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Marja Van den Heuvel-Panhuizen *Editor*

National Reflections on the Netherlands Didactics of Mathematics

Teaching and Learning in the Context of Realistic Mathematics Education

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Editor

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Preface

This volume is part of the ICME-13 Monographs and is a spin-off of the Netherlands strand of the ICME-13 Thematic Afternoon on "European Didactic Traditions" held in Hamburg in 2016. In this session, four European countries—France, Italy, Germany, and the Netherlands—presented their approach to teaching and learning mathematics in school and in research and development. The session inspired mathematics didacticians familiar with Dutch mathematics education to reflect on the approach to teaching and learning mathematics education in the Netherlands and the role of the Dutch domain-specific instruction theory of Realistic Mathematics Education. This resulted in two volumes: *International Reflections on the Netherlands Didactics of Mathematics—Visions on and Experiences with Realistic Mathematics Education* and *National Reflections on the Netherlands Didactics of Mathematics—Teaching and Learning in the Context of Realistic Mathematics Education.*

The current volume is the *National Reflections* book. The volume describes the Dutch approach to teaching and learning mathematics and is written by Dutch people. The authors of these "reflections from inside" have in various ways built up a hoard of expertise on this, either as a mathematics teacher, a mathematics teacher educator, a school advisor, or as a developer and researcher of instructional material, textbooks, teaching–learning trajectories, curricula, and examinations and tests. In 17 chapters, 28 authors reflect on mathematics education in the Netherlands and when doing this they have a broad scope. Several chapters discuss aspects of the theoretical underpinnings of the Dutch approach that, starting some 50 years ago, became rather dominant in the Netherlands, and that is known as Realistic Mathematics Education. Other chapters go back further in history or use history in their teaching of mathematics, or zoom in on changes in particular subject matter domains and in use of technology. One chapter shines a light on the relationship between Dutch mathematicians and mathematics education. Other chapters give a glimpse into the process of innovation and how the Dutch and in particular one Dutch institute have worked on the reform. To place these reflections from inside in the context of the Dutch educational system, the volume also contains chapters that explain how teacher education and testing in mathematics education are organised in the Netherlands.

Of course, all the chapters in this volume together are not enough to give a full picture of the Netherlands didactic tradition. Other people might have told other experiences and might have other views, but the authors of this volume shared their knowledge about mathematics education in the Netherlands by writing a chapter about it. Thanks to their inspiring pieces of work, the volume could come into existence. However, especially instrumental for making this happening was Nathalie Kuijpers, who together with me checked and double-checked all the texts. Many, many thanks for this.

Utrecht, The Netherlands Marja Van den Heuvel-Panhuizen
May 2019

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Chapter 1
A Spotlight on Mathematics Education in the Netherlands and the Central Role of Realistic Mathematics Education

Marja Van den Heuvel-Panhuizen

Abstract In this introductory chapter I give a preview of the landscape of issues concerning mathematics education in the Netherlands and the role of Realistic Mathematics Education (RME) that one can come across in this volume, which contains the reflections of twenty-eight Dutch mathematics didacticians on teaching and learning mathematics in the Netherlands. Although all chapters have their own focus and mostly only discuss one particular aspect, together they provide a rich inside view into what is worth knowing of Dutch mathematics education and RME. The preview highlights some significant topics from these chapters, such as what tasks are preferred in RME to elicit students' mathematical thinking, RME's focus on the usefulness of mathematics, the role of common sense and informal knowledge, changes over time in the content of the mathematics curriculum, aspects of the Dutch educational system, including teacher education and assessment, the implementation of RME, and the context of developing RME.

1.1 Introduction

The 13th International Congress on Mathematical Education (ICME-13) held in Hamburg, Germany, in 2016, and in particular the ICME-13 Thematic Afternoon session "European Didactic Traditions," was a trigger for Dutch mathematics didacticians to reflect on what is typical for mathematics education in their country. In this session, the Dutch approach to teaching and learning mathematics in school, in research, and in development was presented, together with the approaches in France, Italy, and Germany. The aim of the session was to delve into what the four countries

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have in common despite the differences in the cultural, historical, and political circumstances in which their positions and methods regarding mathematics education were developed. The common characteristics that came to the fore and that can be considered as distinctive features of the European didactics of mathematics, were: “a strong connection with mathematics and mathematicians, the key role of theory, the key role of design activities for learning and teaching environments, and a firm basis in empirical research” (Blum et al., 2019, p. 2). These are also the features that recur in the reflections on mathematics education in the Netherlands as described by the twenty-eight Dutch mathematics didacticians in this volume. This places the Dutch didactic tradition inalienably inside the European didactic tradition. Yet within this overarching European framework, Dutch mathematics education and its theoretical grounding have their peculiarities. In the Netherlands, the teaching and learning of mathematics cannot be seen separate from Realistic Mathematics Education (RME), the domain-specific instruction theory that has determined Dutch mathematics education in the last half-century. Therefore, in the reflections presented in this volume, the defining characteristics of RME have a prominent place. In addition to this, ample background information is provided about the educational system in which RME has come into being. In their descriptions, the authors have each their own focus in addressing particular aspects of mathematics education in the Netherlands, and of course, their reflections resonate their own views on RME. They gave their own accentuations and interpretations, which is fully in line with the idea that RME is not a fixed and unified theory of mathematics education.

As an introduction to this multifaceted portrayal of mathematics education in the Netherlands and the central role of Realistic Mathematics Education, in this preview I highlight some of the main thoughts that emerge from the chapters. Underlining these thoughts does not in any way imply that what is characterised as typical for the Dutch approach, is unique in the world of mathematics education. All over the world reforms of mathematics education have taken place and are still happening, and the innovations in the Netherlands have very much in common with those in other countries. In this sense the Dutch reformed ideas on mathematics education are not special.

1.2 The Focus on a Particular Type of Tasks

Several chapters in this volume discuss tasks that should be given to students to elicit mathematical thinking. Preferably, these are tasks that provide students with opportunities to creatively solve unfamiliar open-ended problems, to model, structure and represent problems and solutions, and to work collaboratively and to communicate about mathematics. Tasks that are exemplary for making this happen are described by Wijers and De Haan (Chap. 2). Their experience is that such tasks should be rich, meaning that there is not just one way to come to a solution. Further requirements are that the solutions can vary in mathematical depth, that the tasks build on knowledge students already have and that they offer students opportunities to extend their knowledge. Also important is that higher-order questions are used which ask *how* and *why*, encouraging reasoning rather than getting an answer. In all these requirements, the very nature of RME is clearly apparent, but what Wijers and De Haan also point out is that these requirements not only apply to problems that are close to the real world, but also to assignments that are situated more within the world of mathematics. Besides tasks in which students, for example, reason about the productivity of workers in a factory in connection with the hours they work without having a break, students also work on tasks in which they have to deal with formulas in a quite abstract context, such as dots moving on a grid. The latter type of tasks can, in RME, also be called context problems.

The broad meaning of context problems is clarified in full detail by Vos (Chap. 3). In her fine-grained categorisation of tasks, she distinguishes, apart from bare tasks (tasks without contexts), tasks with mathematical contexts (e.g., matchstick pattern problems), dressed-up tasks (tasks with a pointless question behind which a mathematical question is hidden), tasks with realistic contexts (which are experientially real or imaginable for the students) and tasks with authentic contexts (which use photos, data, and situations from the real world). What the two last types of tasks have in common is that the context justifies the questions that are asked and that the answers to these questions are useful within the described context. For Vos the 'usefulness' of tasks means that they lead to developing the competence and understanding required for using and applying mathematics in future practices as professional or as citizen.

1.3 Usefulness as a Key Concept

The idea of teaching mathematics to be useful was and is a strong driving force for developing mathematics education in the Netherlands. Even before there was RME, Freudenthal made a strong plea for this idea in his article "Why to Teach Mathematics So As to Be Useful" published in 1968 in *Educational Studies in Mathematics*. As De Lange (Chap. 17) underlines, at the time of the rise of New Math—which was around the late 1960s—this was a very relevant question. Yet putting usefulness in the centre of our thinking on mathematics education was not new. The culture of usefulness of mathematics as a curricular emphasis has already existed in the Netherlands for five hundred years, and may, according to Vos (Chap. 3), have created a fertile ground for RME.

Concrete examples of the propensity to adhere to the usefulness aspect of mathematics and instances of the deep historical roots of this tendency are presented by Kool (Chap. 7). Her chapter goes back to Dutch arithmetic education in the 16th century. In that century, calculations were initially made with coins and a counting board, but as the result of the more complex trading methods that entered the market then, this cumbersome way of calculating was gradually replaced by a more advanced written calculation method. Many manuscripts and books were published to teach this new method to future merchants, moneychangers, bankers, bookkeepers, and craftsmen. By means of many tasks about all kinds of commercial transaction and other calculations to be done in various workplace situations, students could learn to solve arithmetical problems of their future profession. This was the main goal of arithmetic education in those days, which was accompanied by devoting much attention to memorising rules and recipes, tables of multiplication and other number relations. When comparing this approach to mathematics education with the current Dutch approach, Kool concludes that teachers of the 16th and the 21st century both want to teach their students the arithmetic they need in daily life and their future profession. As in the 16th century, today's students in the Netherlands need to have knowledge about number relations and arithmetical rules, but different is that they have to learn to apply this knowledge in a flexible way, whereas in the 16th century it was all about using ready-made solution methods.

The relation between mathematics and its usefulness in real-world situations is also shown in the teaching experiment on measurement carried out by Van Gulik-Gulikers, Krüger, and Van Maanen (Chap. 13). What is more, the tasks they have designed for teaching this topic to eight- and ninth-grade students demonstrate that the contexts can also date from three centuries ago. The teaching material they used for this experiment is based on the professional context of a Dutch land surveyor in the 18th century measuring the height of buildings and the width of rivers. Comparable to the surveyors in those times, the students had to use the theory of similar triangles. Of course, nowadays it is common in such situations to use GPS, from which the students can learn as well, but the experiment showed that using the history of mathematics as a didactical tool had a positive effect on the students' motivation and on their conceptual understanding. In particular, the authors found that the transparency of this old-fashioned measurement method made discussions about mathematics accessible.

1.4 Common Sense and Informal Knowledge

The RME characteristic of connecting mathematics education to reality is closely related to the reinforcement of the role of common sense and using informal mathematical knowledge from daily-life experiences as a starting point for teaching. Dekker (Chap. 4) calls this 'the Dutch school' and describes a silent revolution that has taken place at this point in the Netherlands. There is a large difference between what she remembers from the start of her first mathematics lesson as a secondary school student and what students often hear nowadays. Then it was 'forget what you know, here you will learn all sorts of new things', whereas now the motto is 'use your common sense'. Students acquire a lot of mathematical knowledge in the realistic context of their life, and education should make use of this informal knowledge. In this respect, Dekker refers to the pioneering work of Ehrenfest-Afanassjewa, a Russian mathematician who worked in the Netherlands and in 1932 published a course on geometry based on the idea that students have already developed intuitive geometrical notions in reality. These intuitive notions were taken as the starting point of this course. Dekker describes that many people involved in mathematics education were shocked by Ehrenfest's radical ideas. However, this was not true for Freudenthal who was impressed by her revolutionary approach, and stimulated developers of instructional materials to take over these ideas. Also, several other chapters make a point of this shift in teaching geometry, and mention the important role of Ehrenfest-Afanassjewa for Dutch geometry education (see Chaps. 5, 9, 11 and 15).

A question that is inevitable here and asks for discussion is where these intuitive notions and informal knowledge come from, or what common sense is. De Lange (Chap. 17) gives a first-hand peek into Freudenthal's thoughts about this, when he describes a discussion that took place at the Freudenthal Institute between Freudenthal and a number of staff members. Freudenthal was writing a new article meant for what would become his last book. According to *the professor* mathematics is rooted in common sense; for example, your common sense reasons that $2 + 3$ is 5 and that the area of a rectangle is $h \times b$. After he said this, the discussion continued. Someone questioned whether it is really true that '$2 + 3 = 5$' and 'area is length × width' are common sense. Finally, it was concluded: common sense is local, both in time and place, and it includes reasoning. Freudenthal mumbled something, not audible for the others, and decided that he would rewrite his draft.

1.5 Mathematical Content Domains Subject to Innovation

As a constituent of the reform that took place in the Netherlands, the content of the mathematics curriculum changed in many respects. Several chapters pay attention to these changes. For example, Doorman, Van den Heuvel-Panhuizen, and Goddijn (Chap. 15) shed light on the change that happened in geometry education. Here an axiomatic approach to teaching geometry was gradually superseded by an intuitive and meaningful approach focussed on spatial reasoning. Supported by Freudenthal—who was in his turn inspired by Ehrenfest-Afanassjewa and Van Hiele-Geldof—from the 1970s on, experiments were carried out within a new content domain, called 'vision geometry'. Characteristic of this RME-based geometry education is that, together with the introduction of this new content, the structure of the geometry trajectory was also changed. Traditionally, the structure in a teaching-learning trajectory for geometry was provided by a deductive system starting with formal definitions and basic axioms. This 'anti-didactical inversion' of the learning sequence, as Freudenthal called it, means that the final state of the work of mathematicians is taken as a starting point for mathematics education. In RME, the reverse order is followed, in which geometry education starts with offering students geometrical experiences based on observing phenomena in reality. Through explorative activities, geometrical intuitions develop further, and mathematisation is elicited, resulting in the development of situation models like vision lines, which eventually bring the students from informal to more formal geometry. The concepts and reasoning schemes that emerge from this 'local organisation'—again a term introduced by Freudenthal—have the potential to create, for students in the more advanced levels of secondary education, the need for axioms, definitions and mathematics as a logic-deductive system.

Another content domain that was subject to innovation in the Netherlands was calculus. Kindt (Chap. 14), who takes the reader along the history of how calculus developed over time, characterises this innovation process as balancing between conceptual understanding and knowing algebraic techniques—a process which is in fact indicative for the development of RME as a whole. Starting in the 1960s, attempts have been made to develop calculus courses that start with an introduction that is meaningful for the students. The idea was to give students a broadly oriented entrance to differential calculus by starting with a problem about rate of change in a context that made sense to the students, such as a cheetah and a horse that were both running. The students had to answer the question: Does the cheetah overtake the horse? Later on, this RME approach in which a long conceptual introduction with open tasks precedes the teaching of algebraic rules, did not always appear in the textbooks, which were mostly more structured and less challenging than the experimental units. Nevertheless, the current situation is that important elements of this approach, in which attention is paid to exploring linear and exponential relationships in meaningful contexts with tables and difference diagrams, can still be found in Dutch textbooks.

The implementation of the RME-based reform in lower and pre-vocational secondary education described by Hoogland (Chap. 11) which began in the 1990s, and which was meant to move from mathematics for a few to mathematics for all, also implied many changes in the curriculum. The reform affected all elements of mathematics education in secondary schools, including a new and broader curriculum, alternative ways to approach students, fostering students to develop more and other skills such as problem solving, and using different assessment formats such as contextual and open-ended problems. Within the domain of algebra, the emphasis shifted from algebraic and computational manipulation to reasoning on the relationships between variables and to flexibility in switching between different types of representations of relations. In geometry, there was a change from two-dimensional plane geometry with a strong calculational approach, towards two- and three-dimensional geometry with a focus on 'vision geometry'. Numeracy was introduced as a new domain in secondary education, as were data handling, and statistics containing data collecting and visualisation to be used in decision making.

Apart from changes in the mathematical content that occur together with a new RME-based thinking about teaching and learning mathematics, changes, or at least prompts to rethink the practice and theory of mathematics education, were also induced by the new technologies that became available for education. This issue is addressed by Drijvers (Chap. 10), who discusses the relationship between mathematics education in the Netherlands and digital tools. He shows what it means to implement new technologies in RME-based education and concludes that the match between the two is not self-evident. Technology puts the teaching of mathematics in another perspective. Among other things, Drijvers points out that the phenomena that in RME form the point of departure for the learning of mathematics may change in a technology-rich classroom. Also, the teaching approach of guided reinvention may be challenged by the often rigid character of the digital tools. And finally, the use of digital tools for higher-order thinking was found to be more complex than foreseen. According to Drijvers, to realise mathematics education as intended by RME, it is necessary to have a digital mathematics environment that allows the teacher to design open and engaging tasks, and enables students to explore and express mathematical ideas in accessible and natural ways.

The complexity of the issue of what mathematics should be taught, and changing ideas about this are signified by Treffers and Van den Heuvel-Panhuizen (Chap. 15) by retracing the content of the domain of number in two centuries of Dutch primary school mathematics textbooks. In their chapter, in which they cover the period from 1800 to 2010, they describe the longitudinal process featuring seemingly inevitable pendulum movements of procedural versus conceptual textbooks. Generally speaking, in the procedural textbooks the focus is on practising calculation procedures with less attention paid to conceptual understanding of number. Operations have to be carried out in a fixed way. Smart, flexible (mental) calculations and estimating are mostly absent in this approach. Finally, in the main, applications are not used until the very end of the teaching trajectory. The RME-based textbooks that appeared in the 1980s belong to the conceptual textbooks, and are the opposite of the procedural textbooks. Although the distinction between these two textbook types is rather coarse-grained, in most cases, RME-based textbooks start teaching in the domain of numbers and operations with applications and the use of contexts that evolve into models to support the development of calculation strategies. Number sense, number relations, flexible (mental) calculation, and estimation have a central place in the programme next to algorithmic calculations, which are introduced by transparent predecessors of the algorithms. This means, for example, that the digit-based algorithm of long division is prepared through a whole-number-based repeated subtraction approach. Contrary to the commonly held thought that mathematics education of some 100 years ago implies a traditional approach to teaching which focusses on drill-and-practise and fixed rule-governed solution strategies, the analysis of two centuries of mathematics textbooks reveals that this assumption is not correct. Already in 1875, Versluys, a mathematics educator who is considered the founding father of the Dutch didactics of mathematics, published a textbook in which the focus was on insightful, self-inquiry-based learning of mathematics within a whole-class setting guided by the teacher. Also, the way Versluys treated calculations up to one hundred has a lot in common with how this is now dealt with in RME textbooks. Furthermore, to a certain degree similar to RME, Versluys' textbook series contains a large amount of word problems and a rather small number of bare number problems. For Versluys, arithmetic is in the first place applied arithmetic. Again, the deep roots of RME are shown here. What is now considered new in some forums (and is therefore sometimes rejected) is, in some way, in essence not new at all. This is also enlightened by Kool (Chap. 7).

1.6 The Systemic Context of Dutch Education

To comprehend the nature of a country's mathematics education, it is necessary to view this education in its national context and have knowledge about how that country's school system is structured, how teachers are educated, how assessments and evaluations are organised, what the role is of the government and the institutions that deliver support services to schools, what the contribution is of teacher associations and what the position is of the publishers of educational material. It goes beyond this volume to give a complete picture of the Netherlands for all these systemic issues, but two issues which are specifically addressed are teacher education (in Chap. 8 by Oonk et al. and in Chap. 9 by Daemen et al.), and assessment in mathematics education (in Chap. 16 by Scheltens et al.). Furthermore, spread out across the volume other aspects of how education is organised in the Netherlands are also discussed. Without being exhaustive, it can be mentioned that information is provided about: the school system of the Netherlands (in Chap. 9 by Daemen et al. and in Chap. 11 by Hoogland), the different mathematics curricula for different school levels (in Chap. 2 by Wijers et al., Chap. 3 by Vos, and Chap. 11 by Hoogland), examination in secondary education (in Chap. 2 by Wijers et al. and in Chap. 14 by Kindt), the textbooks that are used (in Chap. 3 by Vos and in Chap. 6 by Treffers et al.), and about governmental committees and teacher associations (in Chap. 5 by Smid).

If we look at teacher education, we see a dynamic relationship between the approach to educating teachers and the reform movement in the Netherlands. This particularly applies to the primary school level of mathematics education, because primary school teacher educators were heavily involved in the development of the reform. Therefore, parallel to the changes in primary mathematics education, the curricula of primary mathematics teacher education have drastically changed since the 1970s. What this change means is thoroughly outlined by Oonk, Keijzer, and Van Zanten (Chap. 8). They point out that, with respect to mathematics, primary school teacher education, where students are educated to teach all subjects in primary school, can be characterised as including both a focus on the interconnection between mathematics and didactics, and on the integration of theory and practice. What is more, the developed teacher education theory for primary school mathematics teacher education is largely in line with the RME theory for teaching students in school. This parallelism comes to the fore in the approach to teaching teacher students and teaching students in primary school. For both, concrete mathematical situations are taken as a starting point. For primary school students it means to activate their intuitive notions and start with informal procedures, which, under the guidance of the teacher, can evolve to more formal mathematics. The teacher students start their learning to teach mathematics by carrying out mathematical activities at their own level. Subsequently, their own experiences in learning mathematics are combined with reflections on the learning processes of students. Together, these give them a basis for teaching mathematics. By analysing and discussing real teaching practices and describing their own reflections on these practices, student teachers are prompted to use theoretical ideas and terminology from the didactics of mathematics, and teach mathematics in a professional way. As a result, practical knowledge can develop into so-called ‘theory-enriched practical knowledge’.

