On Arrangements of Orthogonal Circles

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Dedicated to Honza Kratochvíl on his 60th birthday.

Abstract. In this paper, we study arrangements of *orthogonal circles*, that is, arrangements of circles where every pair of circles must either be disjoint or intersect at a right angle. Using geometric arguments, we show that such arrangements have only a linear number of faces. This implies that *orthogonal circle intersection graphs* have only a linear number of edges. When we restrict ourselves to orthogonal *unit* circles, the resulting class of intersection graphs is a subclass of penny graphs (that is, contact graphs of unit circles). We show that, similarly to penny graphs, it is NP-hard to recognize orthogonal unit circle intersection graphs.

1 Introduction

For the purpose of this paper, an *arrangement* is a (finite) collection of curves such as lines or circles in the plane. The study of arrangements has a long history; for example, Grünbaum [15] studied arrangements of lines in the projective plane. Arrangements of circles and other closed curves have also been studied extensively [1,2,13,19,22]. An arrangement is *simple* if no point of the plane belongs to more than two curves and every two curves intersect. A *face* of an arrangement $\mathcal{A}$ in the projective or Euclidean plane P is a connected component of the subdivision induced by the curves in $\mathcal{A}$, that is, a face is a component of $P \setminus \bigcup \mathcal{A}$.

For a given type of curves, people have investigated the maximum number of faces that an arrangement of such curves can form. In 1826, Steiner [23] showed that a simple arrangement of straight lines can have at most $\binom{n}{2} + \binom{n}{1} + \binom{n}{0}$ faces while an arrangement of circles can have at most $2\left(\binom{n}{2} + \binom{n}{0}\right)$ faces.

Alon et al. [2] and Pinchasi [22] studied the number of *digonal* faces, that is, faces that are bounded by two edges, for various kinds of arrangements of
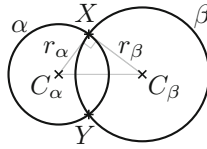
Fig. 1. Circles α and β are orthogonal if and only if $\triangle C_\alpha X C_\beta$ is orthogonal.

circles. For example, any arrangement of n unit circles has $O(n^{4/3} \log n)$ digonal faces [2] and at most $n + 3$ digonal faces if every pair of circles intersects [22], whereas arrangements of circles with arbitrary radii have at most $20n - 2$ digonal faces if every pair of circles intersects [2].

The same arrangements can, however, have quadratically many *triangular* faces, that is, faces that are bounded by three edges. A lower bound example with quadratically many triangular faces can be constructed from a simple arrangement $\mathcal{A}$ of lines by projecting it on a sphere (disjoint from the plane containing $\mathcal{A}$) and having each line become a great circle. This is always possible since the line arrangement is simple; for more details see [12, Section 5.1]. In this process we obtain $2p_3$ triangular faces, where p_3 is the number of triangular faces in the line arrangement. The great circles on the sphere can then be transformed into a circle arrangement in a different plane using the stereographic projection. This gives rise to an arrangement of circles with $2p_3$ triangular faces in this plane. Füredi and Palásti [14] provided simple line arrangements with $n^2/3 + O(n)$ triangular faces. With the argument above, this immediately yields a lower bound of $2n^2/3 + O(n)$ on the number of triangular faces of arrangements of circles. Felsner and Scheucher [13] showed that this lower bound is tight by proving that an arrangement of *pseudocircles* (that is, closed curves that can intersect at most twice and no point belongs to more than two curves) can have at most $2n^2/3 + O(n)$ triangular faces.

One can also specialize circle arrangements by fixing an angle (measured as the angle between the two tangents at either intersection point) at which each pair of intersecting circles intersect; this was recently discussed by Eppstein [10]. In this paper, we consider arrangements of circles with the restriction that each pair of circles must intersect at a right angle. An arrangement of circles in which each intersecting pair intersect at a right angle is called *orthogonal*. We make the following simple observation regarding orthogonal circles; see Fig. 1.

Observation 1. *Let α and β be two circles with centers C_α, C_β and radii r_α, r_β, respectively. Then α and β are orthogonal if and only if $r_\alpha^2 + r_\beta^2 = |C_\alpha C_\beta|^2$.*

We discuss further basic properties of orthogonal circles in Sect. 2. In particular, in an arrangement of orthogonal circles no two circles can touch and no three circles can intersect at the same point.

