A Note on Aggregation of Intuitionistic Values

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Abstract. Atanassov's intuitionistic fuzzy set theory is based on the lattice L^* of intuitionistic values and their aggregation. There are lots of works dealing with this topic, but mostly considering some particular cases. In this contribution, we offer a rather general view on aggregation of intuitionistic values with transparent proofs of several properties which significantly shorten the related proofs for particular cases known from the literature.

Keyword: Aggregation function, Intuitionistic values, Representable aggregation function, t-norm, t-conorm

1 Introduction

The first connectives/aggregation functions for intuitionistic fuzzy logic/theory were proposed by Atanassov [1,2]. They were related to the basic aggregation functions on $[0,1]$, i.e., minimum, maximum, product and probabilistic sum of aggregation functions. Later, Deschrijver and Kerre [11] introduced a class of intuitionistic connectives based on t-norms on $[0,1]$ and the corresponding dual t-conorms. Xu [19] introduced several functions dealing with intuitionistic values, including aggregation functions based on the standard product, such as intuitionistic fuzzy weighted averaging (IFWA) operator. Similar functions/aggregation functions dealing with intuitionistic values but based on the Einstein t-norm (a particular t-norm from the Hamacher family [16]) were proposed in [18] and consequently mentioned in several other papers. In papers [15,17], Hamacher's product or Dombi's t-norms were considered for deriving aggregation functions acting on intuitionistic values.

In all mentioned cases, one can find long and tedious proofs of some properties of the introduced functions, in particular, the proofs of the basic fact that (in some cases) they are aggregation functions on the lattice L^* of intuitionistic values.

M.-J. Lesot et al. (Eds.): IPMU 2020, CCIS 1238, pp. 411–418, 2020.
https://doi.org/10.1007/978-3-030-50143-3_31

The main aim of this contribution is to propose a unified approach to intuitionistic aggregation based on strict t-norms and provide short and transparent proofs of important properties of the introduced functions. As a byproduct, our results can help to the researchers who work in intuitionistic fuzzy set theory to avoid introducing "new" intuitionistic aggregation functions whose only novelty is that they are based on t-norms which have not been explicitly mentioned in the literature yet. Moreover, we show several techniques for making proofs of new results in aggregation domain shorter and more transparent.

2 Preliminaries

We expect that the readers are familiar with the basics of fuzzy set theory [20] and intuitionistic fuzzy set theory [1–3]. Similarly, the basic knowledge concerning triangular norms [16] and aggregation functions [4,6,13] is expected. Here we only recall some most relevant basic notions.

Note that the concept of Atanassov's intuitionistic fuzzy sets was introduced to generalize the concept of Zadeh's fuzzy sets [20], and the original definition [1] of an Atanassov intuitionistic fuzzy set (AIFS for short) in a universe X can be recalled as follows:

Definition 2.1. *Let $X \neq \emptyset$ be any set. An Atanassov intuitionistic fuzzy set A in X is defined as a set of triplets*

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\},$$

where $\mu_A, \nu_A \colon X \to [0,1]$, and for all $x \in X$, also $\mu_A(x) + \nu_A(x) \in [0,1]$.

Note that, for each element $x \in X$, $\mu_A(x)$ and $\nu_A(x)$ are called the membership and the non-membership degrees of x to A, respectively, and the pair $(\mu_A(x), \nu_A(x))$ is called an Atanassov intuitionistic fuzzy value [18].

Denote by L^* the set of all possible intuitionistic values, i.e.,

$$L^* = \{(a,b) \in [0,1]^2 \mid a + b \leq 1\}.$$

In particular, the set L^* with the partial order $\leq_{L^*}$, given by

$$(a,b) \leq_{L^*} (c,d) \text{ if and only if } a \leq c \text{ and } b \geq d,$$

form a bounded lattice with the top element $\mathbf{1}_{L^*} = \top = (1,0)$ and bottom element $\mathbf{0}_{L^*} = \bot = (0,1)$.

Then an Atanassov intuitionistic fuzzy set A in a universe X can be seen as a mapping $A \colon X \to L^*$. To define basic operations of union and intersection of AIFSs, Atanassov [1] has proposed to apply pointwisely some appropriate binary functions mapping $(L^*)^2$ onto L^*. Due to their associativity, the n-ary forms of these operations are uniquely defined.

