Chapter 7
Sixteenth Century Reckoners Versus Twenty-First Century Problem Solvers

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Abstract In this chapter, the focus is on arithmetic which for the Netherlands as a trading nation is a crucial part of the mathematics curriculum. The chapter goes back to the roots of arithmetic education in the sixteenth century and compares it with the current approach to teaching arithmetic. In the sixteenth century, in the Netherlands, the traditional arithmetic method using coins on a counting board was replaced by written arithmetic with Hindu–Arabic numbers. Many manuscripts and books written in the vernacular teach this new method to future merchants, moneychangers, bankers, bookkeepers, etcetera. These students wanted to learn recipes to solve the arithmetical problems of their future profession. The books offer standard algorithms and many practical exercises. Much attention was paid to memorising rules and recipes, tables of multiplication and other number relations. It seems likely that the sixteenth century craftsmen became skilful reckoners within their profession and that was sufficient. They did not need mathematical insight to solve new problems. Five centuries later we want to teach our students mathematical skills to survive in a computerised and globalised society. They also need knowledge about number relations and arithmetical rules, but they have to learn to apply this knowledge flexibly and meaningfully to solve new problems, to mathematise situations, and to evaluate, interpret and check output of computers and calculators. The twenty-first century needs problem solvers, but to acquire the skills of a good problem solver a firm knowledge base—comparable with that of the sixteenth century reckoner—is still necessary.

7.1 Introduction

Over many centuries teaching arithmetic has played an important part in Dutch education. Interest in this subject started to grow in the sixteenth century when the Netherlands began to develop into an important trade nation and arithmetic finally got

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its own Dutch name: '*Rekenen*'. At that time the aims, content, organisation, teachers and students differed a lot from what is going on in Dutch arithmetic education of the twenty-first century. This chapter makes a comparison between now and then. The differences are large and plentiful, but there are also some remarkable similarities that we can perhaps learn from.

7.2 Arithmetic in the Sixteenth Century

7.2.1 Merchants, the New Rich of the Sixteenth Century

An early medieval Dutch merchant's life was not very complicated. He wandered around, visiting towns and villages, trying to barter his goods. He was not schooled in bookkeeping and commercial arithmetic, but that was not a problem. Over time, in the fifteenth and sixteenth century, when the Netherlands grew more prosperous and more goods were produced, merchants were no longer simple wandering adventurers. They stayed in their offices and sent out their traveling salesmen. Business journeys became longer, merchants travelled to different countries, they had to pay salaries, customs rights, costs of transport, assurances of goods, etcetera. They needed to change money in many different ways, because each city had its own money system. They visited exchange banks where moneychangers took care of their affairs. Bankers and bookkeepers were needed. Many merchants earned a lot of money and they spent it on building houses and filling these with luxury goods; so, they needed carpenters, bricklayers, gold and silversmiths and other craftsmen. As trading methods grew more complex, a more advanced arithmetical method was needed, and written records of all commercial transactions and calculations (Swetz, 1989).

7.2.2 Traditional Arithmetic on the Counting Board

In the Netherlands of the early Middle Ages, arithmetic was traditionally done on a counting board with horizontal lines. Each line has a certain value and by placing coins on or between the lines people could express numbers and do calculations. This counting board is a variation of the ancient Greek and Roman abacus with vertical lines and counters of ivory, bones or glass (Mazur, 2014).

Traveling merchants did not always drag along their counting board. Instead, they left it at home and drew chalk lines on a table to do their calculations. Some merchants even omitted the lines. Figure 7.1a shows a picture from the French arithmetic book *Le Livre de Chiffres et de Getz* (Anonymous, 1501). Three merchants are calculating with coins without using lines. In this method (Fig. 7.1b), a decimal system is created by placing coins on a vertical line. These are the so-called 'layers'. The first layer indicates the ones, the second layer indicates the tens, the next one the hundreds,

