# Between Proper and Strong Edge-Colorings of Subcubic Graphs

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**Abstract.** In a proper edge-coloring the edges of every color form a matching. A matching is *induced* if the end-vertices of its edges induce a matching. A *strong edge-coloring* is an edge-coloring in which the edges of every color form an induced matching. We consider intermediate types of edge-colorings, where some of the colors are allowed to form matchings, and the remaining form induced matchings. Our research is motivated by the conjecture proposed in a recent paper on $S$-packing edge-colorings (N. Gastineau and O. Togni, On S-packing edge-colorings of cubic graphs, Discrete Appl. Math. 259 (2019)). We prove that every graph with maximum degree 3 can be decomposed into one matching and at most 8 induced matchings, and two matchings and at most 5 induced matchings. We also show that if a graph is in class I, the number of induced matchings can be decreased by one, hence confirming the conjecture for this class of graphs.

**Keywords:** Strong edge-coloring · $S$-packing edge-coloring · Induced matching

## 1 Introduction

A *proper edge-coloring* of a graph $G = (V, E)$ is an assignment of colors to the edges of $G$ such that adjacent edges are colored with distinct colors. Due to a remarkable result of Vizing [22], we know that the minimum number of colors needed to color the edges of a graph $G$, the *chromatic index* of $G$ (denoted by $\chi'(G)$), is either $\Delta(G)$ or $\Delta(G)+1$, $\Delta(G)$ being the maximum degree of $G$. The graphs with the former value of the chromatic index are commonly said to be in *class I*, and the latter in *class II*.

In this paper, we are interested in graphs with maximum degree 3, to which we will refer as *subcubic graphs*. We need at most 4 colors to color such graphs; the complete graph on four vertices with one edge subdivided being the smallest representative of a class II subcubic graph, and the Petersen graph being the smallest 2-connected class II cubic graph. For subcubic graphs of class II,


L. Gąsieniec et al. (Eds.): IWOCA 2020, LNCS 12126, pp. 355–367, 2020.
https://doi.org/10.1007/978-3-030-48966-3_27

it has been shown that they can be colored in such a way that one of the colors (usually denoted $\delta$) is used relatively rarely (cf. [1,6]). This motivates the question if the edges of color $\delta$ can be pairwise distant. Note that we consider the distance between edges as the distance between the corresponding vertices in the line graph, i.e. adjacent edges are said to be at distance 1. Payan [17] and independently Fouquet and Vanherpe [6] proved that every subcubic graph with chromatic index 4 admits a proper edge-coloring such that the edges of one color are at distance at least 3, i.e. the end-vertices of those edges induce a matching in the graph.

Gastineau and Togni [7] investigated a generalization of edge-colorings with the property described above. For a given non-decreasing sequence of integers $S = (s_1, \dots, s_k)$, an $S$-packing edge-coloring of a graph is a decomposition of edges into disjoint sets $X_1, \dots, X_k$, where the edges in the set $X_i$ are pairwise at distance at least $s_i + 1$. A set $X_i$ is called an *$s_i$-packing*; a 1-packing is simply a matching, and a 2-packing is an induced matching. To simplify the notation, we denote repetitions of same elements in $S$ using exponents, e.g. $(1, 2, 2, 2)$ can be written as $(1, 2^3)$.

The notion of $S$-packing edge-colorings is motivated by its vertex counterpart, introduced by Goddard and Xu [9] as a natural generalization of the packing chromatic number [8]. In [7], the authors consider $S$-packing edge-colorings of subcubic graphs with prescribed number of 1's in the sequence. Vizing's result translated to $S$-packing edge-coloring gives that every subcubic graph admits a $(1, 1, 1, 1)$-packing edge-coloring, while class I subcubic graphs are $(1, 1, 1)$-packing edge-colorable. Moreover, by Payan's, Fouquet's and Vanherpe's result, we have that there is a $(1, 1, 1, 2)$-packing edge-coloring for any subcubic graph.