The first intuitionistic connectives proposed in [1] are just the join $\vee_{L^*}$ and meet $\wedge_{L^*}$, inducing the union and intersection of AIFSs. Clearly,

$$(a,b) \vee_{L^*} (c,d) = (a \vee c, b \wedge d), \tag{1}$$

and

$$(a, b) \wedge_{L^*} (c, d) = (a \wedge c, b \vee d). \tag{2}$$

As another type of operations on L^*, Atanassov [1] proposed the operations given by

$$(a, b) \oplus (c, d) = (a + c - ac, bd), \tag{3}$$

and

$$(a, b) \otimes (c, d) = (ac, b + d - bd). \tag{4}$$

Obviously, all four operations $\vee_{L^*}, \wedge_{L^*}, \oplus$ and $\otimes$ are binary aggregation functions on L^* [5,18], i.e., they are non-decreasing in each variable with respect to the order $\leq_{L^*}$, and they aggregate $\top$ and $\top$ into $\top$ and also $\bot$ and $\bot$ into $\bot$. Moreover, they are commutative and associative, $\top$ is a neutral element of $\wedge_{L^*}$ and $\otimes$, whereas $\bot$ is a neutral element of $\vee_{L^*}$ and $\oplus$. Hence, $\wedge_{L^*}$ and $\otimes$ can be seen as t-norms, and $\vee_{L^*}$ and $\oplus$ as t-conorms on L^*. In all four cases we see the connection with the standard t-norms T_M and T_P and the standard t-conorms S_M and S_P, defined on the real unit interval $[0, 1]$ by

$$\begin{aligned} T_M(a, b) &= \min\{a, b\}, \qquad T_P(a, b) = ab, \\ S_M(a, b) &= \max\{a, b\}, \qquad S_P(a, b) = a + b - ab. \end{aligned}$$

For more details see, e.g., [16]. Formally, $\vee_{L^*} \approx (S_M, T_M)$, i.e.,

$$(a, b) \vee_{L^*} (c, d) = (S_M(a, c), T_M(b, d)).$$

Further, $\wedge_{L^*} \approx (T_M, S_M)$, and thus it can be seen as a dual operation to $\vee_{L^*}$ on L^*. Similarly, $\oplus \approx (S_P, T_P)$ and $\otimes \approx (T_P, S_P)$ are dual operations.

This observation has been generalized by Deschrijver and Kerre [11]. For an arbitrary t-norm $T\colon [0, 1]^2 \to [0, 1]$ and the corresponding dual t-conorm $S\colon [0, 1]^2 \to [0, 1]$, satisfying, for each $(a, b) \in [0, 1]^2$, $S(a, b) = 1 - T(1 - a, 1 - b)$, the mappings $\mathbf{T}_T, \mathbf{S}_T\colon (L^*)^2 \to L^*$,

$$\mathbf{T}_T \approx (T, S) \text{ and } \mathbf{S}_T \approx (S, T), \tag{5}$$

are called a representable t-norm and a t-conorm on L^*, respectively.

For possible applications of AIFS theory some other operations/aggregation functions have also been proposed. For example, Xu [19] has proposed to define the product "$\cdot$", $\cdot\colon]0, \infty[\times L^* \to L^*$ by

$$\lambda \cdot (a, b) = (1 - (1 - a)^\lambda, b^\lambda) \tag{6}$$

and the power on the same domain by

$$(a, b)^\lambda = (a^\lambda, 1 - (1 - b)^\lambda). \tag{7}$$

Based on these functions, in [19] some new n-ary aggregation functions on L^* were proposed and studied, for example, $IFWA_{\mathbf{w}}$ (intuitionistic fuzzy weighted average) given by

$$IFWA_{\mathbf{w}}((a_1, b_1), \dots, (a_n, b_n)) = \left(1 - \prod_{i=1}^{n} (1 - a_i)^{w_i}, \prod_{i=1}^{n} b^{w_i}\right), \tag{8}$$