The main result of our paper is that arrangements of n orthogonal circles have at most $14n$ intersection points and at most $15n + 2$ faces; see Theorem 1 (in Sect. 3). This is different from arrangements of orthogonal circular arcs, which

can have quadratically many quadrangular faces; see the arcs inside the blue square in Fig. 5. In Sect. 3.2 we also consider small (that is, digonal and triangular) faces and provide bounds on the number of such faces in arrangements of orthogonal circles.

Given a set of geometric objects, their *intersection graph* is a graph whose vertices correspond to the objects and whose edges correspond to the pairs of intersecting objects. Restricting the geometric objects to a certain shape restricts the class of graphs that admit a representation with respect to this shape. For example, graphs represented by disks in the Euclidean plane are called *disk intersection graphs*. The special case of *unit disk graphs*—intersection graphs of unit disks—has been studied extensively. Recognition of such graphs as well as many combinatorial problems restricted to these graphs such as coloring, independent set, and domination are all NP-hard [6]; see also the survey of Hliněný and Kratochvíl [17]. Instead of restricting the radii of the disks, people have also studied restrictions of the type of intersection. If the disks are only allowed to touch, the corresponding graphs are called *coin graphs*. Koebe's classical result says that the coin graphs are exactly the planar graphs. If all coins have the same size, the represented graphs are called *penny graphs*. These graphs have been studied extensively, too [4,8,11]. For example, they are NP-hard to recognize [3,7].

As with the arrangements above, we again consider a restriction on the intersection angle. We define the *orthogonal circle intersection graphs* as the intersection graphs of arrangements of orthogonal circles. In Sect. 4, we investigate properties of these graphs. For example, similar to the proof of our linear bound on the number of intersection points for arrangements of orthogonal circles (Theorem 1), we observe that such graphs have only a linear number of edges.

We also consider *orthogonal unit circle intersection graphs*, that is, orthogonal circle intersection graphs with a representation that consists only of unit circles. We show that these graphs are a proper subclass of penny graphs. It is NP-hard to recognize penny graphs [9]. We modify the NP-hardness proof of Di Battista et al. [7, Section 11.2.3], which uses the *logic engine*, to obtain the NP-hardness of recognizing orthogonal unit circle intersection graphs (Theorem 4).

2   Preliminaries

We will use the following type of Möbius transformation [20]. Let α be a circle having center at C_α and radius r_α. The *inversion* with respect to α is a mapping that maps any point $P \neq C_\alpha$ to a point P' on the ray $C_\alpha P$ so that $|C_\alpha P'| \cdot |C_\alpha P| = r_\alpha^2$. Inversion maps each circle not passing through C_α to another circle and a circle passing through C_α to a line; see Fig. 2. Inversion and orthogonal circles are closely related. For example, in order to construct the image P' of some point P that lies inside the inversion circle α, consider the intersection points X and Y of α and the line that is orthogonal to the line through C_α and P in P; see Fig. 2c. The point P' then is simply the center of the circle β that is orthogonal to α and goes through X and Y. This follows from the similarity of the orthogonal triangles $\triangle C_\alpha X P'$ and $\triangle C_\alpha X P$. A useful property of inversion,
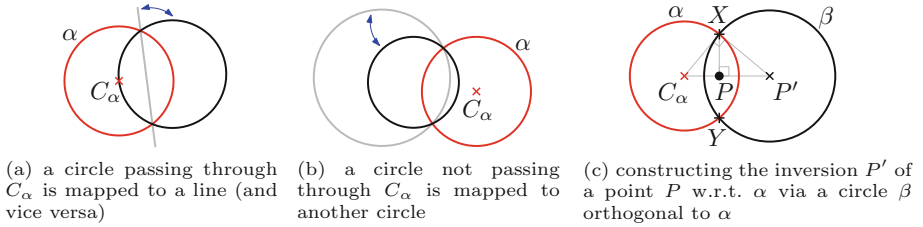
(a) a circle passing through C_α is mapped to a line (and vice versa)

(b) a circle not passing through C_α is mapped to another circle

(c) constructing the inversion P' of a point P w.r.t. α via a circle β orthogonal to α
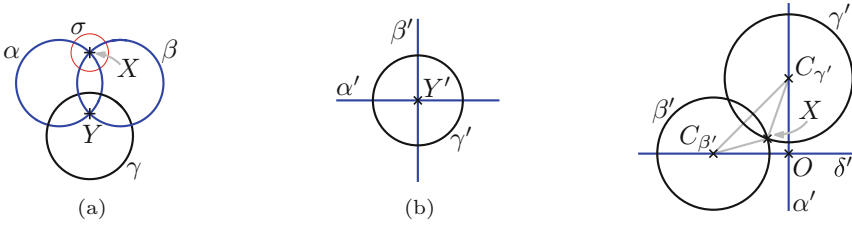
Fig. 2. Examples of inversion

(a)

(b)

Fig. 3. (a) Three pairwise intersecting circles, the red inversion circle is centered at X; (b) image of the inversion. (Color figure online)

Fig. 4. Illustration for the proof of Lemma 2

as of any other Möbius transformation, is that it preserves angles. Using inversion we can easily show several properties of orthogonal circles.

