



Equational Theories of Scattered and Countable Series-Parallel Posets

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Abstract. In this paper we consider two classes of posets labeled over an alphabet A . The class $SP^\circ(A)$ is built from the letters and closed under the operations of series finite, ω and $\bar{\omega}$ products, and finite parallel product. In the class ${}^\omega SP(A)$, ω and $\bar{\omega}$ products are replaced by ω and $\bar{\omega}$ powers. We prove that $SP^\circ(A)$ and ${}^\omega SP(A)$ are freely generated in their respective natural varieties of algebras \mathcal{V} and \mathcal{V}' , and that the equational theory of \mathcal{V}' is decidable.

Keywords: Transfinite N-free posets · Series-parallel posets · Variety · Free algebra · Series product · Parallel product · ω -power · $\bar{\omega}$ -power · Decidability

1 Introduction

In his generalization of the algebraic approach of recognizable languages from finite words to ω -words, Wilke [22] introduced *right binoids*, that are two-sorted algebras equipped with a binary product and an ω -power. The operations are linked together by equalities reflecting their properties. These equalities define a variety of algebras. This algebraic study of ω -words have since been extended to more general structures, such as for example partial words (or equivalently, labeled posets) or transfinite strings (long words). In [8], *shuffle binoids* are right binoids equipped with a shuffle operation that enables to take into consideration N-free posets with finite antichains and ω -chains instead of ω -words. In [3, 5], the structure of right binoids in two parts is modified in order to enable products to extend over ω , ie. small ordinals ($\leq \omega^n$, $n \in \mathbb{N}$) and countable ordinals. The latter algebras are enriched in [10, 11] with operations such as for example reverse ω -power in order to take into account countable linear orderings (scattered in some cases). Some of the previous algebraic enrichments were also applied to shuffle binoids [4, 12]. The motivations in [3–5, 10, 11, 17, 22] are mainly the study of the links between automata, rational expressions, algebraic recognition and monadic second-order logic. In [7–9, 12, 22] the authors focus essentially on varieties of algebras; for example, free algebras are characterized in the corresponding varieties, and decisions algorithms for equivalence of terms are provided.

Let us denote by $\bar{\omega}$ the reverse ordering of ω . In this paper we focus on algebras equipped with a parallel product, series product, and either ω and $\bar{\omega}$

products or ω and $\bar{\omega}$ powers. For example, the class $SP^\circ(A)$ of N-free posets in which antichains are finite and chains are scattered and countable orderings lies in this framework. In [2,6] this class has been studied from the point of view of automata, rational expression and logic. We prove here that $SP^\circ(A)$ is the free algebra in a variety \mathcal{V} of algebras equipped with a parallel product, series product, and ω and $\bar{\omega}$ products. By removing the parallel product, it follows that A° , the class of scattered and countable words over A , is also a free algebra in the corresponding variety. We also consider the class ${}^\omega SP(A)$ where the ω and $\bar{\omega}$ products are replaced by ω and $\bar{\omega}$ powers, and show that it is freely generated in the corresponding variety \mathcal{V}' . Relying of decision results of [2] we prove that the equality of terms of \mathcal{V}' is decidable.

2 Linear Orderings and Posets

We let $|E|$ denote the cardinality of a set E , and $[n]$ the set $\{1, \dots, n\}$, for any non-negative integer $n \in \mathbb{N}$.

Let J be a set equipped with a strict order $<$. The ordering J is *linear* if either $j < k$ or $k < j$ for any distinct $j, k \in J$. We denote by \bar{J} the backward linear ordering obtained from the set J with the reverse ordering. A linear ordering J is *dense* if for any $j, k \in J$ such that $j < k$, there exists an element i of J such that $j < i < k$. It is *scattered* if it contains no infinite dense sub-ordering. The ordering ω of natural integers is scattered as well as the ordering ζ of all integers (negative, 0 and positive). Ordinals are also scattered orderings. We let \mathcal{N} , \mathcal{O} and \mathcal{S} denote respectively the class of finite linear orderings, the class of countable ordinals and the class of countable scattered linear orderings. We also let 0 and 1 denote respectively the empty and the singleton linear ordering. We refer to [20] for more details on linear orderings and ordinals.

A *poset* $(P, <)$ is a set P partially ordered by $<$. For short we often denote the poset $(P, <)$ by P . The *width* of P is $\text{wd}(P) = \sup\{|E| : E \text{ is an antichain of } P\}$ where \sup denotes the least upper bound of the set. In this paper, we restrict to posets with finite antichains and countable and scattered chains.