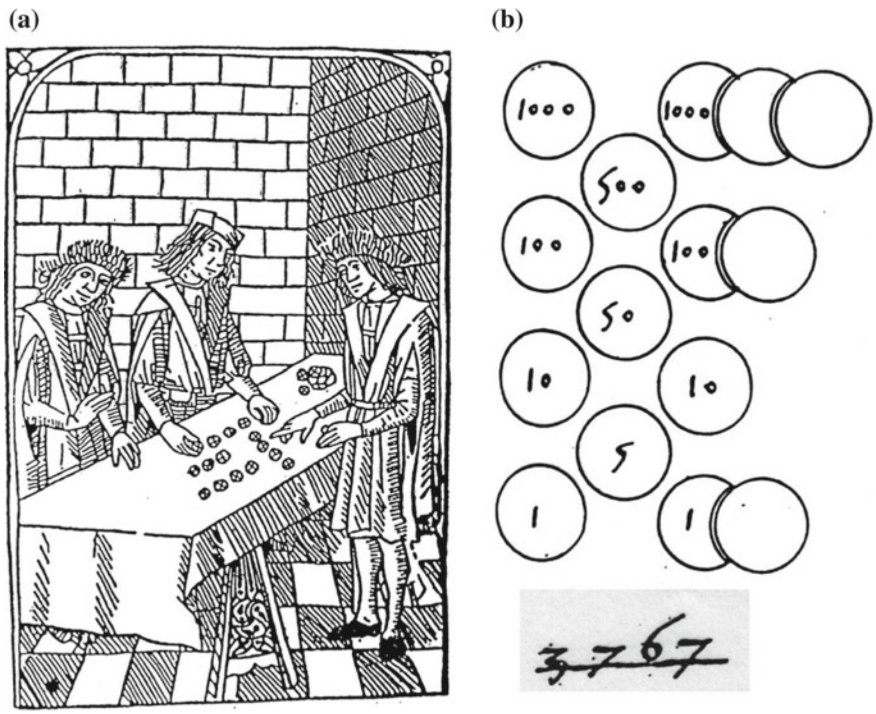

Fig. 7.1 **a** In the French arithmetic book *Le Livre de Chiffres et de Getz* (Anonymous, 1501; picture retrieved from Menninger, 1969, p. 367) the method of calculating with coins without using lines is explained. **b** Calculating $3 \times 1000 + 500 + 2{\times}100 + 50 + 1{\times}10 + 5{+}2 \times 1 = 3767$ by using coins (this picture is from a book of Van Varenbraken, 1532; Ghent, University Library, ms. 2141. fol. 187r.; picture retrieved from Kool, 1988, p. 170)

etcetera. The value of the fields between the layers increases from 5 to 50, 500, etcetera. The number 3767 is expressed in coins. After you have learned to represent numbers with coins the next step is doing calculations. It is quite easy to add and subtract because you only have to add or remove coins, then rearrange the coins and read the result. Doing multiplication and division is a little bit more complicated, but it is doable. So, this traditional way of doing arithmetic sufficed for quite a long time.

7.2.3 A New Written Arithmetic Method with Hindu–Arabic Numbers

At the end of the 12th and the beginning of the thirteenth century a new arithmetic method appeared in Southern Europe. This method was spread in Arabic manuscripts that reached Spain and Sicily via trade routes. The best-known manuscript is the ninth century arithmetic manuscript of al-Khwarizmi (ca. 780–850), a Muslim mathematician, astronomer, geographer at the court of al-Mansur in Baghdad. His arithmetic manuscript has been lost, but Latin translations still exist. In his work, he describes the Hindu system of numeration and a method to do written calculations using this number system. Several Latin translations and adaptations were made of this manuscript. Inspired by these works, thirteenth-century European scholars like John of Sacrobosco and Alexander of Villa Dei wrote their own arithmetic books. These academic Latin treatises may have been intended for a learned audience (Folkerts & Kunitzsch, 1997).

The Italian Leonardo of Pisa (also called Fibonacci, ca. 1170–1240) learned the new arithmetic method during business journeys with his father in North Africa. In 1202, he wrote the *Liber Abaci*. In this book, he applied the new arithmetic method on a great many commercial problems. This practical part of his work was copied by the authors of dozens of Italian arithmetic books. Translations and adaptations of these books in several languages were made and the new method became popular in many other European countries including the Netherlands. The audience of these practical books was not academic.

7.2.4 The Rise of the New Arithmetic Method in the Netherlands

As far as we know now, the oldest arithmetic manuscript in the Dutch language teaching the new arithmetic method appeared in 1445. Two other Dutch arithmetic manuscripts were written in the fifteenth century. From the sixteenth century, 9 Dutch manuscripts and 24 Dutch printed books on written arithmetic with Hindu–Arabic numbers are in existence. If you take into account that arithmetic books were consumables used by teachers and traveling merchants, many more books and manuscripts must have been published at that time (Kool, 1999).