Lemma 1. *No orthogonal circle intersection graph contains a K_4. In other words, in an arrangement of orthogonal circles there cannot be four pairwise orthogonal circles.*

Proof. Assume that there are four pairwise orthogonal circles α, β, γ, and δ. Let X and Y be the intersection points of α and β. Consider the inversion with respect to a circle σ centered at X. The images of α and β are orthogonal lines α' and β' that intersect at Y', which is the image of Y; see Fig. 3. The image of γ is a circle γ' centered at Y' but so is the image δ' of δ. Thus γ' and δ' are either disjoint or equal, but not orthogonal to each other, a contradiction.   □

Lemma 2. *No orthogonal circle intersection graph contains an induced C_4. In other words, in an arrangement of orthogonal circles there cannot be two pairs of circles such that each circle of one pair is orthogonal to each circle of the other pair and the circles within the pairs are not orthogonal.*

Proof. Assume there are two pairs (α, β) and (γ, δ) of circles such that the circles within each pair do not intersect each other and each circle of one pair intersects both circles of the other pair. Consider an inversion via a circle σ centered at one of the intersection points of the circles α and δ. In the image they will
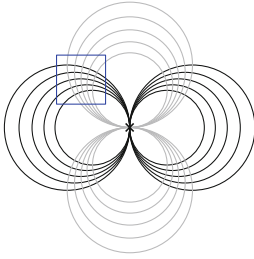
Fig. 5. Apollonian circles consisting of two parabolic pencils of circles (one in black, the other in gray). (Color figure online)
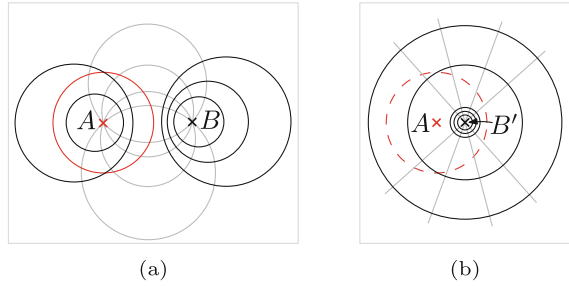
Fig. 6. (a) Apollonian circles consisting of an elliptic (in gray) and hyperbolic (in black) pencil of circles; (b) its inversion via a circle centered at A (in red). (Color figure online)

become lines α' and δ'. The image β' of the circle β must intersect δ' but not α', therefore, its center must lie on the line δ' and it should be to one side of the line α'; see Fig. 4. Similarly the center of the image γ' of the circle γ must lie on the line α' and γ' should be to one side of the line δ'. Shift the drawing so that the intersection of α' and δ' is at the origin O and observe that the triangle $\triangle C_{\beta'} O C_{\gamma'}$ is orthogonal, where $C_{\beta'}$ and $C_{\gamma'}$ are the centers of the circles β' and γ'. Let X be the intersection point of these circles that is closer to the origin. This point X is contained in the triangle $\triangle C_{\beta'} O C_{\gamma'}$. Therefore the triangle $\triangle C_{\beta'} X C_{\gamma'}$ cannot be orthogonal—a contradiction.   □

A *pencil* is a family of circles who share a certain characteristic. In a *parabolic* pencil all circles have one point in common, and thus are all tangent to each other; see Fig. 5. In an *elliptic* pencil all circles go through two given points; see the gray circles in Fig. 6a. In a *hyperbolic* pencil all circles are orthogonal to a set of circles that go through two given points, that is, to some elliptic pencil; see the black circles in Fig. 6a.

For an elliptic pencil whose circles share two points A and B and the corresponding hyperbolic pencil, the circles in the hyperbolic pencil possess several properties useful for our purposes [20]. Their centers are collinear and they consist of non-intersecting circles that form two nested structures of circles, one containing A, the other one containing B in its interior; see Fig. 6a.