Let $(P, <_P)$ and $(Q, <_Q)$ be two disjoint posets. The *union* (or *parallel composition*) $P \cup Q$ of $(P, <_P)$ and $(Q, <_Q)$ is the poset $(P \cup Q, <_P \cup <_Q)$. The *sum* (or *sequential composition*) $P + Q$ is the poset $(P \cup Q, <_P \cup <_Q \cup P \times Q)$. The sum of two posets can be generalized to a J -sum of any linearly ordered sequence $((P_j, <_j))_{j \in J}$ of pairwise disjoint posets by $\sum_{j \in J} P_j = (\bigcup_{j \in J} P_j, (\bigcup_{j \in J} <_j) \cup (\bigcup_{j, j' \in J, j <_j j'} P_j \times P_{j'}))$. The sequence $((P_j, <_j))_{j \in J}$ is called a J -*factorization*, or (sequential) *factorization* for short, of the poset $\sum_{j \in J} P_j$. A poset P is *sequential* if it admits a J -factorization where J contains at least two elements $j \neq j'$ with $P_j, P_{j'} \neq 0$, or P is a singleton. It is *parallel* when $P = P_1 \cup P_2$ for some $P_1, P_2 \neq 0$. A poset is *sequentially irreducible* (resp. *parallelly irreducible*) when P is either a singleton or a parallel poset (resp. a singleton or a sequential poset). A sequential factorization $((P_j, <_j))_{j \in J}$ of $P = \sum_{j \in J} P_j$ is *irreducible* when all the P_j are sequentially irreducible. It is *non-trivial* if all the P_j are non-empty.

The notions of irreducible and non-trivial parallel factorization are defined similarly. A poset is *scattered* if all its chains are scattered. The class SP° of *series-parallel scattered and countable posets* is the smallest class of posets containing 0, the singleton, and closed under finite parallel composition and sum indexed by countable scattered linear orderings. By extension of a well-known result on finite posets [18, 21], it has a nice characterization in terms of graph properties: SP° coincides with the class of scattered countable \mathbb{N} -free posets without infinite antichains [6]. Recall that $(P, <)$ is *\mathbb{N} -free* if there is no $X = \{x_1, x_2, x_3, x_4\} \subseteq P$ such that $< \cap X^2 = \{(x_1, x_2), (x_3, x_2), (x_3, x_4)\}$.

When $P \in SP^\circ$ and $P = R + P' + S$ or $P = P' \cup R$ for some $R, S, P' \in SP^\circ$ then P' is a *factor* of P ; the factors of P' , R and S are also factors of P .

F. Hausdorff proposed in [16] an inductive definition of scattered linear orderings. In fact, each countable and scattered linear ordering is obtained using sums indexed by finite linear orderings, ω and $\bar{\omega}$. This has been adapted in [6] to SP° .

We let $\mathcal{C}_{\cup,+}(E)$ denote the closure of a set E of posets under finite disjoint union and finite disjoint sum.

Definition 1. *The classes of countable and scattered posets (equivalent up to isomorphism) V_α and W_α are defined inductively as follows:*

$$\begin{aligned} V_0 &= \{0, 1\} \\ W_\alpha &= \mathcal{C}_{\cup,+}(V_\alpha) \\ V_\alpha &= \left\{ \sum_{i \in J} P_i : J \in \{\omega, \bar{\omega}\} \text{ and } \forall i \in J, P_i \in \bigcup_{\beta < \alpha} W_\beta \right\} \cup \bigcup_{\beta < \alpha} W_\beta \text{ when } \alpha > 0 \end{aligned}$$

and the class S_{sp} of countable and scattered posets by $S_{sp} = \bigcup_{\alpha \in \mathcal{O}} W_\alpha$.

The following theorem extends a result of Hausdorff on linear orderings [16].

Theorem 1 ([6]). $S_{sp} = SP^\circ$.

For every $\alpha \in \mathcal{O}$, W_α can be decomposed as the closure of V_α by finite disjoint union and finite disjoint sum:

Theorem 2 ([6]). *For all $\alpha \in \mathcal{O}$, $i \in \mathbb{N}$, let*

$$\begin{aligned} X_{\alpha,0} &= V_\alpha \\ Y_{\alpha,i} &= \left\{ P : \exists n \in \mathbb{N} P = \sum_{j \leq n} P_j \text{ such that } P_j \in X_{\alpha,i} \text{ for all } j \leq n \right\} \\ X_{\alpha,i+1} &= \left\{ P : \exists n \in \mathbb{N} P = \bigcup_{j \leq n} P_j \text{ such that } P_j \in Y_{\alpha,i} \text{ for all } j \leq n \right\} \end{aligned}$$

Then $W_\alpha = \bigcup_{i \in \mathbb{N}} X_{\alpha,i}$.

Example 1. W_0 is the set of all finite \mathbb{N} -free posets. Its subset $Y_{0,0}$ is the set of all finite linear orderings. The linear orderings ω and $\bar{\omega}$ are contained in V_1 . Each poset of $V_1 \setminus W_0$ has some chain isomorphic to either ω or $\bar{\omega}$, but can not have a chain isomorphic to ω and another isomorphic to $\bar{\omega}$. The ordering ζ of all integers is in $Y_{1,0}$. For all $\alpha \in \mathcal{O}$, $\omega^\alpha \in V_\alpha$.

Define a well-ordering on $\mathcal{O} \times \mathbb{N}$ by $(\beta, j) < (\alpha, i)$ if and only if $\beta < \alpha$ or $\beta = \alpha$ and $j < i$. As a consequence of Theorems 1 and 2, for any $P \in SP^\diamond$ there exists a unique pair $(\alpha, i) \in \mathcal{O} \times \mathbb{N}$ as small as possible such that $P \in X_{\alpha,i}$.

