Variants of the Segment Number of a Graph

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Abstract. The *segment number* of a planar graph is the smallest number of line segments whose union represents a crossing-free straight-line drawing of the given graph in the plane. The segment number is a measure for the visual complexity of a drawing; it has been studied extensively.

In this paper, we study three variants of the segment number: for planar graphs, we consider crossing-free polyline drawings in 2D; for arbitrary graphs, we consider crossing-free straight-line drawings in 3D and straight-line drawings with crossings in 2D. We first construct an infinite family of planar graphs where the classical segment number is asymptotically twice as large as each of the new variants of the segment number. Then we establish the $\exists\mathbb{R}$-completeness (which implies the NP-hardness) of all variants. Finally, for cubic graphs, we prove lower and upper bounds on the new variants of the segment number, depending on the connectivity of the given graph.

1 Introduction

When drawing a graph, a way to keep the visual complexity low is to use few geometric objects for drawing the edges. This idea is captured by the *segment number* of a (planar) graph, that is, the smallest number of crossing-free line segments that together constitute a straight-line drawing of the given graph. The *arc number* of a graph is defined analogously with respect to circular-arc drawings. So far, both numbers have only been studied for planar graphs. Two obvious lower bounds for the segment number are known [5]: (i) $\eta(G)/2$, where $\eta(G)$ is the number of odd-degree vertices of G, and (ii) the planar slope number of G, that is, the smallest number k such that G admits a crossing-free straight-line drawing whose edges have k different slopes.

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Dujmović et al. [5], who introduced segment number and planar slope number, showed among others that trees can be drawn without crossings such that the optimum segment number and the optimum planar slope number are achieved simultaneously. In fact, any tree T admits a drawing with $\eta(T)/2$ segments and $\Delta(T)/2$ slopes, where $\Delta(T)$ is the maximum degree of T. Unfortunately, these drawings need exponential area. Therefore, Schulz [19] suggested to study the arc number of planar graphs. Among other things, he showed that any n-vertex tree can be drawn on a polynomial-size grid ($O(n^{1.81}) \times n$) using at most $3n/4$ arcs.

Another measure for the visual complexity of a drawing of a graph is the minimum number of *lines* whose union contains a straight-line crossing-free drawing of the given graph. This parameter is called the *line cover number* of a graph G and denoted by $\rho^1_2(G)$ for 2D (where G must be planar) and $\rho^1_3(G)$ for 3D. Together with the plane cover number $\rho^2_3(G)$ and other variants, these parameters have been introduced by Chaplick et al. [2]. They also showed that both line cover numbers are $\exists\mathbb{R}$-hard to compute [3]. (For background on $\exists\mathbb{R}$, see Schaefer's work [18]).

Upper bounds for the segment number and the arc number (in terms of the number of vertices, n, ignoring constant additive terms) are known for series-parallel graphs ($3n/2$ vs. n), planar 3-trees ($2n$ vs. $11n/6$), and triconnected planar graphs ($5n/2$ vs. $2n$) [5,19]. The upper bound on the segment number for triconnected planar graphs has been improved for the special cases of triangulations and 4-connected triangulations (from $5n/2$ to $7n/3$ and $9n/4$, respectively) by Durocher and Mondal [6]. For the special case of triconnected cubic graphs, Dujmović et al. [5] showed that the segment number is upperbounded by $n + 2$. (A cubic graph with n vertices has $3n/2$ edges.) The result of Dujmović et al. was improved by Mondal et al. [16] who gave two linear-time algorithms based on cannonical decompositions; one that uses at most $n/2+3$ segments for $n \geq 6$ and one that uses $n/2+4$ segments but places all vertices on a grid of size $n \times n$. Both algorithms use at most six different slopes. Note that $n/2 + 3$ segments are optimal for cubic planar graphs since in every vertex at least one segment must end and in the at least three vertices on the convext hull all three incident segments must end. Igamberdiev et al. [12] fixed a bug in the algorithm of Mondal et al., presented two conceptually different (but slower) algorithms that meet the lower bound and compared them experimentally in terms of common metrics such as angular resolution.

Hültenschmidt et al. [11] provided bounds for segment and arc number under the additional constraint that vertices must lie on a polynomial-size grid. They also showed that n-vertex triangulations can be drawn with at most $5n/3$ arcs, which is better than the lower bound of $2n$ for the segment number on this class of graphs. For 4-connected triangulations, they need at most $3n/2$ arcs. Kindermann et al. [13] recently strengthened some of these results by showing that many classes of planar graphs admit nontrivial bounds on the segment number even when restricting vertices to a grid of size $O(n) \times O(n^2)$. For drawing n-vertex trees with at most $3n/4$ segments, they reduced the grid size to

$n \times n$. Among other things, Durocher et al. [7] showed that the segment number is NP-hard to compute *with respect to a fixed embedding*, even in the special case of arrangement graphs. They also showed that the following partial representation extension problem is NP-hard: given an outerplanar graph G, an integer k, and a straight-line drawing δ of a subgraph of G, is there a k-segment drawing that contains δ? It is still open, however, whether the segment number is fixed-parameter tractable.

In this paper, we consider several variants of the planar segment number seg_2 that has been studied extensively. In particular, we study the *3D segment number* seg_3, which is the most obvious generalization of the planar segment number. It is the smallest number of straight-line segments needed for a crossing-free straight-line drawing of a given graph in 3D. We also study the *crossing segment number* $\mathrm{seg}_\times$ in 3D, where edges are allowed to cross, but they are not allowed to overlap or to contain vertices in their interiors. In this case, by Lemma 1, the minimum number of segments constituting a drawing of a given graph can be achieved by a plane drawing. Finally, for planar graphs, we study the *bend segment number* $\mathrm{seg}_\angle$ in 2D, which is the smallest number of straight-line segments needed for a crossing-free polyline drawing of a given graph in 2D.

