Line and Plane Cover Numbers Revisited

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Abstract. A measure for the visual complexity of a straight-line crossing-free drawing of a graph is the minimum number of lines needed to cover all vertices. For a given graph G, the minimum such number (over all drawings in dimension $d \in \{2,3\}$) is called the *d-dimensional weak line cover number* and denoted by $\pi_d^1(G)$. In 3D, the minimum number of *planes* needed to cover all vertices of G is denoted by $\pi_3^2(G)$. When edges are also required to be covered, the corresponding numbers $\rho_d^1(G)$ and $\rho_3^2(G)$ are called the *(strong) line cover number* and the *(strong) plane cover number*.

Computing any of these cover numbers—except $\pi_2^1(G)$—is known to be NP-hard. The complexity of computing $\pi_2^1(G)$ was posed as an open problem by Chaplick et al. [WADS 2017]. We show that it is NP-hard to decide, for a given planar graph G, whether $\pi_2^1(G) = 2$. We further show that the universal stacked triangulation of depth d, G_d, has $\pi_2^1(G_d) = d+1$. Concerning 3D, we show that any n-vertex graph G with $\rho_3^2(G) = 2$ has at most $5n - 19$ edges, which is tight.

1 Introduction

Recently, there has been considerable interest in representing graphs with as few objects as possible. The idea behind this objective is to keep the visual complexity of a drawing low for the observer. The types of objects that have been used are straight-line segments [5,8,14,15] and circular arcs [14,16].

Chaplick et al. [3] considered *covering* straight-line drawings of graphs by lines or planes and defined the following new graph parameters. Let $1 \leq l < d$, and let G be a graph. The *l-dimensional affine cover* number of G in $\mathbb{R}^d$, denoted by $\rho_d^l(G)$, is defined as the minimum number of l-dimensional planes in $\mathbb{R}^d$ such that G has a crossing-free straight-line drawing that is contained in the union of these planes. The *weak l*-dimensional affine cover number of G in $\mathbb{R}^d$, denoted by $\pi_d^l(G)$, is defined similarly to $\rho_d^l(G)$, but under the weaker restriction that only the vertices are contained in the union of the planes. Clearly, $\pi_d^l(G) \leq \rho_d^l(G)$,

and if $l' \leq l$ and $d' \leq d$ then $\pi_d^l(G) \leq \pi_{d'}^{l'}(G)$ and $\rho_d^l(G) \leq \rho_{d'}^{l'}(G)$. It turns out that it suffices to study the parameters ρ_2^1, ρ_3^1, ρ_3^2, and π_2^1, π_3^1, π_3^2:

Theorem 1 (Collapse of the Affine Hierarchy [3]**).** *For any integers $1 \leq l < 3 \leq d$ and for any graph G, it holds that $\pi_d^l(G) = \pi_3^l(G)$ and $\rho_d^l(G) = \rho_3^l(G)$.*

Disproving a conjecture of Firman et al. [12], Eppstein [10] constructed planar, cubic, 3-connected, bipartite graphs on n vertices with $\pi_2^1(G) \geq n^{1/3}$. Answering a question of Chaplick et al. [3] he also constructed a family of subcubic series-parallel graphs with unbounded π_2^1-value. Felsner [11] proved that, for every 4-connected plane triangulation G on n vertices, it holds that $\pi_2^1(G) \leq \sqrt{2n}$. Chaplick et al. [4] also investigated the complexity of computing the affine cover numbers. Among others, they showed that in 3D, for $l \in \{1, 2\}$, it is NP-complete to decide whether $\pi_3^l(G) \leq 2$ for a given graph G. In 2D, the question has still been open, but a related question was raised by Dujmović et al. [7] already in 2004. They investigated so-called *track layouts* which are defined as follows. A graph admits a k-track layout if its vertices can be partitioned into k ordered independent subsets such that any pair of subsets induces a plane graph (w.r.t. the order of the subsets). The track number of a graph G, $\mathrm{tn}(G)$, is the smallest k such that G admits a k-track layout. See also [6] for some recent developments. Note that in general $\pi_2^1(G) \neq \mathrm{tn}(G)$; for example, $\pi_2^1(K_4) = 2$, whereas $\mathrm{tn}(K_4) = 4$. Note further that a 3-track layout is necessarily plane (which is not the case for k-track layouts with $k > 3$). Dujmović posed the computational complexity of k-track layout as an open question.

