Small Trees for High Order Whitney Elements

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1 Introduction

We aim at determining in a constructive way, for the high order case, the finite element solutions of $\operatorname{grad}\phi = \mathbf{E}$, $\operatorname{curl}\mathbf{A} = \mathbf{B}$, $\operatorname{div}\mathbf{D} = \rho$, namely, of the equations linking the electric field $\mathbf{E}$, the magnetic induction $\mathbf{B}$, and the electric charge density ρ, to their potentials ϕ, $\mathbf{A}$ and $\mathbf{D}$, respectively. Stating the necessary and sufficient conditions for assuring that a function defined in a bounded set $\Omega \subset \mathbb{R}^3$ is the gradient of a scalar potential, the curl of a vector potential or the divergence of a vector field is one of the most classical problem of vector analysis (see for example [3, 6, 8]). We aim at providing an explicit and efficient procedure to construct a finite element solution. For example, div-free fields, $\mathbf{W}$, are implicitly characterized in terms of a vector $\mathbf{w}$ of degrees of freedom of $\mathbf{W}$ by the algebraic constraint $\mathsf{D}\,\mathbf{w} = \mathbf{0}$, with D the matrix of the div operator between finite elements spaces. The same fields, in the case of a domain with connected boundary, are explicitly defined by $\mathbf{w} = \mathsf{R}\,\mathbf{a}$, with no constraint on $\mathbf{a}$, where R is the matrix of the curl operator between finite elements spaces and $\mathbf{a}$ collects the degrees of freedom of the vector potentials $\mathbf{A}$. Similarly, one can wish to compute a vector potential $\mathbf{a}$ such that $\mathsf{R}\,\mathbf{a} = \mathbf{b}$, for a given field $\mathbf{b}$ verifying $\mathsf{D}\,\mathbf{b} = \mathbf{0}$. As explained in [5], these bases can be constructed by the help of "trees" and "co-trees", which are at the core of this contribution. The case $r = 0$ is largely treated in the literature for different types of topological domains (see for example [2]). In these pages, we develop the

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tree and co-tree approaches for $r > 0$ when fields in the high order Whitney spaces are represented on the basis of their weights on small simplices [7, 9, 10]. With this choice of degrees of freedom, the tree and co-tree concepts extend from $r = 0$ to $r > 0$ straightforwardly.

2 Basic Concepts

Let $\Omega \subset \mathbb{R}^3$ be a bounded polyhedral domain with Lipschitz boundary $\partial\Omega$ and $\mathcal{M}$ a simplicial mesh of $\bar{\Omega}$. We denote by $|A|$ the cardinality of the set A. For $0 \le k \le 3$, let $\Delta_k(T)$ (resp. $\Delta_k(\mathcal{M})$) be the set of k-simplices of a mesh tetrahedron T (resp. of the mesh $\mathcal{M}$). Note that $\Delta_k(\mathcal{M}) = \cup_{T \in \mathcal{M}} \Delta_k(T)$. If $\Delta_0(\mathcal{M}) = \{\mathbf{v}_i\}_i$, with $i = 1, \dots, N_v$, being $N_v = |\Delta_0(\mathcal{M})|$, then each k-simplex $S \in \Delta_k(\mathcal{M})$ has associated an increasing map $m_S : \{0, \dots, k\} \rightarrow \{1, \dots, N_v\}$. This map induces an (inner) orientation on S (i.e., a way to run along S if $k = 1$, through S if $k = 2$, in S if $k = 3$).

If we assign to each $S \in \Delta_k(\mathcal{M})$ a real number c_S we can define the k-chain $c = \sum_{S \in \Delta_0(\mathcal{M})} c_S S$, i.e. a formal weighted sum of k-simplices S in $\mathcal{M}$. One can add k-chains, namely $(c + \tilde{c}) = \sum_S (c_S + \tilde{c}_S) S$, and multiply a k-chain by a scalar p, namely $p\, c = \sum_S (p\, c_S) S$. The set of all k-chains in $\mathcal{M}$, here denoted $\mathcal{C}_k(\mathcal{M})$, is a vector space, in one-to-one correspondence with the set of real vectors $c = (c_S)_{S \in \Delta_k(\mathcal{M})}$. Each k-simplex $S \in \Delta_k(\mathcal{M})$, can be associated with the elementary k-chain c with entries $c_S = 1$ and $c_{\tilde{S}} = 0$ for $\tilde{S} \ne S$. In the following we will use the same symbol S to denote the oriented k-simplex and the associated elementary k-chain.