Compared to primary school teacher education, teacher education for secondary mathematics teachers is far more complex. In this respect, the overview given by Daemen, Konings, and Van den Bogaart (Chap. 9) speaks volumes. Although in one respect secondary teacher education is less complicated than teacher education for primary school, because the focus can be on one subject, the complicating factor comes with the situation that in secondary education there are different school levels and different types of schools, including general education and all kinds of vocational education. This means that there are different routes for qualifying as a secondary education mathematics teacher. For the highest levels of secondary education student teachers go to university. For the other levels they go—like most student teachers for primary school—to colleges for higher vocational education, nowadays called universities for applied sciences. All school levels have their own teacher education programme, which has to prepare student teachers for teaching secondary school students of different capability levels and teaching, to a certain degree, different mathematical content. To prevent the learning process to be too fragmented, much effort is put into working with profession-related tasks which follow a ‘whole-task’ model. Such a task could include, for example, designing a lesson or a test, or designing a lesson series that one has to carry out. Through these profession-related tasks, the aim is to achieve coherence between theoretical courses and practice-oriented activities.

A determining element of the systemic context of Dutch education is the system of assessment and evaluation. This is highlighted by Scheltens, Hollenberg, Limpens, and Stolwijk (Chap. 16), who are affiliated to Cito, the Netherlands national institute for educational measurement. In their chapter, they provide an outline of the tools that are available in the Netherlands for informing schools, teachers, and students about the learning achievements in mathematics for both formative and summative purposes. They describe the content and goals of the various national primary and secondary standardised tests, and illustrate their descriptions with samples of test items. Moreover, they also include examples of examination tasks, for which they also offer the marking guidelines. The overview shows that the picture of official assessment in the Netherlands—that means the assessment commissioned by the government—looks rather diverse. The tests and examinations contain context-based open tasks, but also multiple-choice tasks and bare mathematical tasks. Similarly to what can be seen in the textbooks, the reality of assessment shows a quite moderate version of the big ideas of RME. This, again, is an act of balancing between different approaches to mathematics education and between different interpretations of RME.

1.7 The Implementation of RME

Although the government to a certain degree facilitated the development of RME by establishing institutions and commissions and by giving grants for projects for doing research on mathematics education, developing new instructional materials, and organising professional development for teachers, the reform cannot be labelled as a government-instigated enterprise. This is at least not the case for primary school mathematics education. In secondary education, there was more government interference in connection with decisions made about the central examinations at the end of secondary school.

A major government-paid implementation project in lower and pre-vocational secondary education taking place in the 1980s and 1990s is described by Hoogland (Chap. 11). In this project, the Ministry of Education made funds available for pilot schools and the development of experimental teaching materials, as well as making possible the change of the formal curriculum and the final examinations for secondary vocational education in the examination year 1996, which they did with broad support from parliament. For the teachers in the pilot schools, the most common way to communicate the curriculum changes was through discussing exemplary tasks of the final examinations and comparing 'old' tasks with 'new' tasks. Characteristic of the whole implementation project was the broad involvement of all relevant stakeholders. In addition to teachers, students, parents, editors, curriculum and assessment developers, teacher educators, publishers, media and policy makers were also part of it, and a continuous and extensive dialogue took place among them. Also, the spirit of that time was an important factor in this implementation process. In education and society there was a general feeling that change was necessary. There was an agreed focus on equity and basic education for all, including mathematics, and at the same time there was a commitment not to waste the human potential in mathematics, in particular not that of girls. Another factor that contributed to the implementation was the use of so-called 'advocate teachers'. These were teachers at the pilot schools who acted as advocates for the reform and had an important role in the professional development activities. Other important change agents were the in-service and pre-service teacher education institutions, the publishers, and the education inspectorate, who all supported the chosen vision or were at least benevolent to the change. As Hoogland indicates, the intended changes have proven to be quite sustainable, since the current mathematics textbook series and final examinations still reflect essential tenets of the original vision. At the same time, however, he makes it clear that the change is very vulnerable, by referring to the debate and the framing in social media that started in the first decade of this century, which claim that the educational change in the 1990s is to blame for the alleged low level of mathematics of today's students.

Besides large projects purposely set up to introduce RME in school practice, Wijers and De Haan (Chap. 2) illustrate that extra-curricular mathematics competitions and events, such as the Mathematics A-lympiad, the Mathematics B-day, the Lower-Secondary-Mathematics-Day, and the National Mathematics Day for primary education, can also form a springboard for innovation. For example, when teachers have to prepare their students for the Mathematics A-lympiad competition by giving them opportunities to get experience in working in groups on rich open-ended unfamiliar problems which require mathematical reasoning and modelling, the influence can also work in the other direction. Experience with these competitions which contain other types of problems than the regular textbook problems can prompt teachers to change their regular teaching of mathematics. This means that in this way these competitions and events can become an implementation instrument.

Speaking about ways to implement RME raises the question of what was accomplished of the ideas of RME in Dutch classrooms. Similar to other questions that can emerge when thinking about mathematics education in the Netherlands, this volume cannot give a full answer. In general, most authors indicate that the ideas of RME are unmistakably recognisable in Dutch mathematics education, but in a number of chapters, there are also clear concerns about deficiencies in the implementation. One thing that is rather often mentioned is the difference between what are considered good tasks to elicit mathematical thinking in students and the tasks which can regularly be found in textbooks, the production of which is left to the market in the Netherlands. As Wijers and De Haan (Chap. 2) describe, if open problems are included in textbooks, these mostly refer to the core content of the lesson or the chapter at hand. This means that students do not need to model the problem situation to find a strategy for solving the problems, because the strategy is the one that has been treated in the chapter. The findings of Vos (Chap. 3) when she analysed a textbook chapter and a sample of examination tasks also highlighted that quite a number of tasks in the textbook were dressed-up tasks offering students training to find formulae. Also, many artificial contexts were used, which contrasted with the finding that the examination tasks contained authentic contexts more often. The difference between what RME stands for and what is offered in textbooks was already clearly brought to the fore in the first decade of this century, when it was found that primary school textbooks mostly contain straightforward calculation problems and that opportunities for real problem solving and mathematical reasoning are almost completely lacking. To the same conclusion Gravemeijer (Chap. 12) came. He observed that advanced conceptual mathematical understandings are not formulated as instructional goals, neither in the textbooks, nor in official curriculum documents, and that textbooks capitalised on procedures that can quickly generate correct answers, instead of investing in the underlying mathematics. It is clear that the ideal situation differs from what is actually realised in reality!

This discrepancy also applies to another essential requirement that should be fulfilled in order to realise RME in practice and bring it to fruition, namely a change in classroom culture. One of the cornerstones of RME is that a learning environment should be created that makes guided reinvention possible, in which students can come up with their own solutions and discuss these with other students. Offering students rich open-ended problems that they can work on collaboratively and through which they have opportunities to express their thinking, only works when there is a classroom atmosphere which really stimulates students to communicate about mathematics. Implementing RME in class requires that justice is done to RME's activity principle (treating students as active participants in the learning process) and its interactivity principle (using interaction to evoke reflection and bring students to a higher level of understanding).

However, as indicated by some of the authors, this RME classroom culture has not been entirely successfully implemented. Kool (Chap. 7) explains that in practice it has turned out that it is quite challenging to stimulate students to join actively in interactive problem solving and reasoning, and it places high demands on teachers. Providing students with ready-made solution methods will no longer do. Instead, teachers have to ask their students thought-provoking questions such as “Why does this work?” and “Does it always work?”. At the same time however, the teacher should work on a classroom atmosphere in which the students feel confident enough to explain and justify their solutions, to try and understand other students’ reasoning, and to ask questions when they do not understand something, and challenge arguments they do not agree with.

Also, Van Gulik-Gulikers et al. (Chap. 13) experienced in their teaching experiment about the 18th century land surveyor that the students were not used to a situation in which they had to delve deeply into problems that require more fundamental thinking, broader exploration and endurance. According to Van Gulik-Gulikers et al., this unfamiliarity with such problems may be because, in their regular classes, students often work independently in their textbook, which makes that these complex tasks are often skipped or split into a number of small parts that are easy to digest. This kind of practice is not what one would expect when thinking of RME-based teaching.

The strongest concern about the implementation of a new classroom culture as one of the core aspects of RME is voiced by Gravemeijer (Chap. 12). Also, he thinks that the innovative point of RME to offer students an inquiry-oriented learning environment with many opportunities for interaction and collaboration did not have a systematic elaboration at classroom level. Based on what recent research has revealed about the instructional practice in the Netherlands, according to Gravemeijer the question can even be asked how RME actually works out in Dutch classrooms. For him the solution is that RME should adopt a socio-constructivist approach.

1.8 The Context of Creating a New Approach to Mathematics Education

In the Netherlands, compared to other countries, the reform of mathematics education that started at the end of the 1960s and eventually resulted in RME was mainly a bottom-up process with low government interference. That this reform happened in this way is in essence a consequence of the Dutch constitutional ‘freedom of education’ that is laid down in the Constitution of 1917. This law was originally meant to give parents the right to found schools in accordance with their religious views. Nowadays, this law implies also that schools can be founded based on particular pedagogical and instructional approaches. Another result of this freedom of education is that the government is rather hesitant in giving instructional prescriptions. In fact, the Ministry of Education can only prescribe the subject matter content to be taught and not the way in which this content is taught. This means that textbook authors and publishers have much opportunity to include their own views and ideas on teaching mathematics. Moreover, there is no authority which recommends, certifies or approves Dutch textbook series before they are put on the market.

What is also different in the Netherlands than in most other countries is the position of mathematicians. As is clearly underlined by Smid (Chap. 5), Dutch mathematicians have a rather problematic relationship with mathematics education. This means that on this point the Netherlands deviates from what is considered a distinctive feature of the European tradition. Except for Freudenthal, mathematicians did not have a determining role in the mathematics curriculum. From the 1970s on, the role of the mathematicians and their organisations in school mathematics was minimal, and they hardly seemed interested. This changed only in the first decade of this century, when mathematicians discerned a lack of algebraic skills in first-year university students. Moreover, due to unsatisfactory achievement scores of Dutch primary and secondary school students in national and international studies, a public debate emerged about the quality of education, which caused that the government took on more of a steering role. One measure that was taken to assure that all students acquired a certain basic level in mathematics and particularly in arithmetic, was that the government decided that both secondary education students and primary school teacher students had to do a compulsory arithmetic test. Furthermore, recently the Ministry of Education installed a platform and a number of development teams with representatives from primary and secondary education for enacting a society-broad reconsideration of what students should learn in school to equip them for the future society, their later profession and their personal development. Asking people from school practice, along with other experts, to think about the future curriculum is again a kind of bottom-up approach, yet it is different from what begun half a century ago.

The reform that started at the end of the 1960s was in many ways a child of its time. Just as the society of that time was ripe for a change, meaning that the existing values and way of living were turned upside down, the renewal of Dutch mathematics education also showed characteristics of a certain anarchist stance. In the initial period of the reform, this manifested itself for example in the production of texts in which an alternative spelling was used. ‘Equivalentie-klassen’ (equivalence classes) became ‘ekwivalentie-klassen’ and ‘mate van exactheid’ (degree of exactness) became ‘mate van eksaktheid’, and capitals were left out in names and titles of books and chapters. This atmosphere of wanting to be innovative that was characteristic for IOWO (Institute for the Development of Mathematics Education) and OW&OC (Mathematics Education Research and Educational Computer Centre) has lingered long in the Freudenthal Institute. De Lange’s (Chap. 17) reflection unmistakably shows the traces of this ambiance. He characterises the institute as different, sometimes provocative, but often innovative with vision and carefully bombarding the Ministry of Education with an array of novel ideas such as new curricula, new software, mathematics for all, A-lympiads, cutting edge conferences, and international collaboration. In the words of De Lange, there was never a dull moment. In this way the Freudenthal Institute and its predecessors were for a long time the epicentre of the reform, with Freudenthal as the authority to make it all happen.

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Chapter 2
Mathematics in Teams—Developing Thinking Skills in Mathematics Education

Monica Wijers and Dédé de Haan

Abstract Mathematics is more than just basic skills. Mathematical thinking should be an important aspect of mathematics education. In the Netherlands, higher-order thinking skills like mathematical problem solving, reasoning, modelling and communicating mathematics have been part of the examination program since 1989. To assess these skills in an authentic and open way, the Mathematics A-lympiad, a competition for teams in upper secondary school, was designed. Shortly hereafter a Mathematics B-day was developed which showed that open-ended tasks for teams can also be designed within the domain of pure, formal mathematics. As a result of the success of the Mathematics A-lympiad, similar activities have been created for lower secondary and for primary school. The Mathematics A-lympiad assignments fulfil specific requirements, such as being accessible for all students, eliciting mathematical thinking and providing opportunity for different strategies and solutions. In the wake of these events more attention is paid to higher-order thinking skills in regular mathematics education as well.

2.1 Introduction

To survive in modern society, the emphasis of education should be on learning what to do with knowledge, rather than on what knowledge to learn—this shift is referred to as the essence of 21st century skills (Silva, 2009). It implies a focus on skills like critical thinking, problem solving, inquiry, creativity, communication and cooperation. These skills are not only related to the 21st century, however. Problem solving and mathematical thinking have been part of mathematics education in several countries around the world for decades (Törner, Schoenfeld, & Reiss, 2007).

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Since 1989, there has been a radical change in thinking about the question ‘what mathematics for whom?’ in upper secondary pre-university education in the Netherlands. This resulted in two different types of mathematics curricula: Mathematics A and Mathematics B. Mathematics B with calculus as core component, is suitable for students who will attend scientific/technical/mathematical (STEM) studies; Mathematics A with core topics discrete mathematics, statistics and probability and a little bit of calculus, is meant for students who prepare for academic studies in social or economic sciences. More important however than the differences in topics in these two types of mathematics curricula, were the different and new ideas that guided the design of Mathematics A.

> Mathematics A is intended for students who will have little further education in mathematics in their academic studies, but who must be able to use mathematics as an instrument to a certain extent. In particular, we have in mind those who have to prepare themselves for the fact that subjects outside the traditional sciences are more frequently being approached with the use of mathematics. This means that students must learn to be able to assess the value of a mathematically tinged presentation in their education. To do this they must become familiar with the current mathematical use of language, with formulations in formula language, and with divergent forms of mathematical representation. Furthermore, they must learn to work with mathematical models and be able to assess the relevance of these models. (HEWET report, 1980, p. 19)

The emphasis in Mathematics A is on applying mathematics and on the process of modelling and problem solving, more than on the product. This has greatly influenced the type of problems. Instead of just formal mathematical tasks, in Mathematics A real life situations are used as a context for mathematical modelling and problem solving. Some of the contexts also guide students to develop and reinvent mathematical tools and concepts. In Fig. 2.1 the context of the helix of a propeller is used to introduce the concept of the sine graph (Lange, 1982).

Since this shift in 1989, several curriculum changes have been implemented in Dutch education. As a result, a focus on mathematical thinking and reasoning is visible in all standards for mathematics in primary education (Wit, 1997) and in lower secondary education (Bos, Braber, Gademan, & Wijk, 2010) and in the examination syllabi for upper secondary schools.

This focus has been inspired by the view on the teaching and learning of mathematics in the Netherlands which was initiated in the early 1980s and has evolved as the theory of Realistic Mathematics Education (RME) (Heuvel-Panhuizen, 1998, 2000).

Although general mathematical (thinking) skills, and more broadly 21st century skills, are considered important in society and in mathematics education, it is not easy to realise these in educational practice. The teaching and assessing of problem solving, mathematical modelling, communicating and critical thinking requires other types of problems than the regular textbook problems. Furthermore, it needs a teacher facilitating the process, rather than just explaining mathematical concepts. In the following section, we will describe how these other types of problems came into being.

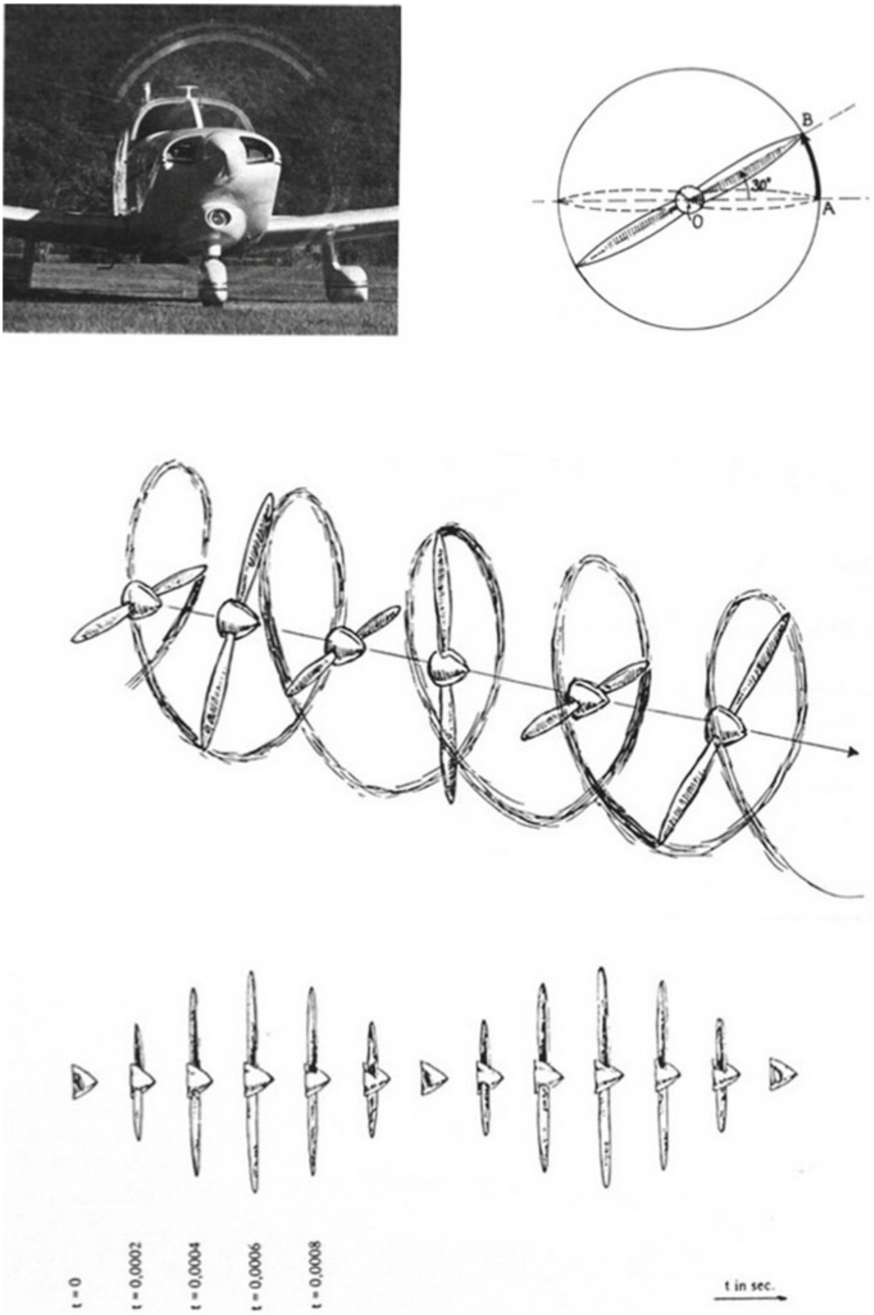

Fig. 2.1 Developing the concept of sine (Lange, 1982)

2.2 The Emergence of Mathematics in Teams to Develop Mathematical Thinking

Although RME can be seen as the leading approach on learning and teaching mathematics in the Netherlands since the 1980s (Heuvel-Panhuizen, 2000), an ever-changing balance exists—especially in assessment—between the emphasis on problem solving and using mathematical thinking skills on the one hand and reproducing basic skills (knowledge and procedures) on the other hand. Being an institute focusing on innovation, the Freudenthal Institute (FI) aims to keep mathematical thinking at the heart of mathematics education and assessment. This needs to be done within the—also changing—constraints of the central examination programs and curricula for higher secondary education and the core standards and curricula for primary and lower secondary education. In this section, we present a brief historical overview of the way in which assessment of problem solving and mathematical thinking has been put into practice in the Netherlands. We focus on secondary education, but when appropriate we discuss similar developments in primary education.