In some of these books both arithmetic methods are explained, the traditional one with the coins as well as the new written Hindu–Arabic one. Both arithmetic methods stayed in use over a long time. In Fig. 7.2 you see the two methods being practised together at the same table, on the left the modern method and in the middle the traditional one. This picture is from the title page of the arithmetic book written by the German Ries (1533).

Ries explains that learning the traditional arithmetic method with coins is a good preparation for learning the new method with pen and paper. In his book, he describes both methods. It seems that quite a few people in sixteenth century Europe could use both methods. The mathematician Peter Ramus used the new arithmetical method in his *Arithmeticae Libri Tres* (1555), but in private, he said, he preferred the traditional way with coins (Verdonk, 1966). There was no competition between the two methods, as is sometimes wrongly suggested (Boyer, 1968; Swetz, 1989).

Fig. 7.2 The traditional arithmetic method with coins (middle) and the new one with Hindu–Arabic numbers (left) on the same table; title page of the arithmetic book written by Ries (1533) (picture retrieved from Swetz, 1989, p. 32)

In the end, the modern way of calculating with a pen was preferred to the old manner. But this happened only after a rather long period of time. In 1689 calculation coins were still struck in the Southern Netherlands (Barnard, 1916).

Why did it take such a long time before the new method was accepted everywhere? For us it is obvious that it has many advantages as compared with the old one. For example, you can easily check your written calculation afterwards. In arithmetic with coins, the numbers you start with disappear from your counting board during your calculation. Of course, people could check their final result by using the 'check of nines', but it is impossible to read over the process afterwards. In the new method, you can. This new method has more advantages. Using Hindu–Arabic numbers extends your mathematical options. It is easy to write big numbers, to extract roots and to calculate with fractions. Using the traditional method people did their calculations with coins and then used a pen to write down their result in Roman numerals. In the new method, the same instrument—the pen—and the same number system—Hindu–Arabic—are used for both calculating and recording the result.

Yet, in spite of these advantages it is understandable why the traditional method with the coins survived for such a long time. Most of the people at that time could not write. Around 1600 in the Netherlands only 40% of women and 60% of men were able to sign their marriage certificate (Dodde, 1997). Perhaps more people could read, because in sixteenth century Dutch education reading was taught before writing and many students left school at the time that writing education started, because they had to work and earn money. The Dutch arithmetician Christianus van Varenbraken explained in his arithmetic manuscript of 1532 that he describes the traditional method with coins for people who cannot write. Another advantage of calculating with coins is that one visualises calculations with concrete objects. And finally, you do not need a zero. It is easy to understand that an empty place on your counting board means nothing. In the new written number system, you need a zero to indicate an empty space. You have to write a sign, although this sign means 'nothing'. And at the same time this magical sign can changes the value of a number when it is added to it. 4 does not mean the same as 40! People found this difficult to understand. Authors gave long explanations about the function of zero. Van Varenbraken (1532, cited in Kool, 1988) wrote about the zero:

> *This 0 means nothing, he has no value of his own, but 0 gives a value to the other 9 number symbols. And he makes their value ten times more than the value they have of their own.*

Some people were even opposed to the new number system because of the zero. In Florence, the Arte del Cambio, the guild of money changers, forbade its members to use the new numbers in their cash books for fear of fraud (Pullan, 1968).

Arithmetic books in the Dutch language, that had been available since the fifteenth century, were not used in the traditional Latin schools, because in these schools all teaching was done in Latin and arithmetic hardly played a part. During the sixteenth century, so-called 'French schools', in whose curriculum the town government did not have a say, were founded by private initiative. Merchants, bankers and other financial and administrative practitioners sent their sons to these schools to study subjects like French, bookkeeping and arithmetic. French was the most important business language at the time. The other subjects at the French schools were taught in the vernacular. It is clear that these schools were good 'nurseries' for future merchants, bankers and money changers. The arithmetic books in Dutch were used in these schools. Some teachers wrote and used their own arithmetic book.