**Theorem 1 (Payan [17], and Fouquet & Vanherpe [6]).** *Every subcubic graph admits a $(1, 1, 1, 2)$-packing edge-coloring.*

Here 2 cannot be changed to 3, due to the Petersen and the Tietze graphs (depicted in Fig. 1): they both have chromatic index 4, and we need at least two edges of each color. Since every pair of edges is at distance at most 3, we have the tightness. However, Gastineau and Togni do believe the following conjecture is true.

*Conjecture 1 (Gastineau and Togni [7]).* Every cubic graph different from the Petersen and the Tietze graph is $(1, 1, 1, 3)$-packing edge-colorable.

Clearly, reducing the number of 1's in sequences increases the total number of needed colors, i.e. the length of the sequence. In fact, if there is no 1 in a sequence, i.e. the edges of every color class induce a matching, the coloring is called a *strong edge-coloring*. It has been proved by Andersen [3] and independently by Horák, Qing, and Trotter [11] that every subcubic graph admits a strong edge-coloring with at most 10 colors, i.e. a $(2^{10})$-packing edge-coloring. The number of colors is tight, e.g. the Wagner graph in Fig. 2 needs 10 colors for a strong edge-coloring. Let us remark here that the Wagner graph is in class I, meaning that smallest chromatic index does not necessarily mean less number of colors for a strong edge-coloring of a graph.

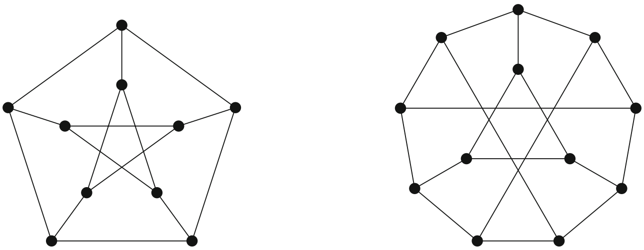

**Fig. 1.** The Petersen (left) and the Tietze graph (right) admit a $(1, 1, 1, 2)$-packing edge-coloring, and 2 cannot be increased to 3.

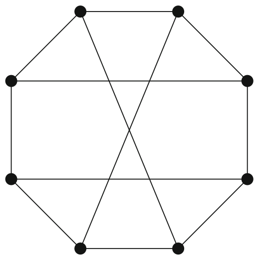

**Fig. 2.** The Wagner graph is the smallest cubic graph which needs 10 colors for a strong edge-coloring.

Proper and strong edge-coloring of subcubic graphs have been studied extensively already in the previous century. In [7], Gastineau and Togni started filling the gap by considering $(1^k, 2^\ell)$-packing edge-colorings for $k \in \{1, 2\}$. They proved that every cubic graph with a 2-factor admits a $(1, 1, 2^5)$-packing edge-coloring, and the number of required 2-packings reduces by one if the graph is class I. For the case with one 1-packing, they remark that using the bound for the strong edge-coloring one obtains that every subcubic graph admits a $(1, 2^9)$-packing edge-coloring. These bounds are clearly not tight, and they propose a conjecture (the items $(a)$ and $(c)$ in Conjecture 2), which motivated the research presented in this paper. The case $(b)$ has been formulated as a question, and we added the case $(d)$, due to affirmative results of computer tests on subcubic graphs of small orders.

*Conjecture 2.* Every subcubic graph $G$ admits:

(a) a $(1, 1, 2^4)$-packing edge-coloring [7];
(b) a $(1, 2^7)$-packing edge-coloring [7];
(c) a $(1, 1, 2^3)$-packing edge-coloring if $G$ is in class I [7];
(d) a $(1, 2^6)$-packing edge-coloring if $G$ is in class I.

The conjectured bounds, if true, are tight. For the cases $(a)$ and $(b)$ a subcubic graph that achieves the upper bound is the complete bipartite graph $K_{3,3}$ with one subdivided edge (the left graph in Fig. 3). Recall that this graph is also class II and needs 10 colors for a strong edge-coloring, hence achieving the upper bound for all four types of colorings considered in this paper. For each 1-packing, we have at most three edges, and there remain 4 and 7, respectively, to be in a separate 2-packing each. An analogous argument holds for the cases ($c$) and ($d$) on the complete bipartite graph $K_{3,3}$.