Durocher et al. [7] were also interested in the 3D segment number. They stated that their proof of the NP-hardness of the above-mentioned partial representation problem can be adjusted to 3D. They suspected that the 3D segment number remains NP-hard to compute even if the given graph is subcubic. Instead, they showed that a variant of the 3D segment number is NP-hard where one is given a 3D drawing and additional co-planarity constraints that must be fulfilled in the final drawing.

Our Contribution. First, we establish some relationships between the variants of the segment number; see Sect. 2. Then we turn to the complexity of computing the new variants of the segment number; see Sect. 3. By re-using ideas from the $\exists\mathbb{R}$-completeness proof of Chaplick et al. [3] regarding the computation of the line cover numbers ρ^1_2 and ρ^1_3, we establish the $\exists\mathbb{R}$-completeness (and hence the NP-hardness) of all variants of the segment number – seg_2, seg_3, $\mathrm{seg}_\times$, and $\mathrm{seg}_\angle$ – even for graphs of maximum degree 4. Thus, we nearly answer the open problem of Durocher et al. [7] concerning the computational complexity of the 3D segment number for subcubic graphs. Note that Hoffmann [10] recently established the $\exists\mathbb{R}$-hardness of computing the slope number $\mathrm{slope}(G)$ of a planar graph G.

Our main contribution consists in algorithms and lower-bound constructions for connected ($\gamma = 1$), biconnected ($\gamma = 2$), and triconnected ($\gamma = 3$) cubic graphs; see Table 1. To put these results into perspective, recall that any cubic graph with n vertices needs at least $n/2 + 3$ and at most $3n/2$ segments to be drawn, regardless of the drawing style. (In contrast, four slopes suffice for cubic graphs [17]). We prove our bounds in Sect. 4. Note that for cubic graphs, vertex- and edge-connectivity are the same [4, Thm. 2.17].

Before we start, we introduce the following notation. For a given polyline drawing δ of a graph in 2D or 3D, we denote by $\mathrm{seg}(\delta)$ the number of (inclusion-wise maximal) straight-line segments of which the drawing δ consists.

Table 1. Overview over existing and new bounds on variants of the segment number of cubic graphs. The upper bounds hold for all n-vertex graphs of a certain vertex connectivity γ. The lower bounds are existential; there exist graphs for which they hold. Note that seg_2 and $\mathrm{seg}_\angle$ are defined only for planar graphs. We skip more specialized known results (e.g., concerning grid size [11] or triangulations [6]).

γ	$\mathrm{seg}_2(G)$	$\mathrm{seg}_3(G)$	$\mathrm{seg}_\angle(G)$	$\mathrm{seg}_\times(G)$
1	$\geq 5n/6$ [Prp. 2]	$\geq 5n/6$ [Prp. 2]	$\geq 5n/6$ [Prp. 2]	$\geq 5n/6$ [Prp. 2]
2		$\leq n+2$ [Th. 5]	$\leq n+1$ [Th. 4]	$\leq n+2$ [Th. 5]
	$\geq 3n/4$ [Prp. 4]	$\geq 5n/6$ [Prp. 3]	$\geq 3n/4$ [Prp. 4]	$\geq 3n/4$ [Prp. 4]
3	$= n/2+3$ [12,16]	$\leq n+2$ [Th. 5]		$\leq n+2$ [Th. 5]
	(except for $G = K_4$)	$\geq 7n/10$ [Prp. 5]	$\mathrm{seg}_\angle \equiv \mathrm{seg}_2$	

2 Relationships Between Segment Number Variants

Lemma 1. *Given a graph G and a straight-line drawing δ of G in 3D with the property that no two edges overlap and no edge contains a vertex in its interior, then there exists a plane drawing δ' of G with $\mathrm{seg}(\delta') \leq \mathrm{seg}(\delta)$ and with the same property as δ. (Note that both in δ and δ' edges may cross).*

Proof. For each triplet u, v, w of points in δ that correspond to three distinct vertices of G, let $P(u, v, w)$ be the plane or line spanned by the vectors $\overrightarrow{uv}$ and $\overrightarrow{wv}$, and let $\mathcal{P}$ be the set of all such planes or lines. Choose a point A in $\mathbb{R}^3 \setminus \bigcup \mathcal{P}$ that does not lie in the xy-plane. Let δ' be the drawing that results from projecting δ parallel to the vector OA onto the xy-plane. Due to the choice of the projection, δ' may contain crossings, but no edge contains a vertex to which it is not incident, and no two edges overlap. By construction, $\mathrm{seg}(\delta') \leq \mathrm{seg}(\delta)$. $\square$

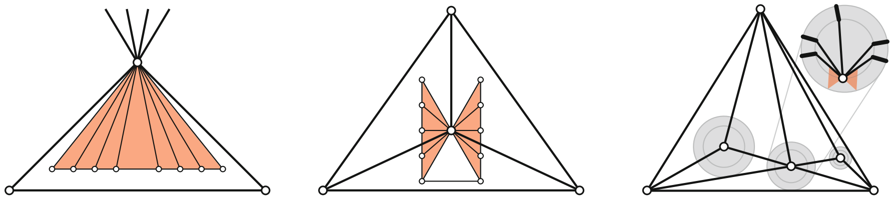
Corollary 1. *For any graph G it holds that $\mathrm{seg}_\times(G) \leq \mathrm{seg}_3(G)$.*