7.2.5 The Content of the Dutch Arithmetic Books from the Sixteenth Century

The authors generally teach the basics of arithmetic, which means that they deal with the reading and writing of Hindu–Arabic numerals including zero, and the arithmetical operations: addition, subtraction, multiplication and division. Some authors also dealt with halving and doubling, which they considered as separate operations. The arithmetic algorithms they teach largely correspond to those in use nowadays. Only the division algorithm shows some differences. First calculating with whole numbers is taught, followed by fractions. To practise the algorithms many examples, worked out in detail, are presented. Most of these examples deal with money, weights and measures. In the sixteenth century, each city had its own system of money and

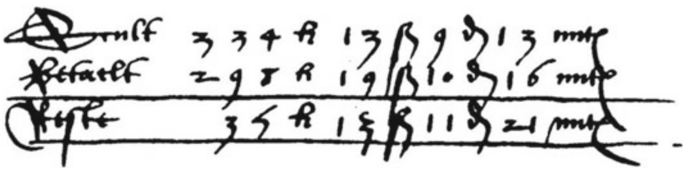

Fig. 7.3 Subtraction (including a mistake) with two amounts of money from the Dutch arithmetic manuscript of Christianus van Varenbraken (1532); Ghent, University Library, ms. 2141. fol. 135r (picture retrieved from Kool, 1999, p. 72)

measures which could make calculations rather complicated. In Fig. 7.3 you see a subtraction with two amounts of money from the arithmetic manuscript of Van Varenbraken (1532) from Ghent: 298 lb, 19 shillings, 10 pennies and 16 mites are subtracted from 334 lb, 13 shillings, 9 pennies and 13 mites. You have to know the Ghent system in which: 1 lb equals 20 shillings, 1 shilling equals 12 pennies and 1 penny equals 24 mites. It is clear that this complicated calculations and many mistakes were made, as you see in the final result of the example: 11 pennies ought to be 10 pennies.

Authors teach their readers to check their calculations, especially the check of nines appears often, but apparently this example was not checked. In the first part of the books sometimes extracting roots is dealt with also, and as said before, calculating on a counting board.

In the second part of the books elementary arithmetic is applied to solve all kinds of practical problems, on buying, selling or exchanging of goods, partnerships, changing money, calculating interest, insurance, profit, loss, etcetera. It is clear that it is useful for future merchants and technical, administrative or financial practitioners to learn to solve these. The most important rule to solve these practical arithmetical problems is the rule of three. This rule is used to find the fourth number in proportion to three given numbers. Because of its importance some authors introduce this rule in a richly decorated frame, see Fig. 7.4. This picture is from the arithmetic book by Van Halle (1568). The text says: 'The rule of three, how you can find the fourth number out of three numbers'. The other arithmetical rules are mostly variants of the rule of three.

If you want to solve a problem with the rule of three, you have to place the three given numbers on a line, multiply the last two numbers and divide the product by the first one. In Fig. 7.5 you see one of the many problems that is solved by the rule of three from the arithmetic book by Van Halle (1568). The problem is: "*If nine seamstresses can make fifteen shirts within one day, how many shirts can six seamstresses make*?" Van Halle places 9, 15, and 6 on a line and calculates $(15 \times 6) \div 9 = 10$ shirts.

Of course, there is a more appropriate way to find the solution of this problem. You can even solve this by doing mental calculations: if nine seamstresses can make fifteen shirts, three seamstresses can make five shirts and six seamstresses can make ten shirts. This is much easier, but this kind of clever alternative solution methods is hard to find in the old arithmetic books. The authors give only one solution method for each problem. They present standard algorithms that always work in the same

Fig. 7.4 The exuberant introduction of the important rule of three in the arithmetic manuscript of Van Halle (1568); Brussels, Royal Library, ms. 3552. fol. 60v (picture retrieved from Kool, 1999, p. 133)

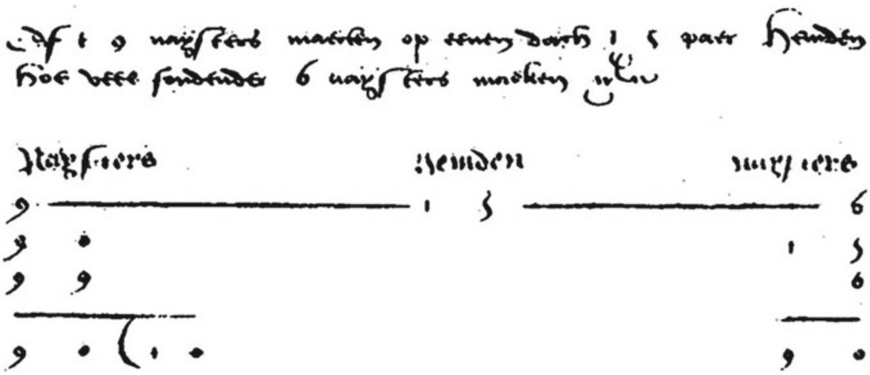

Fig. 7.5 One of the problems that is solved by the rule of three in the arithmetic manuscript by Van Halle (1568); Brussels, Royal Library, ms. 3552. fol. 70v (picture retrieved from Kool, 1999, p. 134)

way, followed by many problems to practise these fixed recipes. There are a few exceptions, which I will discuss later on.