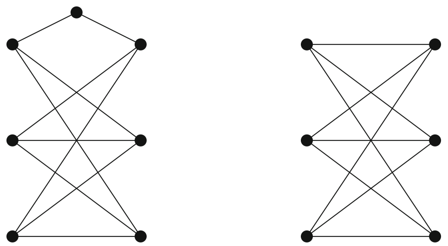

**Fig. 3.** The smallest subcubic graph which does not admit a $(1, 1, 2^3)$-packing edge-coloring nor a $(1, 2^6)$-packing edge-coloring (left), and the smallest class I subcubic graph which does not admit a $(1, 1, 2^2)$-packing edge-coloring nor a $(1, 2^5)$-packing edge-coloring (right).

Conjecture 2 bridges two of the most important edge-colorings, proper and strong, basicly claiming that each 1-packing can be replaced by three 2-packings. Note that this does not apply to subclasses of graphs, e.g. the Wagner graph needs 10 colors for a strong edge-coloring and it is in class I.

This paper contributes to answering the conjecture by providing upper bounds with one additional color for all four cases of Conjecture 2.

**Theorem 2.** *Every subcubic graph $G$ admits:*

*(a) a $(1, 1, 2^5)$-packing edge-coloring;*
*(b) a $(1, 2^8)$-packing edge-coloring;*
*(c) a $(1, 1, 2^4)$-packing edge-coloring if $G$ is in class I;*
*(d) a $(1, 2^7)$-packing edge-coloring if $G$ is in class I.*

The structure of the paper is the following. We begin by presenting notation, definitions and auxiliary results in Sect. 2. In Sect. 3, we give proofs of the cases ($a$) and ($c$) of Theorem 2. In Sects. 4 and 5, we proof the cases ($b$) and ($d$) of Theorem 2 by proving stronger statements of both. We conclude with an overview of open problems on this topic.

## 2 Preliminaries

We call a vertex of degree $k$, at most $k$, and at least $k$ a *$k$-vertex*, a *$k^-$-vertex*, and a *$k^+$-vertex*, respectively. We denote the graph obtained from a graph $G$ by removing a set of vertices $X$ as $G \setminus X$. When $X = \{v\}$ is a singleton, we simply write $G - v$. An analogous notation is used for sets of edges.

As usual, the set of vertices adjacent to a vertex $v$ is denoted $N(v)$, and called the *neighborhood of* $v$. For a vertex $v$, we denote the set of edges incident to $v$ by $N'(v)$, and the edges incident to the neighbors of $v$ (including the edges in $N'(v)$) by $N''(v)$. We refer to the former as the *edge-neighborhood of* $v$ and to the latter as the 2-*edge-neighborhood of* $v$. Analogously, we define the edge-neighborhood and the 2-edge-neighborhood of an edge $e$.

When coloring the edges, we deal with two types of colors. The ones allowing the edges of those colors to be at distance at least 2 we call the 1-*colors*, and the one requiring the edges to be at distance at least 3 are called the 2-*colors*. An edge colored with a 1-color (resp. a 2-color) is a 1-*edge* (resp. a 2-*edge*). For an edge $uv$, we denote by $A_2(uv)$ the number of available 2-colors, i.e., the 2-colors with which the edge can be colored without violating the coloring assumptions.

In our proofs, we will often put lists of colors on some uncoloured edges and try to find a valid assignment that satisfy the color lists. For example, if $e$ has a list $L$ of size $k$ such that all colors of $L$ are available for $e$ and has at most $k-1$ uncolored neighbors then we can ignore $e$ when coloring as there will always be one free color for $e$ in $L$ after coloring all other edges.