The boundary operator ∂ takes a k-simplex S and returns the sum of all its $(k-1)$-faces f with coefficient 1 or -1 depending of whether the orientation of the $(k-1)$-face f matches or not with the orientation induced by that of the simplex S on f. Since the boundary operator is a linear mapping from $\mathcal{C}_k(\mathcal{M})$ to $\mathcal{C}_{k-1}(\mathcal{M})$, it can be represented by a matrix $\mathfrak{d}$ of dimension $|\Delta_{k-1}(\mathcal{M})| \times |\Delta_k(\mathcal{M})|$, which is rather sparse, gathering the coefficients 0, -1, or $+1$. Note that in three dimensions, there are three nontrivial boundary operators acting, respectively, on edges, triangles and tetrahedra: ∂_1 represented by the matrix $\mathsf{G}^\top$, ∂_2 represented by $\mathsf{R}^\top$, and ∂_3 represented by $\mathsf{D}^\top$. To fully specify ∂, we need to specify the boundary of each simplex S. By definition, we have

$$\partial_1 e = \sum_{n \in \Delta_0(\mathcal{M})} \mathsf{G}_{e,n}\, n, \qquad \partial_2 f = \sum_{e \in \Delta_1(\mathcal{M})} \mathsf{R}_{f,e}\, e, \qquad \partial_3 T = \sum_{f \in \Delta_2(\mathcal{M})} \mathsf{D}_{T,f}\, f,$$

for any $e \in \Delta_1(\mathcal{M})$, any $f \in \Delta_2(\mathcal{M})$ and any $T \in \Delta_3(\mathcal{M})$. For $e = [\mathbf{v}_0, \mathbf{v}_1]$, $f = [\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2]$ and $T = [\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$, we have, respectively,

$$\partial_1[\mathbf{v}_0, \mathbf{v}_1] = \mathbf{v}_0 - \mathbf{v}_1, \qquad \partial_2[\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2] = [\mathbf{v}_0, \mathbf{v}_1] - [\mathbf{v}_0, \mathbf{v}_2] + [\mathbf{v}_1, \mathbf{v}_2],$$
$$\partial_3[\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = [\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2] - [\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3] + [\mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_3] - [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3].$$

The subscript is removed when there is no ambiguity, since the operator needed for a particular operation is indicated from the type of the operand (e.g., ∂_3 when ∂ applies to tetrahedra). The notion of boundary can be extended to k-chains by linearity, $\partial c = \partial(\sum_{S \in \Delta_k(\mathcal{M})} \mathbf{c}_S \, S) = \sum_{S \in \Delta_k(\mathcal{M})} \mathbf{c}_S \, \partial S$.

We say that a k-chain c is closed if $\partial_k c = 0$. Non-trivial closed k-chains are called k-cycles and constitute the subspace $Z_k(\mathcal{M}) = \ker(\partial_k; \mathcal{C}_k(\mathcal{M}))$. A k-chain c is a boundary if it exists a $(k+1)$-chain γ such that $c = \partial_{k+1}\gamma$. The k-boundaries constitute the subspace $B_k(\mathcal{M}) = \partial_{k+1}\mathcal{C}_{k+1}(\mathcal{M})$. From the property $\partial\partial = 0$, we know that boundaries are cycles but not all cycles are boundaries, and we have $B_k(\mathcal{M}) \subset Z_k(\mathcal{M})$. The quotient space $\mathcal{H}_k(\mathcal{M}) = [Z_k(\mathcal{M})/B_k(\mathcal{M})]$ is the homology spaces of order k of the mesh $\mathcal{M}$, and the Betti's number $b_k = \operatorname{rank}[\mathcal{H}_k(\mathcal{M})]$. The presence of curl-free fields (resp. div-free fields) that are not the gradient of a scalar field (resp. the curl of a vector field) is indicated from the fact that $b_1 \neq 0$ (resp. $b_2 \neq 0$). We recall that Betti's numbers are topological invariants (i.e., they depend on the domain Ω up to a homeomorphism) and do not depend on the mesh $\mathcal{M}$ on $\bar{\Omega}$ that is used to compute them (see [12] and an application in [11]).

For the high order case, we need to introduce some concepts of relative homology. Let $\mathcal{K}_k(\mathcal{M})$ be subspaces of $\mathcal{C}_k(\mathcal{M})$ with $\partial_k \mathcal{K}_k(\mathcal{M}) \subset \mathcal{K}_{k-1}(\mathcal{M})$. We thus say that $c \in \mathcal{C}_k(\mathcal{M})$ is closed [modulo $\mathcal{K}_k(\mathcal{M})$] if $\partial c \in \mathcal{K}_k(\mathcal{M})$. A $(k-1)$-chain c bounds [modulo $\mathcal{K}_k(\mathcal{M})$] if there exists a k-chain γ such that $c - \partial\gamma \in \mathcal{K}_{k-1}(\mathcal{M})$. We thus talk about relative homology groups.