2.2.1 Secondary Education

When the Mathematics A curriculum was formally introduced in 1989, the need for changes in assessment was felt. The emphasis on higher-order mathematical skills, mathematical modelling and the use of mathematics to solve real world problems had to be reflected in the assessment of Mathematics A as well. Furthermore, cooperating and problem solving in small groups was seen as an important aspect of Mathematics A, since it would contribute to the development of communicating and mathematical reasoning. In the research on the pilot of the small-scale implementation of Mathematics A it was found that working in groups added to the quality of the process as well of the product (Lange, 1987). This is in line with later findings from research by Dekker and Elshout-Mohr (1998), which showed how working in small groups on mathematical problems stimulates mathematical level raising for each individual group member.

The written central examination did not seem the appropriate way to assess these higher-order skills. Although this examination is made up of problems in context, the questions are often closed and focussed on specific mathematical skills. Modelling and problem solving are hardly ever needed, and teamwork is not possible.

In 1989 a pilot was carried out to design a different type of open assignment for teams of students, to assess the new goals of Mathematics A. This resulted in the Mathematics A-lympiad, a mathematical real-world-problem-solving competition for teams, as a way to assess what we now call ‘21st century skills for mathematics’ in an authentic open way. Since the first pilot in 1989 this competition, which consists of two rounds—a qualifying preliminary round in the participating schools and an international final round taking a whole weekend in a conference centre—has been organised yearly. All assignments are designed by the Mathematics A-lympiad committee, a committee residing at the FI consisting of teachers, teacher educators, mathematicians and educational designers.

Participation has grown from 14 schools in 1989 to over 170 schools in 2007. Since then there is a slow but gradual decline resulting in about 100 participating schools in 2014, which is about 15% of all upper secondary schools in the Netherlands. At each school, an average of 40 students—10 teams of four students—participate.

In 1999 the curriculum for Mathematics B—aimed at students with ambitions to continue in STEM studies—was changed to include more modelling and applications. This was a result of a larger educational reform, in which new standards were formulated for all subjects. Higher-order thinking skills were also included in Mathematics B. The curricular changes implied that these skills should be assessed in school examinations, through big, mostly open-ended tasks or projects, of which at least one should be done by a group of three students. Because schools were already familiar with the assignments of the Mathematics A-lympiad, Mathematics B teachers asked for a similar assignment for their students, and this resulted in the Mathematics B-day. The experiences with the assignments of the Mathematics B-day showed that open-ended tasks for teams can also be designed within the domain of pure, formal mathematics.

Participation in the Mathematics B-day rose fast from 22 schools in 1999 to almost 160 in 2010. Since then we have seen the same phenomenon as for the Mathematics A-lympiad: a slow but gradual decline, resulting in 110 participating schools in 2014. The participation of schools in the Mathematics A-lympiad and in the Mathematics B-day is illustrated in Fig. 2.2.

The decline in participation in both competitions (which have no overlap in participating students) started around 2007 when a new curricular change led to more emphasis on basic algebraic skills—both in Mathematics A and Mathematics B. At the same time the explicit link between the choice for a type of mathematics (A or B) and the overall orientation on future studies was abandoned, as well as the obligation to include at least one big open-ended task in a school examination. Also, the recent renewed emphasis on mathematical thinking in the curricula, as

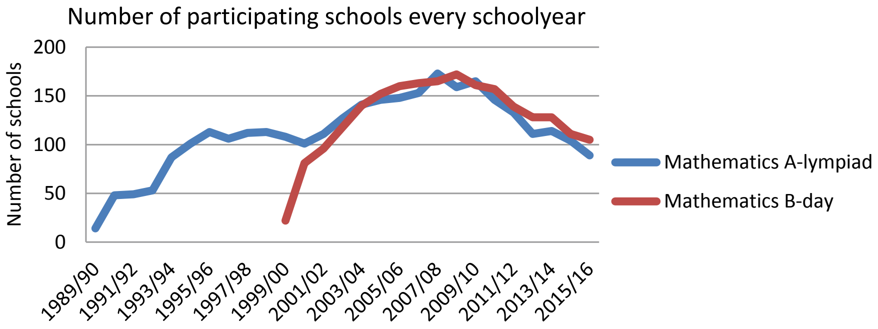

Fig. 2.2 Participation of schools in the Mathematics A-lympiad and in the Mathematics B-day

well as in the assessments for upper and lower secondary education (Commissie Toekomst Wiskundeonderwijs, cTWO, 2012) has not yet led to a higher participation rate in both competitions. However, to ensure more continuous attention for the development of students' mathematical thinking during their full secondary education, in 2012 the FI started to design an activity similar to the Mathematics A-lympiad and the Mathematics B-day for lower secondary education (Grade 9): the Lower-Secondary-Mathematics-Day.

2.2.2 Primary Education

For a long time, RME has had a significant influence on mathematics curriculum standards as well as on the textbooks in primary education. However, despite this, the focus in the commercial textbooks is not on mathematical thinking and reasoning (Kolovou, Heuvel-Panhuizen, & Bakker, 2009). To counter this approach and present primary school teachers and students with a different and broader view on mathematics, in 2003 the Grote Rekendag[1] for primary education was initiated. This is a full day with thematic mathematical activities for students in all grades in primary school (for students aged 4–12). The open activities are mostly performed in small groups and ask for inquiry and creativity by students and focus on mathematical thinking, modelling and communicating. In this respect, the activities are comparable with those described for secondary education. For primary education however, the Grote Rekendag is not a competition and instead of one large open assignment it is made up of a number of smaller activities connected by the theme.

2.3 Characteristics of the Mathematics A-lympiad and the Mathematics B-day Assignments

In the previous section, we described how various open-ended assignments to assess mathematical thinking and problem solving came into being. In this section, we will focus on the characteristics of these assignments and the specific requirements they need to fulfil in order to do what they are meant to do: elicit students to think mathematically, to creatively solve open-ended unfamiliar problems, to model, structure and represent problems and solutions, to work collaboratively and to communicate about mathematics.[2]

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[1] Big Mathematics Day.

[2] References to similar characteristics can also be found in Dan Meyer's 'Three acts' problem (http://blog.mrmeyer.com/2011/the-three-acts-of-a-mathematical-story/) and in Lange (1987).

2.3.1 Example from the Mathematics A-lympiad: 'Working with Breaks'

An example of a Mathematics A-lympiad task is 'Working with breaks'.[3] The complete task is based on one graph only (see Fig. 2.3).

This graph, from a German study, relates the productivity of workers in a factory to the hours they work without a break. Furthermore, in this model there are some rules of thumb relating productivity to the length of the break:

- After a break within the first five hours of working (that is non-stop working) productivity will be back at the level that the productivity was '3.5 times the length of the break' before the start of the break.
- After a break that is taken after more than five working hours the productivity will be back at the level that the productivity was '3 times the length of the break' before the start of the break.

The main question that students have to answer for the board of directors of the company is: how to get 'maximum productivity' in the factory by scheduling breaks in the most effective way. To make calculations easier, the so-called 'work production-units' (wpu) per hour (per worker) are introduced in the assignment with 600 wpu being the maximum productivity.

In the first part of the assignment, students are asked to use the graph and the two rules of thumb to estimate the productivity for one day, in two conditions: without a break, and with one break. In the middle part of the assignment, a linear approximation of the graph is introduced, and students are asked to investigate a

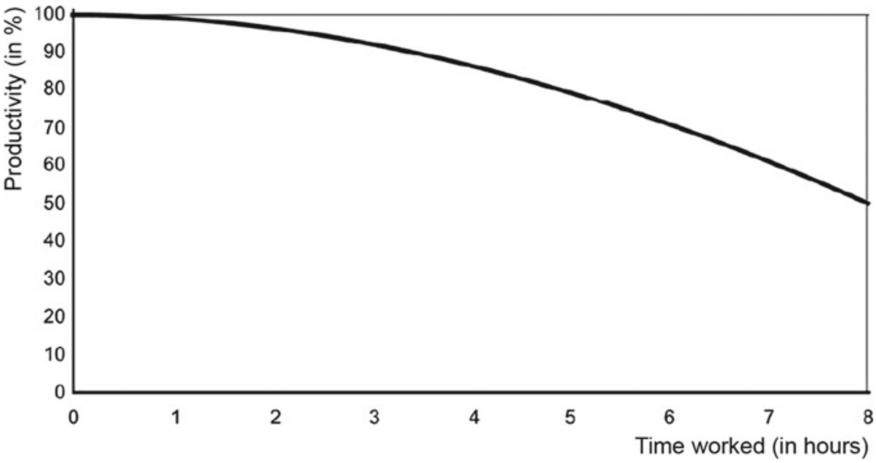

Fig. 2.3 Productivity graph

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[3] This task, from the preliminary round of the Mathematics A-lympiad 2007–2008, can be found at: http://www.fi.uu.nl/alympiade/en/opgaven2007-2008/WorkingWithBreaks.pdf.

few different models (working with one break, with more breaks) and extend their calculations from one day to one week. They also have to work within restraints: the company production must reach a certain (minimum) number of wpu per week, and the workers want to have as much free time as possible. The final part of the assignment asks for at least two well-founded proposals for a daily schedule for the workers. The workers' council and the board of directors together must be able to choose between these proposals, while taking into account:

- The interest of both employer and employee (worker)
- Health and safety rules
- The minimum of 19200 wpu per week.

The health and safety rules are a new, authentic, component in the task, at this stage. Of course, all consequences, choices and assumptions, must be described and justified by the teams in their proposals.

Students work on the assignments in teams for one full day and produce a report. This report is first judged (and sometimes graded as well) by their own teacher. Then, the best ones are judged by a teacher from a different school or by the jury of members of the committee. This evaluation results in a winning team.

As said before, the assignments need to be designed in such a way that they elicit problem solving and mathematical thinking. An important characteristic of the assignments is that they are new to students, which means that the problems are non-routine and non-trivial (Doorman et al., 2007). Schoenfeld (2007) states that these types of problems are needed for problem solving to happen. The absence of a known procedure forces students to come up with new strategies, that need to be tested, compared and evaluated. Other requirements of the assignments are that:

- They should be rich, meaning that there is not only one way to come to a solution, and solutions can vary in mathematical depth
- They should build on knowledge students already have, and extend it
- They should use higher-order questions (how? why?) and encourage reasoning rather than 'answer getting' (Swan, 2005).

Besides general characteristics that elicit problem solving and mathematical thinking, it is important that the assignments are suitable for a competition. An important condition for a competition is that the teacher has a minimal role. He or she facilitates the organisation and the process, but provides no content-related guidance. This asks for a well-structured, but open assignment. All teams must be able to enter the problem without help from a teacher, and on the other hand—in order to determine a winner—the problem has to allow for different approaches and strategies based on decisions by the students and for solutions that differ in quality. Swan (2005), when describing characteristics of rich collaborative tasks, speaks of tasks being 'accessible and extendable'.

The accessibility and extendibility of the assignments for the Mathematics A-lympiad, which are situated in a real-life context, is realised by a more or less fixed structure (Haan & Wijers, 2000):

- The first part is an introduction with some smaller, less open problems to get to understand the context. This ensures that the assignment is accessible for all students.
- The middle part often asks for an analysis of data, of a model, or of a solution that is presented in the assignment.
- The final part asks for creativity in designing, comparing and evaluating a new approach, system, model, solution or product.

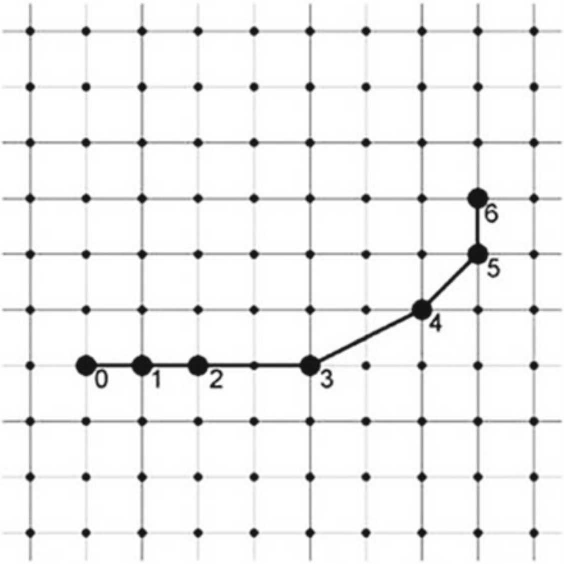
The example ‘Working with breaks’, discussed in this section, illustrates how this structure is concretised in the assignment.

2.3.2 Example from the Mathematics B-day: ‘How to Crash a Dot?’

An example of a Mathematics B-day task is ‘How to crash a dot?’[4] The assignment is based on one of the first computer games in the 1970s.

The route of the dot (see Fig. 2.4) is determined by using buttons that make the dot move in a certain direction (N = north, S = south, E = east, W = west) with a certain increasing speed. Furthermore, there is a button (P = pass), which means that the same direction and speed is kept. For example, when E is used the first time, the dot moves one unit to the east. When you use the button P the next time, the dot

Fig. 2.4 Route of a dot



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[4] The complete task from schoolyear 2009–2010 can be found at http://www.fisme.science.uu.nl/toepassingen/28174/.

moves in the same direction with the same speed. When you use E again however, the dot moves two units two the east.

This example makes clear that the assignments for the Mathematics B-day are situated within the mathematical world itself. Often new mathematical content is addressed, for which a longer and more guided introduction is needed. Therefore, in the first part of this assignment, the rules of the game that let students move a dot along grid points are formulated and students learn how to use the rules. In the middle part of the assignment students first explore movements in one dimension. They investigate for example how to use the rules to make a dot move along a straight horizontal line. Next, the students study movements in both horizontal and vertical directions. They make use of the results found when exploring movements in one direction. In the last part three different final questions are formulated, letting students make a choice between doing all three with the risk that they can only report superficially on them, or making a wise choice and report fully, carefully and in-depth. This final part asks for mathematical creativity. Here extendibility is realised when some of the teams go further and deeper in designing and investigating their mathematical ideas and hypotheses.

2.4 The Role of the Teacher

The assignments discussed so far are meant mainly for assessment and not primarily for learning. They are not part of the regular mathematics classes. During the competition day, the teacher has a very small role. He or she facilitates the process and keeps the teams going, but has no role in providing help (Dekker & Elshout-Mohr, 1998; Haan & Wijers, 2000). One could argue, as Kirschner, Sweller, and Clark (2006) do, that this minimal guidance does not work, but in this case, the aim is not instruction and the teacher can still give process help.

To prepare teachers for using and grading these open, non-routine large assignments, a workshop is offered each year to all teachers who have students participating in one of the competitions. In this three-hour workshop, teachers get to know part of the assignment that will be used later that year. They can work on it in teams themselves and discuss with colleagues their experiences, findings and the problems they foresee. Members of the committee can use the comments to improve the assignments. An important topic in this workshop is how to evaluate and grade student work. Experienced teachers share their tips and tricks with teachers who are new to the competition. The workshop proved to be useful for both novice and experienced teachers: it is a way of preparing for process-guidance during the competition, and it is a way to get a grip on the core content of the assignments. A teacher said once:

> When I don't know the assignment very well, I tend to 'help' students in giving answers to their questions; attending the workshop helps me in getting a grip on the assignment, so when a student now asks me about the content, I know what guidance question I can ask to help them.

Prior to the competition the teacher has an important role in preparing students for this type of assignments. The preparation can be done in different ways. One way is having students practise with old assignments from previous competitions. Although all assignments are available on the web, this approach is seldom used. The principal objection is that it takes a lot of time at the expense of the time available for teaching. To finish one full assignment takes about one whole day, which is equivalent to about five lessons.

Often teachers give a form of preparation (Dijk, 2014) in which they present organisational information on how to deal with this type of assignments as a team. They often have students read one assignment as an example and work on it for half an hour and then discuss ways of working and the product requirements that are listed in an addendum to the assignment. Although the assignments are quite different every year, the criteria for assessing the quality of the reports, and more specific the higher-order general mathematical skills, are more or less constant–apart, of course, from the specific mathematical content. The reports are graded based on aspects such as:

- Quality of argumentation and justification of choices being made
- Use of mathematics
- The (mathematical) creativity in strategies and solutions
- Quality and extensiveness of (mathematical) reasoning and modelling
- The presentation: including form, readability, clarity, completeness, structure, use and function of appendices.

Teachers can also prepare their students for this type of open assignments in their regular mathematics classes. They can do so by creating a classroom culture in which students are used to listening to each other, asking each other questions and writing down their own thinking before they share it. Teachers who do so also help students by orchestrating their thinking (Drijvers, 2015) and evoking mathematical discussions in their regular mathematics classes.

2.5 The Student Perspective

For students in upper secondary the assignment in the competition is often their first experience with this type of large open problems for teams. Textbooks rarely offer this type of problems, and if open problems are included in the textbooks they mostly refer to the core content of the lesson or the chapter at hand, which means that part of the strategy is known or obvious. In this case less mathematical thinking is needed and there is no need for creativity and ‘real’ problem solving in the sense of Schoenfeld (2007).

To illustrate the experiences of the students, some quotes from students, taken from several reports from different assignments, are presented in Fig. 2.5. These quotes show that the students discover the fun of doing mathematics, they are allowed to do their own investigations, and sometimes they surpass themselves and exceed their teachers’ expectations!

"*This was a special day. We learned things and it was fun. We were free to plan the work ourselves. In the introductory task, we tried to explain the methods that were presented. After 'hard thinking' we understood what was going on. In the beginning we were frustrated, but after we found out 'how it worked', it became much more fun.*"

"*In the introductory tasks, we were confronted with mathematics we didn't fully understand. We kept to our initial problem-solving strategies throughout the tasks and we believe this led to a very good outcome. By struggling through the introductory tasks, we got more and more familiar with the context and the mathematics. In the end, we had enough knowledge to complete the final part of the assignment.*"

"*At a certain moment, we understood how everything worked out and from that moment on we 'raced' through the tasks. Because we had divided the work efficiently we could finish the tasks fast and at the same time keep up the fun.* [..] *There was a relaxed atmosphere and we were better at math than we thought and that is worth something as well.*"

Fig. 2.5 Quotes from students

As discussed in the previous section, teachers can prepare their students in several ways. A small-scale study on a comparison of schools participating in the Mathematics A-lympiad (Dijk, 2014) showed that students of teachers who put more effort in creating a investigative classroom culture in which mathematical thinking and creativity are stimulated and who prepare students by introducing them to the ideas and goals of the competition are more often among the winning teams of the preliminary round in the Mathematics A-lympiad. In this case the regular classroom teaching lays a foundation for the students' higher-order thinking skills and thus for their successes in the competitions.

The influence can also work the other way around. Experiences with these competitions can prompt changes in teachers' regular teaching. For example, we noticed that students who participate in the competition of the Lower-Secondary-Mathematics-Day (Grade 9), often struggle with the openness of the task. Although the results do show creativity and mathematical thinking, it is clear that a lot of the students lack a structured approach of formulating hypotheses and systematically investigating these by varying the variables, constraints, representations, models or other aspects. For their teachers, this may be a reason to start paying more attention to how to handle unstructured problems and to stimulate modelling and investigations by their students. In this respect, the competitions function as an entrance into a more inquiry-based way of teaching mathematics.

2.6 The Future of Mathematical Thinking in Secondary Mathematics Education

As a consequence of the recent curriculum change that was fully implemented in 2017, mathematical thinking activities were embedded in the standards for mathematics in upper secondary (cTWO, 2012). Never before have these higher-order thinking skills been described in the standards in such detail. They are characterised in connection to the content domains and include:

- Modelling and algebraisation
- Ordering and structuring
- Analytical thinking and problem solving
- Manipulating formulas
- Abstracting
- Reasoning and proving.