Sometimes, we will need a more careful analysis of choosing colors from the lists of available colors. For that purpose, we will use the classical result due to Hall [10].

**Theorem 3 (Hall's Theorem).** *Let $\mathcal{A} = (A_i, i \in I)$ be a finite family of (not necessarily distinct) subsets of a finite set $A$. A system of representatives (SDR) for the family $\mathcal{A}$ is a set $\{a_i, i \in I\}$ of distinct elements of $A$ such that $a_i \in A_i$ for all $i \in I$. $\mathcal{A}$ has a system of representatives if and only if $|\bigcup_{i \in J} A_i| \geq |J|$ for all subsets $J$ of $I$.*

Perhaps the strongest tool for determining if one can always choose colors from the lists of available colors such that given conditions are satisfied is the following result, due to Alon [2].

**Theorem 4 (Combinatorial Nullstellensatz).** *Let $\mathbb{F}$ be an arbitrary field, and let $P = P(X_1, \ldots, X_n)$ be a polynomial in $\mathbb{F}[X_1, \ldots, X_n]$. Suppose the coefficient of a monomial $X_1^{k_1} \ldots X_n^{k_n}$, where each $k_i$ is a non-negative integer, is non-zero in $P$ and the degree $\deg(P)$ of $P$ equals $\sum_{i=1}^{n} k_i$. If moreover $S_1, \ldots, S_n$ are any subsets of $\mathbb{F}$ with $|S_i| > k_i$ for $i = 1, \ldots, n$, then there are $s_1 \in S_1, \ldots, s_n \in S_n$ so that $P(s_1, \ldots, s_n) \neq 0$.*

In short, $P$ being the chromatic polynomial of a graph $G$, if there is a monomial (of proper degree) of $P$ with non-zero coefficient, then there exists a coloring of $G$.

When considering lists of available colors for an edge, we are in fact dealing with the list version of a coloring. We say that $L$ is an *edge-list-assignment* for a graph $G$ if it assigns a list $L(e)$ of possible colors to each edge $e$ of $G$. If $G$ admits a strong edge-coloring $\sigma$ such that $\sigma(e) \in L(e)$ for all edges in $E(G)$, then we say that $G$ is *strong $L$-edge-colorable* or $\sigma$ is a *strong $L$-edge-coloring* of $G$. The graph $G$ is *strong $k$-edge-choosable* if it is strong $L$-edge-colorable for every edge-list-assignment $L$, where $|L(e)| \geq k$ for every $e \in E(G)$. The *list strong chromatic index* $\chi'_{\rm ls}(G)$ of $G$ is the minimum $k$ such that $G$ is strong $k$-edge-choosable.

We will use the following result, due to Zhang, Liu, and Wang [24] which established a result on an adjacent vertex-distinguishing list edge-coloring of cycles, i.e. proper list edge-coloring where the sets of colors for every pair of adjacent vertices are distinct. It is easy to see that such a coloring is also a strong edge-coloring of a cycle, and we write the statement in this language.

**Theorem 5 (Zhang, Liu & Wang, 2002).** *Let $n$ be an integer with $n \geq 3$. Then,*

*(i)* $\chi'_{ls}(C_n) = 5$ *if* $n = 5$*;*
*(ii)* $\chi'_{ls}(C_n) = 4$ *if* $n \not\equiv 0 \bmod 3$*;*
*(iii)* $\chi'_{ls}(C_n) = 3$ *if* $n \equiv 0 \bmod 3$.

## 3 Proof of the Cases (*a*) and (*c*) of Theorem 2

We begin with the cases of Theorem 2 using two 1-colors. These two cases simply provide a straightforward extensions of the results due to Gastineau and Togni [7], who established them for bridgeless cubic graphs.

The extension comes from the following easy proposition for which we omit the proof (*c.f.* full version of the paper).