where $\mathbf{w} = (w_1, \ldots, w_n) \in [0,1]^n$, $\sum_{i=1}^{n} w_i = 1$, is a weighting vector. Evidently, formulas (6), (7) and (8) are based on the product t-norm. Similarly, modifying formulas (6), (7) and (8), Wang and Liu [18] have proposed functions dealing with values from L^* based on the Einstein t-norm $T_E\colon [0,1]^2 \to [0,1]$ given by $T_E(a,b) = \frac{ab}{2-a-b+ab}$ and its dual, Einstein t-conorm, $S_E\colon [0,1]^2 \to [0,1]$, $S_E(a,b) = \frac{a+b}{1+ab}$ (note that the formula defining S_E is just the Einstein formula for summing relative velocities). For example, the product $\cdot_E\colon]0,\infty[\times L^* \to L^*$ they have defined as follows:

$$\lambda \cdot_E (a,b) = \left(\frac{(1+a)^\lambda - (1-a)^\lambda}{(1+a)^\lambda + (1-a)^\lambda}, \frac{2b^\lambda}{(2-b)^\lambda + b^\lambda} \right). \tag{9}$$

3 Representable Aggregation Functions on L^*

All till now discussed aggregation functions on L^* have been based on some aggregation function $A\colon [0,1]^n \to [0,1]$ and its dual $A^d\colon [0,1]^n \to [0,1]$,

$$A^d(a_1, \ldots, a_n) = 1 - A(1-a_1, \ldots, 1-a_n)$$

and defined as follows: $\mathbf{A}\colon (L^*)^n \to L^*$, $\mathbf{A} \approx (A, A^d)$,

$$\mathbf{A}((a_1,b_1), \ldots, (a_n,b_n)) = (A(a_1, \ldots, a_n), A^d(b_1, \ldots, b_n)). \tag{10}$$

It is obvious that for any aggregation functions $A, B\colon [0,1]^n \to [0,1]$ such that $B \le A^d$, the mapping $\mathbf{A}\colon (L^*)^n \to L^*$, $\mathbf{A} \approx (A, B)$,

$$\mathbf{A}((a_1,b_1), \ldots, (a_n,b_n)) = (A(a_1, \ldots, a_n), B(b_1, \ldots, b_n)) \tag{11}$$

is an aggregation function on L^*. Moreover, $\mathbf{A}$ inherits all common properties of A and B. In particular, if both A and B are associative (symmetric/idempotent or if they have a neutral element/an annihilator) then $\mathbf{A} \approx (A, B)$ also possesses this property. Such types of intuitionistic aggregation functions are called representable intuitionistic aggregation functions [10,12]. They also cover intuitionistic aggregation functions mentioned in (1)–(5) and (8).

As an example when $B \ne A^d$ consider $A = T_P$ and $B = S_M$. Clearly, $S_M \le T_P^d = S_P$ and thus $\mathbf{T} \approx (T_P, S_M)$ given by

$$\mathbf{T}((a,b),(c,d)) = (ac, b \vee d)$$

is a representable t-norm on L^*.

Note that there are also non-representable aggregation functions on L^*, including t-norms. For example, consider the mapping $\mathbf{T}\colon (L^*)^2 \to L^*$ given by

$$\mathbf{T}((a,b),(c,d)) = (a \wedge c, (b \vee (1-c)) \wedge ((1-a) \vee d)).$$

Then $\mathbf{T}$ is a t-norm on L^* which is not representable. For more examples see [10].

4 Main Results

Formulas (3), (4), (6), (7) and (8) are related to the product t-norm. Similarly, (9) and several other formulas in [15] and [18] are related to the Hamacher and Einstein t-norms, respectively, and formulas based on Dombi's t-norms have been discussed in [17]. In all mentioned cases, a strict t-norm $T\colon [0,1]^2 \to [0,1]$ and its dual t-conorm $S\colon [0,1]^2 \to [0,1]$ have been considered. The main advantage of strict t-norms is the fact that they are generated by an additive generator $t\colon [0,1] \to [0,\infty]$ which is a decreasing bijection, i.e.,

$$T(a,b) = t^{-1}(t(a) + t(b)).$$

To have a unique correspondence between strict t-norms and their additive generators, one can always choose the generator satisfying $t(0.5) = 1$. Then the dual t-conorm S is generated by an additive generator $s\colon [0,1] \to [0,\infty]$, $s(x) = t(1-x)$,