Although the assignments of the mathematical competitions stimulate mathematical thinking, they do not reach all students and teachers. It is not always possible in a school to dedicate a full day to mathematics, in which the content may even be outside the core curriculum. Furthermore, to really implement mathematical thinking for all students and help them develop the appropriate skills this should be part of the regular curriculum, which means that suitable assignments and problems are needed that fit within regular 50-min mathematics lessons. To realise this, two movements are currently ongoing: textbooks authors start inserting so called ‘mathematical thinking problems’ in their textbooks, but since it takes time until a new generation of textbooks is published, mathematics teachers themselves also design problems, often as a result of professional development on this topic. These problems are often smaller, open problems, that are non-routine and evoke students’ mathematical thinking, reasoning and creativity and that help students and teachers to make the shift towards ‘relational understanding’ of mathematics, instead of keeping the focus on the (more common) ‘instrumental understanding’ (Skemp, 1976). An example of such a problem, designed by a teacher,[5] is presented in Fig. 2.6.

Usually in the textbooks the scale of the axes is given, and students have to come up with the formula using this information. In this problem, students have to show understanding of the concept of the linear formula, in order to find out the scaling of the axes.

All in all, we may conclude that the time seems right for a shift in mathematics education towards a more inquiry-based 21st century fitting approach. Not all requirements are met, of course, but the necessary conditions seem to be established: standards, examinations, textbooks and teachers are being prepared for such an approach. Professional development of teachers is organised, in which mathematical thinking activities are designed and implemented by teachers in their classrooms. In these courses, they learn how to implement a classroom culture in which they

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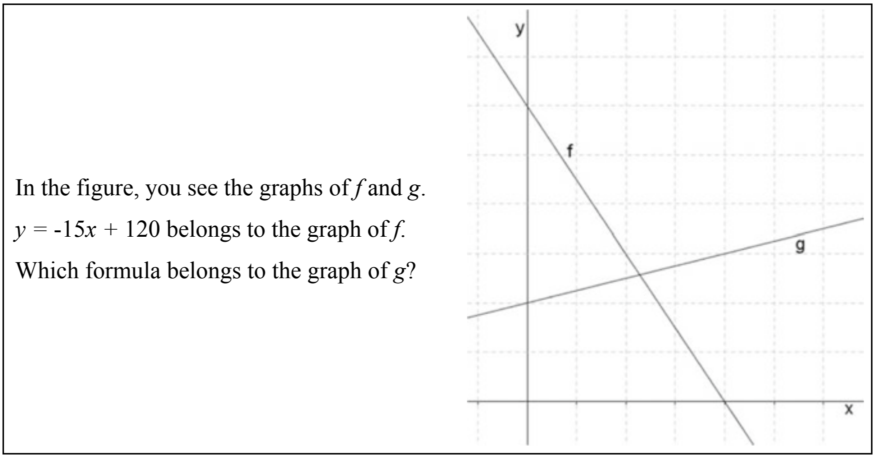
[5] Egbert Jan Jonker, mathematics teacher at Roelof van Echten College in Hoogeveen, the Netherlands.



In the figure, you see the graphs of f and g.

$y = -15x + 120$ belongs to the graph of f.

Which formula belongs to the graph of g?

Fig. 2.6 Problem that stimulates mathematical thinking

stimulate mathematical thinking and problem solving. Research into mathematical thinking is carried out and is disseminated in journals and in research-meets-practice conferences. Especially dissemination of research through databases with assignments and guidelines for teachers (in text and through videos) are used to increase the incorporation of mathematical thinking activities in the classroom.

The assignments for teams discussed in this chapter will cause the stream of development of mathematical thinking to continue flowing. We hope this stream will grow, supported by classroom environments with the right tasks and the appropriate teacher and student attitudes.

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Chapter 3
Task Contexts in Dutch Mathematics Education

Pauline Vos

Abstract This chapter offers a description of task contexts in mathematics education in the Netherlands. International comparative studies show that the Dutch average percentage of mathematics tasks with real-life connections per lesson exceeds any other country by far. This tradition goes back more than 500 years, when the earliest mathematics textbooks in the Dutch language consisted entirely of tasks set in commercial, naval and building contexts. To analyse and characterise the task contexts, I use the notion of usefulness, which is a perception by students on future practices outside school. A distinction is made between bare tasks (without contexts), tasks with mathematical contexts (e.g., matchstick pattern problems), dressed-up tasks (hiding a mathematical question), tasks with realistic contexts with questions that are useful within the context, and tasks with authentic contexts. The empirical part of this chapter contains an analysis of a mathematics textbook chapter and a sample of examination tasks. This analysis shows that Dutch mathematics education contains many links to real-life, which is not just verbally presented, but also visually with drawings, photos, diagrams and other visualisations. The contexts are drawn from a wide spectrum of areas in real-life, reflecting that mathematics can be found anywhere in society. The examinations contain more authentic aspects than the textbook, and the higher-level examinations have more authentic aspects than the lower-level examinations. Nevertheless, contexts both in the examinations and in the textbook can still be artificial, with questions which would not be asked by actors within the context. Task contexts often come from recreational or professional practices, demonstrating to students the usefulness of mathematics in their future lives beyond school.

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3.1 The Prevalent Use of Real-Life Contexts in Dutch Mathematics Tasks

In 1999, an international study was carried out in which teaching practices in mathematics classrooms at Grade 8 level in seven countries were analysed: the TIMSS Video Study (Hiebert et al., 2003). The participating countries were Australia, the Czech Republic, Hong Kong, Japan, the Netherlands, Switzerland and the United States. In each country 84 random lessons were video-captured, transcribed and quantitatively analysed. The report offers many tables and bar graphs with descriptive statistics for a variety of mathematics classroom aspects. We can see, for example, the percentages of lessons in each country in which the teacher made a goal statement or summarised the taught content, the amount of time spent on whole-class discussion, the role of homework, the complexity of the taught mathematics content, the type of reasoning asked in the activities, and so forth. The chapter "Instructional Practices: How Mathematics was Worked On" contains findings on the mathematical procedures used to solve tasks, the mathematical representations used, the frequency of students being allowed to select their own methods for solving the mathematical tasks and the number of tasks embedded in real-life situations.

In many of the report's statistics, the Netherlands does not stand out at all. Many of the Dutch frequencies in the tables and bar graphs are similar to those of the other countries, in particular to those from Australia, the Czech Republic or Switzerland. Just like in other countries, the Dutch teacher sits and stands in front of the class, discusses homework, presents new content to the whole class, and the students individually do exercises to practise the mathematics taught.

However, there is one result in which the Netherlands distinguishes itself from the other six countries in the study, see Fig. 3.1. This involves how tasks are presented to the students. In the study, a distinction is made in two categories. The first category contains tasks which are presented by using mathematical language only, such as "Graph the equation: $y = 3x + 7$" or "Find the volume of a cube whose side measures 3.5 cm." The tasks in this category are given in numbers, mathematical operations and symbols, and the verbal expressions relate to mathematical objects only. The second category contains tasks which are presented to students within a real-life context, such as "Estimate the surface area of the frame in the picture below" or "Samantha is collecting data on the time it takes her to walk to school. A table shows her travel times over a two-week period; find the mean." Whether teachers brought in real-life connections at a later stage of the lesson was not included in the statistics. The categorisation only deals with the set-up of the tasks.

In Fig. 3.1, we see that in the 84 randomly selected mathematics lessons in Grade 8 in the Netherlands, there is a relatively smaller percentage of tasks which was set up using mathematical language or symbols only, on average 40% of the tasks per lesson. In the other countries, approximately 70–90% of the tasks was set up only with numbers and symbols. The frequencies for the second category show a reverse picture. In the Netherlands, the percentage of tasks per lesson that started from real-life connections is 42%, while in the other six countries this percentage

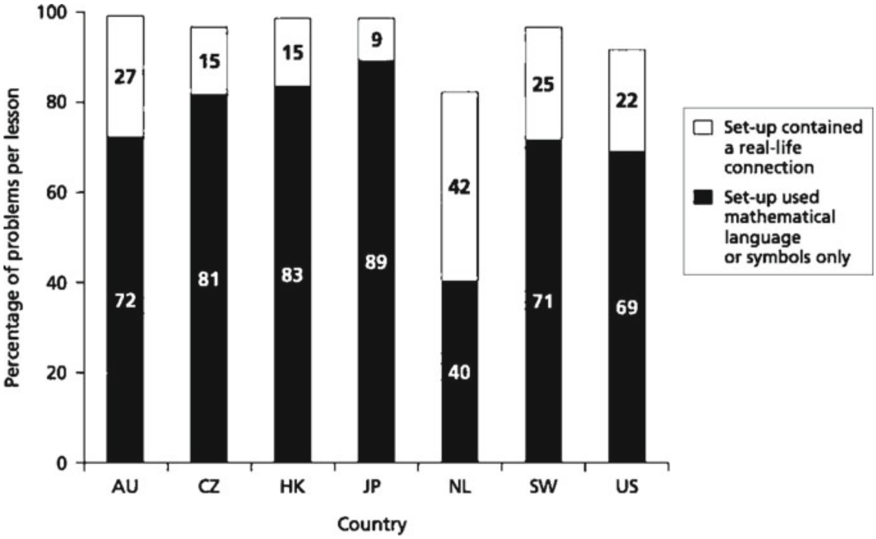

Fig. 3.1 Percentage of problems per mathematics lesson in Grade 8 that was set up with and without the use of a real-life connection (from Hiebert et al., 2003, p. 85); AU = Australia CZ = Czech Republic, HK = Hong Kong, JP = Japan, NL = the Netherlands, SW = Switzerland, US = United States

ranged from 9 to 22%. Another thing that Fig. 3.1 shows is that the percentages for the Netherlands do not add up to 100. The report does not give much explanation for this. There is only a footnote, saying that the researchers were not able to code all tasks. It means that apparently 18% of the tasks in the Dutch classrooms could not be coded as either having mathematical language and symbols only, or having a real-life connection. I will later come back to this issue.

In Fig. 3.1 the Netherlands was compared to only six other countries. When comparing to a larger number of countries we do see a similar phenomenon. The TIMSS 2003 Mathematics report (Mullis, Martin, Gonzalez, & Chrostowski, 2004) contains the results of an analysis of where the emphasis is placed in the intended mathematics curriculum in Grade 8. Educational authorities in all participating countries were asked, for example, whether the focus of mathematics education is on mastering basis skills, on understanding mathematical concepts and principles, on applying mathematics in real-life contexts, on communicating mathematically, on reasoning mathematically, on integrating mathematics with other subjects or on incorporating experiences of ethnic/cultural groups. Again, the Netherlands stands out.

> Applying mathematics in real-life contexts was given a lot of emphasis in the intended eighth-grade curriculum of 17 participants. Botswana, the Netherlands and South Africa reported placing more emphasis on this approach than on mastering basic skills or understanding mathematics concepts. (Mullis et al., 2004, p. 177)

In most countries, there was more emphasis on mastering basic skills or understanding mathematics concepts, but this was not found for the Netherlands.

The results from the TIMSS Video Study and from TIMSS 2003 do not reflect a new phenomenon. Dutch mathematics education has had an emphasis on the usefulness of mathematics for centuries already. This emphasis distinguished Dutch mathematics education from that of many other nations, where the classical Greek mathematics dominated the curriculum, with topics such as two-dimensional geometry, arithmetic, harmonics (the study of structures in music) and astronomy. Greek philosophers such as Plato considered every kind of skill connected with daily needs as ignoble and vulgar, and they praised mathematics for its purity. The justification for teaching esoteric mathematics was aesthetical: it would raise the learners' spirits (Kline, 1953). In the past two thousand years in Europe, this Greek emphasis was upheld in mainstream mathematics education. The higher classes (nobility and clergy) in the aristocratic European societies had an esoteric activity in studying mathematical deductive reasoning through Euclid's *Elements* (Dunham, 1990). It established mathematics as a deductive science, in which axioms and definitions lead to a hierarchy of theorems, and the truth of theorems was established through proofs.

In the 16th century Netherlands, citizens started a revolt against the Habsburg Monarchy, which led to the proclamation of an independent republic, the Republic of the Seven United Netherlands. The Dutch elite consisted of rich, Calvinistic patricians with a pragmatic mind and an aversion to vanity (Schama, 1991). These powerful, urban merchants needed worldly mathematical applications to organise their society, their businesses and their lives. They needed practical mathematics, which did not come from Greece and which was not written in Latin. Already in the late Middle Ages, many Dutch artisans and merchants, including women, gained mathematical competencies. In an early manuscript in the Dutch language, dating back to 1445 and in which calculations with the Hindu-Arabic numerals were taught, mathematics was presented and practised through tasks with commercial contexts, such as converting measures (weights, lengths) and money (pounds, shillings, pennies) (Kool, 1999). Also, the mathematics taught at the citizen's universities was fairly practical, linking to the training of future civil engineers. The mathematics at 17th century universities contained many applications from navigation and the architecture of fortifications (Van Maanen, 1987).

The Dutch curricular emphasis in which mathematics teaching and learning is connected to real-life contexts can be traced back to a mix of Calvinism (abstaining from esoterica), medieval democracy (accessibility of mathematics education to many), and civil societal needs (engineering, commerce). This utilitarian emphasis was already present centuries before the domain-specific instruction theory of Realistic Mathematics Education (RME) (Van den Heuvel-Panhuizen & Drijvers, 2014) was developed by Hans Freudenthal and his colleagues in the 1970s. In fact, the culture of usefulness of mathematics as a curricular emphasis which already existed for 500 years may have created a fertile ground for RME.

Tasks in mathematics education generally consist of a text (whether with mathematical language or not) and a question, or a sequence of questions. The questions within tasks are meant to make student carry out mathematical activities. As the TIMSS Video Study (Hiebert et al., 2003) and the TIMSS 2003 study (Mullis et al., 2004) demonstrated, many tasks in Dutch mathematics education are presented to students within a real-life context. In this chapter I will describe context tasks in Dutch mathematics education as found in textbooks and examinations, and I will focus on the contexts, the questions, and their relationship to reality. However, in this chapter I will not:

– Discuss students' and teachers' perceptions, appreciation or dissatisfaction of contexts in Dutch tasks
– Analyse instructional settings for context tasks (e.g., whether, how and why used for group work, homework, tests, etc.)
– Describe curricular goals of using contexts within tasks, namely to introduce students to a mathematical concept, allowing them to use out-of-school knowledge and informal procedures, from which they can develop more abstract knowledge and more formal procedures.

In Sects. 3.3 and 3.4, I will qualitatively and quantitatively analyse context tasks in a sample chapter of a textbook and in a sample of national examination tasks. To describe distinguishing features of contexts, I will first explain a framework for categorising task contexts.

3.2 Categories for Mathematical Tasks and Their Relation to Reality

Tasks in mathematics classes may or may not have a link to real-life. The TIMSS Video Study (Hiebert et al., 2003) distinguished two categories: (1) tasks, which are presented by using mathematical language only, and (2) tasks which are presented to students within a real-life context. To describe contexts in Dutch mathematics tasks, I will make a more fine-grained categorisation.

When tasks are presented by using mathematical language only, some researchers speak of abstract tasks. I will call them 'bare tasks', following Van den Heuvel-Panhuizen (2005). For example, the exercise

$$3\frac{1}{2} \div \frac{1}{4}$$

is a bare task; the numbers have no other meaning than being numbers. In many textbooks throughout the world, one may see rows of such tasks. The repetitive nature of such tasks is intended to train students to memorise and practice the rule for the division of a mixed number by a simple fraction: dividing by a fraction means to multiply by its inverse. Rows of such bare tasks are part of mechanistic drill-and-practice.

Drill-and-practice may be useful for instilling automated competencies. However, in this chapter I will use the term ‘useful’ in the way Freudenthal (1968) meant in his seminal article “Why to Teach Mathematics So As to Be Useful”, where usefulness of mathematics means that an individual student manages to flexibly and practically apply the mathematics learned in a rich variety of new situations. Wigfield and Eccles (2000) explained usefulness as a motivator, when students expect and value learned content as something that will help them do things better outside of class. Most students are aware that drill-and-practice tasks are useful to pass tests (and that the skills may be forgotten thereafter). Therefore, Williams (2012) specified usefulness as: (1) having ‘exchange’ value (relating to the possibility that a mark can be used to enter a next level of learning) and (2) having ‘use’ value (relating to the competence and understanding required to use and apply mathematics in future practices, as professional or as citizen). Bare tasks clearly have ‘exchange’ value, but their ‘use’ value is not easily perceived by students. In this chapter, I will refer to ‘use value’ when speaking of ‘usefulness’.

When tasks for students in mathematics classes are presented within a real-life context, there are many words to describe such tasks: word problems, story problems, context(ual) problems, real-world problems, work-related problems, situated problems, and so forth. In this section I will use the terms ‘tasks’[1] and ‘contexts’. The term context refers to a situation or event in the task, which often is from real-life or from imaginary situations (e.g., fairy tales). Essentially such contexts look quite unmathematical. Contexts in tasks are also referred to as ‘figurative contexts’ or ‘problem situations’. Below I will discuss sub-categories of context tasks.

Task designers (textbook authors, teachers) can opt to adapt the above division 3½ ÷ ¼ = … into the following task:

> “How many quarters of an hour go into three and a half hours?”

In this reformulation, the fraction exercise is given meaning, with all numbers becoming time chunks and the dimension unit is an hour. This is a contextualisation of the original bare task. A contextualised task has little mathematical language and few symbols. One may observe that the task is connected to an unspecified time situation, as it is not clarified what the quarters of an hour and the three and a half hours are part of, nor is any reason given why the question should be answered.

The bare division exercise 3½ ÷ ¼ = … could also have been contextualised into another unspecified context, for example into a pizza situation:

> “How many quarters of a pizza go into three and a half pizzas?”

Again, this context is unspecified, as it is not clarified what the quarter pizzas are needed for and where the three and a half pizzas come from. Furthermore, the bare division exercise could also have been contextualised into money units:

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[1] In this chapter, the term ‘problem’ is used for non-routine, problem-solving tasks. Therefore, terms such as ‘practice problems’ and (standard) ‘word problems’ are avoided.

> "How many quarters of a dollar go into three and a half dollars?"

Again, this is an unspecified context, as it is not clarified what the quarter coins and the three and a half dollars are used for. Moreover, one may notice that the exercise cannot be contextualised well in money units for some countries; for example, the Euro has no quarter coins.

With the above contextualisations, the bare division of fractions acquires a certain meaning, because the numbers become concrete. Amongst others, Clausen-May and Vappula (2005) and Palm (2002) have convincingly demonstrated that such contextualisations change the cognitive demand of bare tasks, for a variety of reasons. First, the adapted task requires students to read words instead of symbols. Second, many students are discouraged by symbolical tasks and more motivated for contextualised tasks. Also, most students are able to mobilise knowledge acquired outside school and use it for solving the task. For example, they may use the idea that a quarter of an hour equals 15 min, and then use the fact that four times 15 min make an hour. Or, in the pizza situation, they may use the idea that four quarter pizzas make one pizza; or, in the dollar coins situation they may use the idea that four quarter dollars make one dollar. In this way, the divisor is no longer a simple fraction, and the fraction task loses one of the fractions. Thus, contextualisation may increase the task's accessibility and support students' understanding that a division by ¼ can be translated into a multiplication by 4. Such contextualisations could be used in the introduction to a teaching sequence to assist students in understanding mathematical rules for fraction operations, allowing them to use their out-of-school knowledge to first develop informal procedures, from which they can later develop more formal procedures.

By contextualising a task, the numbers get a meaning (in units and dimensions). However, this does not mean that the exercise becomes useful (meaningful or interesting) to all students. Why should anyone calculate the number of quarters of an hour that go into three and a half hours? Why should anyone calculate the number of quarter dollars that go into three and a half dollars? Why should anyone calculate the number of quarter pizzas that go into three and a half pizza? What is the justification for the calculation? In particular, if there are no credible actors described within the context: people or institutions with a problem that needs to be solved. Thus, a context does not imply that the question posed to the students has justification. Therefore, it is important to consider whether the posed question would be asked within the context. If there is no clear need to perform the mathematical activities, other than a didactical need to get a correct answer within the discourse of mathematics learning, the contextualised task is as a 'dressed-up' task, hiding a mathematical task (Blum & Niss, 1991). Many so-called 'word problems' are dressed-up tasks with pointless questions. The task in which is asked "How many quarters of an hour go into three and a half hours?" is a dressed-up mathematical task.