**Proposition 1.** *Let $G$ be a subcubic graph and let $X$ be a set of edges in $G$ such that every two edges in $X$ are at distance exactly* 2*. Then, $X$ contains at most* 5 *edges. Moreover, if $|X| = 5$, then $G$ is cubic with* 10 *vertices.*

*Proof (Theorem 2(a) and (c)).* We begin with the case $(a)$. Let $G$ be a connected subcubic graph and let $\pi$ be a $(1,1,1,2)$-packing edge-coloring of $G$ which exists by Theorem 1. To establish the statement, we only need to replace one 1-color in $\pi$ with four 2-colors. Let $X$ be a set of all the edges in $G$ colored by one 1-color in $\pi$, and $H$ be the subgraph of $G$ induced by $X$. Let $G^*$ be the graph obtained from $H$ by contracting all the edges in $X$. Clearly, $G^*$ has maximum degree at most 4, and is 4-vertex-colorable by the Brooks' Theorem, unless it is isomorphic to $K_5$. Observe that vertex coloring of $G^*$ induces a strong edge-coloring of the edges in $X$. Furthermore, by Proposition 1, the only graphs in which it may happen that five colors are needed to color $G^*$, are cubic with 10 vertices. For these small graphs we have even determined that they admit a $(1,1,2^4)$-packing edge-coloring computationally, and thus establish the case $(a)$.

The case $(b)$ follows immediately from the argument above, since we do not have an extra 2-color in the coloring $\pi$. □

## 4 Proof of the Case (*b*) of Theorem 2

In order to prove Theorem 2$(b)$, we prove a bit stronger result. We say that a $(1,2^8)$-packing edge-coloring of a subcubic graph $G$ with the color set

$\{0, 1, \ldots, 8\}$, where 0 is a 1-color and the others are 2-colors, is a *good* $(1, 2^8)$-*packing edge-coloring* if no $2^-$-vertex of $G$ is incident with a 0-edge.

**Theorem 6.** *Every subcubic graph admits a good* $(1, 2^8)$*-packing edge-coloring.*

*Proof.* We prove Theorem 6 by contradiction. Let $G$ be a minimal counterexample to the theorem in terms of $|V(G)| + |E(G)|$. Clearly, $G$ is connected and has maximum degree 3. In the following lemma, we establish some structural properties of $G$ which will eventually yield a contradiction on the existence of $G$. In most of the cases, we consider a graph $G'$ smaller than $G$, which, by the minimality of $G$, admits a good $(1, 2^8)$-packing edge-coloring $\pi$, and we show that $\pi$ can be extended to $G$ by recoloring some edges of $G'$ and coloring the edges of $G$ not being colored by $\pi$.

**Lemma 1.** *The graph $G$ verifies the following properties:*

1. *$G$ is simple,*
2. *$G$ is cubic,*
3. *$G$ is* 2*-connected,*
4. *$G$ does not contain* 3*-cycles,*
5. *$G$ does not contain* 4*-cycles and*
6. *$G$ contains no cycle of length at least* 5*.*

Due to size constraint we do not give the proof of Lemma 1 (*c.f.* full version of the paper) except for Lemma 1.6. The main techniques used in the proof of Lemma 1 consist in removing part of the graph and coloring it by minimality. In some cases, we need to use Hall's Theorem.

We nonetheless present the proof of Lemma 1.6 to show why we need the stronger statement of Theorem 6.

*Proof (Lemma 1.6).* Suppose the contrary, and let $C = u_1 u_2 \ldots u_n$ be a minimal induced $n$-cycle in $G$, with $n \geq 5$. For every $i$, $1 \leq i \leq n$, let $u'_i$ be the neighbor of the vertex $u_i$ not in $C$, and let $G' = G \setminus V(C)$. Note that the $u'_i$ are pairwise distinct by the minimality of $C$, 1.4 and 1.5. Then, by the minimality of $G$, there is a good $(1, 2^8)$-packing edge-coloring $\pi$ of $G'$. Since $\pi$ is good, no $u'_i$ is incident with the color 0. So, in the coloring $\varphi$ of $G$ induced by $\pi$, we can color every edge $u_i u'_i$ with 0. In this way, only the edges of $C$ are left non-colored. Observe that each of those edges has at least 4 available 2-colors. If $n \geq 6$, then we can complete the coloring by Theorem 5, a contradiction.