A k-cochain w (over the mesh $\mathcal{M}$) is a linear mapping from $\mathcal{C}_k(\mathcal{M})$ to $\mathbb{R}$. They are discrete analogues to differential forms. For $k > 0$, the exterior derivative of the $(k-1)$-form w is the k-form $\mathrm{d}w$ such that $\int_s \mathrm{d}w = \int_{\partial s} w$ for all $s \in \mathcal{C}_k(\mathcal{M})$. With this simple equation relating the evaluation of $\mathrm{d}w$ on a simplex s to the evaluation of w on the boundary of this simplex, the exterior derivative is readily defined. We can naturally extend the notion of evaluation of a differential form w on an arbitrary chain by linearity: $\int_{\sum_i \mathbf{c}_i s_i} w = \sum_i \mathbf{c}_i \int_{s_i} w$. Thus

$$\int_{\sum_i \mathbf{c}_i s_i} \mathrm{d}w = \int_{\partial(\sum_i \mathbf{c}_i s_i)} w = \int_{\sum_i \mathbf{c}_i \partial s_i} w = \sum_i \mathbf{c}_i \int_{\partial s_i} w.$$

The operator d is the dual of the boundary operator ∂. As a corollary of the boundary operator property $\partial\partial = 0$, we have that $\mathrm{dd} = 0$. Since we used arrays of dimension $|\Delta_k(\mathcal{M})|$ to represent a k-cochain, the operator d can be represented by a matrix $\mathbf{d}$ of dimension $|\Delta_k(\mathcal{M})| \times |\Delta_{k-1}(\mathcal{M})|$, $1 \leq k \leq 3$. Again, we have one matrix for the exterior derivative operator for each simplex dimension. When a metric is introduced on the ambient affine space, the exterior derivative operator d stands for grad, curl, div, according to the value of k from 1 to 3, and it is represented by, respectively, G, R, D, the connectivity matrices of the mesh $\mathcal{M}$.

3 Small Simplices, Weights and Potentials

We introduce the multi-index $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_s)$ of $s+1$ integers $\alpha_i \geq 0$ and weight $|\boldsymbol{\alpha}| = \sum_{i=1}^s \alpha_i$. The set of multi-indices $\boldsymbol{\alpha}$ with $s+1$ components and weight r is denoted $\mathcal{I}(s+1, r)$. We denote by $\mathbf{v}_i$ the (Cartesian) coordinates of the node n_i in $\mathbb{R}^3$. Given a multi-index $\boldsymbol{\alpha} \in \mathcal{I}(4, r)$, and a k-subsimplex S of T, the small simplex $\{\boldsymbol{\alpha}, S\}$ is the k-simplex that belongs to the small tetrahedron with barycenter at the point of coordinates $\sum_{i=0}^3 [(\frac{1}{4} + \alpha_i)\mathbf{v}_{\sigma_T^0(i)}]/(r+1)$, which is parallel and $1/(r+1)$-homothetic to the (big) sub-simplex S of T. The notation $\{\boldsymbol{\alpha}, S\}$ was first defined in [9]. The set of small tetrahedra of order $r+1 > 1$ can be visualized starting from the principal lattice $L_{r+1}(T)$ in the simplex $T = \{n_{\sigma_T^0(0)}\, n_{\sigma_T^0(1)}\, n_{\sigma_T^0(2)}\, n_{\sigma_T^0(3)}\}$ defined as

$$L_{r+1}(T) = \left\{ \mathbf{x} \in T \,:\, \lambda_{\sigma_T^0(i)}(\mathbf{x}) \in \{0, \frac{1}{r+1}, \frac{2}{r+1}, \ldots, \frac{r}{r+1}, 1\},\ 0 \leq i \leq 3 \right\}.$$

and connecting its points by edges parallel to those of T. (See, e.g., Fig. 1.)