While it is easy to decide efficiently whether a graph admits a 2-track layout, Bannister et al. [1] answered the open question of Dujmović et al. already for 3-track layouts in the affirmative. They first showed that a graph has a leveled planar drawing if and only if it is bipartite and has a 3-track layout. Combining this results with the NP-hardness of level planarity, proven by Heath and Rosenberg [13], immediately showed that it is NP-hard to decide whether a given graph has a 3-track layout. For $k > 3$, deciding the existence of a k-track layout is NP-hard, too, since it suffices to add to the given graph $k - 3$ new vertices each of which is incident to all original vertices of the graph [1].

Our contribution. We investigate several problems concerning the *weak line cover number* $\pi_2^1(G)$ and the *strong plane cover number* $\rho_3^2(G)$:

- We settle the open question of Chaplick et al. [4, p. 268] by showing that it is NP-hard to test whether, for a given planar graph G, $\pi_2^1(G) = 2$; see Sect. 2.
- We show that G_d, the universal stacked triangulation of depth d, (which has treewidth 3) has $\pi_2^1(G_d) = d + 1 = \log_3(2n_d - 5) + 1$, where n_d is the number of vertices of G_d; see Sect. 3.
- Eppstein has identified classes of treewidth-2 graphs with unbounded π_2^1-value. We give an easy direct argument showing that some 2-tree H_d with n'_d vertices has $\pi_2^1(H_d) \in \Omega(\log n'_d)$; see the full version [2].
- Concerning 3D, we show that any n-vertex graph G with $\rho_3^2(G) = 2$ has at most $5n - 19$ edges; see Sect. 4. This bound is tight.

2   Complexity of Computing Weak Line Covers in 2D

In this section we investigate the computational complexity of deciding whether a graph can be drawn on two lines.

Theorem 2. *It is NP-hard to decide whether a given plane (or planar) graph G admits a drawing with $\pi_2^1(G) = 2$.*

Proof. Our proof is by reduction from the problem LEVEL PLANARITY, which Heath and Rosenberg [13] proved to be NP-hard. The problem is defined as follows. A planar graph G is *leveled-planar* if its vertex set can be partitioned into sets $V_1, \ldots, V_m$ such that G has a planar straight-line drawing where, for every $i \in \{1, \ldots, m\}$, vertices in V_i lie on the vertical line $\ell_i \colon y = i$ and each edge $v_j v_k$ of G connects two vertices on consecutive lines (that is, $|j - k| = 1$).

Chaplick et al. [3] have shown that every leveled-planar graph can be drawn on two lines. The converse, however, is not true. For example, K_4 is not leveled-planar, but $\pi_2^1(K_4) = 2$. Therefore, we modify the given graph in three ways. (a) We replace each edge of G by a $K_{2,4}$-gadget where the two nodes in one set of the bipartition replace the endpoints of the former edge; see Fig. 1a. (b) We add to the graph G' that resulted from the previous step a new subgraph G_0 (two copies of K_4 sharing exactly two vertices), which we connect by a path to a vertex on the outer face of G. (If the outer face is not fixed, we can try each vertex.) In Fig. 1b, G_0 is yellow and the path is red. The length L of the path is any upper bound on the number of levels of G', e.g., the diameter of G' (plus 1). (c) We attach to G_0 a triangulated spiral S (dark green in Fig. 1b). The spiral makes $L+2$ right turns; its final vertex is identified with the outermost vertex of the previous turn. Hence, apart from its many triangular faces, the graph $S+G_0$ has a large inner face F of degree $2(L+2)$ and a quadrangular outer face. Let G'' be the resulting graph. It remains to show that G is leveled-planar if and only if $\pi_2^1(G'') = 2$.

"$\Rightarrow$": Fix a leveled-planar drawing of G. By doubling the layers and using the new layers to place the large sides of $K_{2,4}$'s, one easily sees that G' is also leveled-planar, see Fig. 1a. As shown in Fig. 1b, the large inner face F of $S + G_0$ can be drawn so that it partitions the halflines emanating from the origin into L levels. (It is no problem that consecutive levels are turned by 90°.) Since we chose L large enough (in particular $L \geq 2m - 1$), we can easily draw G' inside F. Note that the red path attached to G_0 is long enough to reach any vertex on the outer face of G'. Hence, $\pi_2^1(G'') = 2$.