Proposition 1. *There is an infinite family of planar graphs $(\mathcal{S}_i)_{i\geq 3}$ such that $\mathcal{S}_i$ has $n_i = i^3 - i + 6$ vertices and the ratios $\mathrm{seg}_2(\mathcal{S}_i)/\mathrm{seg}_3(\mathcal{S}_i)$, $\mathrm{seg}_2(\mathcal{S}_i)/\mathrm{seg}_\angle(\mathcal{S}_i)$, and $\mathrm{seg}_2(\mathcal{S}_i)/\mathrm{seg}_\times(\mathcal{S}_i)$ all converge to 2 with increasing i.*

Proof. We construct, for $i \geq 3$, a triangulation $\mathcal{T}_i$ with maximum degree 6 and $t_i = i^2 - 2i + 3$ vertices (and, hence, $3t_i - 6$ edges and $2t_i - 4$ faces), as follows. Take two triangular grids of side length $i - 1$ (a single triangle is a grid of side length 1) and glue their boundaries, identifying corresponding vertices and edges. Clearly, the result is a (planar) triangulation. Let $s_i = \mathrm{seg}_2(\mathcal{T}_i)$. Then, by the result of Dujmović et al. [5], $s_i \leq 5t_i/2$.

We assume that i is even. To each vertex v of the triangulation, we attach an *i-fan*, that is, a path of length i each of whose vertices is connected to v. Let $\mathcal{S}_i$ be the resulting graph, which has $n_i = t_i(i + 2)$ vertices.

In 2D, no matter how the triangulation is drawn, only three vertices lie on the outer face. Consider an i-fan incident to one of the $t_i - 3$ inner vertices; see



(a) planar case (b) with crossings or in 3D (c) with bends

Fig. 1. Attaching a fan (thin edges) to a vertex of a triangulation (thick edges) of maximum degree 6

Fig. 1a. Each such i-fan must be placed into a triangular face and needs at least $i-3$ segments that are disjoint from the drawing of the triangulation. (Here we use that every vertex has degree at most 6.) Hence, $\mathrm{seg}_2(\mathcal{S}_i) \geq (t_i - 3) \cdot (i-3) = i^3 - O(i^2)$.

In 3D on the other hand, we can draw every fan in a plane different from the triangulation such that the fan's path lies on three segments and the remaining edges are paired such that each pair shares a segment; see Fig. 1b. Hence, $\mathrm{seg}_3(\mathcal{S}_i) \leq t_i \cdot (i/2+3) + s_i = i^3/2 + O(i^2)$. Due to Corollary 1, $\mathrm{seg}_\times(\mathcal{S}_i) \leq \mathrm{seg}_3(\mathcal{S}_i)$.

To bound $\mathrm{seg}_\angle(\mathcal{S}_i)$, observe that we can modify the layout of the triangulation as in Fig. 1c such that every vertex is incident to an angle greater than π without any incoming edges. This can be achieved as follows. On each inner vertex v, place a disk D_v whose radius is (slightly smaller than) the minimum over the lengths of the incident edges divided by 2 and over the distances to all non-incident edges. The resulting disks have positive radii and are pairwise disjoint. Now we go through all vertices. Let v be the current vertex and let ∂D_v be the boundary of D_v. We bend all edges incident to v at ∂D_v and place v on some unused point on ∂D_v. As a result, every vertex is incident to an angle greater than π without any incoming edges. In this area (marked red in Fig. 1c), we can place the corresponding fan. The modification introduces at most two bends in every edge of the triangulation. Hence, $\mathrm{seg}_\angle(\mathcal{S}_i) \leq t_i \cdot (i/2+3) + 3 \cdot (3t_i - 6) = i^3/2 + O(i^2)$. □

Open Problem 1. *What are upper bounds for the ratios* $\mathrm{seg}_2(G)/\mathrm{seg}_3(G)$, $\mathrm{seg}_2(G)/\mathrm{seg}_\angle(G)$, *and* $\mathrm{seg}_2(G)/\mathrm{seg}_\times(G)$ *with* G *ranging over all planar graphs?*

3 Computational Complexity

Chaplick et al. [3, Theorem 1] showed that it is $\exists\mathbb{R}$-hard to decide for a planar graph G and an integer k whether $\rho^1_2(G) \leq k$ and whether $\rho^1_3(G) \leq k$. We follow their approach to show the hardness of all variants of the segment number that we study in this paper.

A *simple line arrangement* is a set $\mathcal{L}$ of k lines in $\mathbb{R}^2$ such that each pair of lines has one intersection point and no three lines share a common point. We define the *arrangement graph* for a set of lines as follows [1]: The vertices correspond to the intersection points of lines and two vertices are adjacent in the graph if and only if they lie on the same line and no other vertex lies between them. The ARRANGEMENT GRAPH RECOGNITION problem is to decide whether a given graph is the arrangement graph of some set of lines.

Bose et al. [1] showed that this problem is NP-hard by reduction from a version of PSEUDOLINE STRETCHABILITY for the Euclidean plane, whose NP-hardness was proved by Shor [20]. It turns out that ARRANGEMENT GRAPH RECOGNITION is actually an $\exists\mathbb{R}$-complete problem [8, p. 212]. This stronger statement follows from the fact that the Euclidean PSEUDOLINE STRETCHABILITY is $\exists\mathbb{R}$-hard as well as the original projective version [15,18].