We denote by $\Lambda^k(\Omega)$ the space of all smooth differential k-forms on Ω. The completion of $\Lambda^k(\Omega)$ in the corresponding norm defines the Hilbert space $L^2\Lambda^k(\Omega)$. Let $\mathcal{P}_{r+1}^-\Lambda^k(T)$ be the space of so-called trimmed polynomial k-forms of degree $r+1$ on T, with $r \geq 0$, (as in [7]), and we define
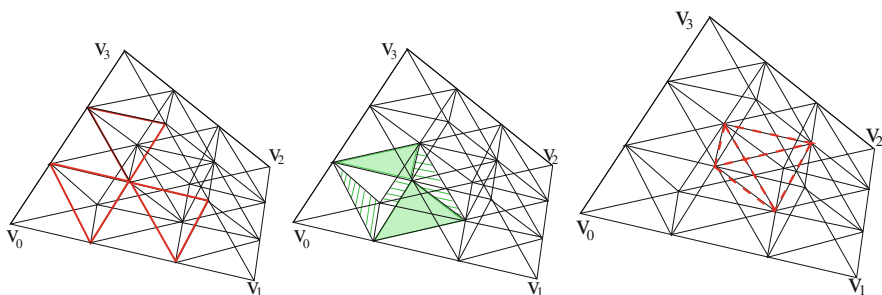
$$\mathcal{P}_{r+1}^-\Lambda^k(\mathcal{M}) = \{\omega \in H\Lambda^k(\Omega) \,:\, \omega_{|T} \in \mathcal{P}_{r+1}^-\Lambda^k(T),\ T \in \mathcal{M}\}$$

where $H\Lambda^k(\Omega) = \{\omega \in \Lambda^k(\Omega) \,:\, d\omega \in \Lambda^k(\Omega)\}$ is a Hilbert space (see [4]).

Definition 1 The weights of a polynomial k-form $u \in \mathcal{P}_{r+1}^-\Lambda^k(T)$, with $0 \leq k \leq 3$ and $r \geq 0$, are the scalar quantities

$$\int_{\{\boldsymbol{\alpha}, S\}} u, \tag{1}$$

on the small simplices $\{\boldsymbol{\alpha}, S\}$ with $\boldsymbol{\alpha} \in \mathcal{I}(4, r)$ and $S \in \Delta_k(T)$.

Fig. 1 From the principal lattice of degree $r+1 = 3$ in a tetrahedron T, we define a decomposition of T into 10 small tetrahedra, 4 octahedra O and 1 reversed tetrahedron. Each face on ∂T is decomposed into 6 small faces and 3 reversed triangles, in solid red line (Left)

We now list some remarkable properties of the small simplices which are useful in the tree construction.

Property 1 The weights (1) of a Whitney k-form $u \in \mathcal{P}_{r+1}^- \Lambda^k(T)$ on all the small simplex $\{\boldsymbol{\alpha}, S\}$ of T are unisolvent, as stated in [7, Proposition 3.14]. The small simplices can thus support the degrees of freedom for fields $u \in \mathcal{P}_{r+1}^- \Lambda^k(T)$, with $0 \leq k \leq 3$ and $r \geq 0$. Since the result on unisolvence holds true also by replacing T with $F \in \Delta_{n-1}(T)$ then $\mathrm{Tr}_F u \in \mathcal{P}_{r+1}^- \Lambda^k(F)$ is uniquely determined by the weights on small simplices in F. It thus follows that a locally defined u, with $u_{|T} \in \mathcal{P}_{r+1}^- \Lambda^k(T)$ and single-valued weights, is in $H \Lambda^k(\Omega)$. We thus can use the weights on the small simplices $\{\boldsymbol{\alpha}, S\}$ as degrees of freedom for the fields in the finite element space $\mathcal{P}_{r+1}^- \Lambda^k(\mathcal{M})$ being aware that their number is greater than the dimension of the space.

Property 2 The weights given in Definition 1 have a meaning as cochains and this relates directly the matrix describing the exterior derivative with the matrix of the boundary operator. The key point is the Stokes' theorem $\int_C du = \int_{\partial C} u$, where u is a $(k-1)$-form and C a k-chain. More precisely, if $u \in \mathcal{P}_{r+1}^- \Lambda^k(\mathcal{M})$ then $z = du \in \mathcal{P}_{r+1}^- \Lambda^{k+1}(\mathcal{M})$ and

$$\int_{\{\boldsymbol{\alpha}, S\}} z = \int_{\{\boldsymbol{\alpha}, S\}} du = \int_{\partial \{\boldsymbol{\alpha}, S\}} u = \sum_{\{\boldsymbol{\beta}, F\}} \mathrm{B}_{\{\boldsymbol{\alpha}, S\}, \{\boldsymbol{\beta}, F\}} \int_{\{\boldsymbol{\beta}, F\}} u$$