Definition 2. *The rank $r(P)$ of $P \in SP^\diamond$ is the smallest pair $(\alpha, i) \in \mathcal{O} \times \mathbb{N}$ such that $P \in X_{\alpha,i}$.*

Example 2. The linear ordering ζ has rank $r(\zeta) = (1, 1)$. Each linear ordering I of \mathcal{S} has rank $r(I) \in \{(\alpha, 0), (\alpha, 1)\}$ for some $\alpha \in \mathcal{O}$. For all $\alpha \in \mathcal{O}$, $r(\omega^\alpha) = \alpha$.

Remark 1. Let $P \in SP^\diamond$ with $r(P) = (\alpha, 0)$, $\alpha > 0$. Assume that $P = \sum_{j \in J} P_j$ is a non-trivial J -factorization of P for some $J \in \{\omega, \bar{\omega}\}$. If $J = \omega$ (resp. $J = \bar{\omega}$), then, for all $j \in J$, $r(P_j) < r(P)$. In addition, for all $(\beta, i) \in \mathcal{O} \times \mathbb{N}$ such that $(\beta, i) < (\alpha, 0)$, for all $j \in J$ there exists $k \in J$ such that $k > j$ (resp. $k < j$) and

$$(\beta, i) \leq r(P_k) < r(P)$$

This implies that, for all $j \in J$, $\sum_{j' \geq j} P_j$ (resp. $\sum_{j' \leq j} P_j$) is of rank $(\alpha, 0)$.

Lemma 1. *Let $P \in SP^\diamond$ be a sequential poset such that $r(P) = (\alpha, 0)$, $\alpha > 0$. Let $\sum_{j \in J} P_j$ and $\sum_{j \in J'} P'_j$ be some non-trivial J - and J' -factorizations of P where $J, J' \in \{\omega, \bar{\omega}\}$. Then $J = J'$.*

Proof. Assume by contradiction that $J \neq J'$. Assume wlog that $J = \omega$ and $J' = \bar{\omega}$. Let $L = \sum_{j \leq k} P_j$ and $R = \sum_{k < j < \omega} P_j$ for some $k \in \omega$. Then $P = L + R$. As a consequence of Remark 1, $r(R) = (\alpha, 0)$. Observe that there exists $k' \in \bar{\omega}$ such that R is a sequential factor of $R' = \sum_{j' \geq k'} P'_{j'}$. Let $L' = \sum_{\bar{\omega} < j' < k'} P'_{j'}$. As a consequence of Remark 1, $r(L') = r(R) = (\alpha, 0)$. Furthermore, $r(R) \leq r(R') \leq r(P)$. Thus $r(R') = (\alpha, 0)$ too. We have $P = L' + R'$, and by Theorem 2, $r(P) = (\alpha, 1)$, which is a contradiction.

In [6] an equivalence relation \sim over the elements of a poset of SP^\diamond is given, such that P/\sim is isomorphic to a countable and scattered linear ordering (Lemma 9), and such that each equivalence class is a sequentially irreducible factor of P (Lemma 10). This leads to the following proposition.

Proposition 1 ([6]). *Each poset of SP^\diamond admits a unique irreducible sequential factorization.*

Definition 1 and Theorem 1 provide a well-founded definition of SP^\diamond which we consider from now as a set, although originally defined as a class.

3 Labeled Posets

An *alphabet* A is a non-empty set (not necessarily finite) whose elements are called *letters* or *labels*. In the literature a *word* over A is a totally ordered sequence of elements of A . The sequence may have properties depending on the context, for example it can be finite, an ordinal, or a countable scattered linear ordering. The notion of a finite word has early been extended to partial orderings (finite *partial words* or *pomsets* [14, 15, 23]). In this paper we consider a mixture between the notions of finite partial words and words indexed by scattered and countable linear orderings.

A poset P is *labeled* by A when it is equipped with a *labeling* total map $l: P \rightarrow A$. Also, the finite labeled posets of width at most 1 correspond to the usual notion of words. We let ϵ denote the empty labeled poset. For short, the singleton poset labeled by $\{a\}$ is denoted by a , and we often make no distinction between a poset and a labeled poset, except for operations.

The *sequential product* (or *concatenation*, denoted by $P \cdot P'$ or PP' for short) and the *parallel product* $P \parallel P'$ of two labeled posets are respectively obtained by the sequential and parallel compositions of the corresponding (unlabeled) posets. By extension, the sequential product $\prod_{j \in J} P_j$ of a linearly ordered sequence of labeled posets is the poset $\sum_{j \in J} P_j$ in which the label of the elements is kept. In particular, the ω -*product* (resp. $\bar{\omega}$ -*product*) of an ω -sequence (resp. $\bar{\omega}$ -sequence) of labeled posets $(P_i)_{i \in \omega}$ (resp. $(P_i)_{i \in \bar{\omega}}$) is denoted by $\prod_{i \in \omega} P_i$ (resp. $\prod_{i \in \bar{\omega}} P_i$). The ω -*power* (resp. $\bar{\omega}$ -*power*) P^ω (resp. $P^{\bar{\omega}}$) of the poset P is the ω -product (resp. $\bar{\omega}$ -product) of an ω -sequence (resp. $\bar{\omega}$ -sequence) of posets that are all isomorphic to P . As usual, in this paper we consider two labeled posets to be identical if they are isomorphic. By extension, the rank $r(P)$ of a labeled poset P is the rank of its underlying unlabeled poset.