Theorem 1. *Given a planar graph G of maximum degree 4 and an integer k, it is $\exists\mathbb{R}$-hard to decide whether $\mathrm{seg}_2(G) \le k$, whether $\mathrm{seg}_\angle(G) \le k$, and whether $\mathrm{seg}_\times(G) \le k$.*

Proof. Similarly to Chaplick et al. [3, proof of Theorem 1], we first observe that if G is an arrangement graph, there must be an integer ℓ such that G has $\ell(\ell-1)/2$ vertices (of degree $d \in \{2,3,4\}$) and $\ell(\ell-2)$ edges. This uniquely determines ℓ. We set the parameter k from the statement of our theorem to this value of ℓ. Again, as Chaplick et al., we construct a graph G' from G by appending a tail (i.e., a degree-1 vertex) to each degree-3 vertex of G and two tails to each degree-2 vertex of G.

We claim that the following five conditions are equivalent: (i) G is an arrangement graph on k lines, (ii) $\rho^1_2(G') \le k$, (iii) $\mathrm{seg}_2(G') \le k$, (iv) $\mathrm{seg}_\angle(G') \le k$, and (v) $\mathrm{seg}_\times(G') \le k$. Once the equivalence is established, the $\exists\mathbb{R}$-hardness of deciding (i) implies the $\exists\mathbb{R}$-hardness of deciding any of the other statements.

Indeed, according to Chaplick et al. [3, proof of Theorem 1], G is an arrangement graph if and only if $\rho^1_2(G') \le k$, that is, (i) and (ii) are equivalent.

Assume (i). If G corresponds to a line arrangement of k lines, all edges of G lie on these k lines and the tails of G' can be added without increasing the number of lines. This arrangement shows that $\mathrm{seg}_2(G') \le k$, that is, (i) implies (iii).

Assume (iii), i.e., $\mathrm{seg}_2(G') \le k$. Then $\mathrm{seg}_\angle(G') \le k$ (iv) and $\mathrm{seg}_\times(G') \le k$ (v).

Assume (iv), i.e., $\mathrm{seg}_\angle(G') \le k$. Let Γ' be a polyline drawing of G' on $\mathrm{seg}_\angle(G')$ segments. The graph G' contains $\binom{k}{2}$ degree-4 vertices. As each of these vertices lies on the intersection of two segments in Γ', we need k segments to get enough intersections, that is, $\mathrm{seg}_\angle(G') \ge k$. Thus $\mathrm{seg}_\angle(G') = k$ and each intersection of the segments of Γ' (in particular, each bend) is a vertex of G'. Therefore edges in Γ' do not bend in interior points and Γ' witnesses that $\mathrm{seg}_2(G) \le k$. Thus (iv) implies (ii).

Finally, assume (v), i.e., $\mathrm{seg}_\times(G') \le k$. Let Γ be a straight-line drawing with possible crossings on $\mathrm{seg}_\times(G')$ segments. Again, we need k segments to get enough intersections, that is, $\mathrm{seg}_\times(G') \ge k$. Thus $\mathrm{seg}_\times(G') = k$ and each intersection of the segments of Γ' is a vertex of G'. Therefore edges in Γ' do not cross and Γ' witnesses that $\mathrm{seg}_2(G) \le k$. Thus (v) implies (ii).

Summing up, (iii) implies (iv) and (v), which both imply (ii), which implies (i), which implies (iii). Hence, all statements are equivalent. $\square$

Theorem 2. *Given a graph G of maximum degree 4 and an integer k, it is $\exists\mathbb{R}$-hard to decide whether* $\mathrm{seg}_3(G) \le k$.

Proof. Chaplick et al. [3, proof of Theorem 1] argued that for the graph G' constructed in the proof of Theorem 1 above, it holds that $\rho^1_2(G') = \rho^1_3(G')$. Then, by the proof of Theorem 1, we have $\rho^1_3(G') = \mathrm{seg}_\times(G')$.

By definition, we immediately obtain $\mathrm{seg}_3(G') \le \rho^1_3(G')$. By Corollary 1, we have that $\mathrm{seg}_\times(G') \le \mathrm{seg}_3(G')$. Therefore, $\mathrm{seg}_\times(G') = \mathrm{seg}_3(G')$. Together with the arguments in the proof of Theorem 1, this implies the theorem. □

Theorem 3. *Given a planar graph G and an integer k, it is $\exists\mathbb{R}$-complete to decide whether* $\mathrm{seg}_2(G) \le k$, *whether* $\mathrm{seg}_3(G) \le k$, *whether* $\mathrm{seg}_\angle(G) \le k$, *and whether* $\mathrm{seg}_\times(G) \le k$.

Proof. Given the hardness results in Theorems 1 and 2, it remains to show that each of the four problems lies in $\exists\mathbb{R}$. Chaplick et al. [3] [ArXiv version, Sect. 2] have shown that deciding whether $\rho^2_1(G) \le k$ and $\rho^3_1(G) \le k$ both lie in $\exists\mathbb{R}$. To this end, they showed that these questions can be formulated as first-order existential expressions over the reals. We now show how to extend their expression for deciding whether $\rho^2_1(G) \le k$ to an expression for deciding whether $\mathrm{seg}_2(G) \le k$. The expressions for the other variants can be extended in a similar way.

Their existential statement over the reals starts with the quantifier prefix $\exists v_1 \dots \exists v_n \exists p_1 \exists q_1 \dots \exists p_k \exists q_k$, where quantification $\exists a$ over a point $a = (x, y)$ means the quantifier block $\exists x \exists y$, the points $v_1, \dots, v_n$ are the points to which the vertices of G, $\{1, \dots, n\}$, are mapped, and the pairs $(p_1, q_1) \dots, (p_k, q_k)$ define the k lines that cover the drawing of G. The expression Π over which they quantify uses a subexpression that takes as input three points in $\mathbb{R}^2$; for a, b, and c, they define the expression $B(a, b, c)$ such that it is true if and only if a lies on the line segment $\overline{bc}$.