being B the boundary matrix with as many rows as small simplices of dimension k and as many columns as small simplices of dimension $k-1$. The small simplices $\{\boldsymbol{\alpha}, S\}$ inherit the orientation of the simplex S so the coefficient $\mathrm{B}_{\{\boldsymbol{\alpha}, S\}, \{\boldsymbol{\beta}, F\}}$ is equal to the coefficient $\mathrm{B}_{S, F}$ of the boundary of the simplex S if $\boldsymbol{\beta} = \boldsymbol{\alpha}$. This is straightforward if $\dim(F) > 0$ and when $\dim(F) = 0$, providing that small nodes in T are given in the notation $\{\boldsymbol{\alpha}, n\}$ according to their position in the small simplices when fragmented (see Fig. 1 in [1]).

Property 3 The generated $\binom{r+2}{2}$ small faces on each face F of T, pave F together with the $\binom{r+1}{2}$ reversed triangles, denoted by ∇, contained in F. Similarly, the generated $\binom{r+3}{3}$ small tetrahedra contained in T pave T together with the $\binom{r+2}{3}$ octahedra, denoted by O, and the $\binom{r+1}{3}$ reversed tetrahedra, denoted by $\perp$, contained in T, as shown in Fig. 1. Reversed octahedra and reversed tetrahedra are examples of "holes" in T (see [9, 10]).

Property 4 Since homology is preserved by homotopy, in [10, Section 3.4], it is discussed the fact that the relative homology (i.e., the homology [modulo the holes' boundaries]), of the complex of small simplices is the same of the homology of $\mathcal{M}$. This property is fundamental to build the tree for high order potentials when working with small simplices. The homology [modulo the holes' boundaries] can be translated in matrix notation, by showing that the boundary matrices associated with

the small simplices, "modified" and "completed" (in a sense that we explain in the next section) by the relations [10, Proposition 3.5] are incidence matrices of a graph. To apply the theory presented in [10, Section 3.4] in a tetrahedron $T \in \Delta_3(\mathcal{M})$, we need to introduce, for $r > 0$, two sets $\mathcal{K}_1$ and $\mathcal{K}_2$ of chains generated by the small simplices that belong to the boundary of some hole in T as follows:

- $\mathcal{K}_1$ are the chains generated by the boundary of the $\binom{r+1}{2}$ reversed triangle $\nabla \subset F$ and that for each $F \in \Delta_2(T)$, and the boundary of the three faces out of four on the boundary $\partial\perp$ of each of the $\binom{r}{2}$ reversed tetrahedra $\perp$ in T;
- $\mathcal{K}_2$ are the chains generated by 4 out of 8 faces of the $\binom{r+2}{3}$ octahedra O in T. The involved faces are the small faces belonging to the boundary ∂O privated of $\partial O \cap (\Delta_2(T) \cup \partial\perp)$.

The two sets $\mathcal{K}_1$ and $\mathcal{K}_2$ satisfy the property $\partial\mathcal{K}_2 \subset \mathcal{K}_1$, decisive to conclude that the relative homology [modulo the holes' boundaries] of the complex of the small simplices is the same as the homology of the original mesh $\mathcal{M}$ [10].

4 Trees and Graphs

As stated in [12], a directed graph $\mathcal{G}$ consists of two sets $\mathcal{N}$ and $\mathcal{A}$ of nodes and arcs, respectively, subjected to certain incidence relations, collected in the all-vertex incidence matrix $\mathtt{M}^{\mathcal{G}} \in \mathbb{Z}^{|\mathcal{N}| \times |\mathcal{A}|}$ as follows:

$$\mathtt{M}^{\mathcal{G}}_{n,a} = \begin{cases} -1 \,, & \text{if } a \text{ starts from } n, \\ +1 \,, & \text{if } a \text{ ends in } n, \\ 0 \,, & \text{if } a \text{ does not contain } n. \end{cases}$$

An incidence matrix $\mathtt{M}$ of the graph $\mathcal{G}$ is any sub-matrix of $\mathtt{M}^{\mathcal{G}}$ with $|\mathcal{N}| - 1$ rows and $|\mathcal{A}|$ columns. The node that corresponds to the row of $\mathtt{M}^{\mathcal{G}}$ that is not in $\mathtt{M}$ will be indicated as the reference node of $\mathcal{G}$. A graph $\mathcal{G}$ is connected if there is a path between any two of its nodes. A tree $\mathcal{T}$ of a graph $\mathcal{G}$ is a connected acyclic subgraph of $\mathcal{G}$. A spanning tree $\mathcal{T}_s$ is a tree of $\mathcal{G}$ visiting all its nodes. Any connected graph $\mathcal{G}$ admits a spanning tree $\mathcal{T}_s$. We have now to particularize these notions for small simplices. In each tetrahedron T of the oriented mesh $\mathcal{M}$, we consider the small mesh associated with $L_{r+1}(T)$ composed only of small tetrahedra, for a given r uniform all over the mesh $\mathcal{M}$. The union of the small meshes for all $T \in \Delta_3(\mathcal{M})$ is denoted $\mathcal{M}_{all}$.