Two pencils of circles such that each circle in one pencil is orthogonal to each circle in the other are called *Apollonian circles*. There can be two such combinations of pencils, that is, one with two parabolic pencils and one with an elliptic and a hyperbolic pencil. We focus on the latter since such Apollonian circles contain arbitrarily large arrangements of orthogonal circles, that is, two orthogonal circles from the elliptic pencil and arbitrary many circles from the hyperbolic pencil. Equivalently, such Apollonian circles are an inversion image of a family of concentric circles centered at some point X and concurrent lines passing through X; see Fig. 6b. We use this equivalence in the next proof.

Lemma 3. *Three circles such that one is orthogonal to the two others belong to the same family of Apollonian circles. Two sets of circles such that each circle in one set is orthogonal to each circle in the other set and each set has at least two circles belong to the same family of Apollonian circles. In particular the set belonging to the elliptic pencil can contain at most two circles.*

Proof. Consider three circles such that one is orthogonal to two others. If all three are pairwise orthogonal, then their inversion via a circle centered at one of their intersection points (see Fig. 3a) is two perpendicular lines and a circle centered at their intersection point (see Fig. 3b), therefore, they belong to the same family of Apollonian circles. If two circles do not intersect, then by [20, Theorem 13], it is always possible to invert them into two concentric circles. Since inversion preserves angles, the image of the third circle must be orthogonal to both concentric circles and therefore it must be a straight line passing through the center of both circles. Therefore, the three circles belong to the same family of Apollonian circles.

Consider now two sets S_1 and S_2 of circles such that each circle in one set is orthogonal to each circle in the other set and each set has at least two circles. By Lemma 2 there must be two circles α and β in one of the sets, say S_1, that are orthogonal. Consider an inversion via a circle σ centered at one of the intersection points X of the circles α and β. In the image they will become orthogonal lines α' and β' intersecting at a point Y. Because inversion preserves angles, the image of each circle in S_2 is a circle centered at Y. Since S_2 contains at least two circles, the image of each circle in S_1 must be orthogonal to two circles centered at Y, therefore, it must be a straight line passing through Y. Thus, the circles in S_1 and S_2 belong to the same family of Apollonian circles and S_1 contains at most two circles.   □

Because each triangular or quadrangular face consists of either three circles such that one is orthogonal to two others or two pairs of circles such that each circle in one pair is orthogonal to each circle in the other pair, we obtain the following observation from Lemma 3.

Observation 2. *In any arrangement of orthogonal circles, each triangular and each quadrangular face is formed by Apollonian circles.*

3   Arrangements of Orthogonal Circles

In this section we study the number of faces of an arrangement of orthogonal circles. In Sect. 3.1, we give a bound on the total number of faces. In Sect. 3.2, we separately bound the number of faces formed by two and three edges.

Let $\mathcal{A}$ be an arrangement of orthogonal circles in the plane. By a slight abuse of notation, we will say that a circle α *contains* a geometric object o and mean that the disk bounded by α contains o. We say that a circle $\alpha \in \mathcal{A}$ is *nested* in a circle $\beta \in \mathcal{A}$ if α is contained in β. We say that a circle $\alpha \in \mathcal{A}$ is nested *consecutively* in a circle $\beta \in \mathcal{A}$ if α is nested in β and there is no other
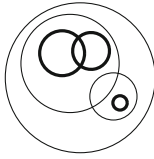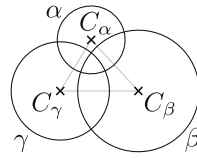
Fig. 7. Deepest circles in bold

Fig. 8. $\angle C_\beta C_\alpha C_\gamma \geq \pi/3$

circle $\gamma \in \mathcal{A}$ such that α is nested in γ and γ is nested in β. Consider a subset $S \subseteq \mathcal{A}$ of maximum cardinality such that for each pair of circles one is nested in the other. The innermost circle α in S is called a *deepest* circle in $\mathcal{A}$; see Fig. 7.

Lemma 4. *Let α be a circle of radius r_α, and let S be a set of circles orthogonal to α. If S does not contain nested circles and each circle in S has radius at least r_α, then $|S| \leq 6$. Moreover, if $|S| = 6$, then all circles in S have radius r_α and α is contained in the union of the circles in S.*

Proof. Let C_α be the center of α. Consider any two circles β and γ in S with centers C_β and C_γ and with radii r_β and r_γ, respectively. Since $r_\beta \geq r_\alpha$ and $r_\gamma \geq r_\alpha$, the edge $C_\beta C_\gamma$ is the longest edge of the triangle $\triangle C_\beta C_\alpha C_\gamma$; see Fig. 8. So the angle $\angle C_\beta C_\alpha C_\gamma$ is at least $\pi/3$. Thus, $|S| \leq 6$.