To the expression Π we simply add a term that ensures that, for each pair of consecutive points v_i and v_j on the same line, vertices i and j are adjacent in G:

$$\bigwedge_{l \in \{1,\dots,k\}, i,j,k \in V} B(v_i, p_l, q_l) \wedge B(v_j, p_l, q_l) \wedge \neg B(v_k, v_i, v_j) \Rightarrow \{i, j\} \in E,$$

where V is the vertex set and E is the edge set of the graph G. □

4 Algorithms and Lower Bounds for Cubic Graphs

Consider a polyline drawing δ of a cubic graph (in 2D or 3D). Note that there are two types of vertices; those where exactly one segment ends and those where three segments end. We call these vertices *flat vertices* and *tripods*, respectively. Let $f(\delta)$ be the number of flat vertices, $t(\delta)$ the number of tripods, and $b(\delta)$ the number of bends in δ.

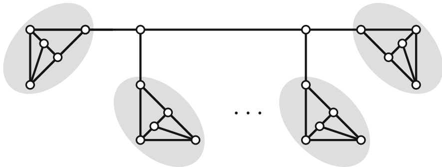


Fig. 2. The graph G_k (here $k = 4$) is a caterpillar with $k - 2$ inner vertices of degree 3 where each leaf has been replaced by a copy of the 5-vertex graph K_4' (shaded gray).

Lemma 2. *For any straight-line drawing δ of a cubic graph with n vertices,* $\mathrm{seg}(\delta) = 3n/2 - f(\delta) + b(\delta) = n/2 + t(\delta) + b(\delta)$.

Proof. Clearly, $n = f(\delta) + t(\delta)$. The number of "segment ends" is $3t(\delta) + f(\delta) + 2b(\delta) = 3n - 2f(\delta) + 2b(\delta) = n + 2t(\delta) + 2b(\delta)$. The claim follows since every segment has two ends. □

4.1 Singly-Connected Cubic Graphs

Proposition 2. *There is an infinite family $(G_k)_{k \geq 1}$ of connected cubic graphs such that G_k has $n_k = 6k - 2$ vertices and* $\mathrm{seg}_2(G_k) = \mathrm{seg}_3(G_k) = \mathrm{seg}_\angle(G_k) = \mathrm{seg}_\times(G_k) = 5k - 1 = 5n_k/6 + 2/3$.

Proof. Let K_4' be the graph K_4 with a subdivided edge. Consider the graph G_k depicted in Fig. 2 (for $k = 4$). It consists of a caterpillar with $k - 2$ inner vertices (of degree 3) where each of the k leaf nodes is replaced by a copy of K_4'. The convex hull of every polyline drawing of K_4' has at least three extreme points. One of these points may connect K_4' to $G_k - K_4'$, but each of the remaining two must be a tripod or a bend. This holds for every copy of K_4'. Hence, for any drawing δ of G, $t(\delta) + b(\delta) \geq 2k$. Now Lemma 2 yields that $\mathrm{seg}(\delta) \geq 5k - 1$. For the drawing in Fig. 2, the bound is tight. □

4.2 Biconnected Cubic Graphs

Proposition 3. *There is an infinite family of Hamiltonian (and hence biconnected) cubic graphs $(H_k)_{k \geq 3}$ such that H_k has $n_k = 6k$ vertices,* $\mathrm{seg}_3(H_k) = 5k = 5n_k/6$, *and* $\mathrm{seg}_\times(H_k) = 4k = 2n_k/3$.

Proof. Consider the graph H_k depicted in Fig. 3 (for $k = 4$). It is a k-cycle where each vertex is replaced by a copy of a 6-vertex graph K ($K_{3,3}$ minus an edge). The graph H_k has $n_k = 6k$ vertices and is not planar.

In any 2D drawing of the subgraph K, at least three vertices lie on the convex hull of the drawing of K. Two of these vertices may connect K to $H_k - K$, but at least one of the convex-hull vertices is a tripod. This holds for every copy of K. Hence, for any (3D) drawing δ of H_k, $t(\delta) \geq k$. Now Lemma 2 yields that $\mathrm{seg}(\delta) \geq n_k/2 + k = 2n_k/3$. The same bound holds for $\mathrm{seg}_\times(H_k)$.

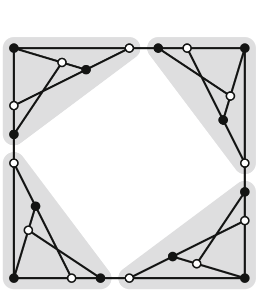


Fig. 3. The cubic graph H_k (here $k = 4$) is a k-cycle whose vertices are replaced by $K_{3,3}$ minus an edge (shaded).

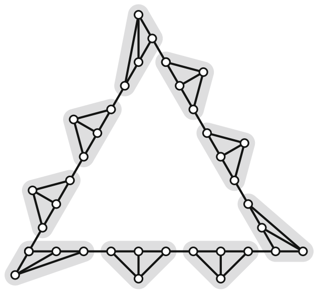


Fig. 4. The planar cubic graph I_k (here $k = 9$) is a k-cycle whose vertices are replaced by K_4 minus an edge (shaded).