A (Primal) Small Tree for the Gradient Problem

For $r = 0$, the graph $\mathcal{G}^1$ has $\mathcal{N} = \Delta_0(\mathcal{M})$ and $\mathcal{A} = \Delta_1(\mathcal{M})$. The boundary matrix $\mathtt{G}^{\top}$ is the all-vertex incidence matrix of the graph $\mathcal{G}^1$. Extracting a spanning 1-tree $\mathcal{T}_s^1$ from $\mathcal{G}^1$ is equivalent to finding in $\mathtt{G}^{\top}$, minus one row, a submatrix of maximal rank (see [11] for a suitable and easy way of constructing $\mathcal{T}$). For $r > 0$, we have
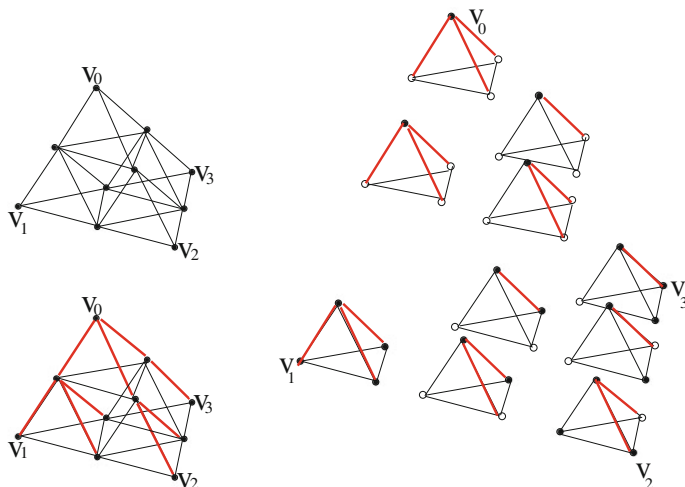
Fig. 2 (Left) The graph $\mathcal{G}^1$ and a spanning tree in thick line, for $r = 1$. (Right) A spanning tree for $r = 2$ in a fragmented layout

to consider the new graph $\mathcal{G}^1$ with $\mathcal{N} = \Delta_0(\mathcal{M}_{all})$ and $\mathcal{A} = \Delta_1(\mathcal{M}_{all})$. Let $\mathsf{G}_{all}^\top$ be the all-vertex incidence matrix of this new graph $\mathcal{G}^1$. Note that $\mathsf{G}_{all}^\top$ results from the boundary operator ∂_1 on the elementary 1-chains from $\mathcal{M}_{all}$. Extracting a spanning 1-tree $\mathcal{T}_s^1$ from $\mathcal{G}^1$ is equivalent to finding in $\mathsf{G}_{all}^\top$, minus one row, a submatrix of maximal rank. Example of spanning 1-tree $\mathcal{T}_s^1$ for $r+1 = 2$ in the right part of Fig. 2 and for $r + 1 = 3$ in Fig. 5 (fragmented visualization). Note that we can repeat this construction in the two-dimensional case.
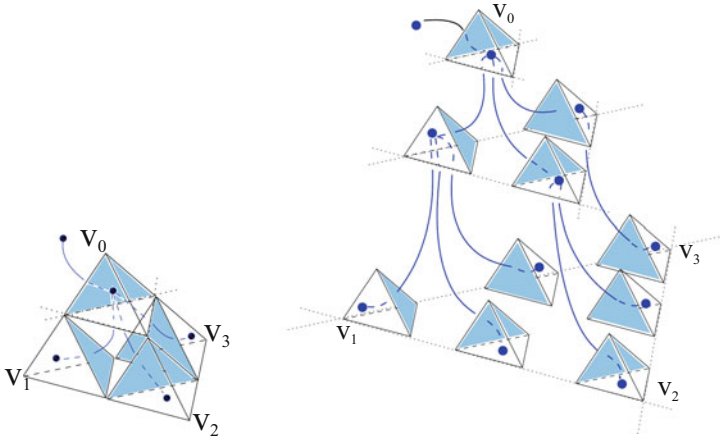
A (Dual) Small Tree for the Divergence Problem