To improve dressed-up tasks, a task designer can make the context more realistic. I use the term 'realistic' here as being related to real. The relationship between real and realistic is considered parallel to the relationships absolute-absolutistic, central-centralistic, dual-dualistic, ideal-idealistic, material-materialistic, natural-naturalistic, and so forth. Realistic means: as if from real-life, close to reality, or could be imagined as real. The term ‘realistic’ is used in this chapter for contexts in tasks only, and it needs to be distinguished from the meaning of ‘realistic’ in RME, which refers to the use of a certain sequence of activities, starting from more concrete tasks, for which students use common knowledge and after a carefully designed sequence of activities, the students are guided towards more formal mathematical thinking. Thus, in RME the curriculum may contain tasks with realistic contexts, but there may also be bare tasks. The adjective ‘realistic’ in RME is not the adjective for all tasks within that approach. Moreover, I would like to emphasise that in this chapter I only describe contexts for tasks in Dutch mathematics education, and not the philosophy for including or sequencing different sorts of tasks.

The dressed-up task “How many quarters of an hour go into three and a half hours?” can be contextualised with a realistic context. A first example is:

> A doctor in a health centre has consultations in the morning from 8.30 to 12.00 h. The patients have consultation visits of a quarter of an hour. How many patients can the doctor see?

In this task, again the students have to calculate how many quarters of an hour go into three-and-a-half hours. However, in this doctor context the justification for the calculation is to know the maximum number of patients, excluding the options for coffee breaks or speedy five-minute consultations. The question in the task is a question that may be asked of a medical doctor in a real situation. An answer to the question is useful for planning purposes within a professional practice. Additionally, assuming that most students know the system of medical consultations by appointment, the division exercise becomes experientially real (Gravemeijer, Cobb, Bowers, & Whitenack, 2000). A second example of a realistic context for this task is:

> A team of whale watchers (biologists) has a boat in a coastal area of a deep ocean. One day they pursue one animal for observations. Their boat has fuel that will last for a trip of three and a half hours. In between breathing the animal plunges into the deep water and then it cannot be observed. The animal plunges for a quarter of an hour before it needs to breathe air again. How many times can the whale watchers see the mammal?

The question in the task is a question that might be asked in a real whale watching situation, and the answer to the division gives an estimate for the maximum number of sightings. Thus, in this context such questions are asked, or in other words, the context of whale watching justifies the division calculation making the mathematical calculation useful. The whale watching context is an example of a context that the students may never have experienced in their lives, unlike the context of the medical doctor and his consultation slots. However, many students may have heard of the experience of whale watching, or may have seen it on television. This makes the task imaginable for students, without being experientially real. With this whale watching context, one may also observe, that adding realism implies adding complexity. It is a realistic context, and not a real, authentic context. In a truly authentic situation, the whale watchers need extra fuel for returning home, for possible bad weather, and whales do not surface exactly at the beginning of a trip. In real life, the question may require a more complex calculation than a mere fraction division.

In the above text, I have described a designer's hypothetical road, thereby distinguishing between possible contexts for one and the same bare task. This creates a more fine-grained categorisation of the category of tasks which are presented to students within a real-life context that has been distinguished in the TIMSS Video study (Hiebert et al., 2003). I now have the following sub-categories for mathematical tasks and their relation to reality:

- 'Bare tasks', which are presented in mathematical language and symbols
- 'Dressed-up tasks', which hide a mathematical task; they have a certain context and a pointless question; this category includes tasks with realistic contexts, in which the need for answering the question is not justified through the context
- 'Tasks with a realistic context' (experientially real or imaginable), in which the question makes sense within the context, and an answer to this question has use value within the context.

In addition to these categories, I will introduce two more categories of mathematical tasks. The bare task on the division of a mixed number by a simple fraction 3½ ÷ ¼ = … can be embedded into a mathematical context, by showing a bar, which consists of three-and-a-half units (see Fig. 3.2).

Without using (much) mathematical language, a task designer can ask for the number of small units that would fit into the larger, or can ask how many of the small units would make the same area as the larger one. This yields another category:

- 'Tasks with mathematical contexts'.

Tasks with mathematical contexts do not contain (much) mathematical language, but they are about mathematical objects and their properties. Such tasks can be found in geometry and are often visual. They can be, for example, about tiling. Also, tasks on matchstick patterns or growing patterns of triangular shapes, as used in early algebra (see, for example, Radford, 2006) have mathematical contexts, which are not encountered in real-life. The need to answer the question is never justified by the context, because mathematical contexts do not have actors who need solutions. For students, it will often be hard to perceive any 'use value' to an answer. Tasks with

Fig. 3.2 Bar visualisation for the fractions 3½ and ¼

a mathematical context can be distinguished from a bare task through the language used. Bare tasks contain mainly mathematical language and symbols, while tasks with a mathematical context have more informal language. A task with a mathematical context contains descriptions that give a certain meaning to mathematical concepts. In some cases, the mathematical context can even be associated to real-life objects. In the example above, the rectangular bar can be associated to a chocolate bar, without this explicitly being mentioned in the task.

I will use one more category to describe the context in mathematical tasks. Among others, Dierdorp, Bakker, Eijkelhof, and Van Maanen (2011), Palm (2002), Vos (2011, 2015) and Wijers, Jonker, and Kemme (2004) have used the term 'authenticity' when describing a context of a task. This term refers to being a genuine (true, honest) context, not being a copy or a simulation. Such a context may be related to practices outside school (e.g., the workplace). Authenticity is a characteristic that requires clear evidence, for example, through photos (as opposed to drawings), or when governmental datasets are used in a statistical task. Thus, I add another category of tasks and their contexts:

- 'Tasks with authentic contexts', in which the origin of the context is explained through convincing resources. In this category, also, the context justifies the question, and an answer is useful within the described context.

It remains to be noted that not all tasks with an authentic context contain meaningful questions. For example, the context of the 'Big foot' task (Blum, 2011), in which a giant shoe is depicted by a photo, is authentic. The photo is the proof of its existence in real-life. The question here is to calculate the height of a person who fits this shoe. This task matches the curricular concept of mathematical similarity and proportionality, which makes the task relevant to mathematics teachers. However, depending on one's background, one may raise other questions. A shoemaker may ask how much leather is needed for such a giant shoe. A thief may ask how much the statue weighs. An art student may ask into what artistic tradition the statue fits. In other words, the task resources may be authentic, but the mathematical question that is posed in the task is only useful to practise mathematical operations. It is not a question that would emerge from people working with statues, nor from people admiring art. Blum's (2011) classroom experiment also showed that the question did not make sense to students. They just took the numbers out of the text and performed erratic operations. Therefore, the task is a dressed-up mathematical task on similar triangles, only distinguished from an ordinary word problem by the authenticity of its context, but not of its question. In an interesting alternative to the 'Big foot' task, Biccard and Wessels (2011) designed a task to assist the police in relating foot prints found at crime scenes to the possible size (height and weight) of suspects. In this way, this task did not merely ask for finding a number, but became realistic (not fully authentic, but imaginable) and the posed question was a useful component of crime scene investigation.

In sum, the above categorisation contains five categories: the first is bare tasks and then I listed four categories for context tasks, including the category of tasks with mathematical contexts. This categorisation will assist in describing tasks and their contexts in Dutch mathematics education.

3.3 Tasks Contexts in a Dutch Secondary Education Mathematics Textbook

In this section I will provide the results of an analysis of tasks in a mathematics textbook following the previously outlined categorisation. For this analysis, I used the textbook series *Getal & Ruimte* (Reichard et al., 2006) which is most widely used in the Netherlands. The analysis is based on the textbook for students in Grade 10 in HAVO (general secondary education)[2] and within this textbook I chose the chapter “Working with Formulae”, which I consider as representative for this textbook in particular, and for textbooks used in mathematics education in the Netherlands in general. This chapter has four sections, each of which can be covered within approximately two to three hours. The chapter is introduced with a page-wide photograph of four students doing a physics experiment and a text stating that it takes a number of measurements to create a formula, which can be used to predict where a moving object will halt. This introduction (without task) has a realistic context, which is experientially real, as many students have done experiments in physics classes. It explains the usefulness of mathematical models for making predictions. The photograph showing four students doing a speed experiment in a laboratory creates an aspect of authenticity.

The first section is about creating and working with formulae in two variables. In all tasks and worked examples[3] the variables are x and y. Out of the 18 tasks, there are 14 bare tasks and 4 context tasks (22%). Out of the six worked examples there is one worked context task (17%). To give an idea of the contexts, the first context task is about a school class going on a weekend camping trip and Peter (an unspecified boy) organises the shopping to the bakery, buying only loaves and buns, each of which has a unit price. The question is to create a formula for the total costs in two variables (x for loaves of bread, y for buns). The worked example is about a concert hall, which has two price levels with tickets being €12 or €15, and the total income will be $12x + 15y$. The created formulae are simplified versions of more complex price models, which are used for economic decisions. However, the contexts of a certain Peter buying bread or a certain concert hall selling tickets do not provide any evidence whatsoever that there is a need for creating such formulae within such simplified contexts. Also, the formulae in two variables are not used for any further problem solving related to the context. Thus, the tasks are dressed-up tasks offering students training to find formulae.

The second section is about using given equations in two variables. It starts with five bare tasks and a bare worked example on finding intersection points of two

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[2] Grants admission to higher vocational education.

[3] A worked example is a task with a complete explanation showing how to solve the task.

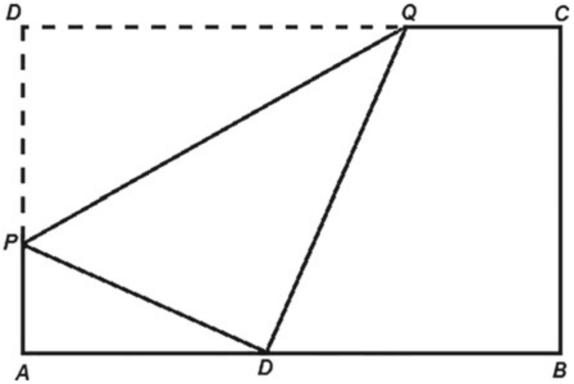


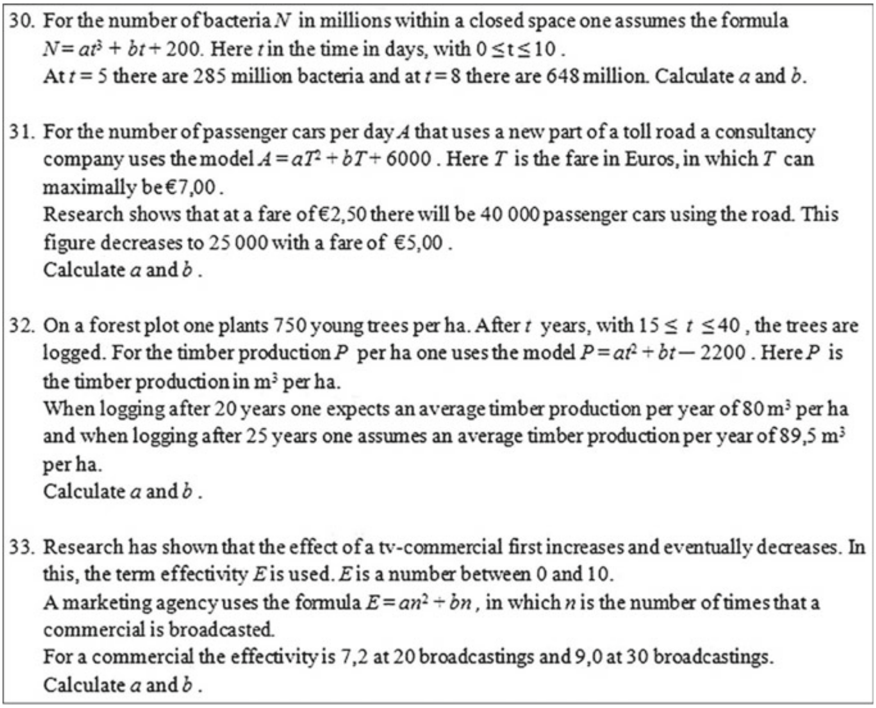
Fig. 3.3 Folding task on the position of *P* to make *AP* equal to *AD*

graphs, or on determining the parameters of a parabola $y = ax^2 + bx + c$ passing through three given points. Then, there are five tasks with mathematical contexts. The first is on paper folding (see Fig. 3.3). The context is a rectangular piece of paper *ABCD* of size 20 cm by 30 cm and point *D* is folded onto side *AB*. The question is to find the position of point *P* on *AD*, which will make *AP* equal to *AD*. The students are invited to try the folding physically first. Through some scaffolding, the students are guided to take $AP = x$, determine the quadratic equation $x^2 + x^2 = (20 - x)^2$ and from there find point *P*. The mathematical context is described in limited mathematical language.

The worked example also has a mathematical context and asks: "How can a letter T be drawn inside a circle with radius 6, with the restriction that the vertical bar of the T must be equally long to its horizontal bar?" Thus, out of 10 tasks in the second section, there are five tasks with mathematical contexts (50%). Out of the two worked examples there is one with a mathematical context (50%). This section does not have a single realistic context.

The third section contains 12 tasks, all of which are tasks with contexts. A sample of four tasks is shown in Fig. 3.4, and these tasks illustrate the others in this section. The tasks have a repetitive format. First a context is described and a formula with parameters *a* and *b* is given, then two data points are given that need to be fitted into the formula, and from there the parameters can be calculated. All tasks end with the same small sentence: "Calculate *a* and *b*."

The contexts in these four tasks present problem situations in which phenomena have to be modelled mathematically: the growth of bacteria, the density of traffic related to the tariffs of toll roads, the productivity of timber production depending on the growth time of trees before chopping them, and the effectiveness of TV-commercials. The other tasks in the section additionally have contexts of balls in sports, packaging of tin cans and wooden boxes, and farmers enclosing paddocks. The contexts are realistic and imaginable, assuming that most Grade 10 students have little personal experience with these areas, but a certain notion that such areas could exist. The mathematical models of the problem situations resonate with the text on laboratory research at the beginning of the chapter, in which it was explained that



30. For the number of bacteria N in millions within a closed space one assumes the formula $N = at^3 + bt + 200$. Here t in the time in days, with $0 \leq t \leq 10$.
At $t = 5$ there are 285 million bacteria and at $t = 8$ there are 648 million. Calculate a and b.

31. For the number of passenger cars per day A that uses a new part of a toll road a consultancy company uses the model $A = aT^2 + bT + 6000$. Here T is the fare in Euros, in which T can maximally be €7,00.
Research shows that at a fare of €2,50 there will be 40 000 passenger cars using the road. This figure decreases to 25 000 with a fare of €5,00.
Calculate a and b.

32. On a forest plot one plants 750 young trees per ha. After t years, with $15 \leq t \leq 40$, the trees are logged. For the timber production P per ha one uses the model $P = at^2 + bt - 2200$. Here P is the timber production in m^3 per ha.
When logging after 20 years one expects an average timber production per year of 80 m^3 per ha and when logging after 25 years one assumes an average timber production per year of 89,5 m^3 per ha.
Calculate a and b.

33. Research has shown that the effect of a tv-commercial first increases and eventually decreases. In this, the term effectivity E is used. E is a number between 0 and 10.
A marketing agency uses the formula $E = an^2 + bn$, in which n is the number of times that a commercial is broadcasted.
For a commercial the effectivity is 7,2 at 20 broadcastings and 9,0 at 30 broadcastings.
Calculate a and b.

Fig. 3.4 Tasks from the chapter "Working with Formulae" in the textbook series *Getal & Ruimte, Wiskunde HAVO B, Part 2* (Reichard et al., 2006, p. 22) (translated from Dutch by the author)

mathematical models are used for making predictions. However, the usefulness of calculating *a* and *b* is nowhere explained.

The textbook authors have kept the contexts vague. There is no evidence of an authentic resource of the contexts. They describe the contexts in impersonal terms by using the pronoun 'one', like in 'one can assume' or 'one uses a model' or by referring to unspecified consultancy companies and marketing agencies. In the other tasks a photo was added (the packaging task shows a tin can) and people's names were invented (the farmer who encloses a paddock is a certain farmer Wunderink and the tennis player is a certain Richard) to improve the realism. Also, it is highly unlikely that the mathematical formulae are authentic. Instead they are quadratic or cubic polynomials to fit the cognitive level of Grade 10. If the formulae had originated from real research, this could have been mentioned.

In the above analysed section of the chapter, students encounter twelve different real-world situations, in which mathematical formula are used to model phenomena. However, these contexts are not to be considered for answering the posed question to calculate the parameters. What students need to do is lift the formula from the text, identify the two variables and, in the text, find the two appropriate data points that should be inserted into the formula. This results in two equations with two unknowns, which can be solved for *a* and *b* respectively. The answers that the student will give are either correct or incorrect, and these answers have nothing to do with the contexts. In not one task, the created formula is put to use and given meaning within the described context. These tasks can also be termed as ‘reproductive mathematising’ (Vos, 2013). They are dressed-up tasks for solving two variables from two equations. In fact, the sequence of twelve tasks in this section is a dressed-up drill-and-practise activity.

The fourth section of this textbook chapter contains ten tasks and two worked examples, all of which are contextualised. The contexts are on the trajectory of a football, a physics experiment with a cart moving over a rough surface, water running out of a bath tub, a biologist counting crane-flies, labour productivity, and a meteorologist measuring weather temperatures. In these tasks, a context is described realistically (imaginable and not authentically), a formula is given with two parameters, and two data points are offered so students can determine the parameters *a* and *b* (just like in Fig. 3.4). The difference with the previous section is that there are additional questions to put the formulae to use to make predictions or to find maximal values. In all tasks, the questions are useful to imaginable actors in the context. Therefore, I evaluate these tasks as having realistic contexts, and the questions make sense within the context.

Summing up, in this average Dutch mathematics textbook chapter there were 50 tasks, out of which 19 (38%) were bare tasks posed in mathematical language only, 5 (10%) had mathematical contexts, and a little more than half of the tasks were related to real-life. Of these tasks 16 (32%) were dressed-up tasks and there were 10 (20%) tasks with realistic contexts and questions that made sense within the context. There were no authentic contexts. As a whole, this textbook chapter confirms the high level of contextualised activities in Dutch mathematics classrooms.

The analysis of this exemplary chapter may also offer a possible explanation for the tasks that the TIMSS Video Study (Hiebert et al., 2003) could not code, which means that according to this study 18% of the tasks in the Dutch classrooms had neither mathematical symbols only nor a real-life connection. It is possible that such tasks had informal language only for describing a mathematical context. As we have seen, such tasks can be found to a small extent in Dutch mathematics education (10% of the tasks in the analysed textbook chapter). Therefore, the TIMSS Video Study found tasks in the Netherlands that did not match either of the categories of (1) having mathematical symbols only, or (2) having a real-life connection.

Another observation that has to be made is that judging tasks individually within a chapter has its limitations. The tasks in the third section, which all asked: “Calculate a and b” were judged as dressed-up, because the answers for the parameters *a* and *b* were not useful at the very moment of doing the tasks. However, this judgement should be nuanced when the chapter is observed as a whole. The chapter starts with an explanation that gives a connecting thread until the end of the chapter. The introduction to the chapter highlights the importance of mathematical models for making predictions, and in the final section this is practised through scaffolded tasks, in which formulae are created to make predictions. Thus, the practise-and drill questions “Calculate a and b” become useful as an intermediate step for follow-up activities. So, what in the short term may look as dressed-up may be a stepping stone towards a realistic task where the context justifies the question, and where an answer is useful within the described context.

3.4 Contexts in Dutch Secondary Education National Mathematics Examinations

In this section I discuss the results from analysing a sample of mathematics tasks from the Dutch national examinations at the end of secondary education. The Netherlands has a system of exit examinations, which implies that all students who want to enter tertiary education (higher vocational education or university) have to pass one of the national examinations. Since teachers at secondary schools have to prepare students for these examinations, the national examinations have quite an impact on mathematics classroom practice, including the role of contexts in mathematics tasks. The purpose of my analysis of the examination tasks was: (1) to verify earlier claims on the high frequency of mathematics tasks with a real-life connection in Dutch education, and (2) to characterise the tasks.

For this analysis, I selected the examinations from 2010 for all available secondary school levels: pre-vocational (VMBO, examination at the end of Grade 10), general (HAVO, examination at the end of Grade 11), and pre-university (VWO, examination at the end of Grade 12). Each of these secondary school levels (pre-vocational, general and pre-university) has several examinations, depending on the track that students follow. At the pre-vocational level, there are three different examinations (for the tracks KB, BB, and GLTL), which differ mainly on cognitive demand (KB for the lowest achievers, and GLTL for the highest achievers in pre-vocational education). At general and pre-university level, there are two different examinations, for Mathematics A and for Mathematics B. The subject of Mathematics A is meant for students who are more interested in the social and economic sciences, while Mathematics B is for students interested in natural sciences and technology. In the analysis, I did not include the experimental computer-based examinations and the re-examinations, because the characterisation of contexts in these was not expected to differ from the regular examinations.