$$S(a,b) = s^{-1}(s(a) + s(b)) = 1 - t^{-1}(t(1-a) + t(1-b)).$$

Clearly, a representable intuitionistic t-norm $\mathbf{T}_T \approx (T,S)$ and t-conorm $\mathbf{S}_T \approx (S,T)$ in their n-ary form are given by

$$\mathbf{T}_T((a_1,b_1),\dots,(a_n,b_n)) = \left(t^{-1}\left(\sum_{i=1}^{n} t(a_i)\right), 1 - t^{-1}\left(\sum_{i=1}^{n} t(1-b_i)\right)\right)$$

and

$$\mathbf{S}_T((a_1,b_1),\dots,(a_n,b_n)) = \left(1 - t^{-1}\left(\sum_{i=1}^{n} t(1-a_i)\right), t^{-1}\left(\sum_{i=1}^{n} t(b_i)\right)\right).$$

Now, fix a strict t-norm T having an additive generator t, and introduce the mappings:

$$\lambda \cdot_T (a,b) = \left(1 - t^{-1}\left(\lambda t(1-a)\right), t^{-1}\left(\lambda t(b)\right)\right),\ \lambda > 0,\ (a,b) \in L^*, \tag{12}$$

$$(a,b)^{\lambda_T} = \left(t^{-1}\left(\lambda t(a)\right), 1 - t^{-1}\left(\lambda t(1-b)\right)\right),\ \lambda > 0,\ (a,b) \in L^*, \tag{13}$$

and for all $(a_1,b_1),\dots,(a_n,b_n) \in L^*$,

$$IFWA_{\mathbf{w}}^T((a_1,b_1),\dots,(a_n,b_n)) = \mathbf{S}_T(w_1 \cdot_T (a_1,b_1),\dots,w_n \cdot_T (a_n,b_n)). \tag{14}$$

Then we have

$$IFWA_{\mathbf{w}}^T((a_1,b_1),\dots,(a_n,b_n)) = \left(1 - t^{-1}\left(\sum_{i=1}^{n} w_i t(1-a_i)\right), t^{-1}\left(\sum_{i=1}^{n} w_i t(b_i)\right)\right), \tag{15}$$

and the following results hold:

Theorem 4.1. *Let $T\colon [0,1]^2 \to [0,1]$ be a strict t-norm with an additive generator t. Then*

(i) $\mathbf{S}_T((a,b),(a,b)) = 2 \cdot_T (a,b)$;
(ii) $\mathbf{T}_T((a,b),(a,b)) = (a,b)^{2_T}$;
(iii) $IFWA_{\mathbf{w}}^T$ *is an idempotent representable* n*-ary aggregation function on* L^**, which is* $\oplus_T$*-additive where* $\oplus_T = S_T$.

Proof: The proofs of (i) and (ii) are only a matter of simple calculations, e.g.,

$$\mathbf{S}_T((a,b),(a,b)) = \left(1 - t^{-1}\left(2t(1-a)\right), t^{-1}\left(2t(b)\right)\right) = 2 \cdot_T (a,b).$$

To prove (iii), first observe that for any $b_1, \ldots, b_n \in [0,1]$ and a weighting vector $\mathbf{w}$, the formula

$$t^{-1}\left(\sum_{i=1}^{n} w_i t(b_i)\right) = T_{\mathbf{w}}(b_1, \ldots, b_n)$$

defines a weighted t-norm $T_{\mathbf{w}}$, see [7]. Obviously, for the dual t-conorm S to T and $a_1, \ldots, a_n \in [0,1]$, the formula

$$1 - t^{-1}\left(\sum_{i=1}^{n} w_i t(1-a_i)\right) = S_{\mathbf{w}}(a_1, \ldots, a_n) = T_{\mathbf{w}}^d(a_1, \ldots, a_n)$$

gives the weighted t-conorm $S_{\mathbf{w}}$, and the weighted aggregation functions $T_{\mathbf{w}}$ and $S_{\mathbf{w}}$ based on the same weighting vector are dual. Then $IFWA_{\mathbf{w}}^T \approx (S_{\mathbf{w}}, T_{\mathbf{w}})$ is a representable aggregation function on L^*. As both $S_{\mathbf{w}}$ and $T_{\mathbf{w}}$ are idempotent, $IFWA_{\mathbf{w}}^T$ is also idempotent.