The problem of the seamstresses is quite simple, but the books contain many problems that are (much) more complicated. As you can see in the following example from the arithmetic book by Van Halle (1568):

Three merchants are at sea and suddenly a violent storm arises. They have to throw overboard a part of their cargo. The value of this part is 100 guilders. In the end, they come home safely where they have to divide the loss. The first merchant had 300 guilders worth of cargo on the ship, the second had 400 guilders worth of cargo on the ship and the third one had 500 guilders worth of cargo in the ship. The cargo had a total value of 1200 guilders, of which 100 guilders was thrown overboard. What is the loss of each individual merchant? (Fig. 7.6).

Money changers had to solve problems like the one in Fig. 7.7, from the arithmetic book by Van der Gucht (1569):

A merchant from Florence went to the exchange bank in London in order to change $120\frac{1}{2}$ *ducats at* $42\frac{1}{4}$ *pennies each into angelots at* $66\frac{1}{2}$ *pennies each. The question is: How many angelots will he get in London? The calculation here is:* $(120\frac{1}{2} \times 42\frac{1}{4}) \div 66\frac{1}{2} = 76$ *angelots and the remainder is 594.*

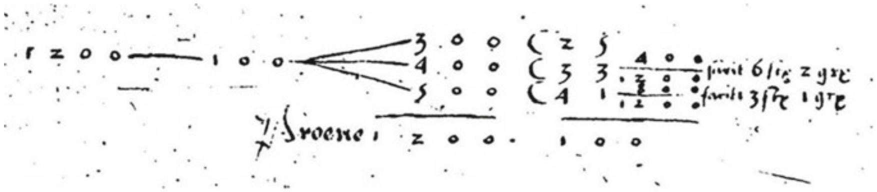

Fig. 7.6 Solution of the problem about three merchants who share the loss they had in a violent storm at sea; this problem is from the arithmetic manuscript of Van Halle (1568); Brussels: Royal Library, ms. 3552. fol. 97r (picture retrieved from microfilm)

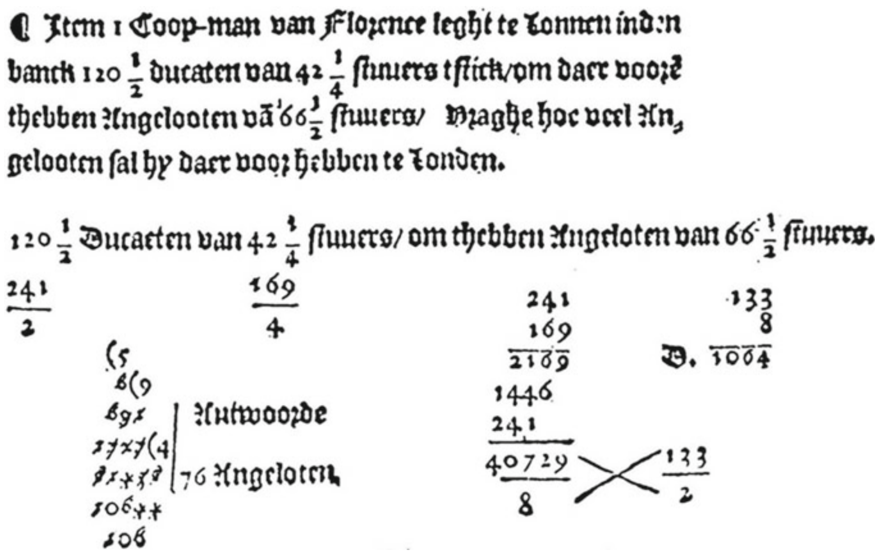

Fig. 7.7 A problem about changing money, from the arithmetic book by Van der Gucht (1569); Ghent, University Library, Acc. 1463. fol. 96r (picture retrieved from microfilm)

The authors of the sixteenth century arithmetic books only use words and numbers to describe problems and solution methods. In the first parts of these books the solution descriptions are very long and cumbersome, but further on in the books, as you can see in the Figs. 7.5, 7.6 and 7.7, authors use more concise, symbolic notations and try to limit the number of words. They use lines, crosses and other graphical means, and signal words with a special meaning, for example, the word ‘proeve’, which means ‘check’. These schematic presentations increase readability, are easier to learn by heart and reduce the risk of making mistakes. These efforts to shorten the presentation of calculations prepare the way for the later symbolic mathematical notation.