Moreover, if $|S| = 6$ then, for each pair of circles β and γ in S that are consecutive in the circular ordering of the circle centers around C_α, it holds that $\angle C_\beta C_\alpha C_\gamma = \pi/3$. This is only possible if $r_\beta = r_\gamma = r_\alpha$. Thus, all the circles in S have radius r_α and α is contained in the union of the circles in S; see Fig. 9b. □

3.1   Bounding the Number of Faces

Theorem 1. *Every arrangement of n orthogonal circles has at most $14n$ intersection points and $15n + 2$ faces.*

The above theorem (whose formal proof is at the end of the section) follows from the fact that any arrangement of orthogonal circles contains a circle α with at most seven *neighbors* (that is, circles that are orthogonal to α).

Lemma 5. *Every arrangement of orthogonal circles has a circle that is orthogonal to at most seven other circles.*

Proof. If no circle is nested within any other, Lemma 4 implies that the smallest circle has at most six neighbors, and we are done.

So, among the deepest circles in $\mathcal{A}$, consider a circle α with the smallest radius. Let r_α be the radius of α. Note that α is nested in at least one circle. Let β be a circle such that α and β are consecutively nested. Denote the set of all circles in $\mathcal{A}$ that are orthogonal to α but not to β by S_α. All circles in S_α are nested in β. Since α is a deepest circle, S_α contains no nested circles; see Fig. 9a. Since the radius of every circle in S_α is at least r_α, Lemma 4 ensures that S_α

(a) the circles of S_α are in bold black

(b) $i = 4$, $j = 5$

(c) $i = 3$, $j = 5$
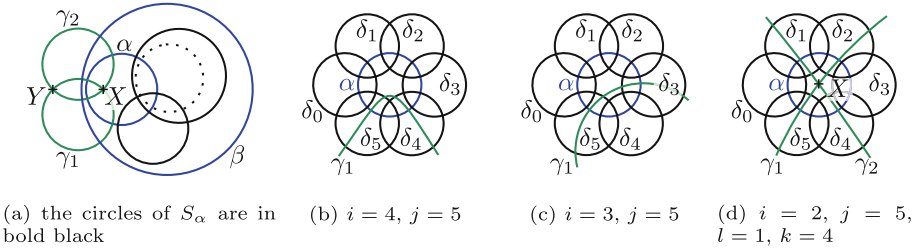
(d) $i = 2$, $j = 5$, $l = 1$, $k = 4$

Fig. 9. Illustrations to the proof of Lemma 5

contains at most six circles. Given the structure of Apollonian circles (Lemma 3), there can be at most two circles that intersect both α and β. This together with Lemma 4 immediately implies that α cannot be orthogonal to more than eight circles. In the following we show that there can be at most seven such circles.

If there is only one circle intersecting both α and β, then α is orthogonal to at most seven circles in total, and we are done.

Otherwise, there are two circles orthogonal to both α and β. Let these circles be γ_1 and γ_2. We assume that S_α contains exactly six circles. Hence, by Lemma 4, all circles in S_α have radius r_α. Let $S_\alpha = (\delta_0, \ldots, \delta_5)$ be ordered clockwise around α so that every two circles δ_i and δ_j with $i \equiv j + 1 \bmod 6$ are orthogonal.

Let X and Y be the intersection points of γ_1 and γ_2; see Fig. 9a. Note that, by the structure of Apollonian circles, one of the intersection points, say X, must be contained inside α, whereas the other intersection point Y must lie in the exterior of β. Since the circles in S_α are contained in β, none of them contains Y. Further, no circle δ_i in S_α contains X, as otherwise the circles δ_i, α, γ_1, and γ_2 would be pairwise orthogonal, contradicting Lemma 1. Recall that, by Lemma 4, α is contained in the union of the circles in S_α. Since X is not contained in this union, γ_1 intersects two different circles δ_i and δ_j, and γ_2 intersects two different circles δ_k and δ_l. Note that γ_1 and γ_2 cannot intersect the same circle ε in S_α, because ε, α, γ_1, and γ_2 would be pairwise orthogonal, contradicting Lemma 1. Therefore, the indices i, j, k, and l are pairwise different.