Each analysed examination paper is 11–13 pages long and contains much text, in which contexts are described, often accompanied by illustrations, diagrams or photographs. Students' reading time for these examinations must be considerable. All questions are grouped under a theme, which is indicated by a clear title. For example, the following titles are used:

- At pre-vocational level (VMBO GLTL 2010) there are 25 tasks grouped under the following headers: 'Pita bread', 'Quetelet index', 'From Betancuria to Antigua', 'Magnetic', 'Façade flag', 'Thunder and lightning'
- At general level (HAVO Mathematics A 2010) there are 23 tasks grouped under the following headers: 'A game of tennis', 'China's defence budget', 'Gas transport', 'Bullet proof vests', 'Fuel consumption by airplanes'

- At pre-university level (VWO Mathematics B 2010) there are 18 tasks grouped under the following headers: ‘Equal surfaces’, ‘Trivet’, ‘Rectangles touching a circle’, ‘Condensators’, ‘A rectangle in pieces’, ‘Logarithm and 4th power’, ‘A geo triangle’.

The titles indicate a wide variety of areas where mathematics can be used. Under each title a context is described, which serves as a context for several questions. In this way, each question does not have its own context, which reduces the reading time. Questions belonging to a title are independent of one another, that is, if students cannot answer one question, they can still complete ensuing questions that have the same context.

For example, the examination at the pre-vocational level includes questions grouped under the title ‘Pita bread’. The context is an event in the city of Eindhoven on 24 December 2004 where a huge pita bread was baked. The questions are about the diameter and the area of the baking tray, the required amount of flour, and the number of normal-size sandwiches that could be cut from it. The context is clearly authentic, as testified by a given date and an existing Dutch city (allowing for verification of the event), and the questions make sense within the context. The task shows clearly how mathematics can be useful outside school within recreational domains.

In this examination at the pre-vocational level, I also found a number of questions grouped under the title ‘Façade flags’, which contains the illustrations shown in Fig. 3.5. The text explains that there are three possible models. The students are asked to make a drawing in which Model 1 is mirrored, to calculate the lengths of sides *c* (in Model 1) and *d* (in Model 2), and to calculate the area of Model 3. As

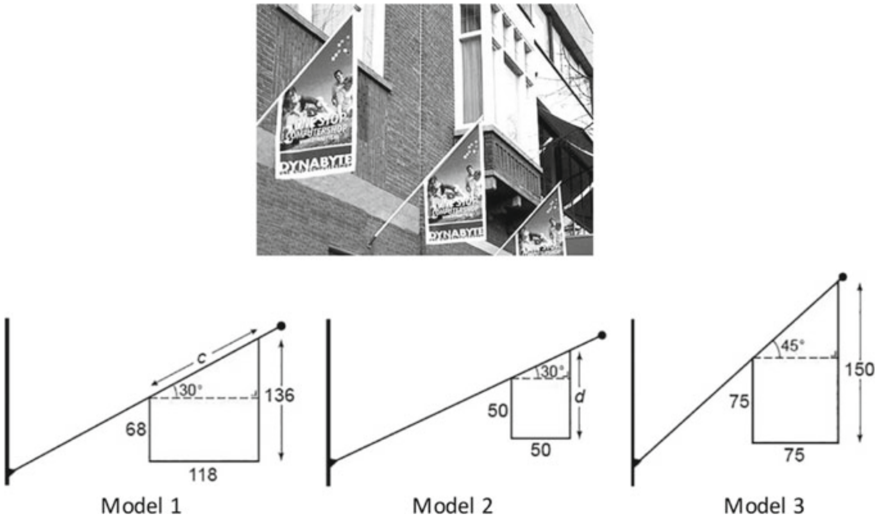


Fig. 3.5 Illustrations that go with the ‘Façade flags’ task (VMBO GLTL Examination, 2010)

there is no clear reason given why the drawing and calculations are needed, I coded the questions as ‘dressed-up’.

The examination for Mathematics A at the general level (HAVO) starts with two groups of questions clustered by authentic contexts. The context of the first group is a tennis match between Roger Federer and Fernando Gonzalez at the Australian Open Championships of 2007. The statistics of the match are given (points played, points won on first service, points on second service, and so on) and the questions are about the probabilities of winning points, which are meaningful for sports fans (and betting companies). The context of the second group of questions is China’s defence budget according to the Pentagon and according to the Chinese government information. The questions are about trends in the data, which are meaningful for critical observers of political information.

The third group of questions is clustered under a context of a company that transports gas, which is a context related to the fact that the Netherlands has a natural gas reserve and exports gas. The text explains that in the Netherlands there is a network of gas pipes bringing gas to families and businesses for heating and cooking. If it is very cold, then customers will need more gas and the maximum capacity of the network is reached. A certain unidentified company for gas transport uses the formula $P = 5.5 + \frac{18-T}{30} \cdot 94.5$, where P is the percentage of the capacity used, and T is the temperature. First, the students are asked about the properties of the formula (the range for T). Then some data are given on the occurrence of temperatures below −12 °C over the past 100 years, and the students are asked for a probability that such low temperatures occur on a day within a three-month winter season. Finally, the students are told that the above stated formula can be re-written in the shape $P = aT + b$ and do they have to calculate a and b.

The above questions are all set within an industrial context, implying that companies use mathematical formulae for their planning. However, the context is artificial, the given formula lacks credibility for real-life use, the probability question is not used within the context, and the final question to calculate parameters is pointless. All questions were therefore coded as dressed-up tasks.

In the Mathematics B examination at pre-university level I found a number of bare mathematics tasks, mainly on calculus. There is a task in which a trivet is shown (see Fig. 3.6), which consists of bars that can hinge. The text explains that this trivet has 19 equal rhombuses, and that the thickness of the bars will be ignored for creating a mathematical model for this trivet. The leftmost hinging point is indicated with P, the midpoint of the middle rhombus with O. The inner angle at P is α (in radians), and for the side of a rhombus length 1 is taken. Length l and width w of the model are functions of α, whereby $0 \leq \alpha \leq \pi$, and it is given that $l = 10\cos(½\alpha)$ and $w = 6\sin(½\alpha)$. The question then is: “Show that the formula for l and w are correct.” In the following questions these formulae have to be used for calculating angle α, at which w increases with the same rate as l decreases, for reconstructing a given formula for distance OQ, and for calculating angle α at which the trivet fits within a circle. I coded the task as a task with a mathematical context.

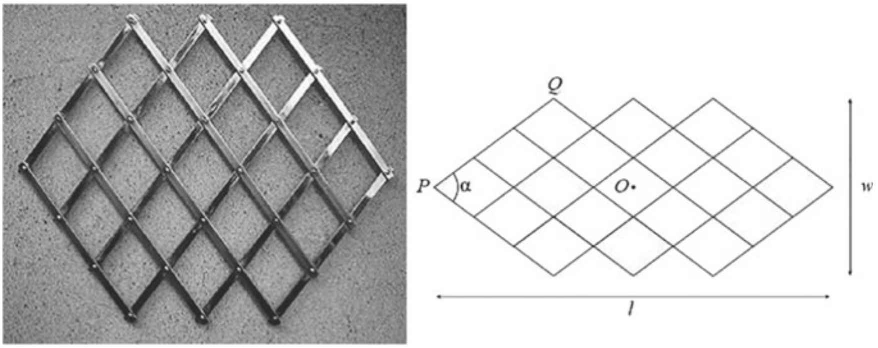


Fig. 3.6 Illustrations that go with the ‘Trivet’ task (Pre-university secondary education, Mathematics B Examination, 2010)

To give an overview of the context characteristics of the tasks in the Dutch mathematics examinations, Table 3.1 displays the types of tasks involved at each school level and the proportion of points that can be earned for each type of task.

From Table 3.1, we see that many of the tasks in the examinations at the vocational level and for Mathematics A at the general secondary education level, were set in contexts, in particular in realistic contexts. There were quite some dressed-up tasks (realistic descriptions, but questions that were not justified through the context), such as the ‘Façade flags’ task or the ‘Gas company’ task. All tasks contained some authentic contexts and questions that were relevant in such contexts, such as the ‘Pita bread’, the ‘Game of tennis’ and the ‘China’s defence budget’ tasks. The authentic contexts were mostly used in Mathematics A in general and pre-university secondary education and not that much in vocational education, meant for students who generally have a lower level of learning. Obviously, authentic contexts are more complex and require mathematics with higher demands. The examination papers for

Table 3.1 Context characteristics of the tasks in the Dutch secondary education mathematics examinations in 2010

Secondary school type (total amount of points)	Proportion of points that can be earned in				
	Bare tasks (%)	Mathematical context tasks (%)	Dressed-up tasks (%)	Realistic context tasks (%)	Authentic context tasks (%)
Vocational	0	14	16	57	13
Mathematics A					
General	0	0	18	37	45
Pre-university	0	9	38	0	52
Mathematics B					
General	29	5	56	0	10
Pre-university	25	57	18	0	0

Mathematics B (for students aspiring natural sciences and technology) contain more bare tasks than any other examination paper. This can be explained, because Mathematics A is a subject that aims more at modelling competencies and the practical use of mathematics, while Mathematics B aims more at conceptual understanding of mathematical concepts, such as the derivative or trigonometric functions. When we take the Mathematics A and B examinations together and compare the examinations at the general secondary education level with those at the pre-university secondary education level we see that the former has more dressed-up tasks, while the latter has more mathematical contexts (such as the 'Trivet ask').

Overall, the Dutch mathematics examinations of 2010 contain many context tasks, whether dressed-up, realistic or authentic, confirming the Dutch emphasis on connecting mathematics to real-life contexts. The contexts in the examinations were primarily from recreational practices (sports and leisure) or professional practices (commerce, research). As a driving force in classroom practice, the examinations clearly set out that mathematics is useful in many real-life situations, and that students can expect to encounter unexpected areas of mathematics application in the examinations.

3.5 Conclusion on Contexts in Dutch Mathematics Education

In this chapter I have described characteristics of contexts in mathematics tasks in the Netherlands. The underlying frame was the notion of usefulness as a subjective perception by students on future practices outside school. In analysing the tasks used in Dutch mathematics education, I made a distinction between bare tasks (without contexts), tasks with mathematical contexts (e.g., matchstick pattern tasks), dressed-up tasks (a context with a pointless question that hides a mathematical question), tasks with realistic or authentic contexts with questions that are useful within the context. I analysed a chapter of a mathematics textbook and a sample of examination tasks, confirming that, indeed, Dutch mathematics education contains many links to real-life, which are not just presented verbally, but also visually with drawings, photos, diagrams and other visualisations. The contexts are drawn from a wide spectrum of areas in real-life, reflecting that mathematics can be found anywhere in society. Most task contexts come from recreational or professional practices (economy, research), demonstrating to students the usefulness of mathematics in their future lives beyond school.

Many contexts can be said to be realistic (imaginable or experientially real), without being authentic. It was observed that the analysed examinations contained more authentic aspects than the textbook chapter, and the higher-level examinations have more authentic aspects than the lower level examinations. Nevertheless, there were also many artificial contexts in which the posed questions would not be asked by possible actors in these contexts.

Finally, a consequence of the typical Dutch feature of offering mathematics tasks with a relation to real-life is that the attribution of tasks to subjects is not always clear. This means that a task which in other countries is considered a task belonging to science education, can in the Netherlands be considered a mathematics task. This is what I experienced when I offered one of the physics tasks of TIMSS 1999 (see Fig. 3.7) to a number of mathematics teachers, and all of them said that they con-

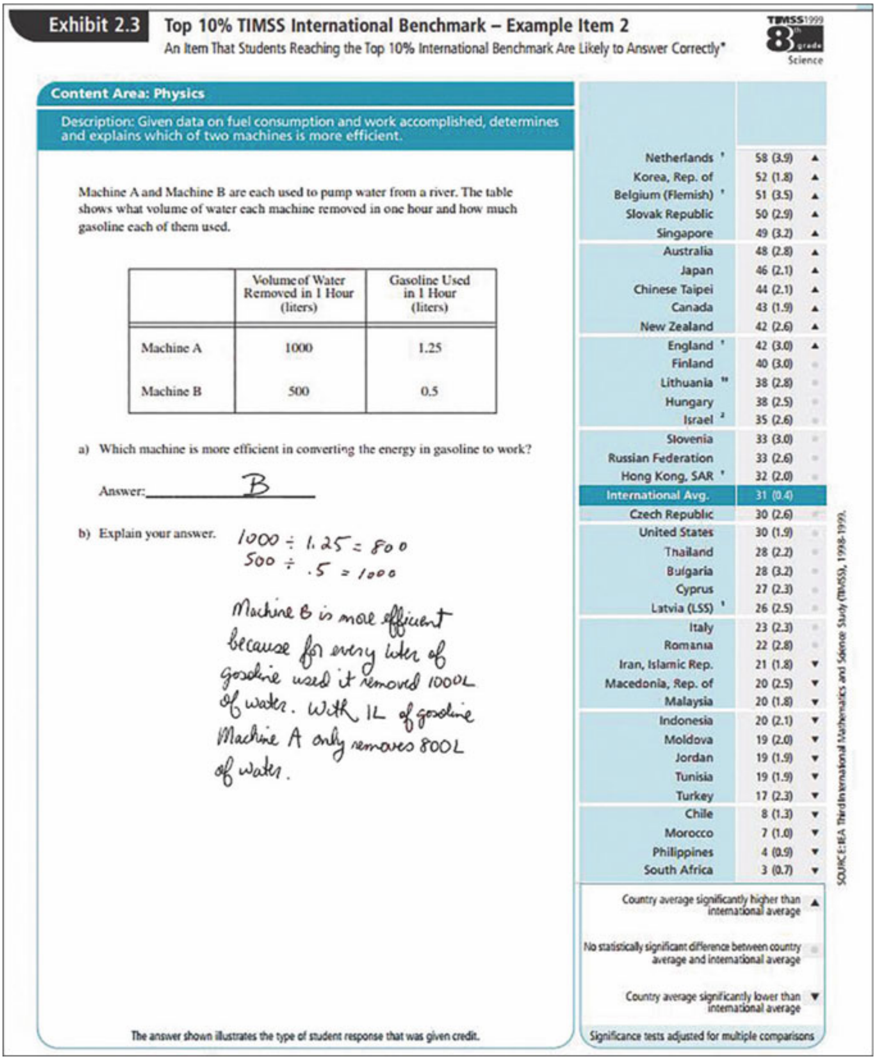


Exhibit 2.3 **Top 10% TIMSS International Benchmark – Example Item 2**

An Item That Students Reaching the Top 10% International Benchmark Are Likely to Answer Correctly*

TIMSS 1999 8th grade Science

Content Area: Physics

Description: Given data on fuel consumption and work accomplished, determines and explains which of two machines is more efficient.

Machine A and Machine B are each used to pump water from a river. The table shows what volume of water each machine removed in one hour and how much gasoline each of them used.

	Volume of Water Removed in 1 Hour (liters)	Gasoline Used in 1 Hour (liters)
Machine A	1000	1.25
Machine B	500	0.5

a) Which machine is more efficient in converting the energy in gasoline to work?

Answer: B

b) Explain your answer.

$1000 \div 1.25 = 800$
$500 \div .5 = 1000$

Machine B is more efficent because for every liter of gasoline used it removed 1000L of water. With 1L of gasoline Machine A only removes 800L of water.

The answer shown illustrates the type of student response that was given credit.

Netherlands †	58 (3.9)	▲
Korea, Rep. of	52 (1.8)	▲
Belgium (Flemish) †	51 (3.5)	▲
Slovak Republic	50 (2.9)	▲
Singapore	49 (3.2)	▲
Australia	48 (2.8)	▲
Japan	46 (2.1)	▲
Chinese Taipei	44 (2.1)	▲
Canada	43 (1.9)	▲
New Zealand	42 (2.6)	▲
England †	42 (3.0)	▲
Finland	40 (3.0)	■
Lithuania ‡	38 (2.8)	■
Hungary	38 (2.5)	■
Israel [2]	35 (2.6)	■
Slovenia	33 (3.0)	■
Russian Federation	33 (2.6)	■
Hong Kong, SAR †	32 (2.0)	■
International Avg.	31 (0.4)	
Czech Republic	30 (2.6)	■
United States	30 (1.9)	■
Thailand	28 (2.2)	■
Bulgaria	28 (3.2)	■
Cyprus	27 (2.3)	■
Latvia (LSS) [1]	26 (2.5)	■
Italy	23 (2.3)	■
Romania	22 (2.8)	■
Iran, Islamic Rep.	21 (1.8)	▼
Macedonia, Rep. of	20 (2.5)	▼
Malaysia	20 (1.8)	▼
Indonesia	20 (2.1)	▼
Moldova	19 (2.0)	▼
Jordan	19 (1.9)	▼
Tunisia	19 (1.9)	▼
Turkey	17 (2.3)	▼
Chile	8 (1.3)	▼
Morocco	7 (1.0)	▼
Philippines	4 (0.9)	▼
South Africa	3 (0.7)	▼

Country average significantly higher than international average ▲

No statistically significant difference between country average and international average ■

Country average significantly lower than international average ▼

Significance tests adjusted for multiple comparisons

SOURCE: IEA Third International Mathematics and Science Study (TIMSS), 1998-1999.

Fig. 3.7 'Fuel consumption of pumping machines' task from TIMSS 1999 (Martin et al., 2000, p. 65; copyright 2000 by International Association for the Evaluation of Educational Achievement (IEA), reprinted with permission) sidered it a normal mathematics task. The task was about two machines, which have a different fuel consumption and a different pumping capacity. The question was: "Which one is more efficient?" Such a question within the context of pumping water makes particular sense in a low-lying country which needs to stay dry, and which has a commercial culture in which effectivity and productivity are frequently used concepts. No wonder that on this task the Dutch students had the highest average score.

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Chapter 4
Mathematics and Common Sense—The Dutch School

Rijkje Dekker

Abstract The old illusion in Dutch mathematics education was that the teacher could lead the students into a completely new world, ignoring all their prior knowledge and common sense. Nowadays, in many Dutch mathematics lessons, the teachers encourage their students to use their common sense. This is the result of a silent revolution in mathematics education. In this chapter I will offer a collage of the work of several mathematics educators, who have helped to put the common sense of students in the middle of Dutch mathematics education. We will meet students from age 6 to 16 working with whole numbers, fractions, geometry and exponential functions and we will discover how their common sense plays a crucial role in the development of their mathematical knowledge.

4.1 Introduction

A long time ago, at the start of my first mathematics lesson as a secondary school student, the mathematics teacher told us: "Forget what you know, here you will learn all sorts of new things." As if we could delete all our prior knowledge and would not try to make sense of all these 'new things'. This is an illusion. An echo of this view still survives in places. When I had a discussion with a mathematics teacher at the university and asked him how they orientate themselves on their students' prior knowledge, he answered: "We just build the whole mathematics as new, so prior knowledge is not necessary."

In the meantime, in primary schools and later in secondary schools, a silent revolution in mathematics education has taken place and nowadays the motto in Dutch mathematics education is 'use your common sense!' (see Dekker, 1996).

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4.2 Common Sense of Young Students

What is the common sense mathematical knowledge of students before they get any formal mathematics education? Marja van den Heuvel Panhuizen (1996) researched this question. She developed a paper-and-pencil test to find out how much common sense mathematical knowledge and capabilities young students at the age of 6 already have before they start the formal mathematical course at primary school. She gave the test to 441 students in 22 different primary schools. The teachers only gave a short oral instruction. The test was very visual and consisted for instance of adding up the dots on two dice in the context of a board game, counting down in the context of a missile launch, and subtracting without countable objects in the context of buying sunglasses with a given amount of pocket money. She also gave the test to a group of experts, consisting of school consultants, teacher educators, educational researchers, educational developers and a few primary school teachers. She compared their answers with the answers of the students. The experts clearly underestimated the knowledge of the students. The task of adding up the dots on a dice was expected by the experts to be answered correctly by 45% of the students, while the actual result was 80%. The counting down task was expected to be answered correctly by 25% of the students, while the actual result was 65%. And for the task on buying sunglasses the experts thought that close to 0% of the students would give a good answer, while the actual result was 40%!

The research findings of Marja van den Heuvel-Panhuizen were very striking and gained much attention. Several countries, including Germany, have repeated the research with similar outcomes.