7.2.6 Didactic Principles in Dutch Arithmetic Books from the Sixteenth Century

If you study sixteenth century Dutch arithmetic books you can derive some didactic principles. Arithmetic skills are needed by merchants and financial, administrative and technical practitioners. To develop these skills the authors offer a limited number of standard algorithms to do arithmetic and rules to solve the practical problems they come across in their professions. They present one solution method for each problem type and to practise this method they give many similar problems that differ only in the numbers used. Repetition may help the pupil to remember the solution method. In some situations, alternative and more convenient solution methods are possible, but these are rarely shown. Probably the authors want to achieve that their students can use this method more or less ‘blindly’. They must become skilful reckoners. Repetition, practise and drill were the main principles of this education. You can recognise these principles, for example, when studying the tables of multiplication in the books. In the arithmetic manuscript of Christianus van Varenbraken of 1532, you see a 12 times 12 table with the exhortation to learn these tables “as well as your ‘Ave Maria’ without missing anything”. It shows that learning these tables was a serious matter, as important as learning prayers. An anonymous arithmetic manuscript of 1594 contains tables of multiplication even up to 17×27. The author of this manuscript likewise ordered his students to learn these tables by heart. And they probably did, because in a time without pocket calculators, in a society with very complicated systems of money, weights and measures, it will be useful to have many multiplications in your head, especially when you realise that paper was expensive at the time. Calculations were made on a slate. Arithmetic books were used by the teacher and mostly not available for students.

When considering the standard rules in sixteenth century arithmetic books, the practical problems, the many exercises to apply algorithms and fixed recipes, you can imagine that sixteenth century craftsmen became well trained reckoners within their profession. If they came across a new mathematical problem they probably did not know what to do, but that did not matter because they hardly came across new mathematical problems. They wanted to know the arithmetic of their profession and they had no need for learning mathematics.

7.2.7 Interesting Exceptions

In some of the sixteenth century arithmetic books there are problems that do not fit the previously sketched situation. These problems are not practical at all. They contain unrealistic stories and have nothing to do with money and commerce. For example, in the book written by Van der Gucht (1569) there is the following problem:

> *A man walks 11 miles during the day and at night he walks back for 3 miles. The question is in how many days he will reach Rome, if the distance to Rome is 500 miles.*

It is quite unlikely that a traveller to Rome would walk back three miles each night. How could this problem end up in a book with practical exercises? Tropfke (1980) discovered that variations of this problem already appeared in India in the ninth century, and also in the Arabic manuscript of al-Karagi (late tenth and early eleventh century), and you can find it in several European arithmetic books, including the *Liber Abaci* of Leonardo of Pisa from 1202. It turns out that most of the unrealistic problems in the sixteenth century arithmetic books are very old and appeared in different historical mathematical manuscripts. Their function in the sixteenth century books is not clear. Perhaps it is a matter of tradition, a kind of cultural heritage. Van Egmond (1980) and Tropfke (1980) think that these problems had a recreational function in the serious practical books, to break the routine. That seems plausible, because authors like Van Varenbraken (1532) and Stockmans (1595) call these problems 'problems for pleasure' and 'entertaining problems'. Van den Dijcke (1591) collected all these curious problems in a special chapter at the end of his book. He introduces this collection with: "Here you will find many different beautiful problems to sharpen and enjoy your mind."

Only a few of the arithmetic books have some of these unrealistic traditional problems. It is clear that sharpening and enjoying the mind of the readers was not a common or important purpose of the authors. These problems originally belonged to the old sources of the academic mathematical tradition and arrived perhaps more or less 'accidentally' in some of the commercial arithmetic books. You can imagine that an author saw a source with these entertaining problems and added a few to his own book to bring some variation, but it is clear that these problems may not distract the students too much from the main aim to learn practical and useful arithmetic.

There is a second unexpected phenomenon in some of the arithmetic books. The authors call it French or Italian practice. This is a collection of alternative arithmetic methods with which the arithmetician can speed up and simplify his calculations. But these methods only work in particular cases and with specific numbers. You cannot use them blindly and you need arithmetical insight to judge if it is possible and efficient to use these special strategies.