We now consider possible values of the indices i, j, k, and l, and show that in each case we get a contradiction to Lemma 1 or Lemma 2. If $j \equiv i + 1 \bmod 6$, then γ_1, α, δ_i, and δ_j would be pairwise orthogonal, contradicting Lemma 1; see Fig. 9b. If $j \equiv i + 2 \bmod 6$, then γ_1, δ_i, δ_{i+1}, and δ_j would form an induced C_4 in the intersection graph; see Fig. 9c. This would contradict Lemma 2. If $j \equiv i + 3 \bmod 6$ and $k \equiv l + 3 \bmod 6$, then either $k \equiv i + 1 \bmod 6$ or $i \equiv l + 1 \bmod 6$; see Fig. 9d. W.l.o.g., assume the latter and observe that then γ_2, δ_i, γ_1, δ_l would form an induced C_4, again contradicting Lemma 2.

We conclude that S_α contains at most five circles. Together with γ_1 and γ_2, at most seven circles are orthogonal to α.   □

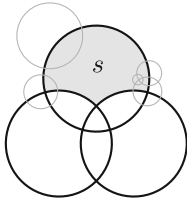Using the lemma above and Euler's formula, we now can prove Theorem 1.

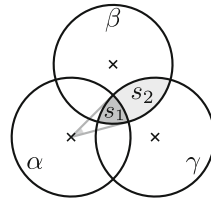**Fig. 10.** Region s is a face in the arrangement of the bold circles

Fig. 11. Angles subtended by the regions s_1 and s_2 in the circle α; $\angle(s_1, \alpha) = -\angle(s_2, \alpha)$

Proof (of Theorem 1). Let $\mathcal{A}$ be an arrangement of orthogonal circles. By Lemma 5, $\mathcal{A}$ contains a circle α orthogonal to at most seven circles. The circle α yields at most 14 intersection points. By induction, the whole arrangement has at most $14n$ intersection points.

Consider the planarization G' of $\mathcal{A}$, and let n', m', f', and c' denote the numbers of vertices, edges, faces, and connected components of G', respectively. Since every vertex in the planarization corresponds to an intersection, the resulting graph is 4-regular and therefore $m' = 2n'$. By Euler's formula, we obtain $f' = n' + 1 + c'$. This yields $f' \leq 15n + 1$ since $n' \leq 14n$ and $c' \leq n$. □

3.2   Bounding the Number of Small Faces

In the following we study the number of faces of each type, that is, the number of digonal, triangular, and quadrangular faces. We begin with some notation. Let $\mathcal{A}$ be an arrangement of orthogonal circles in the plane. Let S be some subset of the circles of $\mathcal{A}$. A face in S is called a region in $\mathcal{A}$ formed by S; see for instance Fig. 10. Note that each face of $\mathcal{A}$ is also a region.

Let s be the region formed by some circular arcs $a_1, a_2, \ldots, a_k$ enumerated in counterclockwise order around s. For an arc a_i with $i \in \{1, \ldots, k\}$, let α be the circle that supports a_i. If $C_\alpha = (x_\alpha, y_\alpha)$ is the center of α and r_α its radius, we can write α as $\{C_\alpha + r_\alpha(\cos t, \sin t) \colon t \in [0, 2\pi]\}$. Let u and v be the endpoints of a_i so that we meet u first when we traverse s counterclockwise when starting outside of a_i. Let $u = C_\alpha + r_\alpha(\cos t_1, \sin t_1)$ and $v = C_\alpha + r_\alpha(\cos t_2, \sin t_2)$. We say that the region s *subtends* an angle in the circle α of size $\angle(s, a_i) = t_2 - t_1$ with respect to the arc a_i. Note that $\angle(s, a_i)$ is negative if a_i forms a concave side of s; see Fig. 11. If the circle α forms only one side of the region s, then we just say that the region s *subtends* an angle in the circle α of size $\angle(s, \alpha) = t_2 - t_1$. Moreover, if s is a digonal region, that is, it is formed by only two circles α and β, then we simply say that β subtends an angle of $\angle(\beta, \alpha) = t_2 - t_1$ in α to mean $\angle(s, \alpha)$.

By *total angle* we denote the sum of subtended angles by s with respect to all the arcs that form its sides, that is, $\sum_{i=1}^{k} \angle(s, a_i)$.

We now give an upper bound on the number of digonal and triangular faces in an arrangement $\mathcal{A}$ of n orthogonal circles. The tool that we utilize in this section

is the Gauss–Bonnet formula [24] which, in the restricted case of orthogonal circles in the plane, states that, for every region s formed by some circular arcs $a_1, a_2, \ldots, a_k$, it holds that

$$\sum_{i=1}^{k} \angle(s, a_i) + \frac{k\pi}{2} = 2\pi.$$

This formula implies that each digonal or triangular face subtends a total angle of size π and of size $\pi/2$, respectively. Thus, we obtain the following bounds.