Finally, verification of the $\oplus_T$-additivity of $IFWA_{\mathbf{w}}^T$, i.e., the property

$$\begin{aligned} &IFWA_{\mathbf{w}}^T\left((a_1,b_1) \oplus_T (c_1,d_1), \ldots, (a_n,b_n) \oplus_T (c_n,d_n)\right) \\ &= IFWA_{\mathbf{w}}^T\left((a_1,b_1), \ldots, (a_n,b_n)\right) \oplus_T IFWA_{\mathbf{w}}^T\left((c_1,d_1), \ldots, (c_n,d_n)\right) \end{aligned}$$

is a matter of an easy computation. □

Note that the proofs of these results realized for particular t-norms known from the literature, see, e.g. [18,19], are rather long and non-transparent.

Remark 4.1. Nilpotent t-norms [16] are also generated by additive generators. In this case, an additive generator is a decreasing bijection $t\colon [0,1] \to [0,1]$, and then

$$T(a,b) = t^{-1}\left(\min\{1, t(a) + t(b)\}\right),$$

and for the corresponding dual t-conorm S we have

$$S(a,b) = t^{-1}\left(\min\{1, t(1-a) + t(1-b)\}\right).$$

Surprisingly, nilpotent t-norms are rarely applied in intuitionistic fuzzy set theory (IFST), possibly except the Łukasiewicz t-norm T_L, $T_L(a,b) = \max\{0, a + b - 1\}$ and its dual t-conorm S_L, $S_L(a,b) = \min\{1, a+b\}$. The corresponding intuitionistic t-norm $\mathbf{T}_L \approx (T_L, S_L)$ and t-conorm $\mathbf{S}_L \approx (S_L, T_L)$ form the basis of probability theory in IFST, see [14] and several Riečan's works [8,9]. Based on a nilpotent t-norm with additive generator t, one can introduce the related product, power and weighted mean, paraphrasing formulas (12), (13) and (15). Considering T_L and its additive generator $t_L\colon [0,1] \to [0,1]$, $t_L(x) = 1 - x$, we obtain:

$$\begin{aligned} \lambda \cdot_{T_L} (a,b) &= \left(1 - t_L^{-1}\left(\min\{1, \lambda t_L(1-a)\}\right), t_L^{-1}\left(\min\{1, \lambda t_L(b)\}\right)\right) \\ &= \left(\min\{1, \lambda a\}, \max\{0, 1 - \lambda(1-b)\}\right); \\ (a,b)^{\lambda_{T_L}} &= \left(\max\{0, 1 - \lambda(1-a)\}, \min\{1, \lambda b\}\right), \end{aligned}$$

and

$$IFWA_{\mathbf{w}}^{T_L}((a_1,b_1),\ldots,(a_n,b_n)) = \left(\sum_{i=1}^{n} w_i a_i, \sum_{i=1}^{n} w_i b_i\right),$$

i.e., $IFWA_{\mathbf{w}}^{T_L}$ is an intuitionistic fuzzy weighted arithmetic mean. Note that considering continuous t-norms and their weighted forms discussed in [7], one can introduce more general types of weighted aggregation functions also in the intuitionistic framework.

5 Concluding Remarks

We have introduced a rather general approach to functions dealing with intuitionistic values from L^*, in particular to functions based on continuous Archimedean t-norms and their additive generators. Our results significantly generalize the related results based on special strict t-norms (e.g., the standard product, Einstein's t-norm, Dombi's t-norms) known from the literature. The main advantage of our approach is its transparentness and obtaining short general proofs of the results in contrast to the long and non-transparent proofs of results in particular cases. We believe that our paper will contribute not only to the theoretical basis of IFST, but in particular, it can make intuitionistic fuzzy sets more efficient for applications. Moreover, it can also help in related domains, such as picture fuzzy sets, neutrosophic fuzzy sets, etc.

Acknowledgement. The authors kindly acknowledge the support of the project APVV-18-0052 and the grant VEGA 1/0614/18.

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