In order to bound $\mathrm{seg}_3(H_k)$ we consider two possibilities for the drawing of the subgraph K; either it lies in a plane or it doesn't. In the planar case, the two vertices that connect K to $H_k - K$ cannot lie in the same face of the planar embedding of K (otherwise we could connect these two vertices without crossings, contradicting the fact that $K_{3,3}$ is not planar). Hence, at least two vertices on the convex hull of K must be tripods. In the non-planar case, the convex hull consists of four vertices. Two of these may connect K to $H_k - K$, but again at least two must be tripods. In both cases we hence have $t(\delta) \geq 2k$ for any 3D drawing δ of H_k. Now Lemma 2 yields $\mathrm{seg}(\delta) \geq n_k/2 + 2k = 5n_k/6$. The same bound holds for $\mathrm{seg}_3(H_k)$.

For the drawing in Fig. 3, the bound for $\mathrm{seg}_\times$ is tight. Lifting the k white vertices that do not lie on the outer face from the xy-plane ($z = 0$) to the plane $z = 1$, yields a crossing-free 3D drawing where the bound for seg_3 is tight. □

Proposition 4. *There is an infinite family of planar cubic Hamiltonian (and hence biconnected) graphs $(I_k)_{k \geq 3}$ such that I_k has $n_k = 4k$ vertices and* $\mathrm{seg}_2(I_k) = \mathrm{seg}_3(I_k) = \mathrm{seg}_\angle(I_k) = \mathrm{seg}_\times(I_k) = 3k = 3n_k/4$.

Proof. Consider the graph I_k depicted in Fig. 4 (for $k = 9$). It is a k-cycle where each vertex is replaced by a copy of the graph K', which is K_4 minus an edge. Therefore, I_k has $4k$ vertices. The depicted drawing consists of $3k$ segments. This yields the upper bounds.

Concerning the lower bounds, note that, in any drawing style, each subgraph K' has an extreme point not connected to $I_k - V(K')$. This point must be a tripod or a bend. Hence, in any drawing δ of I_k, $t(\delta) + b(\delta) \geq k$ and, by Lemma 2, $\mathrm{seg}_2(I_k) = \mathrm{seg}_3(I_k) = \mathrm{seg}_\angle(I_k) = \mathrm{seg}_\times(I_k) \geq 2k + t(\delta) + b(\delta) \geq 3k$. □

Theorem 4. *For any biconnected planar cubic graph G with n vertices, it holds that $\mathrm{seg}_\angle(G) \leq n + 1$. A corresponding drawing can be found in linear time.*

Proof. We draw G using the algorithm of Liu et al. [14] that draws any planar biconnected cubic graph except the tetrahedron orthogonally with at most one bend per edge and at most $n/2+1$ bends in total. It remains to count the number of segments in this drawing. In any vertex exactly one segment ends; in any bend exactly two segments end. In total, this yields at most $n+2\cdot(n/2+1)=2n+2$ segment ends and at most $n+1$ segments.

Concerning the special case of the tetrahedron (K_4), note that it can be drawn with five segments when bending one of its six edges. □

Open Problem 2. *What about 4-regular graphs? They have* $2n$ *edges. If we bend every edge once, we already need* $2n$ *segments – and not all 4-regular graphs can be drawn with at most one bend per edge.*

Every biconnected graph G admits an *st-numbering*, that is, an ordering $\langle v_1,\dots,v_n\rangle$ of the vertex set $\{v_1,\dots,v_n\}$ of G such that for every $j\in\{2,\dots,n-1\}$ vertex v_j has at least one predecessor (that is, a neighbor v_i with $i<j$) and at least one successor (that is, a neighbor v_k with $k>j$). Such a numbering can be computed in linear time [9]. Given a cubic graph with an st-numbering $\langle v_1,\dots,v_n\rangle$, we call a vertex v_j with $j\in\{1,\dots,n\}$ a *p-vertex* if it has p predecessors; $p\in\{0,1,2,3\}$.

Lemma 3. *Given a biconnected cubic graph with an st-numbering* $\langle v_1,\dots,v_n\rangle$*, there is one 0-vertex and one 3-vertex and there are* $(n-2)/2$ *1-vertices and* $(n-2)/2$ *2-vertices.*

Proof. Direct every edge from the vertex with smaller index to the vertex with higher index. In the resulting directed graph, the sum of the indegrees equals the sum of the outdegrees. Hence, the number of 1-vertices (with indegree 1 and outdegree 2) and the number of 2-vertices (with indegree 2 and outdegree 1) must be equal. It is obvious that there is one 0- and 3-vertex each. □

Theorem 5. *For any biconnected cubic graph* G *with* n *vertices,* $\mathrm{seg}_3(G)\le n+2$ *and* $\mathrm{seg}_\times(G)\le n+2$[1].

Proof. We show that $\mathrm{seg}_3(G)\le n+2$. Then Corollary 1 yields $\mathrm{seg}_\times(G)\le n+2$. For two different points x and y in $\mathbb{R}^3$, we denote the line that goes through x and y by xy.

Let $\langle v_1,\dots,v_n\rangle$ be an st-numbering of G. We construct a drawing δ of G, going through the vertices according to the st-numbering and using x-coordinate $j\pm\varepsilon$ for vertex v_j, where $0<\varepsilon\ll 1$. We place v_1 at $(1,1,1)$. At every step $j=2,\dots,n$, we maintain a set $\mathcal{L}$ of lines that are directed to the right such that any two lines in $\mathcal{L}$ are either skew (that is, they don't lie in the same plane) or they intersect and their unique intersection point is the location of a vertex v_k with $k\le j$ (that is, the intersection point is v_j or it lies to the left of v_j). Initially, $\mathcal{L}$ is empty.

[1] After submitting this article, we realized that our proof is incomplete. The correct statement of the theorem and its proof can be found in the full version https://arxiv.org/abs/1908.08871.