Let A and B be two alphabets and let P be a poset labeled by A . For all $a \in A$, let G_a be some poset labeled by B , and let $G = (G_a)_{a \in A}$. The poset labeled by B consisting of P in which each element labeled by the letter a is replaced by G_a , for all $a \in A$, is denoted by $G \circ_A P$. If the underlying posets of P and of all the G_a are in SP^\diamond , then so is $G \circ_A P$.

Definition 3. *Let A be an alphabet. We define:*

- $SP^\diamond(A)$, the smallest set of posets labeled by A containing ϵ , a for all $a \in A$, and closed under operations of sequential, parallel, ω and $\bar{\omega}$ -products. According to Theorem 1, the underlying posets are precisely those of SP^\diamond ;
- ${}^\omega A$, the smallest subset of $SP^\diamond(A)$ containing ϵ , a for all $a \in A$, and closed under operations of sequential product, ω -power and $\bar{\omega}$ -power;
- A^\diamond , the smallest subset of $SP^\diamond(A)$ containing ϵ , a for all $a \in A$, and closed under operations of sequential product, ω -product and $\bar{\omega}$ -product;
- ${}^\omega SP(A)$, the smallest subset of $SP^\diamond(A)$ containing ϵ , a for all $a \in A$, and closed under operations of sequential and parallel product, ω -power and $\bar{\omega}$ -power;

- $\omega SP(A)$, the smallest subset of $SP^\diamond(A)$ containing ϵ , a for all $a \in A$, and closed under operations of sequential product, parallel product and ω -product (note that there is no $\bar{\omega}$ -product here).

Note that ${}^\omega A = \{P \in SP^\diamond(A) : r(P) \in \mathbb{N} \times \{0, 1\}\}$ and ${}^\omega SP(A) = \{P \in SP^\diamond(A) : r(P) \in \mathbb{N} \times \mathbb{N}\}$.

4 Varieties

In this section we define the different varieties studied throughout this paper by listing the axioms they satisfy. The usual notions and results of universal algebra apply to our case, even if we use here for example operations of infinite arity. For more details about universal algebra, we refer the reader to [1]. In the following 1 is considered as a neutral element (the interpretation of a constant).

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \quad (A_1)$$

$$x \parallel (y \parallel z) = (x \parallel y) \parallel z \quad (A_2)$$

$$x \parallel y = y \parallel x \quad (A_3)$$

$$(x \cdot y)^\omega = x \cdot (y \cdot x)^\omega \quad (A_4)$$

$$(x^n)^\omega = x^\omega, \quad n \geq 1 \quad (A_5)$$

$$(x \cdot y)^{\bar{\omega}} = (y \cdot x)^{\bar{\omega}} \cdot y \quad (A_6)$$

$$(x^n)^{\bar{\omega}} = x^{\bar{\omega}}, \quad n \geq 1 \quad (A_7)$$

$$x \cdot 1 = x \quad (A_8)$$

$$1 \cdot x = x \quad (A_9)$$

$$x \parallel 1 = x \quad (A_{10})$$

$$1^\omega = 1 \quad (A_{11})$$

$$1^{\bar{\omega}} = 1 \quad (A_{12})$$

for all ω -sequences $x_0, x_1, \dots, x_i, \dots$ and all decompositions

$$(x_0, \dots, x_{n_0-1}), (x_{n_0}, \dots, x_{n_1-1}), \dots, (x_{n_i}, \dots, x_{n_i-1}), \dots$$

$$\omega((x_0, \dots, x_{n_0-1}), (x_{n_0}, \dots, x_{n_1-1}), \dots) = \omega(x_0, x_1, \dots) \quad (A_{13})$$

$$x_0 \cdot \omega(x_1, x_2, \dots) = \omega(x_0, x_1, x_2, \dots) \quad (A_{14})$$

$$\omega(1, 1, \dots) = 1 \quad (A_{15})$$

for all $\bar{\omega}$ -sequences $\dots, x_i, \dots, x_1, x_0$ and all decompositions

$$\dots, (x_{n_i-1}, \dots, x_{n_i}), \dots, (x_{n_1-1}, \dots, x_{n_0}), (x_{n_0-1}, \dots, x_0)$$

$$\bar{\omega}(\dots, (x_{n_1-1}, \dots, x_{n_0}), (x_{n_0-1}, \dots, x_0)) = \bar{\omega}(\dots, x_1, x_0) \quad (A_{16})$$

$$\bar{\omega}(\dots, x_2, x_1) \cdot x_0 = \bar{\omega}(\dots, x_2, x_1, x_0) \quad (A_{17})$$