"$\Leftarrow$": Fix a drawing of G'' on two lines. The two lines cannot be parallel since G'' contains $K_{2,4}$ and is not outer-planar; so after translation and/or skew we may assume that these two lines are the two coordinate axes. It is not hard to verify that G_0 must be drawn such that the origin is in its interior, at the common edge of the two K_4's. Furthermore, given this drawing of G_0, the 3-connected spiral S must be drawn as in Fig. 1b. Due to planarity and the fact that G' is connected to G_0 via the red path, G' can only be drawn in the interior of F. The drawing of $S + G_0$ partitions the halflines emanating from the origin

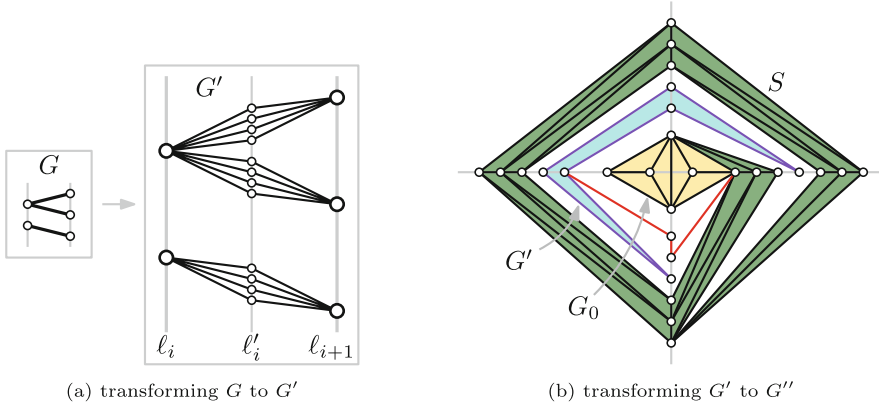(a) transforming G to G' (b) transforming G' to G''

Fig. 1. Our reduction from LEVEL PLANARITY

into levels, which we number $1, 2, \ldots$ starting from the innermost level that contains a vertex of G'. Inside this face, the only way to draw the $K_{2,4}$-gadgets is as in Fig. 1a, spanning three consecutive levels. This forces all vertices of G to be placed on the odd-numbered levels and the vertices in $G' - G$ on the even-numbered levels. Now we can get a level assignment for G by reverting the transformation in Fig. 1a. Hence, G is leveled-planar.

This shows that our reduction is correct. It runs in polynomial time.   □

3   Weak Line Covers of Planar 3-Trees in 2D

In this section we consider the weak line cover number π_2^1 for planar graphs, i.e., we are interested in crossing-free straight-line drawings with vertices located on a small collection of lines. Clearly $\pi_2^1(G) = 1$ if and only if G is a forest of paths. The set of graphs with $\pi_2^1(G) = 2$, however, is already surprisingly rich, it contains all trees, outerplanar graphs and subgraphs of grids [1, 10].

Stacked triangulations, a.k.a. planar 3-trees or Apollonian networks, are obtained from a triangle by repeatedly selecting a triangular face T and adding a new vertex (the *vertex stacked inside T*) inside T with edges to the vertices of T. This subdivides T into three smaller triangles, the *children* of T.

For $d \geq 0$ let G_d be the *universal stacked triangulation of depth d*, defined as follows. The graph G_0 is a triangle T_0, and G_d (for $d \geq 1$) is obtained from G_{d-1} by adding a stack vertex in each bounded face of G_{d-1}. Graph G_d has $n_d = \frac{1}{2}(3^d + 5)$ vertices and 3^d bounded faces. We show that its weak line cover number is $d + 1 = \log_3(2n_d - 5) + 1 \in \Theta(\log n_d)$. (A lower bound of d can also be found in Eppstein's recent book [9, Thm. 16.13].)

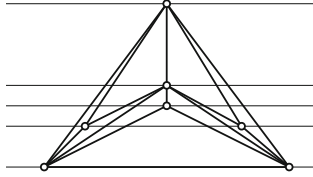**Theorem 3.** *For $d \geq 1$ it holds that $\pi(G_d) = d + 1$.*

Fig. 2. A drawing of G_2 that can be extended to a drawing of G_3 on 5 parallel lines.

Proof. Here we prove only the lower bound; the construction for the upper bound is illustrated in Fig. 2 and given in the full version. Let $\mathcal{L}$ be a family of lines covering the vertices of a drawing of G_d. Let a, b, and c be the vertices of T_0. We first argue that at least d lines are needed to cover $V \setminus T_0$. Let x_1 be stacked into T_0. There is a line $L_1 \in \mathcal{L}$ covering x_1. Note that L_1 can intersect only two of the three child triangles of T_0 (where "intersect" here means "in the interior"). Let T_1 be a child triangle avoided by L_1, and let x_2 be the vertex stacked into T_1. There is a line $L_2 \in \mathcal{L}$ covering x_2. Let T_2 be a child triangle of T_1 avoided by L_2. Iterating this yields d pairwise distinct lines in $\mathcal{L}$.

To find one additional line in $\mathcal{L}$, we distinguish some cases. If a line $L \in \mathcal{L}$ covers two vertices of T_0, then it covers no inner vertex, and we are done.