If v_j is a 1-vertex, we differentiate two cases depending on the unique predecessor v_i of v_j.

Case I: If v_i is the last vertex on a line ℓ in $\mathcal{L}$, we place v_j on the intersection point of ℓ with the plane $x = j$. In this case, the set $\mathcal{L}$ doesn't change.

Case II: Otherwise, we place v_j in the plane $x = j$ such that the line v_iv_j is skew with respect to all lines in $\mathcal{L}$ except for the line ℓ that contains v_i and the unique predecessor of v_i. (Note that the predecessor of v_i and the line ℓ don't exist if $i = 1$). Clearly, v_iv_j and ℓ intersect in v_i and $i < j$. Hence, we can add the line v_iv_j to the set $\mathcal{L}$.

If v_j is a 2-vertex, let v_i and $v_{i'}$ be the two predecessors of v_j. Again, we consider two cases.

Case I': At least one of v_i or $v_{i'}$ is flat (that is, it lies on an inner point of the segment created by its incident edges that have already been drawn) or one of them is the vertex v_1.

In this case, we treat v_j similarly as in Case II above; we make sure that the lines v_iv_j and $v_{i'}v_j$ are skew with respect to all lines in $\mathcal{L}$ except that v_iv_j won't be skew with respect to the at most two lines that connect v_i to its predecessors and $v_{i'}v_j$ won't be skew with respect to the at most two lines that connect $v_{i'}$ to its predecessors. Note that v_iv_j intersects any line through v_i and its neighbors in v_i, and it holds that $i < j$. Similarly, $v_{i'}v_j$ intersects any line through $v_{i'}$ and its neighbors in $v_{i'}$, and it holds that $i' < j$. The lines v_iv_j and $v_{i'}v_j$ intersect in v_j. Hence, we can add the lines v_iv_j and $v_{i'}v_j$ to the set $\mathcal{L}$.

Case II': Both v_i and $v_{i'}$ are the last vertices on their lines ℓ and ℓ', respectively.

If one of them, say v_i, has a successor v_k with $k > j$, we extend the line ℓ of v_i and put v_j on the intersection of ℓ and the plane $x = j$.

Otherwise v_i has a successor v_k with $k < j$ and $v_{i'}$ has a successor $v_{k'}$ with $k' < j$, which both don't lie on the lines ℓ and ℓ'. In this case, we put v_j on one of ℓ and ℓ', say ℓ, and add the line $v_{i'}v_j$ to the set $\mathcal{L}$. Now we pick some $0 < \varepsilon \ll 1$ such that we can place v_j at the intersection of ℓ and $x = j + \varepsilon$. We must avoid to place v_j on a plane spanned by any two non-skew lines in $\mathcal{L}$ (intersecting to the left of $x = j$). With this trick, the invariant for $\mathcal{L}$ still holds since the new line in $\mathcal{L}$, $v_{i'}v_j$, intersects only ℓ' (in $v_{i'}$, hence to the left).

Finally, we place v_n (which is a 3-vertex) at a point in the plane $x = n$ that does not lie on any of the lines spanned by pairs and planes spanned by triples of previously placed vertices.

This finishes the description of the drawing δ of G. Due to our invariant regarding the set $\mathcal{L}$, no two edges of G intersect in δ.

To bound the number of segments in δ, we use a simple charging argument. Each non-first and non-last vertex v has a predecessor which is a flat vertex or v_1. To this predecessor v pays a coin. On the other hand, v_1 receives at most three coins and every flat vertex receives at most two coins. Hence, $f(\delta) \geq (n - 5)/2$. Since n is even, $f(\delta) \geq n/2 - 2$. Now, Lemma 2 yields the claim. □

4.3 Triconnected Cubic Graphs

Proposition 5. *There is an infinite family of triconnected cubic graphs* $(F_k)_{k\geq4}$ *such that* F_k *has* $n_k = 5k$ *vertices and* $\mathrm{seg}_3(F_k) = 3.5k = 7n_k/10$.

Proof. Let G_k be an arbitrary triconnected cubic graph with k vertices (k even). By Steinitz's theorem, there exists a drawing of the graph G_k as a 1-skeleton of a 3D convex polyhedron. Replace each vertex v of G_k by a copy of $K_{2,3}$ as shown in Fig. 5, where v is the central (orange) vertex—a tripod—, all other vertices of the copy are flat, and the three arrows correspond to the three edges of G_k. The resulting geometric graph F_k has $n_k = 5k$ vertices and is not planar. Since F_k has k tripod vertices, by Lemma 2, $\mathrm{seg}_3(F_k) \leq n_k/2 + k = 3.5k = 7n_k/10$.

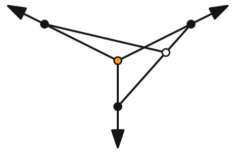


Fig. 5. Gadget for the proof of Proposition 5 (Color figure online)

In order to bound $\mathrm{seg}_3(F_k)$ from below, we consider two possibilities for the drawing of each subgraph $K_{2,3}$; either it lies in a plane or it doesn't. In the planar case, the convex hull of the drawing has at least three extreme points. If none of them was a tripod then there would be exactly three extreme points, each a black vertex. Thus we could place an additional white vertex in the exterior of the convex hull and connect it to all black vertices, obtaining an impossible plane drawing of $K_{3,3}$. In the non-planar case, the convex hull consists of at least four vertices. Three of these may connect $K_{2,3}$ to $F_k - V(K_{2,3})$, but again at least one must be a tripod.

