}(X, \mathcal{N}(Y)) = \mathbf{0}_{\mathbb{H}}$. Then, the following results are verified:

$$\mathcal{I}_{\mathcal{T}}^{\mathcal{N}}(X, X) \overset{Eq.(4)}{=} \mathcal{N}(\mathcal{T}(X, \mathcal{N}(X)) \overset{(\mathcal{T}6)}{=} \mathcal{N}(\mathbf{0}_{\mathbb{H}}) = \mathbf{1}_{\mathbb{H}}.$$

Therefore, Proposition 4 is verified.

Concluding, three examples illustrating such methodology are presented in the following:

Example 5. Based on the methodology established in Theorem 2 and main operators presented in Examples 2 and 4, we construct new $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-implication functions in the class of typical hesitant (T,N)-implication functions. Meaning that, the next three functions $\mathcal{I}_{\mathcal{T}_P}^{\mathcal{N}_S}, \mathcal{I}_{\mathcal{T}_M}^{\mathcal{N}_S}, \mathcal{I}_{\mathcal{T}_L}^{\mathcal{N}_S} : \mathbb{H}^2 \rightarrow \mathbb{H}$, respectively expressed as follows

$$\begin{aligned} \mathcal{I}_{\mathcal{T}_P}^{\mathcal{N}_S}(X, Y) &= \mathcal{N}_S(\mathcal{T}_P(X, \mathcal{N}_S(Y))), \\ \mathcal{I}_{\mathcal{T}_M}^{\mathcal{N}_S}(X, Y) &= \mathcal{N}_S(\mathcal{T}_M(X, \mathcal{N}_S(Y))), \\ \mathcal{I}_{\mathcal{T}_L}^{\mathcal{N}_S}(X, Y) &= \mathcal{N}_S(\mathcal{T}_L(X, \mathcal{N}_S(Y))); \end{aligned}$$

are $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-THIF with respect to the admissible linear $\leq_{\mathbb{H}}^{A_1}$-order and $\leq_{\mathbb{H}}^{A_3}$-order:

5 Final Remarks

Regarding many extensions of multi-valued fuzzy logics, this paper introduces the definition of the class of (T,N)-implications in the context of Typical Hesitant Implication Functions, extending such analysis in order to consider their main properties: left neutrality property, right boundary, law of contraposition and its corresponding right contraposition based on $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-negations, also including the identity and exchange principles. As another important contribution, we investigate the conditions under which the use of admissible orders based on aggregation operators, performed on $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-lattice, allows a comparison of THFS with different cardinalities. Additionally, among several partial orders defined over $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$, the discussed admissible orders on $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$ promote comparisons even between THFS with different cardinalities.

Our results in the class of $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-implication functions, named (T,N)-implications extend the previous study presented in [19]. Thus, this novelty methodology corroborates with the meaning of $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-implication functions, providing semantic interpretation for implications setting found in multi-valued fuzzy logics.

As ongoing work, we are considering to prove some other properties of $\langle \mathbb{H}, \leq_{\mathbb{H}} \rangle$-THIF operators as a support to generate hesitant fuzzy subsethood measures, based on the studied class of (T,N)-implications. This study needs to consider the discussion about how can we have any (generalized) property of typical hesitant (T,N)-implication, for which the standard property is not satisfied by the standard (T,N)-implication.

One can easily observe that fuzzy implications have been used in preference computations also including ordering relations in related works, see e.g. [9]. Following such research approach, further work also intends to apply the present results on typical hesitant (T,N)-implication in order to achieve new results on hesitant-based fuzzy preferences relations.

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