$$\bar{\omega}(\dots, 1, 1) = 1 \quad (A_{18})$$

Definition 4. We define

- \mathcal{V} , the collection of algebras $(S, \cdot, \parallel, \omega, \bar{\omega}, 1)$ satisfying the axioms (A_1) – (A_3) , (A_8) – (A_{10}) and (A_{13}) – (A_{18}) ;
- \mathcal{V}_0 , the collection of algebras $(S, \cdot, \omega, \bar{\omega}, 1)$ satisfying the axioms (A_1) , (A_8) , (A_9) and (A_{13}) – (A_{18}) ;
- \mathcal{V}_1 , the collection of algebras $(S, \cdot, \parallel, \omega, 1)$ satisfying the axioms (A_1) – (A_3) , (A_8) – (A_{10}) and (A_{13}) – (A_{15}) ;
- \mathcal{V}' , the collection of algebras $(S, \cdot, \parallel, \omega, \bar{\omega}, 1)$ satisfying the axioms (A_1) – (A_{12}) ;
- \mathcal{V}'_0 , the collection of algebras $(S, \cdot, \omega, \bar{\omega}, 1)$ satisfying the axioms (A_1) , (A_4) – (A_9) and $(A_{11}), (A_{12})$.

In order to simplify the notation, an algebra whose set of elements is S is sometimes denoted by S when there is no ambiguity.

5 Freeness

Throughout this section, A denotes an alphabet. We start by proving the freeness of $SP^\diamond(A)$.

Theorem 3. $SP^\diamond(A)$ is freely generated by A in \mathcal{V} .

Proof. For all $(\alpha, i) \in \mathcal{O} \times \mathbb{N}$, let $X_{\alpha, i}$ denote the set of posets of $SP^\diamond(A)$ of rank (α, i) or less. Let $\mathcal{M} = (M, \cdot, \parallel, \omega, \bar{\omega}, 1)$ be any algebra of \mathcal{V} and let $h: A \rightarrow M$ be any function. We show that h can be extended into a homomorphism of \mathcal{V} -algebras $h^\sharp: SP^\diamond(A) \rightarrow M$ in a unique way. Define h^\sharp as $h^\sharp = \bigcup_{(\alpha, i) \in \mathcal{O} \times \mathbb{N}} h_{\alpha, i}$

where each $h_{\alpha, i}: X_{\alpha, i} \rightarrow M$ is defined by induction over (α, i) as follows. Let us denote by $h_{<(\alpha, i)} = \bigcup_{(\beta, j) < (\alpha, i)} h_{\beta, j}$. Let $P \in X_{\alpha, i}$. If $r(P) < (\alpha, i)$ then $h_{\alpha, i}(P) = h_{<(\alpha, i)}(P)$. Otherwise

- if $\alpha = 0$ and $i = 0$ then $h_{\alpha, i} = h \cup (\epsilon \rightarrow 1)$;
- if $\alpha > 0$ and $i = 0$ then P admits a non-trivial J -factorization

$$P = \prod_{j \in J} P_j \tag{19}$$

where $J \in \{\omega, \bar{\omega}\}$ (see Remark 1) and $r(P_j) < r(P)$ for all $j \in J$. Define $h_{\alpha, i}(P)$ by

$$h_{\alpha, i}(P) = \prod_{j \in J} h_{<(\alpha, i)}(P_j)$$

- if $i > 0$:
 - if P is a sequential poset then it has a factorization

$$P = \prod_{j \in [n]} P_j \tag{20}$$

where each P_j is a non-empty poset of rank lower than (α, i) and $n \in \mathcal{N} \setminus \{0, 1\}$. Define $h_{\alpha, i}(P)$ by

$$h_{\alpha, i}(P) = \prod_{j \in [n]} h_{\alpha, i-1}(P_j)$$

- otherwise, P is a parallel poset. Write

$$P = \parallel_{s \in [n]} P_s$$

where each P_s is a sequential poset and $n \geq 2$. Then, define $h_{\alpha, i}(P)$ by

$$h_{\alpha, i}(P) = \parallel_{s \in [n]} h_{\alpha, i}(P_s)$$

By Theorem 2, the factorizations used in the definition of $h_{\alpha, i}$ exist. However, observe that the sequential ones ((19) and (20)) are not unique. This would question the fact that $h_{\alpha, i}$ is a well-defined function. For all $P \in SP^\circ(A)$ of rank (α, i) , we show that:

1. $h_{\alpha, i}(P)$ does not depend on the factorization of P and thus is well-defined;
2. $h_{\alpha, i}$ commutes with all the operations of \mathcal{V} :
 - (a) $h_{\alpha, i}(\prod_{j \in J} P_j) = \prod_{j \in J} h_{\alpha, i}(P_j)$, for some $J \in \mathcal{N} \cup \{\omega, \bar{\omega}\}$;
 - (b) $h_{\alpha, i}(\parallel_{s \in [n]} P_s) = \parallel_{s \in [n]} h_{\alpha, i}(P_s)$, for some $n \in \mathbb{N}$.