For example, in Fig. 7.8 you see a problem from the arithmetic book by Van der Gucht (1569): "*How many guilders can you have for 4321 nickels?*" To solve this problem you have to divide 4321 by 20. Van der Gucht advises to put aside the last cipher of the number and halve the remaining number.

Van Halle (1568) deals with problems like, "*If 16 m of cloth cost 99 guilders, what is the price of 128 m?*" Instead of the standard calculation with the rule of

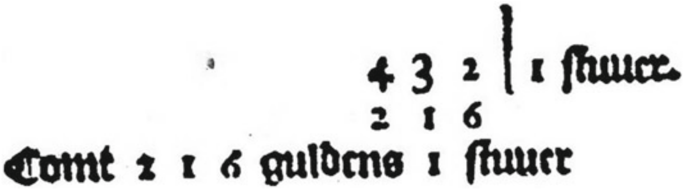

Fig. 7.8 A fast way to change guilders into nickels, from the arithmetic book by Van der Gucht (1569); Ghent, University Library, Acc. 1463, fol. 39v (picture retrieved from Kool, 1999, p. 162)

three $(99 \times 128) \div 16 = 792$, he advises to divide 128 and 16 by 8 first. Because then you have to calculate $(99 \times 16) \div 2 = 792$, which is much easier. He probably did not realise that he could simplify the problem even more by dividing 16 and 128 by 16, because then the remaining calculation is even more easier $(99 \times 8) \div 1 = 792$.

This type of insightful efficient calculation only plays a minor part in some of the arithmetic books. It is conceivable that experienced merchants used many strategies from the French or Italian practice in their daily work, but in the arithmetic books you hardly see them. The core business of the teachers was to practise and drill standard rules and fixed recipes, flexibility was learned during work.

7.3 Arithmetic in the Twenty-First Century

7.3.1 Comparing Sixteenth and Twenty-First Century Education

Let us make a giant leap to education in Dutch schools of the twenty-first century. It is not surprising that the differences with the sixteenth century business schools are huge! In our time, all children, including of course girls, go to school; this is not a privilege for sons of merchants and bankers. All students learn arithmetic as part of mathematics for at least ten years, with books of their own, pen and paper, tablets, laptops, smartboards, computers and calculators. The differences between sixteenth and twenty-first century education are huge and numerous, but there is one similarity between the teachers of the sixteenth and their twenty-first century colleagues, they both want to teach their students the arithmetic they need in daily life, in society and in their future profession. It seems that the teachers of the sixteenth century French schools were quite successful in reaching this aim. But concerning twenty-first century education, there is much discussion about the skills that our students need to acquire and the way that modern education can contribute to them.

7.3.2 Twenty-First Century Skills in General

Wagner (2008) speaks of an achievement gap between what schools (in the United States) are teaching and what is necessary for students to succeed in the current knowledge society. He argues that students have simply not been taught the competences that are most important for the twenty-first century. The skills that current and future professions require, differ significantly from what current education offers. Wagner gives the following list of what he calls, "the new survival skills": (1) critical thinking and problem solving, (2) collaborating and leading by influence, (3) agility and adaptability, (4) initiative and entrepreneurism, (5) effective oral and written communication, (6) accessing and analysing information, (7) curiosity and imagination. Wagner is not the only one who discussed this issue. The twenty-first century skills that we need to survive in our rapidly changing computerised and globalised society are discussed by many experts from inside and outside education, and they give lists comparable to that of Wagner.

7.3.3 Twenty-First Century Skills in Mathematics Education

Making a list of necessary twenty-first century skills is a good thing to start with, but the next question is what such a list means for the organisation and content of education, especially mathematics education. Gravemeijer (2012) concluded that critical thinking, problem solving, collaborating and communicating fit very well with problem-centred, interactive, mathematics education. These are also the aims of Realistic Mathematics Education (Van den Heuvel-Panhuizen & Drijvers, 2014) in which students get the opportunity to work in groups on meaningful problems guided and supported by their teacher. In this way, students try to reinvent parts of mathematics. Interaction, discussion, reasoning, asking questions and understanding are important features of this kind of education. In practice, it turns out that it is quite challenging to stimulate students to join actively in interactive problem solving and reasoning, because they are not familiar with it. Students need time to adopt new classroom social norms and to develop enough self-confidence to explain and justify their solutions, to try and understand other students' reasoning, and to ask questions when they do not understand something, and challenge arguments they do not agree with. It takes time to change the classroom into a research, annex learning, community (Cobb & Yackel, 1996). At the same time, it places high demands on teachers. They can no longer give ready-made solution methods, but have to develop students' reasoning to higher levels of understanding by fostering discussions and asking questions like: What is the general principle here? Why does this work? Does it always work? Can we prove that? Can we describe it in a more precise manner? Can we do this in another way? Etcetera. Creating a classroom atmosphere where students construct knowledge by learning from and with each other demands special qualities, competences and efforts of the teacher, but it is worth it.