So, students acquire more mathematical knowledge in realistic contexts of their life than we are aware of. And as we want to connect the mathematical knowledge we want them to learn with the knowledge they already have, we can use these contexts and make the mathematics in them the object of discussion in the classroom, as you can see in many mathematical learning materials for primary schools in the Netherlands.

4.3 A ‘Math Mom’ at Work with a Small Group

In many primary schools the parents are asked to participate in reading with some children. A mother of one of the children in a primary school in the centre of Amsterdam suggested to the school that she could do something with mathematics as a so-called ‘math mom’. The school reacted very positively and her daughter’s teacher suggested doing something extra with fractions. Many students find that subject problematic. The school has a mathematics textbook series based on Realistic Mathematics Education (RME), but the reality in it is often restricted to pictures and the tasks are often meant for individual work with paper and pencil. To compensate for this, the math mom combined her knowledge about fractions in the context of pizzas (Treffers, Streefland, & De Moor, 1994) with her knowledge about collaborative learning (Dekker & Elshout-Mohr, 2004). On several afternoons, you could see her working with a small group of three children in Grade 4, where they played that they each were going to buy small pizzas to be shared by the four of them, mom included. The pizzeria had a special offer: three small pizzas for 5 euro. So, they had to divide. On the table, there were circles of cardboard of different colours, representing different types of pizzas; yellow could be cheese, red could be tomatoes or salami and green could be basil or olives. There were also scissors, pencils and rulers for the dividing.

The first child chose a type of small pizza and the math mom said that the three pizzas of this type would be served one after the other. The children discussed how to divide the pizzas fairly without having to wait for the next one to be served. The math mom intervened as little as possible. She only did so when she noticed that a child was not taking part in the discussion or was not ‘heard’. The children decided that they would divide each pizza in four pieces and so they did. Each of the four persons ended up with ¾ pizza, and then the math mom asked them to write that down in a fraction problem that reflects this action. Again, they discussed it and then decided to write it as: $¼ + ¼ + ¼ = ¾$.

Then the second child chose another type of pizza and the math mom told them that now two pizzas would be served at the same time and the third one a bit later. Again, the children discussed how to divide fairly without waiting until the third one arrived. They decided that it made sense to give each participant half a pizza and divide the last one in four pieces later. Then they discussed what fraction problem reflected this division and they came up with: $½ + ¼ = ¾$.

The third child also chose a particular type of pizza, and the math mom told them that now all pizzas would be served at the same time. The children decided that they would each take one pizza and that each of them would give the math mom a quarter of their pizza: $1 − ¼ = ¾$.

So, in each situation each person ended up with ¾ of a pizza. But each situation was a different one and so was the fraction problem that reflected this situation.

The children clearly enjoyed this ‘play’, and the math mom did as well. But as she also wanted them to reflect on the work, she asked them at the end to individually describe in a little story one of the situations, visualise it in a drawing and write down the problem that fits this situation. Most children chose their ‘own’ division, but some described them all and with beautiful drawings, even a comic was drawn; and with the appropriate problems (Fig. 4.1). Mathematics can be fun, especially when you bring to life the situations in which it can develop.

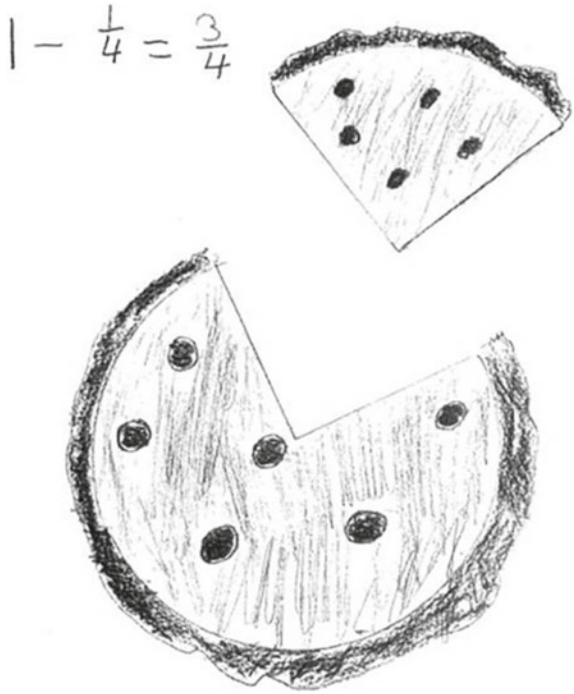

Fig. 4.1 Visualisation of 1 − ¼

4.4 A Russian Pioneer Within the Dutch School

Tatiana Ehrenfest-Afanassjewa (1876–1964), a Russian mathematician, came to the Netherlands in 1912 with her husband Paul Ehrenfest, the physicist, who had got a job at the University of Leiden. Tatiana had a deep interest in mathematics education. In Russia, she had developed an introductory course for geometry. The course was based on the idea that students already have developed intuitive geometrical notions in reality. These notions form the start of the course. In her *Übungensammlung* (Ehrenfest-Afanassjewa, 1931, p. 27), she described all sorts of problems to be analysed by the students. Problems like:

> Warum läuft der Mond uns nach? Warum laufen uns die nahen Gegenstände rascher vorbei, als die entfernten, wenn wir etwa in einem Zuge fahren? – Eine schematische Zeichnung machen.[1]

In the Netherlands, Tatiana started to invite mathematics teachers and mathematicians into her home in Leiden to discuss mathematics education and her ideas about it (De Moor, 1993). Most of her guests were ‘shocked’ by her radical ideas, but some

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[1] Why does the moon run after us? Why do near objects pass us more rapidly than distant ones when we ride, say, in a train? Make a schematic drawing (translated by the author).

of them became really interested and tried to put her ideas into practice. Dina van Hiele-Geldof, for example, developed her lessons on tiles and geometry and analysed the classroom conversations and student work wonderfully in her dissertation (Van Hiele-Geldof, 1957). There was also a German mathematician who had come to the Netherlands and was impressed by Tatiana's revolutionary ideas: Hans Freudenthal. He discussed these ideas in her house and in public (Ehrenfest-Afanassjewa & Freudenthal, 1951). Later he stimulated developers of learning materials to use her ideas and now in many Dutch mathematics school books for secondary school subjects like looking along lines, or vision geometry, form a substantial part of geometry. The following exercise is used even in primary schools. It is taken from the Ehrenfest-Afanassjewa's (1931, p. 25) *Übungensammlung*:

> Es sollen sich drei Schüler längs einer Geraden vor die Klasse aufstellen – ohne irgend welche Hilfsmittel zu gebrauchen; ein vierter Schüler soll sie, ebenfalls ohne Hilfsmittel, kontrollieren. – Worauf beruht die Möglichkeit dieser Aufgabe?[2]

4.5 A World of Packages

Wim Sweers was a teacher at a secondary school where many students had problems with mathematics. The subject he focused on was three-dimensional mathematical shapes, which is always a very difficult subject. Walking through the school and thinking about problems to introduce the subject, he noticed a glass case with many beautiful packages, made by his own students! He realised that they had to do a lot of implicit mathematics to create these packages. He decided to bring the world of packages into the classroom and so he did. He asked his students to collect different empty packages from home and he invited them to sort them according to function, material, colour and also according to the mathematical shapes which he wanted to teach them. Then the students were invited to deconstruct the packages and reconstruct them again, and so they discovered all sorts of characteristics of the shapes. And step by step he guided his students from the world of packages into the world of mathematical shapes, which were now much more concrete to them than they had been. He got the feeling that his students would now be able to develop some spatial insight.

This strong idea has found its way into Dutch mathematics school books and also into mathematics schoolbooks abroad, for instance in Portugal.

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[2] Three students should stand in a straight line in front of the class without using any tools; a fourth student should check them, also without any tools. What makes this problem possible? (translated by the author).

4.6 A Real Problem in the Classroom

During one of my visits to a mathematics lesson by teacher Lidy, she introduced a problem about cleaning a brush after painting. She said that she had painted her garden fence and that she wanted to clean the brush. She found only a little bit of turpentine in a bottle and she wondered what she could do best: pour the turpentine in a jam jar and clean the brush, or first divide the turpentine over two jam jars and clean the brush in the first jar and then in the second. The students worked in small groups and the teacher gave them all the information they wanted. They concluded to their own surprise that it was better to divide the turpentine over two jam jars. Then the teacher went on and asked them what would happen if the turpentine were divided over more cups, and more and more… and so she was planning to draw her students into the world of formal mathematics with the number 'e' at the horizon.

One small group of students was arguing a lot. They wondered whether it would work to clean a brush in just a little bit of turpentine. They remembered a test they did in chemistry where they put sugar in water and noticed at a certain moment that the sugar did not dissolve any more. They thought that could also happen when the quantity of turpentine was too small. I witnessed their discussion. These students clearly stopped in front of the door of formal mathematics. Their common sense prevented them from passing it. Later I told a colleague from abroad about this conversation and she reacted that the teacher should have told the students that in the mathematics lesson it is all about mathematics. I heard an echo of an old message… I also told teacher Lidy about my observation and she answered: "I should have discussed the problem with the chemistry teacher first."

Lidy made my heart glow. She clearly is a member of the Dutch School!

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Chapter 5
Dutch Mathematicians and Mathematics Education—A Problematic Relationship

Harm Jan Smid

Abstract Mathematics as a compulsory school subject was introduced in the Netherlands in the first decades of the 19th century. While in the beginning there was some involvement of Dutch academic mathematicians, later on their engagement with mathematics teaching was only marginal. That changed in the second half of the 20th century. Hans Freudenthal, professor of mathematics in Utrecht, became deeply involved in mathematics teaching. He became the first director of the IOWO, the Institute for the Development of Mathematics Education, that dominated Dutch mathematics teaching from the 1970s on. In the 1960s, under the influence of New Math, other mathematicians had already played a role in the modernisation of the teaching of mathematics, but from the 1970s on, their role became minimal again. In the first decade of the 21st century the dominance of the ideas of Realistic Mathematics Education elicited protests from mathematics departments at several universities. This criticism induced fierce and often heated debates. At the moment, these discussions have calmed down and it seems that a new understanding between the worlds of school and university mathematics is growing.

5.1 Start of a Tradition of Academic Involvement in Mathematics Teaching?

In September 1826, D. J. van Ewijck, one of the highest-ranking government officials of the Ministry of the Interior and in charge of educational affairs, wrote a letter to all Latin schools[1] about the teaching of mathematics. In this letter, he explicitly recommended the use of the textbooks of J. de Gelder, professor of mathematics at the University of Leiden. De Gelder, a former schoolteacher and now a prominent

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[1] Latin schools were grammar schools for boys of approximately 12–18 years old to prepare them for university.

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mathematician, wrote books on arithmetic, algebra and plane geometry, especially destined for use in Latin schools (Smid, 1997).

By advising these books, the Dutch government sailed an intermediate course between Prussia and France. In Prussia, the mathematics teachers in the gymnasia (grammar schools) enjoyed complete freedom in the choice of their textbooks, in France, the government strictly controlled which books were used. In Prussia not seldom the teacher did write his own textbook, or did not use one at all, while in France the government stimulated, or even required the use of textbooks written by eminent mathematicians.

The letter of Van Ewijck, and the books of De Gelder, who did not only write textbooks, but was involved in many ways in the teaching of mathematics, might well have been the start of a tradition of academic involvement in mathematics teaching. To call such a tradition into existence, two conditions had to be fulfilled. The government should stimulate and facilitate an active role for mathematics experts in secondary education. These experts themselves had to be interested in fulfilling such a role, and be willing to spend time on it. In the persons of Van Ewijck and De Gelder, these requirements were met.

5.2 Aloofness of the Government

But the actions of Van Ewijck and De Gelder were not followed by further steps. On the contrary, the growing political influence of liberally orientated politicians, such as J. R. Thorbecke, who wanted government interference with internal school affairs to be as minimal as possible, made these impossible. The laws of 1863, in which the *HBS*,[2] the Dutch variant of the German '*Realschule*' was created, and of 1876, which did the same for the modernised gymnasia, gave only a short outline of the content of the required mathematics curriculum (Krüger, 2014). Schools and teachers were left a great amount of freedom to organise their teaching.

The main advisor for the law of 1863 was D. J. Steyn Parvé, a government official who had been a mathematics teacher at the gymnasium in Maastricht. Steyn Parvé was a competent mathematician, but not a productive researcher and had not had an academic career. Soon after the introduction of the law, he was appointed as one of three *inspectors* for the HBS. The corps of inspectors, three[3] for the HBS, one of whom was always a mathematician, and one for the gymnasia, constituted the intermediary between the government and the schools. They were usually former teachers, who had made a career as headmaster of a HBS or rector of a gymnasium. They were not chosen for their scientific reputation, but for their experience in education. Some inspectors, like Steyn Parvé or, later, E. Jensema, became quite

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[2] Hogere burgerschool; the former Dutch general secondary school for 12- to 17-year-olds intended as a practically oriented education for higher functions in industry and trade.

[3] At the start of the HBS three inspectors were appointed, but the number varied over the years.

influential in the field of mathematics teaching, and the government usually followed their advice.

Towards the end of the 19th century, the political discussions about religious schools and their financing, the so-called '*school war*', forced the government to even greater restraint in internal school affairs. After the compromise that was reached in 1917 ended the conflict. A reserved attitude towards internal school affairs remained the ultimate wisdom. Until the 1960s the Dutch government did not want to play a prominent role in schools. "*No state pedagogics*", was its credo.

5.3 No Role for the Experts

If the experts wanted to play a role of any importance in secondary education or exert serious influence on it, they would have had to achieve this on their own strength, without government support. That would not have been easy.

For example, changes in the curriculum were usually discussed by teacher unions, or special committees of teachers, and then had to pass the corps of inspectors before the government took any decision. University professors did not have any formal role in this process. They played some part in the final school examinations, through taking part in oral examinations and controlling the grading of written assignments. But these written assignments were devised by a select group of teachers and determined by the inspectors, the experts had no say in this. Nor did they have any involvement with the textbooks. The government left the production of textbooks to the market; textbooks were written by teachers and teachers were free to choose those books they liked most.

If the group of university experts had shared some common ideals and goals about mathematics teaching, and if they had combined their efforts, they might still have gained some influence. But then they should have been willing to spend time and energy on a more than incidental basis on questions and problems concerning mathematics education.

That was not the case. Involvement by professional mathematicians in mathematics teaching for secondary schools remained incidental and the concern of isolated individuals. The development of the Wiskundig Genootschap (Mathematical Society) is an example of this. Originally founded in 1779 as a society for mathematics practitioners and schoolteachers, in the 19th century it became a society for research mathematicians, and school teachers hardly played a role anymore. The society never formulated any advice or proposal concerning mathematics education.

An example of an expert who at least was interested in teaching and education was D. Bierens de Haan, professor in mathematics in Leiden in the second half of the 19th century, with a good international reputation. He had been a mathematics teacher in Deventer, and was an elementary school inspector alongside his professorship, he wrote some mathematics textbooks (including an adaptation and translation of Lacroix's *Elements de Geometrie*) and played a minor role as an advisor of Thorbecke concerning the law of 1863. He collected an impressive collection of books concerning (mathematics) education, which is now in the University Library of Leiden. Even so, one cannot say that he had any important or lasting influence on secondary mathematics teaching.

There were more mathematicians who were interested in teaching, like G. Mannoury and D. van Dantzig. Mannoury was a professor in Amsterdam, Van Dantzig was one of Mannoury's students and after World War II also a professor in Amsterdam. In the late 1920s they wrote some articles about mathematics teaching, but the ideas of Mannoury and Van Dantzig differed so much from the mainstream ideas in those days, that already for that reason alone they had hardly any influence (Smid, 2000). They criticised the general formative value of mathematics and transfer to other subjects. Another example of a professional mathematician who was interested in mathematics teaching was F. Schuh, professor in Delft and Groningen. Not only did he publish many articles about mathematics especially for teachers, but he even wrote a book on the didactics and methodology of mathematics (Schuh, 1940). Schuh did not propagate modern mathematics to the community of mathematics teachers but restricted himself to then already outdated 19th century topics. His book on didactics and methodology is not written for secondary education, but for junior students in the sciences and technology and their lecturers. It is mainly a collection of tips and tricks, for example, how to use a textbook or solve certain types of mathematics problems or about the best way to pass an exam. There were more professors in Delft who were interested in mathematics teaching, but as a rule they were even more conservative. *Euclides*, the magazine for mathematics teachers, founded in the early 1920s, regularly published articles by mathematics professors, for example, their inaugural addresses (with a photo of the new professor), but these seldom had to do with the teaching of school mathematics.

A far more interesting person was Tatiana Ehrenfest-Afanassjewa, born in Kiev in Ukraine. She was a physicist in the first place, but had also studied mathematics in Göttingen with Klein and Hilbert. She had no official position at a university, but being the wife of Paul Ehrenfest, the successor of Lorentz, in 1912 she came to live in Leiden. She wrote some interesting articles on mathematics teaching, published her now famous *Übungensammlung* (Ehrenfest-Afanassjewa, 1931), and organised a discussion group about mathematics teaching (De Moor, 1999). Before World War II, this group consisted mainly of outsiders and had little influence, but that changed after the war.

Things might have been different if the only Dutch mathematician of great international reputation, L. E. J. Brouwer, had been interested in teaching. But Brouwer had no interest at all. In the Netherlands, there simply was no Felix Klein, Émile Borel or Guido Castelnuovo.

So, for a long time Dutch mathematics teaching received little influence or impulse from the scientific community. To a large extent, mathematics teachers could settle their own affairs, not only at the HBS and gymnasium level, but even more so at the numerous MULO[4] schools.

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[4] (Meer) Uitgebreid Lager Onderwijs ((Further) Extended Primary Education).

5.4 A Stagnating World

In a way, the system could be called democratic: teachers could have their say concerning changes, and developments and drastic interventions from outside did not occur. But there was a price to be paid for this. After the attempts to renew Dutch mathematics teaching in the 19th century, a long period of conservatism and stagnation followed. The world of Dutch mathematics teaching was trapped in isolation. For instance, the first international reform movement, initiated by Felix Klein, had hardly any influence in the Netherlands (Smid, 2012). In the Commission Internationale de l'Énseignement Mathématique (CIEM), the forerunner of the International Commission on Mathematical Instruction (ICMI), Dutch mathematicians played no role of significance. The most prominent schoolbook author of the first half of the 20th century, and founder of several journals for teachers, P. Wijdenes, claimed that he never consulted a foreign secondary school textbook.

Of course, there were some reform attempts, mainly to include calculus into the curriculum. The attempt that at least had a partial result was the one led in the 1920s by E. J. Dijksterhuis, who later became famous as a historian of science. He was progressive concerning the teaching of calculus, but an outspoken conservative on the teaching of geometry. As a historian, Dijksterhuis, who later became professor in the history of science in Utrecht, but who was an HBS teacher at that time, had ample international contacts, but hardly any concerning mathematics teaching. He was aware of the publications of Klein, and at a conference in Groningen in 1925 he had a public discussion with Walter Lietzmann, rejecting all his ideas about modernising mathematics teaching, but there are no indications that he was really interested in modern developments in teaching abroad. Dijksterhuis' attempts on the introduction of calculus received hardly any support from the academic community. They were even opposed by some academic groups, such as the mathematics professors at the Technical University in Delft, who preferred to keep the HBS as it was in the 19th century.

On the whole, the result was stagnation (Van Berkel, 1996). Around 1955, the world of Dutch mathematics teaching had a surprising resemblance to that of 1875. Change would come in the 1960s, under pressure from the government, and led by experts from outside.

5.5 The Times They Are A-Changin'

In the 1950s, the pressure on changing the *status quo* was slowly building up. First of all, the modernising society created new jobs and roles for mathematicians, and more generally, for experts in the sciences. Statistics became more important, not only in mathematics and the natural sciences, but also in the social sciences. The idea that the teaching of mathematics was only important for most students as '*gymnastics for the mind*' became untenable. Mathematics and good mathematics education became an economic necessity. The idea of *transfer*, that is, the idea that learning mathematics would automatically make you a better thinker, was more and more criticised. On top of that, within the academic community of mathematicians there was a growing concern about the gap between what was essentially 19th century school mathematics and modern, 20th century mathematics. These were not national, but international trends, but unlike in the years before World War II, the Netherlands could no longer afford an isolated position.

A sign of the growing concern and interest for mathematics education within the community of mathematics experts was that in 1954, the Wiskundig Genootschap founded the Nederlandse Onderwijs Commissie Wiskunde (Dutch Committee for Mathematics Education), a committee in which professional mathematicians and mathematics teachers cooperated. The committee, soon chaired by Hans Freudenthal, operated also as a subgroup of ICMI.