We proceed by induction on (α, i) . Let us start by proving that $h_{\alpha, i}$ maps $P \in X_{\alpha, i}$ to the same element of M regardless of the factorization of P . If $(\alpha, i) = (0, 0)$ the theorem follows immediately. Otherwise, assume first that $i = 0$. By Lemma 1, all the possible factorizations of P as in (19) are either all ω -factorizations or all $\bar{\omega}$ -factorizations. Assume wlog that P admits only ω -factorizations as in (19). Let $P = \prod_{j \in \omega} P_j$ and $P = \prod_{j \in \omega} Q_j$ be two different such ω -factorizations. By definition of $h_{\alpha, i}$

$$h_{\alpha, i}(\prod_{j \in \omega} P_j) = \prod_{j \in \omega} h_{<(\alpha, i)}(P_j) \text{ and } h_{\alpha, i}(\prod_{j \in \omega} Q_j) = \prod_{j \in \omega} h_{<(\alpha, i)}(Q_j)$$

There exists a sequence $(R_j)_{j \in \omega}$ of non-empty posets such that $P = \prod_{k \in \omega} R_k$ and for all $j \in \omega$ there exist $k_{P_j}, k'_{P_j}, k_{Q_j}, k'_{Q_j} \in \omega$ such that

$$P_j = \prod_{k_{P_j} \leq l \leq k'_{P_j}} R_l \text{ and } Q_j = \prod_{k_{Q_j} \leq l \leq k'_{Q_j}} R_l$$

By induction hypothesis $h_{<(\alpha, i)}$ commutes with all the operations of \mathcal{V} . Then, we have for all $j \in \omega$:

$$h_{<(\alpha, i)}(P_j) = \prod_{k_{P_j} \leq l \leq k'_{P_j}} h_{<(\alpha, i)}(R_l) \text{ and } h_{<(\alpha, i)}(Q_j) = \prod_{k_{Q_j} \leq l \leq k'_{Q_j}} h_{<(\alpha, i)}(R_l)$$

Thus $\prod_{j \in \omega} h_{<(\alpha, i)}(P_j)$ can be written as

$$\begin{aligned} & \omega(h_{<(\alpha, i)}(R_{k_{P_0}}) \cdots h_{<(\alpha, i)}(R_{k'_{P_0}}), h_{<(\alpha, i)}(R_{k_{P_1}}) \cdots h_{<(\alpha, i)}(R_{k'_{P_1}}), \cdots) \\ & \stackrel{(A_{13})}{=} \omega(h_{<(\alpha, i)}(R_{k_{P_0}}, \dots, h_{<(\alpha, i)}(R_{k'_{P_0}}), h_{<(\alpha, i)}(R_{k_{P_1}}, \dots, h_{<(\alpha, i)}(R_{k'_{P_1}}), \cdots) \\ & \stackrel{(A_{13})}{=} \omega(h_{<(\alpha, i)}(R_{k_{Q_0}}) \cdots h_{<(\alpha, i)}(R_{k'_{Q_0}}), h_{<(\alpha, i)}(R_{k_{Q_1}}) \cdots h_{<(\alpha, i)}(R_{k'_{Q_1}}), \cdots) \end{aligned}$$

We have $\prod_{j \in \omega} h_{<(\alpha, i)}(P_j) = \prod_{j \in \omega} h_{<(\alpha, i)}(Q_j)$. The case where P admits only $\bar{\omega}$ -factorizations as in (19) is proved symmetrically using (A₁₆) instead of (A₁₃). In addition, using (A₁) instead of (A₁₃) and arguments similar to those of the previous case, we prove that when P is sequential and $i > 0$, $h_{\alpha, i}(P)$ does not depend on the factorization of P .

Thus, we have proved that $h_{\alpha, i}$ is well-defined for sequential posets of rank $(\alpha, i) \in \mathcal{O} \times \mathbb{N}$. In addition, the irreducible parallel factorization is unique modulo the commutativity of \parallel . Thus $h_{\alpha, i}$ is well-defined for all posets of rank (α, i) , for all $(\alpha, i) \in \mathcal{O} \times \mathbb{N}$. Furthermore, proving that $h_{\alpha, i}$ commutes with all the operations in $X_{\alpha, i}$ can be done by induction on $r(P)$ too. The arguments are very similar to those used to prove that $h_{\alpha, i}$ is well-defined. It follows that h^\sharp is a homomorphism of \mathcal{V} -algebras. In addition, since h^\sharp relies on h then h^\sharp is unique.

The proofs of the following theorems rely on the same arguments. It suffices to restrict h^\sharp to the operations of the corresponding variety. In particular, this provides a new proof of Theorem 5.

Theorem 4. A^\diamond is freely generated by A in \mathcal{V}_0 .

Theorem 5 ([12]). $\omega SP(A)$ is freely generated by A in \mathcal{V}_1 .

In the remainder of this section, we prove the freeness of $\omega SP(A)$ in \mathcal{V}' . The arguments are similar to those of the proof of Theorem 6.1 in [12] in which the variety considered is \mathcal{V}' without $\bar{\omega}$ -power. We need the following result.

Theorem 6 ([9]). ωA is freely generated by A in \mathcal{V}'_0 .