7.3.4 The Content of the Mathematics Curriculum

Now we know what requirements the classroom culture must meet to develop twenty-first century skills, the next question is: What should be the content of the mathematics curriculum when computers take over mathematical routine tasks? Focusing on standard algorithms seems less important. The rise of computers in society and education places new mathematical demands on students. They have to learn to recognise the mathematical structure of situations and problems, they need to translate these problems into tasks that a computer or calculator can execute; this means quantifying and mathematising reality. So, students must have some idea of what quantifying (measuring) reality entails. Besides that, they have to understand what a variable is, and how to reason about interdependencies between variables, and finally they have to interpret and evaluate the output of the computer. This asks for mathematical topics such as measuring, tables, graphs, variables, models of relationships between variables, and elementary statistics (Gravemeijer, 2012).

The more we leave mathematical work to the computer, the more important it becomes that we control the computer output in a more or less approximate way. This asks for arithmetical skills to estimate calculations, based on networks of number relations and flexible and meaningful use of features of arithmetical operations. For example, if you want to check calculations such as: $4 \times 26 = 104$ and 13% of 888 = 115.44 it is useful to know number relations like $4 \times 25 = 100$ and 12, 5% equals 1/8. And if you want to check $7 \times 99 = 693$ it is good to know the distributive law $7 \times 99 = 7 \times 100 - 7 \times 1$.

You can check $8 \times 1.76 = 14.08$ by calculating 8×1.75. You know $8 \times 2 = 16$ and $8 \times 0.25 = \ldots$? You may think $8 \times 25 = 200$, so $8 \times 0.25 = 2$ or $8 \times ¼=2$, you will find $8 \times 1.75 = 16 - 2 = 14$. But you can also use the arithmetical rule of halving and doubling, like $8\times1.75=4\times3.5=2\times7=14$. It is clear that $8\times1.76=14.08$. These are just a few examples to show how you can use number relations and arithmetical rules in many different ways to check calculations. It is obvious that there is still much work to do in the arithmetic education of the twenty-first century, as it will be a big effort to equip students with sufficient flexible and meaningful arithmetic skills.

The contrast with the educational aims of the sixteenth century arithmetic books is enormous. Instead of recognising the type of problem and choosing the standard recipe to solve it, twenty-first century students have to mathematise a given problem situation, solve it with or without a computer or calculator and interpret and evaluate the output by checking it in an approximate way using flexible knowledge of number relations and arithmetical rules. Instead of recognising a well-known situation, our students need to recognise the mathematical structure of a new situation. Instead of using a ready-made solution method, our students need to construct a new solution method using the arithmetical knowledge and tools they possess.

We may not underestimate the arithmetical skills of the sixteenth century practitioners. They had to deal with complicated money, weight and measure systems. They learned fixed arithmetic recipes at school, and it is plausible that they became experienced in the flexible arithmetical tricks of the French and Italian practice during their working life. They were not taught to deal with new arithmetic problems, but they were experienced, flexible reckoners within the borders of their profession. Learning arithmetic to solve applied problems is part of the Dutch didactic tradition until today, but the nature of the applied problems changed during the years and that asked for different knowledge and skills.

The twenty-first century asks for problem solvers, people who can apply their arithmetical knowledge to unknown problems in new situations. At first glance, the computerisation of society makes life easier and more comfortable compared to the sixteenth century. We no longer have to use long and cumbersome arithmetic algorithms. But when you realise what our society asks from its members it is clear that the aims of arithmetic education are much more challenging than they were in the sixteenth century.

In spite of that, we can learn two important things from the sixteenth century. School is not the only place where you can learn things. After school, in your professional life, learning is still going on. In recent years the lifelong learning concept has gained adherents because people realise that it is impossible to reach all goals at the required level in school. That means that we have to make choices in our arithmetic education. The arithmetic books of the sixteenth century make a suggestion. Equipping students with a solid basis of arithmetical knowledge seems a valuable starting point.

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