If $n = 5$ then we can color $C$, except if all five edges have the same four 2-colors available by Hall's Theorem 3. If we are in this case, then suppose that 1 and 2 are the two colors on the edges incident to $u'_1$, and 3 and 4 are the two colors on the edges incident to $u'_2$. Then $\{1, 2\}$ must also be on the edges incident to $u'_3$, $\{3, 4\}$ on the edges incident to $u'_4$, and again $\{1, 2\}$ on the edges incident to $u'_5$. Thus the edge $u_1 u_5$ has five available 2-colors, a contradiction. ♦

By Lemma 1, we have that $G$ is a cubic bridgeless graph with no cycles. Hence $G$ is a tree, a contradiction with the fact that $G$ is cubic. This concludes the proof of Theorem 6. □

## 5 Proof of the Case (*d*) of Theorem 2

Recall that in the case $(d)$, we assume the graph is in class I. In our proof, this is an important feature which enables us to confirm Conjecture 2$(b)$ for this class of graphs. We again prove a stronger version of the theorem.

**Theorem 7.** *Let $G$ be a graph of class I. Then for every proper 3-edge-coloring $\pi$ with colors $a$, $b$, and $c$, and for every color $\alpha \in \{a, b, c\}$ there exists a $(1, 2^7)$-packing edge-coloring $\sigma$ such that the edges of color $\alpha$ in $\pi$ are colored with 0 in $\sigma$.*

The proof of this theorem is quite involved. Due to size constraints, we only provide the main ideas of the proof of this theorem.

*Proof (Ideas only).* Let $G$ be a minimal counterexample to the theorem minimizing the sum $|V(G)| + |E(G)|$. Let $\pi$ be a proper 3-edge-coloring (using colors $a$, $b$, and $c$) and let the color $a$ be the color class for which there is no $(1, 2^7)$-packing edge-coloring $\sigma$ (using colors $\{0, 1, \ldots, 7\}$, 0 being the 1-color) of $G$ such that all edges colored $a$ in $\pi$ are colored 0 in $\sigma$.

We begin by establishing some structural properties of $G$. First we prove that $G$ is a simple cubic graph. This is done using a case analysis. Recall that $G$ being cubic implies that in $\pi$ every color appears at every vertex. Then we remove short cycles and prove that $G$ has girth at least 5. The proof here is more complex than for their equivalent in the previous section. We use an additional technique for removing cycles of length 4, that is we apply the Combinatorial Nullstellensatz to color some cases.

Finally, we want to remove long cycles. We do not show this exact fact but a similar one. We call a $bc$-cycle, a cycle colored only with the two colors $b$ and $c$ in $\pi$. These $bc$-cycles need to be colored with only 2-colors. If $u$ is a vertex of such $bc$-cycle and $u'$ is one of its neighbour and is not on the cycle then we know that $uu'$ is color with color $a$. Simply uncoloring the $bc$-cycle would yield only three available 2-colors for each edge of the cycle which is not enough.

We separate two cases, chordless $bc$-cycles and $bc$-cycles with chords. In both cases, we reduce the graph to a smaller one by removing some vertices of the cycle and connecting some neighbours to provide useful properties on the coloring obtained by minimality. These properties will allow us to precolor some edges of the cycle in $G$. We color the rest of the cycle with the help of the Combinatorial Nullstellensatz. As $bc$-cycles must be colored with 2-colors it is possible to express the coloring problem as a polynomial to apply the Combinatorial Nullstellensatz. Note that we use the Combinatorial Nullstellensatz in a different way than for small cycles as we have an infinity of cycle lengths. Therefore, we must find a generic non null coefficient in a family of polynomials which depend on the length of the cycle.