For $r = 0$, the graph $\mathcal{G}^2$ is built on $\mathcal{M}^*$, the so-called dual mesh of $\mathcal{M}$, as follows. Let us note that an internal face $F \in \Delta_2(\mathcal{M})$ connects two adjacent tetrahedra $T_1, T_2 \in \Delta_3(\mathcal{M})$ whereas a boundary face $F_b \in \Delta_2(\mathcal{M})$ connects a tetrahedron $T_b \in \Delta_3(\mathcal{M})$ and the boundary $\partial\Omega$. We can construct the following connected (dual) graph $\mathcal{G}^2$: the set of nodes, $\mathcal{N}$, contains the barycenter of any tetrahedron $T \in \Delta_3(\mathcal{M})$ together with one additional exterior node representing $\partial\Omega$; the set of arcs, $\mathcal{A}$, contains any face $F \in \Delta_2(\mathcal{M})$. For $r = 0$, the matrix D associated with the boundary operator ∂_3, acting on $C_3(\mathcal{M})$, is an incidence matrix of the (dual) graph $\mathcal{G}^2$, with reference node the one corresponding to $\partial\Omega$. Extracting a spanning tree $\mathcal{T}_s^2$ from $\mathcal{G}^2$ is equivalent to finding in D a submatrix of maximal rank.

For $r > 0$, let $\mathcal{R}_2$ be the set of small faces chosen as follows: one small face for each octahedron O contained in $\mathcal{K}_2$ (see the right side of Fig. 3 for the dashed small face in $\mathcal{R}_2$ when $r+1 = 2$). To construct the graph $\mathcal{G}^2$ for $r > 0$ we need to consider $\mathcal{M}_{all}^*$, the dual mesh associated to $\mathcal{M}_{all}$, where nodes are the small tetrahedra and the arcs the small faces, apart from the ones in $\mathcal{R}_2$. To understand this, we can reason

Fig. 3 The (dual) graph $\mathcal{G}_*^2$ associated with the small mesh $\mathcal{M}_{all}$ defined in a tetrahedron T for $r = 1$: the black dots are the nodes, and curved lines the arcs (Left). The (dual) graph $\mathcal{G}^2$ obtained from $\mathcal{G}_*^2$ by merging the nodes corresponding to barycenter of $t_0 = \{(1, 0, 0, 0), T\}$ and of O, thus eliminating the arc associated with the shaded small face f_u^O (Right)

as follows. For $r > 0$, we have one arc connecting two small tetrahedra, say $t_\diamond, t_\circ$, when

- either $t_\diamond, t_\circ$ share the *same* small face f, i.e. $\partial t_\diamond \cap \partial t_\circ = f$;
- or $t_\diamond, t_\circ$ have a small face on the boundary of the *same* octahedron O, i.e. $f_\diamond = \partial t_\diamond \cap \partial O$ and $f_\circ = \partial t_\circ \cap \partial O$ for the *same* octahedron O.

See an example of graph $\mathcal{G}^2$ for $\mathcal{M}_{all}$ (here $\mathcal{M} = \{T\}$) in the left part of Fig. 3 for $r + 1 = 2$, where the node associated with the octahedron O is not a node in the graph, but stands to indicate that the four small tetrahedra are connected one to the other by one arc because they all have one small face on ∂O. Naming t_k the small tetra with a vertex in $\mathbf{v}_k$, $k = 0, 3$, and numbering first the 3×4 faces on $t_k \cap \partial T$, called f_i^k for $i = 1, 2, 3$, second those on ∂O (where $f_u^O, f_\ell^O, f_d^O, f_r^O$ are the small faces up, left, down, right of ∂O), we have

$$
\mathrm{D}_{tmp} = \begin{array}{c} \partial T \\ t_0 \\ t_1 \\ t_2 \\ t_3 \\ O \end{array} \left(\begin{array}{cccccccccccc|cccc} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & & & & \\ 1 & 1 & 1 & & & & & & & & & & -1 & & & \\ & & & 1 & 1 & 1 & & & & & & & & -1 & & \\ & & & & & & 1 & 1 & 1 & & & & & & -1 & \\ & & & & & & & & & 1 & 1 & 1 & & & & -1 \\ & & & & & & & & & & & & 1 & 1 & 1 & 1 \end{array} \right)
$$

$$
\begin{array}{cccccccccccccccc} f_1^0 & f_2^0 & f_3^0 & f_1^1 & f_2^1 & f_3^1 & f_1^2 & f_2^2 & f_3^2 & f_1^3 & f_2^3 & f_3^3 & f_u^O & f_\ell^O & f_d^O & f_r^O \end{array}
$$