Lemma 2. Let A and B be two alphabets. Let $S \subseteq \omega SP(B)$ such that S is closed under sequential product, ω -power and $\bar{\omega}$ -power. Let $f: A \rightarrow G$ be some function defined by $f(a) = G_a \in G$ for some $G \subseteq S$. Then, the function $f^\sharp: \omega A \rightarrow S$ extending f defined by $f^\sharp(u) = (G_a)_{a \in A} \circ_A u$, for all $u \in \omega A$, is a homomorphism from $(\omega A, \cdot, \omega, \bar{\omega}, \epsilon)$ to $(S, \cdot, \omega, \bar{\omega}, 1)$.

Furthermore, if f is bijective, S is generated by G , and G contains only sequentially irreducible posets then f^\sharp is bijective.

Proof. Let $u \in \omega A$ whose irreducible sequential factorization is $\prod_{j \in J} u_j$ for some $J \in \mathcal{S}$, where each $u_j \in A$. Note that

$$f^\sharp(u) = (G_a)_{a \in A} \circ_A u = \prod_{j \in J} (G_a)_{a \in A} \circ_A u_j = \prod_{j \in J} f(u_j)$$

Let $v \cdot w, x^\omega$ and $y^{\bar{\omega}}$ be some sequential factorizations of u . Then, one can prove easily that

$$f^\sharp(u) = f^\sharp(v) \cdot f^\sharp(w) = f^\sharp(x)^\omega = f^\sharp(y)^{\bar{\omega}}$$

relying on the uniqueness of the irreducible sequential factorization of u (Proposition 1).

Let us prove now that when f is bijective and S is generated by a set of sequentially irreducible posets then f^\sharp is bijective. Let $u, v \in {}^\omega A$ and assume that $f^\sharp(u) = P$ and $f^\sharp(v) = Q$. Let $\prod_{i \in I} u_i$ and $\prod_{j \in J} v_j$ be the irreducible sequential factorizations of respectively u and v , for some $I, J \in \mathcal{S}$, where each u_i and v_j are in A . By definition of f^\sharp , $P = \prod_{i \in I} P_i$ and $Q = \prod_{j \in J} Q_j$ where each $P_i = (G_a)_{a \in A} \circ_A u_i$ and $Q_j = (G_a)_{a \in A} \circ_A v_j$. Then, for all $i \in I$ and for all $j \in J$, P_i and Q_j are sequentially irreducible posets of G . Assume that $P = Q$. Then $I = J$ and, for all $i \in I$, $P_i = Q_i$. We have, for all $i \in I$, $u_i = v_i$ since f is injective by hypothesis. In addition, as G generates S , each element P of S can be written as $\prod_{j \in J} P_j$ where each $P_j \in G$, for some $J \in \mathcal{S}$. Since f is surjective by hypothesis, for all $j \in J$ there exists $u_j \in A$ such that $f(u_j) = P_j$. Then $f^\sharp(\prod_{j \in J} u_j) = P$.

As a consequence of HSP Birkhoff's Theorem (see eg. [1, Theorem 1.3.8]) and Lemma 2:

Corollary 1. *For all $S \subseteq {}^\omega SP(A)$ closed under sequential product, ω -power and $\bar{\omega}$ -power and generated by a set of sequentially irreducible posets of ${}^\omega SP(A)$, $(S, \cdot, {}^\omega, \bar{\omega}, 1)$ is a \mathcal{V}'_0 -algebra.*

In addition, as a consequence of Theorem 6 and Lemma 2:

Corollary 2. *For all $S \subseteq {}^\omega SP(A)$ closed under sequential product, ω -power and $\bar{\omega}$ -power and generated by a set G of sequentially irreducible posets of ${}^\omega SP(A)$, $(S, \cdot, {}^\omega, \bar{\omega}, 1)$ is freely generated by G in \mathcal{V}'_0 .*

We are now ready to prove the following theorem.

Theorem 7. *${}^\omega SP(A)$ is freely generated by A in \mathcal{V}' .*

Proof. For all $i \in \mathbb{N}$, let ${}^\omega SP(A)_i$ be the subset of ${}^\omega SP(A)$ consisting all its posets of width lower or equal to i . Then ${}^\omega SP(A) = \bigcup_{i \in \mathbb{N}} {}^\omega SP(A)_i$. Note that ${}^\omega SP(A)_0 = \{\epsilon\}$ and ${}^\omega SP(A)_1 = {}^\omega A$. Observe that for all $i \in \mathbb{N}$, ${}^\omega SP(A)_i$ is closed under sequential product, ω -power and $\bar{\omega}$ -power. In addition, for all $i \in \mathbb{N}$, ${}^\omega SP(A)_i$ is generated by its sequentially irreducible posets. By Corollary 1, for all $i \in \mathbb{N}$, ${}^\omega SP(A)_i$ can be considered as a \mathcal{V}'_0 -algebra. In addition, by Corollary 2, for all $i \in \mathbb{N}$, ${}^\omega SP(A)_i$ is freely generated by its sequentially irreducible posets in \mathcal{V}'_0 . Then, for all $i \in \mathbb{N}$ and $S \in \mathcal{V}'_0$, a function $h': A \rightarrow S$ can be extended in a unique homomorphism of \mathcal{V}'_0 -algebras $h'_i: {}^\omega SP(A)_i \rightarrow S$.