Theorem 2. *Every arrangement of n orthogonal circles has at most $2n$ digonal faces and at most $4n$ triangular faces.*

Proof. Because faces do not overlap, each digonal or triangular face uses a unique convex arc of a circle bounding this face. Therefore, the sum of angles subtended by digonal or triangular faces formed by the same circle must be at most 2π. Analogously, the sum of total angles over all digonal or triangular faces cannot exceed $2n\pi$. By the Gauss–Bonnet formula each digonal or triangular face subtends a total angle of size π or $\pi/2$, respectively. This gives an upper bound of $2n$ on the number of digonal faces and an upper bound of $4n$ on the number of triangular faces. □

Theorem 2 can be generalized to all convex orthogonal closed curves since the Gauss–Bonnet formula does not require curves to be circular. In contrast to this, for example, a grid made of axis parallel rectangles has quadratically many quadrangular faces. This makes circles a special subclass of convex orthogonal closed curves. We refer to the full version for more details [5].

The Gauss–Bonnet formula does not help us to get an upper bound on the number of quadrangular faces. Using Observation 2, however, it is possible to restrict the types of quadrangular faces to several shapes and obtain bounds on the number of faces of each type. Apart from being interesting in its own right, such a bound also provides a bound on the total number of faces in an arrangement of orthogonal circles. Namely, since the average degree of a face in an arrangement of orthogonal circles is 4, a bound on the number of faces of degree at most 4 gives a bound on the number of all faces in the arrangement (via Euler's formula). Unfortunately, the bound on the number of quadrangular faces that we achieved was $17n$ and thus higher than the bound $15n + 2$ that we now have for the number of *all* faces in an arrangement of n orthogonal circles.

4   Intersection Graphs of Orthogonal Circles

Given an arrangement $\mathcal{A}$ of orthogonal circles, consider its *intersection* graph, which is the graph with vertex set $\mathcal{A}$ that has an edge between any pair of intersecting circles in $\mathcal{A}$. Lemmas 1 and 2 imply that such a graph does not contain any K_4 and any induced C_4. We show that such graphs can be non-planar (Lemma 6), then we bound their edge density (Theorem 3), and finally we consider the intersection graphs arising from orthogonal *unit* circles (Theorem 4).
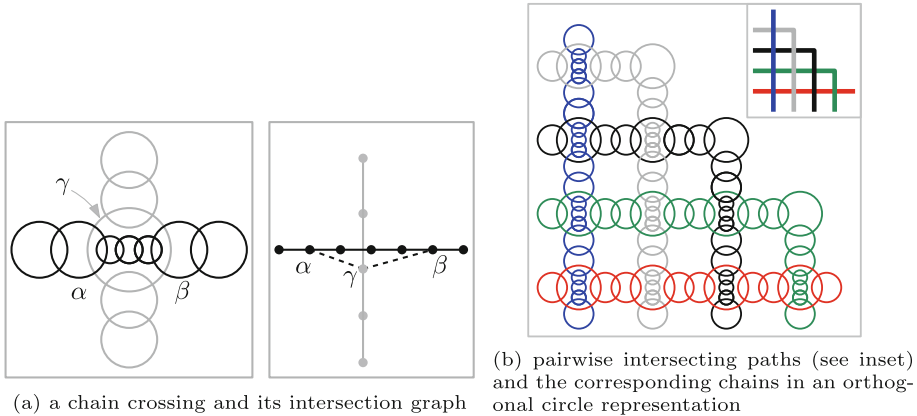
(a) a chain crossing and its intersection graph

(b) pairwise intersecting paths (see inset) and the corresponding chains in an orthogonal circle representation

Fig. 12. Construction of an orthogonal circle intersection graph that contains K_n as a minor (here $n = 5$).

Lemma 6. *For every n, there is an intersection graph of orthogonal circles that contains K_n as a minor. The representation uses circles of three different radii.*

Proof. Let a *chain* be an arrangement of orthogonal circles whose intersection graph is a path. We say that two chains C_1 and C_2 *cross* if two disjoint circles α and β of one chain, say C_1, are orthogonal to the same circle γ of the other chain C_2; see Fig. 12a (left). If two chains cross, their paths in the intersection graph are connected by two edges; see the dashed edges in Fig. 12a (right).