Since the octahedron O is not part of the small mesh $\mathcal{M}_{all}$, we have to imagine that its node collapses with the node of one of its neighbouring small tetrahedron, say t_0 with a vertex in $\mathbf{v}_0$, and thus that the corresponding arc (i.e. the small face $f_u^O = \partial t_0 \cap \partial O$, the dashed one in the right part of Fig. 3) is eliminated. From a

Fig. 4 Example of spanning tree in the (dual) graph $\mathcal{G}^2$, namely a selection of acyclic paths made of arcs, visiting all the nodes of $\mathcal{G}^2$ ($r = 1$, Left and $r = 2$, Right)

matrix point of view, D is obtained by adding the line "O" in D_{tmp} to the line "t_0", and eliminating f_u^O, namely

$$
D = \begin{array}{c} t_0 \\ t_1 \\ t_2 \\ t_3 \end{array}
\left(
\begin{array}{ccccccccccccccc}
\mathbf{1} & \mathbf{1} & \mathbf{1} & & & & & & & & & & \mathbf{1} & \mathbf{1} & \mathbf{1} \\
& & & 1 & 1 & 1 & & & & & & & -\mathbf{1} & & \\
& & & & & & 1 & 1 & 1 & & & & & -\mathbf{1} & \\
& & & & & & & & & 1 & 1 & 1 & & & -1
\end{array}
\right)
$$
$$
\begin{array}{ccccccccccccccc}
f_1^0 & f_2^0 & f_3^0 & f_1^1 & f_2^1 & f_3^1 & f_1^2 & f_2^2 & f_3^2 & f_1^3 & f_2^3 & f_3^3 & f_\ell^O & f_d^O & f_r^O
\end{array}
$$

(in bold font, the submatrix of maximal rank in D for the spanning tree $\mathcal{T}_s^2$ illustrated in Fig. 4, left part for $r + 1 = 2$). To repeat this construction in the two-dimensional case, when T is a triangle, we have to consider the mesh $\mathcal{M}_{all}$ of small triangles in T and the role of the core octahedra O is played by the reversed triangles $\nabla \in T$. The set $\mathcal{R}_2$ is replaced by $\mathcal{R}_1$, composed of one small edge for each reversed triangle $\nabla \in \mathcal{K}_1$. In two dimensions we do not have reversed tetrahedra, therefore no reversed triangles $\nabla_\perp$.

The construction of the spanning tree in $\mathcal{M}_{all}$ can be done by assembling that of the geometrical mesh $\mathcal{M}$, namely a spanning tree for the Whitney forms of lower degree (blue lines in Fig. 5 (Right)), together with local contributions, one from each element (green lines in Fig. 5 (Right)). Each local contribution results from one fixed on a *reference* element which is *mapped* on the current element (respecting the orientation). In Fig. 5 (Left), in green/red thick line we have marked the small edges of a spanning tree in the graph $\mathcal{G}^1$, for $r = 3$, in the reference triangle. The red ones belong to the spanning tree in the reference triangle, but they are in general omitted in the spanning tree of $\mathcal{M}_{all}$, (indeed, they appear only if they are covered by the
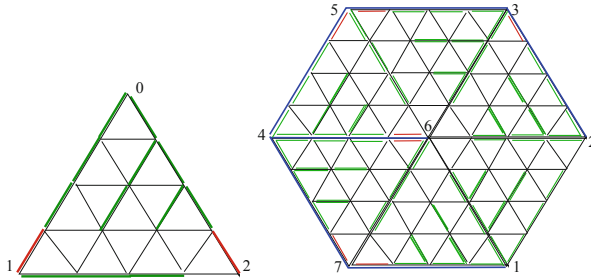
Fig. 5 (Left) In thick colored line, the small edges of the graph $\mathcal{G}^1$, for $r = 3$, that compose a spanning tree in a *reference* triangle. (Right) In thick blue line the contribution of the branches of a spanning tree in a (2D) toy mesh $\mathcal{M}$ reported on $\mathcal{M}_{all}$. In green, the contribution of the small branches *mapped* from the green ones in the reference triangle. It is not necessary to report the red ones since they are either covered by the blue ones or omitted. The co-tree is in black

blue tree). The small co-tree is in black. A similar construction can be repeated in 3D (both for $k = 1$ and $k = 2$) and it reflects the decomposition given, for instance, in [13] (Sect. 5).

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