Combining the previous facts yields a contradiction. □

# 6 Further Work

Conjecture 2 remains open, however, our upper bounds are only by one 2-color off. Unfortunately, we were not able to apply the proving techniques, used to prove tight bounds for proper edge-coloring and strong edge-coloring of subcubic graphs, to the problems considered in this paper. Therefore, since solving Conjecture 2 in the general setting seems to be challenging, we suggest in this section additional problems which arise naturally when dealing with the considered colorings. All of them are supported with computational results on graphs of small orders.

We begin with a general conjecture for strong edge-coloring.

*Conjecture 3.* Every bridgeless subcubic graph $G$, not isomorphic to the Wagner graph or the complete bipartite graph $K_{3,3}$ with one edge subdivided, admits a $(2^9)$-packing edge-coloring.

We proceed with an overview of results in specific graph classes and list open problems for each of them. For that, we follow the conjecture on strong edge-coloring of subcubic graphs proposed by Faudree, Gyárfás, Schelp, and Tuza [5] in 1990.

*Conjecture 4 (Faudree, Gyárfás, Schelp & Tuza [5]).* For every subcubic graph $G$ it holds:

(1) $G$ admits a $(2^{10})$-packing edge-coloring;
(2) If $G$ is bipartite, then it admits a $(2^9)$-packing edge-coloring;
(3) If $G$ is planar, then it admits a $(2^9)$-packing edge-coloring;
(4) If $G$ is bipartite and each edge is incident with a 2-vertex, then it admits a $(2^6)$-packing edge-coloring;
(5) If $G$ is bipartite of girth at least 6, then it admits a $(2^7)$-packing edge-coloring;
(6) If $G$ is bipartite and has girth large enough, then it admits a $(2^5)$-packing edge-coloring.

All the cases of the conjecture, except (5), are already resolved, and we present the results in what follows.

## 6.1 Planar Graphs

It was the well-known connection between edge-coloring of bridgeless cubic planar graphs and the Four Color Problem, established by Tait [21], which initiated the research in this area. By the Four Color Theorem, we thus have that every bridgeless cubic planar graph admits a $(1,1,1)$-edge-coloring. The condition of being bridgeless is necessary, since already $K_4$ with one subdivided edge is in class II. However, not all questions are resolved. The following conjecture of Albertson and Haas [1], which is a special case of Seymour's conjecture [18], is still widely open.

*Conjecture 5 (Albertson & Haas [1]).* Every bridgeless subcubic planar graph with at least two vertices of degree 2 admits a $(1,1,1)$-packing edge-coloring.

The number of required colors for strong edge-coloring of planar graphs is also determined. Just recently, Kostochka et al. [14] proved the following (and resolved the Case (3) of Conjecture 4).

**Theorem 8 (Kostochka et al. [14]).** *Every subcubic planar graph admits a* $(2^9)$*-packing edge-coloring.*

The upper bound is tight and there are infinitely many bridgeless cubic graphs that need nine 2-colors for strong edge-coloring. An example is e.g. the 3-prism, depicted in Fig. 4.

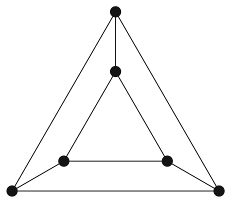

**Fig. 4.** A bridgeless cubic planar graph which needs nine colors for a strong edge-coloring.

On the other hand, there are no results for planar graphs on the colorings with one or two matchings. We propose the following conjecture.

*Conjecture 6.* Every subcubic planar graph admits a $(1,2^6)$-packing edge-coloring and a $(1,1,2^3)$-packing edge-coloring.

The conjectured upper bound, if true, is tight and attained by an infinitely many bridgeless subcubic planar graphs for both values. It also appears to be much more demanding as the result of Theorem 8. Thus, also some partial results, with additional restrictions on the structure of planar graphs, might also be interesting, in order to understand the general problem better.

## 6.2 Bipartite Graphs

In the class of bipartite graphs, the proper and the strong case of the colorings are long solved. In 1916, König [13] proved that every bipartite graph is in class I, and in 1993, Steger and Yu [20] established the following (and resolved the Case (2) of Conjecture 4).