Assume some line $L_a \in \mathcal{L}$ intersects x_1 and one vertex of T_0, say a. Let L_b and L_c be the lines intersecting b and c. The lines L_a, L_b, and L_c are pairwise different, else we are in the previous case. Of the three child triangles of T_0, at most one is intersected by L_a and at most two each are intersected by L_b and L_c. Therefore, some child triangle T_1 of T_0 is intersected by at most one of L_a, L_b, or L_c. The graph G_{d-1} inside T_1 requires at least $d - 1$ lines for its interior points, and at most one of those lines is L_a, L_b, or L_c, so in total at least $d + 1$ lines are needed.

The argument is similar if no line covers two of a, b, c, and x_1. The four distinct lines supporting a, b, c, and x_1 then intersect at most two child triangles each. So one child triangle T_1 is intersected by at most two of these lines. Combining the $d - 1$ lines needed for the interior of T_1 with the two lines that do not intersect it, shows that $d + 1$ lines are needed. □

4 Maximal Graphs on Two Planes in 3D

We now switch to dimension $d = 3$ and the strong cover number. Obviously any graph G with a drawing that is covered by two planes has at most $6n - 12$ edges since it is the union of two planar graphs. Using maximality arguments and counting, we show that in fact G has at most $5n - 19$ edges if $n \geq 7$. (The restriction $n \geq 7$ is required since for $n = 3, 4, 5, 6$ we can have $3, 6, 9, 12$ edges.)

We argue first that our bound is tight. The *spine* is the intersection of two planes A and B. Put a path with $n - 4$ vertices on the spine. Add one vertex in each of the four halfplanes and connect each of these vertices to all vertices on the spine and to the vertex on the opposite halfplane. (We provide a figure in

the full version.) This yields $n - 5$ edges on the path and $2(n - 4) + 1$ edges in each of the two planes, so $5n - 19$ edges in total.

Theorem 4. *Any graph G with $\rho_3^2(G) = 2$ and $n \geq 7$ vertices has at most $5n - 19$ edges.*

Proof. Fix a drawing of G on planes A and B, inducing planar graphs G_A and G_B within those planes. Let G_A^+ and G_B^+ be the graphs obtained from G_A and G_B by adding any edge that can be inserted without crossing, within the same plane, and with at most one bend on the spine. Clearly it suffices to argue that G_A^+ and G_B^+ together have at most $5n - 19$ edges. Let s be the number of vertices on the spine, let a be the number of vertices of G_A^+ not on the spine, and let b be the number of vertices of G_B^+ not on the spine. Clearly, $a + b + s = n$. We may assume $a \leq b$. We also assume that $1 \leq s \leq n - 4$ and that at least one edge of G_A^+ crosses the spine (so $2 \leq a \leq b$); see the full version.

Let t be the number of edges drawn along the spine. These are the only edges that belong to G_A^+ and G_B^+. Since G_A^+ and G_B^+ have at least three vertices each, we can bound the number of edges of G, $m(G)$, as follows:

$$m(G) \leq m(G_A^+) + m(G_B^+) - t \leq 3(s + a) - 6 + 3(s + b) - 6 - t \qquad (1)$$
$$= 3n - 12 + 3s - t \leq 4n - 16 + 2s - t.$$

So we must show that $2s - t \leq n - 3$. Let an *internal gap* be a line segment connecting two consecutive, non-adjacent vertices on the spine. There are $s - t - 1$ internal gaps. Let the *external gap* be the two infinite parts of the spine. Note that at least one edge of G_A^+ must cross the external gap, because G_A^+ has at least one vertex on each side of the gap, and we could connect the extreme such vertices (or re-route an existing edge) to cross the external gap, perhaps using a bend on the spine. We may further assume that even after such re-routing every internal gap is crossed by at least one edge of G_A^+. Otherwise we could delete all edges of G_B^+ passing through the gap, insert the edge between the spine vertices, and re-triangulate the drawing of G_B^+ where we removed edges. This would remove an internal gap, but would not decrease the number of edges. Since no edge can cross two gaps, at least $s - t$ edges of G_A^+ cross gaps. These edges form a planar bipartite graph with at most a vertices; therefore $s - t \leq 2a - 3$.[1] This yields $2s - t \leq s + 2a - 3 \leq s + a + b - 3 = n - 3$ as desired. $\qquad\square$

We conjecture that the following more general statement holds:

   Any n-vertex graph G with $\rho_3^2(G) = k$ has at most $(2k+1)(n-2k)+k-1$ edges, for all large enough n.

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[1] One might be tempted to write a bound of $2a - 4$ here, but we must allow for the possibility of $a = 2$, in case of which the planar bipartite graph may have $1 = 2a - 3$ edges.

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