In both cases we hence have $t(\delta) \geq k$ for any 3D drawing δ of F_k. Now Lemma 2 yields $\mathrm{seg}(\delta) = n_k/2 + t(\delta) \geq 3.5k$. □

5 Open Problems

Apart from improving our bounds, we have the following open problem.

Open Problem 3. *Can we produce drawings in 3D (or with bends or crossings in 2D) that fit on grids of small size?*

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References

1. Bose, P., Everett, H., Wismath, S.K.: Properties of arrangement graphs. Int. J. Comput. Geom. Appl. **13**(6), 447–462 (2003). https://doi.org/10.1142/S0218195903001281

2. Chaplick, S., Fleszar, K., Lipp, F., Ravsky, A., Verbitsky, O., Wolff, A.: Drawing graphs on few lines and few planes. In: Hu, Y., Nöllenburg, M. (eds.) GD 2016. LNCS, vol. 9801, pp. 166–180. Springer, Cham (2016). https://doi.org/10.1007/978-3-319-50106-2_14
3. Chaplick, S., Fleszar, K., Lipp, F., Ravsky, A., Verbitsky, O., Wolff, A.: The complexity of drawing graphs on few lines and few planes. In: Ellen, F., Kolokolova, A., Sack, J.R. (eds.) Algorithms and Data Structures. LNCS, vol. 10389, pp. 265–276. Springer, Cham (2017). https://doi.org/10.1007/978-3-319-62127-2_23. arxiv.org/1607.06444
4. Chartrand, G., Zhang, P.: Chromatic Graph Theory, 1st edn. Chapman & Hall/CRC, Boca Raton (2008)
5. Dujmović, V., Eppstein, D., Suderman, M., Wood, D.R.: Drawings of planar graphs with few slopes and segments. Comput. Geom. Theory Appl. **38**(3), 194–212 (2007). https://doi.org/10.1016/j.comgeo.2006.09.002
6. Durocher, S., Mondal, D.: Drawing plane triangulations with few segments. In: Proceedings Canadian Conference on Computational Geometry (CCCG 2014), pp. 40–45 (2014). http://cccg.ca/proceedings/2014/papers/paper06.pdf
7. Durocher, S., Mondal, D., Nishat, R., Whitesides, S.: A note on minimum-segment drawings of planar graphs. J. Graph Algorithms Appl. **17**(3), 301–328 (2013). https://doi.org/10.7155/jgaa.00295
8. Eppstein, D.: Drawing arrangement graphs in small grids, or how to play planarity. J. Graph Algorithms Appl. **18**(2), 211–231 (2014). https://doi.org/10.7155/jgaa.00319
9. Even, S., Tarjan, R.E.: Computing an *st*-numbering. Theoret. Comput. Sci. **2**(3), 339–344 (1976). https://doi.org/10.1016/0304-3975(76)90086-4
10. Hoffmann, U.: On the complexity of the planar slope number problem. J. Graph Algorithms Appl. **21**(2), 183–193 (2017). https://doi.org/10.7155/jgaa.00411
11. Hültenschmidt, G., Kindermann, P., Meulemans, W., Schulz, A.: Drawing planar graphs with few geometric primitives. In: Bodlaender, H.L., Woeginger, G.J. (eds.) WG 2017. LNCS, vol. 10520, pp. 316–329. Springer, Cham (2017). https://doi.org/10.1007/978-3-319-68705-6_24
12. Igamberdiev, A., Meulemans, W., Schulz, A.: Drawing planar cubic 3-connected graphs with few segments: algorithms and experiments. J. Graph Algorithms Appl. **21**(4), 561–588 (2017). https://doi.org/10.7155/jgaa.00430
13. Kindermann, P., Mchedlidze, T., Schneck, T., Symvonis, A.: Drawing planar graphs with few segments on a polynomial grid. In: Archambault, D., Tóth, C.D. (eds.) GD 2019. LNCS, vol. 11904, pp. 416–429. Springer. Cham (2019). https://arxiv.org/abs/1903.08496
14. Liu, Y., Marchioro, P., Petreschi, R.: At most single-bend embeddings of cubic graphs. Appl. Math. **9**(2), 127–142 (1994). https://doi.org/10.1007/BF02662066
15. Matoušek, J.: Intersection graphs of segments and $\exists\mathbb{R}$. ArXiv report (2014). http://arxiv.org/abs/1406.2636
16. Mondal, D., Nishat, R.I., Biswas, S., Rahman, M.S.: Minimum-segment convex drawings of 3-connected cubic plane graphs. J. Comb. Optim. **25**(3), 460–480 (2013). https://doi.org/10.1007/s10878-011-9390-6
17. Mukkamala, P., Szegedy, M.: Geometric representation of cubic graphs with four directions. Comput. Geom. Theory Appl. **42**(9), 842–851 (2009). https://doi.org/10.1016/j.comgeo.2009.01.005
18. Schaefer, M.: Complexity of some geometric and topological problems. In: Eppstein, D., Gansner, E.R. (eds.) GD 2009. LNCS, vol. 5849, pp. 334–344. Springer, Heidelberg (2010). https://doi.org/10.1007/978-3-642-11805-0_32

19. Schulz, A.: Drawing graphs with few arcs. J. Graph Algorithms Appl. **19**(1), 393–412 (2015). https://doi.org/10.7155/jgaa.00366
20. Shor, P.W.: Stretchability of pseudolines is NP-hard. In: Gritzmann, P., Sturmfels, B. (eds.) Applied Geometry and Discrete Mathematics-The Victor Klee Festschrift. DIMACS Series in Mathematics and Theoretical Computer Science, vol. 4, pp. 531–554. American Mathematical Society (1991)