**Theorem 9 (Steger & Yu [20]).** *Every subcubic bipartite graph admits a* $(2^9)$*-packing edge-coloring.*

Again, these bounds are tight and attained by infinitely many graphs.

Since all bipartite graphs are in class I, the results and conjectures for them apply also in the bipartite case. It is known that as soon as we leave the 'proper' setting, i.e., require some 2-colors instead just 1-colors, the problems become much harder. E.g., a tight upper bound for a strong edge-coloring of bipartite graphs is still not known (c.f. [5,20]). Therefore, the Cases $(c)$ and $(d)$ of Conjecture 2 may be considered just in the bipartite setting. Moreover, we have an infinite number of graphs attaining the conjectured upper bounds also among bipartite graphs.

If we consider subcubic graphs with only edges of weight at most 5, i.e., edges where at least one of the end-vertices is of degree at most 2, the number of required colors decreases substantially. In particular, the Case (4) of Conjecture 4 was resolved by Maydanskiy [16] and independently by Wu and Lin [23].

**Theorem 10 (Maydanskiy [16], and Wu & Lin [23]).** *Every subcubic bipartite graph, in which each edge has weight at most* 5*, admits a* $(2^6)$*-packing edge-coloring.*

Clearly, an analogous question for coloring such graphs with two 1-colors is if they admit a $(1, 1, 2^2)$-packing edge-coloring. It is answered in affirmative [19]. The bound is tight already in the class of trees. On the other hand, we do not have the answer for the following.

*Question 1.* Is it true that every subcubic bipartite graph, in which each edge has weight at most 5, admits a $(1, 2^4)$-packing edge-coloring?

This bound is again attained in the class of trees.

### 6.3 Graphs with Big Girth

Similarly as the bipartiteness, having big girth does not really simplify edge-colorings in which some colors must be 2-colors. Even more, due to Kochol [12] we know, there are graphs with arbitrarily large girth which are in class II! Anyway, if the girth is infinite, i.e., we consider the trees, the following simple observation is immediate.

**Observation 1.** *Every subcubic tree admits:*

*(1) a* $(1, 1, 1)$*-packing edge-coloring;*
*(2) a* $(1, 1, 2, 2)$*-packing edge-coloring;*
*(3) a* $(1, 2^4)$*-packing edge-coloring;*
*(4) a* $(2^5)$*-packing edge-coloring.*

The bounds are tight already if we just consider a neighborhood of one edge with both end-vertices of degree 3.

In the case of strong edge-coloring, the Case (6) of Conjecture 4 was also rejected just recently by Lužar, Mačajová, Škoviera, and Soták [15], who proved that a cubic graph is a cover of the Petersen graph if and only if it admits a $(2^5)$-packing edge-coloring.

Before we consider the intermediate colorings, we first recall the result of Gastineau and Togni [7].

**Proposition 2 (Gastineau & Togni [7]).** *Every cubic graph admitting a* $(1, 1, 2, 2)$*-packing edge-coloring is class I and has order divisible by four.*

Hence, the analogue of the Case (6) of Conjecture 4 when having two 1-colors does not hold. However, the following remains open.

*Question 2.* Is it true that every subcubic bipartite graph with big enough girth admits a $(1, 2^4)$-packing edge-coloring?

To conclude, we believe that studying properties of the considered edge-colorings will have impact to the initial problem of strong edge-coloring, which is in general case still widely open. Namely, the conjectured upper bound for graphs with maximum degree $\Delta$ is $1.25\Delta^2$, while currently the best upper bound is due to Bonamy, Perrett, and Postle [4], set at $1.835\Delta^2$.

**Acknowledgement.** This research has been done in the scope of the bilateral project between France and Slovenia, BI-FR/19-20-PROTEUS-001. The third author was partly supported by the Slovenian Research Agency Program P1–0383 and the Project N1–0057(B).

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