Let S be some \mathcal{V}' -algebra and let $h: A \rightarrow S$ be some function. We show that h can be extended into a homomorphism of \mathcal{V}' -algebras $h^\sharp: {}^\omega SP(A) \rightarrow S$ in a unique way. Indeed, we define h^\sharp as $h^\sharp = \bigcup_{i \in \mathbb{N}} h_i$ where each $h_i: {}^\omega SP(A)_i \rightarrow S$ is defined, by induction on i , as follows:

- when $i = 0$, h_0 is defined by $\epsilon \rightarrow 1$;
- when $i = 1$, h_1 is the unique homomorphism of \mathcal{V}'_0 -algebras $\omega A \rightarrow S$ extending h (Theorem 6);
- when $i \geq 2$, h_i is defined as follows:
 - on posets P of width lower than i , $h_i(P)$ is $h_{i-1}(P)$;
 - on sequential posets P of width i , $h_i(P)$ is $h'_i(P)$;
 - on parallel posets P of width i , $h_i(P)$ is defined relying on the irreducible parallel factorization $\parallel_{j \in [n]} P_j$ of P , for some $n \in \mathbb{N}$, by:

$$h_i(P) = \parallel_{j \in [n]} h_{i-1}(P_j)$$

Proving that h^\sharp is a homomorphism of \mathcal{V}' -algebras is routine. Furthermore, the uniqueness of h^\sharp comes from the facts that h^\sharp extends h and that A is a generating set of $\omega SP(A)$.

6 Decidability

Throughout this section, A denotes an alphabet. The set of terms of some signature over A is the smallest set of finite words built from A using the operations of the corresponding signature. In this section we prove the decidability of the equational theory of \mathcal{V}' .

Let τ be the signature of \mathcal{V}' -algebras. We start by defining the set of terms in which we are interested.

Definition 5. *The set of terms T_A over A is the smallest set satisfying the following conditions:*

- $A \cup \{1\} \subseteq T_A$;
- if $t_1, t_2 \in T_A$ then $t_1 \cdot t_2, t_1 \parallel t_2 \in T_A$;
- if $t \in T_A$ then $t^\omega, t^{\overline{\omega}} \in T_A$.

By equipping T_A with the operations of τ , we define a structure called the term algebra $\mathcal{T}(A) = (T_A, \cdot, \parallel, \omega, \overline{\omega}, 1)$ over A . Note that T_A can be considered also as the set of trees whose leaves are labeled by $A \cup \{1\}$ and whose internal nodes are labeled by the operations of τ where the out-degree of each internal node coincides with the arity of the corresponding operation.

Two terms $t, t' \in T_A$ are equivalent if t' can be derived from t using the axioms which \mathcal{V}' satisfy (denoted $t \equiv t'$). This equivalence relation is actually a congruence. It is well-known that $\mathcal{T}(A)$ is absolutely free i.e. it is freely generated by A in the class containing all the algebras of signature τ . In addition, as a consequence of Theorem 7, $\mathcal{T}(A)/\equiv$ is isomorphic to $\omega SP(A)$ (see eg. [1, Theorem 1.3.2]). This isomorphism can be defined by $\llbracket 1 \rrbracket = \epsilon$ and $\llbracket a \rrbracket = a$ for all $a \in A$.

Then we have:

Proposition 2. *Let $t, t' \in T_A$. Then $\llbracket t \rrbracket = \llbracket t' \rrbracket$ if and only if $t \equiv t'$ holds in \mathcal{V}' .*

As a consequence, proving the decidability of the equational theory of \mathcal{V}' can be reduced to decide whether $\llbracket t \rrbracket = \llbracket t' \rrbracket$.

Theorem 8. *Let $t, t' \in T_A$. It is decidable whether $\llbracket t \rrbracket = \llbracket t' \rrbracket$.*

We now give a quick outline of the proof. The terms t and t' can be interpreted as particular forms of rational expressions over languages of $SP^\circ(A)$, see [6]. By extension of a well-known result of Büchi on ordinals, it is known from [2] that a language of $SP^\circ(A)$ is rational if and only if it is definable in an extension, named *P-MSO*, of the so-called monadic second-order logic. Two P-MSO formulæ ψ_t and $\psi_{t'}$ such that $L(\psi_t) = \llbracket t \rrbracket$ and $L(\psi_{t'}) = \llbracket t' \rrbracket$ can effectively be built from t and t' . We have $L(\psi_t \wedge \psi_{t'}) = \emptyset$ if and only if $\llbracket t \rrbracket \neq \llbracket t' \rrbracket$. Theorem 8 follows from the decidability of the P-MSO theory of $SP^\circ(A)$ [2, Theorem 6].

This decision procedure has a non-elementary complexity. Another proof with an exponential complexity (in the size of t, t') can be derived from the proof of [12, Theorem 7.6], in which the $\bar{\omega}$ -power is not considered, by replacing the use of [12, Theorem 7.3] by [9, Corollary 3.19].

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