Consider an arrangement of n rectilinear paths embedded on a grid where each pair of curves intersect exactly once; see the inset in Fig. 12b. We convert the arrangement of paths into an arrangement of chains such that each pair of chains crosses; see Fig. 12b. Now consider the intersection graph of the orthogonal circles in the arrangement of chains. If we contract each path in the intersection graph that corresponds to a chain, we obtain K_n. ☐

Next, we discuss the density of orthogonal circle intersection graphs. Gyárfás et al. [16] have shown that any C_4-free graph on n vertices with average degree at least a has clique number at least $a^2/(10n)$. Due to Lemma 1, we know that orthogonal circle intersection graphs have clique number at most 3. Thus, their average degree is bounded from above by $\sqrt{30n}$, leading to at most $\sqrt{7.5}n^{\frac{3}{2}}$ edges in total. However, Lemma 5 implies the following stronger bound.

Theorem 3. *The intersection graph of a set of n orthogonal circles has at most $7n$ edges.*

Proof. The geometric representation of an orthogonal circle intersection graph is an arrangement of orthogonal circles. By Lemma 5, an arrangement of n orthogonal circles always has a circle orthogonal to at most seven circles. Therefore, the corresponding intersection graph always has a vertex of degree at most seven. Thus, it has at most $7n$ edges. ☐
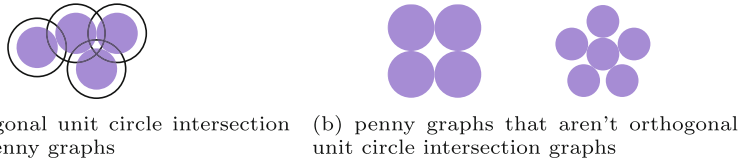
(a) all orthogonal unit circle intersection graphs are penny graphs

(b) penny graphs that aren't orthogonal unit circle intersection graphs

Fig. 13. Penny graphs vs. orthogonal unit circle intersection graphs

(a) $1.5n$ digonal faces

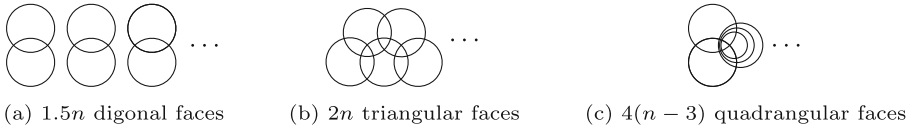(b) $2n$ triangular faces

(c) $4(n-3)$ quadrangular faces

Fig. 14. Arrangements of n orthogonal circles with many digonal, triangular, and quadrangular faces.

The remainder of this section concerns a natural subclass of orthogonal circle intersection graphs, the orthogonal *unit* circle intersection graphs. Recall that these are orthogonal circle intersection graphs with a representation that consists of unit circles only. As Fig. 13a shows, every representation of an orthogonal unit circle intersection graph can be transformed (by scaling each circle by a factor of $\sqrt{2}/2$) into a representation of a *penny graph*, that is, a contact graph of equal-size disks. Hence, every orthogonal unit circle intersection graph is a penny graph – whereas the converse is not true. For example, C_4 or the 5-star are penny graphs but not orthogonal unit circle intersection graphs (see Fig. 13b).

Orthogonal unit circle intersection graphs being penny graphs implies that they inherit the properties of penny graphs, e.g., their maximum degree is at most six and their edge density is at most $\lfloor 3n - \sqrt{12n-6} \rfloor$, where n is the number of vertices [21, Theorem 13.12, p. 211]. Because triangular grids are orthogonal unit circle intersection graphs, this upper bound is tight.

As it turns out, orthogonal unit circle intersection graphs share another feature with penny graphs: their recognition is NP-hard. The hardness of penny-graph recognition can be shown using the *logic engine* [7, Section 11.2], which simulates an instance of the Not-All-Equal-3-Sat (NAE3SAT) problem. We establish a similar reduction for the recognition of orthogonal unit circle intersection graphs; the details are in the full version [5].

Theorem 4. *It is NP-hard to recognize orthogonal unit circle intersection graphs.*

5 Discussions and Open Problems

In Sect. 3 we have provided upper bounds for the number of faces of an orthogonal circle arrangement. As for lower bounds on the number of faces, we found only very simple arrangements containing $1.5n$ digonal, $2n$ triangular, and $4(n-3)$

quadrangular faces; see Figs. 14a, b, and c, respectively. Can we construct better lower bound examples or improve the upper bounds?

Recognizing (unit) disk intersection graphs is ∃ℝ-complete [18]. But what is the complexity of recognizing (general) orthogonal circle intersection graphs?

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