Individual experts also showed their interest. One of them was Van Dantzig, mentioned before as a student of Mannoury. In some articles, he had published around 1930 as a young man he had argued that the way mathematics was taught was quite useless for most students. He had not obtained any hearing, and he remained silent on this subject for over twenty years. He was now professor of mathematics in Amsterdam, had specialised in statistics and was deeply involved in consulting activities for industry and society. In 1955 Van Dantzig wrote a report for ICMI, *The Function of Mathematics in Modern Society and Its Consequences for the Teaching of Mathematics* (Van Dantzig, 1955). A translation was also published in *Euclides*, the Dutch magazine for mathematics teachers. It did not have immediate effects, but it certainly had influence on the long term.

There are more examples of mathematics professors who wrote articles on mathematics teaching in the same spirit. For instance, in 1958, J. C. H. Gerretsen, professor of mathematics in Groningen, wrote an article in *Euclides* about the goals of mathematics education. The article was written on the occasion of the new curriculum of 1958, and it stressed the need of the modernisation of mathematics teaching on account of the now crucial role played by mathematics in modern industrial society. The article ends with the prophetic words that the new curriculum could be seen as a deserving step forwards, but that it should not be seen as a definitive curriculum, not even as a programme for the long term (Gerretsen, 1958).

Not all mathematicians shared this view about the role of mathematics in society. The logician E. W. Beth, known for his cooperation with Jean Piaget (see Piaget et al., 1955), maintained the point of view that the introduction of axiomatic reasoning was the ultimate purpose of mathematical instruction, but he was an exception.

The involvement of most mathematicians like Van Dantzig and others in those years can be characterised in two ways: (1) it had as a starting point their concern about the *content* of school mathematics, and (2) it had an incidental character and did not lead to a *permanent* involvement of the authors in the affairs of mathematics education.

Neither of these two characteristics can be applied to the activities in mathematics education of another mathematician: Hans Freudenthal. He became seriously interested in the didactics of mathematics during World War II, and after that he soon joined the Wiskunde Werkgroep (Mathematics Working Group), a continuation of the discussion group led by Tatiana Ehrenfest-Afanassjewa and now rapidly gaining importance as a focal point for all who had the revitalisation of Dutch mathematics teaching as their goal. Within a short time Freudenthal became the chair of the Mathematics Working Group, which was the starting point of a permanent involvement in mathematics education. Of course, Freudenthal was well aware of the outdatedness of Dutch school mathematics. He and his ideas played an important role in the realisation of the new curriculum of 1958, which introduced calculus to the curriculum and examination programmes, and removed descriptive geometry (La Bastide-van Gemert, 2015).

But unlike other mathematicians who were interested in mathematics education, Freudenthal's focus was primarily on good teaching, not on modernising the curriculum. In a letter, written in August 1961, Freudenthal wrote:

> I have argued several times, as is well known, that I see the modernisation of the curriculum (…) not as an urgent problem, not because I should dislike modern mathematics, but because of the fact that in several proposals the introduction of modern mathematics is seen as a principal goal. On the contrary, I see as the first and only urgency the improvement of mathematics education. (Wijdeveld, 2002, p. 202) (translated from Dutch by the author)

Within ten years, Freudenthal's point of view would become the dominant one. That would have great consequences for Dutch mathematics teaching.

5.6 The Big Bang

The letter by Freudenthal cited in the previous section was addressed to A. F. Monna, then lecturer, later professor of mathematics at the University of Utrecht. Monna, who also had made a career as a government official at the Ministry of Education, was the secretary of the Commissie Modernisering Leerplan Wiskunde (CMLW; Commission Modernisation Mathematics Curriculum).

That commission was a new development in the world of Dutch mathematics teaching. It was appointed in 1961 by the Dutch government and had as its task to advise the government about the modernisation of the mathematics curriculum. The founding of the commission was a direct consequence of the Royaumont conference. Convinced of the urgency of such a modernisation, the government did not want to wait for initiatives from mathematics teachers themselves, and appointed a commission that consisted of ten professors and a lecturer in mathematics, a teacher educator, two inspectors and only four teachers. Its chairman was H. T. M. Leeman, mathematics professor at the University of Amsterdam, who had attended the Royaumont conference. The other two Dutch delegates, L. N. H. Bunt, a teacher educator and P. G. J. Vredenduin, a mathematics teacher and textbook author, who went to Royaumont were also appointed. Freudenthal was one of the committee's members. As the State Secretary for Education stated, the CMLW had the explicit task to advise the government about

> which modern parts of mathematics could, seen from the viewpoint of science, be apt for introduction in schools preparing for university, in view of the reduction of the gap that exists between university and school mathematics", as the deputy minister of education formulated it in his address on occasion of the installation of the committee. (Euclides, 1962, p. 146)

There were some additional questions, about special programmes for mathematically gifted children, possible experiments in schools concerning the teaching of modern mathematics, and courses to introduce older mathematics teachers to modern mathematics, but the main purpose was fairly straightforward: the modernisation of mathematics curricula in schools preparing for university.

Freudenthal had not attended the Royaumont conference as he did not expect it would be very important. That was a mistake, as he admitted later, and due to his stay in the United States, Freudenthal had also missed the first meeting of the new commission, so in a way he was lagging behind. As soon as he had returned to the Netherlands, he took action, as the letter of August 1961 shows. He made it clear that the main and official task of the commission, the modernisation of the curriculum, was not his first priority. He had exerted considerable influence on the curriculum of 1958, with which he was rather content, and he saw no reason for immediate change. In its first meeting, the commission had mainly discussed possible changes in the examination programmes, but Freudenthal had other priorities. He wanted to start with the lower grades.

In his autobiography, Freudenthal (1987) suggests that the CMLW initially viewed him as an 'enfant terrible', and that the subgroup that was created on his request and that should concern itself with the lower grades, was no more than a kind of playground specially created for him. That seems rather unlikely. At that time, Freudenthal was already without a doubt the most outstanding mathematician within the commission, and moreover, he was the only one who had already been deeply involved in the didactics of mathematics, both nationally and internationally, for more than fifteen years. He soon succeeded in bending the commission to his will.

In the first years, most of the commission's work focused on two aspects: developing courses for teachers to make them acquainted with modern mathematics, and carrying out teaching experiments, including one for the lower grades on transformational geometry and one for the higher grades on calculus. The courses for teachers, that attracted a large number of participants, were in line with the terms of reference of the commission, the teaching experiments in fact much less so.

The work of the CMLW was complicated by a development that had nothing to do with the teaching of mathematics itself. Finally, in the 1960s the government was successful in replacing the patchwork of 19th century laws on education with one comprehensive law, creating a complete new school system, with new school types. The Ministry of Education asked the commission to devise curricula for all new school types, a much larger task than foreseen at the start. After some years, even mathematics for elementary schools became a topic of study for the commission. In the end, the commission was able to present a complete set of curricula for all new school types.

To do all this work, the CMLW appointed a substantial group of co-operators, mainly former mathematics teachers. In the early 1970s, this group formed the core of the newly-founded IOWO,[5] now Freudenthal Institute (FI). Freudenthal became its first director.

Halfway through the 1960s, Wimecos,[6] one of the two then existing mathematics teachers' associations, tried to devise a program for one of the new school types on its own, but its effort was soon over taken by the commission (Wijdeveld, Verhage, & Schoemaker, 2000). In the 1970s, the union tried again to gain influence on curriculum formulation by setting up a didactical working group for that purpose, but to no avail. Constructing curricula was definitively out of the hands of teachers and in the hands of the experts.

For most of the mathematics professors, the work in the CMLW must have been a bewildering experience. They were mathematicians in the first place, with some interest in mathematics teaching. They were asked to advise the government in their spare time about the modernisation of the mathematics curriculum of schools preparing for the university, which seemed a pretty simple and straightforward task. But their commission ended in a complete institute with more than a dozen of full-time collaborators, all of them with a primarily didactical orientation, performing tasks with which most of these mathematics professors had hardly any affinity. What was their role? Apart from Freudenthal and Van der Blij, a younger colleague of Freudenthal who shared many of his ideas and who soon succeeded Leeman as chair of the CMLW, did they still have any influence?

The archives of the CMLW have been lost, so it is impossible to reconstruct the complete history of the commission.[7] But one conclusion can be drawn. When the committee started its work, it seemed as if the experts, mainly the mathematics professors, had at last obtained considerable influence in the world of mathematics teaching, at the expense of the traditional organisation of teachers of mathematics.

Then, years later, the picture was completely different. The community of professional mathematicians had again disappeared from the educational scene, and would not return there for decades. The mathematics teachers' unions had indeed lost a great deal of their influence which was now in the hand of another group of experts: full-time didacticians, first only within the IOWO and its successors OW & OC[8] and FI, later also in SLO (Netherlands Institute for Curriculum Development). Of course, the first two directors of the IOWO, Freudenthal and Van der Blij, were excellent mathematicians, but in their role as IOWO directors they acted much more as didacticians than as representatives of the group of mathematics experts. Their successors, J. de Lange and J. van Maanen, had made their career in teaching, not in mathematics research. Didactical experience and expertise had become much more important than mathematical brilliance.

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[5] Instituut voor de Ontwikkeling van het Wiskunde Onderwijs (Institute for the Development of Mathematics Education).

[6] Vereeniging van Leeraren in de Wiskunde, de Mechanica en de Cosmographie aan Hoogere Burgerscholen en Lycea (Association of teachers of mathematics, mechanics, and cosmography).

[7] The archives of the CMLW are recently rediscovered in the Central Archives of the Ministry of Education in The Hague.

[8] Onderzoek Wiskundeonderwijs en Onderwijs Computercentrum (Mathematics Education Research and Educational Computer Centre).

5.7 Return of the Mathematicians

Freudenthal and his institute dominated the Dutch world of mathematics education from the 1970s on. In the 1980s, most of the examination programmes in secondary school had been reformed and contained at least some elements in the spirit of Realistic Mathematics Education, and in the 1990s the same happened with the programmes of the lower grades in secondary education and the textbooks in primary education. All these and other changes to the programmes were of course the cause of some discussion, but on the whole, the world of primary and secondary education agreed with the way things were going.

Professional mathematicians and their organisations seemed hardly interested. Typically, at most universities, teacher education was removed from the mathematics departments and centralised under the authority of general educationalists. The mathematics departments did not protest. Educating a mathematics teacher was not what they were interested in.

That changed in the first decade of the 21st century. Universities, especially the universities of technology, started to complain about the lack of algebraic skills of their first-year mathematics, science and technology students. According to these departments, their students were not able to handle the simplest techniques. They blamed the new programmes and criticised the extensive use of (graphic) calculators in mathematics teaching in secondary schools. Universities started to organise entrance tests for their first-year students, and offered courses to repair their shortcomings, sometimes even obliging their students to follow these courses if they had failed to pass the tests. Surprisingly, the students joined in with the complaints of their lecturers and exerted pressure to the Ministry of Education to put more emphasis on the teaching of algebraic skills in school.

Another point of criticism concentrated on the teaching of arithmetic in primary schools. According to the critics, national and international evaluations showed that the performance of Dutch children was deteriorating, and the critics blamed the influence of Freudenthal and his school for that.

The main initiator of all this criticism was J. van de Craats. He had been lecturer at the University of Leiden, professor at the Military Academy in Breda and then professor at the University of Amsterdam. In the beginning, he had been quite enthusiastic about the Freudenthal reform, and as chairman of one of the committees that were charged with drawing up examinations questions, he had had some responsibility for the developments in the 1980s and early 1990s. But in the long run he became doubtful about their results and he started to sharply criticise Freudenthal's work. Van de Craats gained support from professors of other universities, like F. Keune and K. Landsman from Nijmegen, and M. Pelletier from Eindhoven (Van de Craats, 2008).

The result was a lot of often heated discussions, since the advocates of Realistic Mathematics Education of course defended their positions. The discussion about the teaching of arithmetic, which ran the most heated, led to a request by the State Secretary for Education to the Royal Dutch Academy of Sciences (KNAW) to investigate the matter. The KNAW appointed a commission with some outstanding mathematicians, like J. K. Lenstra and R. Tijdeman, with specialists in educational research like the Belgian L. Verschaffel, and with field specialists in arithmetic teaching like M. Kool. The main conclusion of the report (KNAW, 2009) was that there was indeed reason for concern, but that there is in fact no hard evidence for better results either for the 'realistic' school or for the 'back to basic' movement. Another important conclusion was that the quality and know-how of the teacher are the main factors in explaining teaching results, and that it is beyond doubt that in this respect, in the last decades, students enrolled in teacher education for elementary school teachers who had insufficient knowledge of mathematics and did not receive enough education to remediate their mathematical competence. As a result of this conclusion, the demands on arithmetic for future teachers have been raised considerably, a measure that has the approval of all concerned. Another consequence of the discussions on arithmetic teaching was that the government started to implement compulsory tests on arithmetic in secondary education. However, that measure became subject of severe criticism.

The discussion on algebraic skills also had its effects. During the last decade, the emphasis on skills in the final examinations was gradually raised, which had its influence on teaching in school. In the in 2005 established commission for the revision of the mathematics curriculum, the Commissie Toekomst Wiskundeonderwijs (cTWO; Commission Future Mathematics Education), the problem of algebraic skills received more attention than in the years before. While in former curriculum committees, the dominance of the FI and the SLO was considerable, in cTWO that was not the case. The committee was chaired by D. Siersma, a professor of mathematics at Utrecht University and numbered more professional mathematicians, including M. Pelletier, one of the critics of the FI. The committee counted only one member of the FI, P. Drijvers, who acted as the secretary of the committee. Other didacticians of mathematics were also appointed, as well as teachers and representatives of the Nederlandse Vereniging van Wiskundeleraren (NVvW; the Dutch association of mathematics teachers). In a way, cTWO signified the official return of professional mathematicians and teachers and their union to the discussions about mathematics education in secondary school. The work of the committee was critically followed by another committee explicitly appointed by the minister for that purpose. This committee was the so-called 'Resonance group', chaired by Van de Craats and mainly consisting of university lecturers and students; another indication of the changing situation.

5.8 A New Start?

The heated and sometimes unfriendly discussions about mathematics teaching also had as a result that the official organisations of mathematics teachers and university mathematicians began to see that such discussions did not enhance their prestige, and that it should be wise to seek ways for mutual understanding and even cooperation.

The Wiskundig Genootschap realised that mathematics teaching in secondary education was also their concern, and that they could not restrict themselves to the realms of scientific research. The *NVvW*, on the other hand saw that the demands of the teacher education institutions and universities had to be taken into account and that support of and cooperation with professional mathematicians and their organisation would only strengthen their own position. They formalised their new understanding by jointly establishing *Platform Wiskunde Nederland* (Platform *Mathematics* Netherlands), to promote the position of mathematics as a whole and to function as the common voice for the mathematics community.

Except perhaps for some diehards, nobody wants a return to the 1960s and nobody wants to abolish all that has been introduced since then, including the institutions that have been established. But it is also clear that in the future a new equilibrium must be found, not only in the curricula, but also between all involved in mathematics education: the teachers who have to do the hard work, and the experts on both sides, the didacticians and the mathematicians. The indications for such cooperation do not look too bad.

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Chapter 6
Dutch Didactical Approaches in Primary School Mathematics as Reflected in Two Centuries of Textbooks

Adri Treffers and Marja Van den Heuvel-Panhuizen

Abstract This chapter contains an overview of the most important textbook series used in the Netherlands from 1800 to 2010. We distinguish five time periods, and for each period we highlight the textbook series that are most characteristic. To describe the textbooks that were in fashion in the successive periods we distinguish three categories of textbooks: procedural, conceptual, and dual textbooks. The dual textbooks have elements of the first two. For the procedural textbook series, which are also referred to as 'mechanistic', memorisation of mathematical facts, automatisation on of operational procedures and recognising types of problems are the primary interest. Application is only considered at the very end of the teaching trajectory, and then rarely. Smart, flexible (mental) calculations and estimating are not part of the program. The conceptual textbook series have an opposite approach. In learning mathematical facts and procedures, understanding is highly valued, and applications are included from the start as the basis for this. Number sense, flexible (mental) calculation, and estimation are central, next to algorithmic calculation. Students can design their own problems, develop solution methods and work on their own level. As expected, using different textbook series with different content and teaching methods results in pursuing different goals in mathematics education, which in turn results in different learning outcomes, as has been shown by national evaluations of progress in educational achievement.

This chapter is a revised version of Chap. 7 of Adri Treffers' book *Weg van het Cijferen* (*A-Way With Arithmetic*) (Treffers, 2015).

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6.1 Introduction

Knowing Dutch mathematics textbooks means knowing a lot about Dutch mathematics education. Especially in primary school, mathematics textbooks in the Netherlands are not just books with exercises for students. They go together with extensive teacher guides. In fact, the textbooks determine largely what mathematics is taught and how it is taught. They can be seen as the potentially implemented curriculum (Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002). Of course, in what way mathematics education comes to life in classrooms can differ between teachers and can also not be the same for all types of students, but in general what is in the textbooks is also found in classrooms. Therefore, textbooks are a good source to gain knowledge about the Dutch didactic tradition. They even give us a window to mathematics education in the past that we cannot witness anymore, and for which no video recordings are available.

In the chapter, a tour is made along textbooks for primary school mathematics education from 1800 to 2010. The aim of this tour is to reveal the various approaches towards mathematics education that were in fashion during this period of time, and in this way to learn about how these different approaches evolved and what lessons can be learned from the past for our current mathematics education in primary school. The description of the textbooks that were in fashion in the successive periods is structured by distinguishing three categories of textbooks: procedural, conceptual, and dual textbooks. Because the various views on the teaching mathematics manifest mostly in the domains of flexible calculation, algorithmic calculation and applications, we focus our description on these domains.

6.1.1 Procedural Textbook Series

The procedural textbook series are sometimes also referred to as ‘mechanic’ or ‘mechanistic’. The adage of these textbooks is “First do, then know”. They focus on the mathematical content. Their intention is to line out this content exactly, that is, to split it based on difficulty level and to unravel every part into the tiniest details. In these textbook series, there is a lot of attention for repetition, based on the idea that practise makes perfect, and provides skill. Little value is given to understanding as the basis for practising, for example in the case of tables of multiplication. Memorisation of mathematical facts, automatisation of operational procedures and recognising types of problems are the first and foremost interests. There is a single-track approach, guided by rules, and aiming to achieve an efficient standard method to solve a particular type of problems as quickly as possible. The operations for calculations up to ten, twenty and one hundred are performed according to a fixed rule, starting with splitting at ten for adding up to twenty. Next, the standard recipes for the algorithmic calculations of the four basic operations with whole numbers, decimal numbers and fractions are successively introduced and rehearsed. Application is only considered at the very end of the teaching trajectory, and then only rarely. Smart, flexible (mental) calculations and estimating are not part of the programme.

6.1.2 Conceptual Textbook Series

The conceptual textbook series differ from the procedural textbooks series on almost all the aspects mentioned. The mathematical content is not atomised. Understanding is highly valued in learning mathematical facts and procedures. This is one reason that the most shortened forms of the calculation algorithms are not taught immediately, but that teaching starts with transparent predecessors of these algorithms. Applications are included from the beginning of the teaching trajectory, as the basis for understanding. Number sense, flexible (mental) calculation, and estimation have a central place next to algorithmic calculation. Within a framework of carefully formulated goals, students are given the opportunity to design their own problems, develop solution methods and work on their own level. And finally, the relations between the four basic operations and between the sub-domains ratio, fractions, percentages and measurement are firmly anchored.

Within the category of conceptual textbook series three sub-types can be distinguished. There are textbooks with a heuristic, a functional, and a realistic orientation.

The conceptual textbook series with a heuristic orientation strongly emphasise that understanding must come first. They focus on students' insightful, self-inquiry-based way of dealing with numbers in a whole-class setting led by the teacher. Moreover, these textbook series put great importance on applications being given attention from the very start.

The conceptual textbook series with a functional orientation are characterised by stimulating the understanding of students by, for example, involving them actively in discussions about the adequacy of certain solution methods. Applying learned-by-heart tricks should be avoided as much as possible. The functionality of these methods is mainly reflected by letting the students estimate and check their answers, and the connection that is sought to daily-life related problems.

The conceptual textbook series with a realistic orientation, called 'realistic textbook series'—named after the domain-specific instruction theory of Realistic Mathematics Education (RME)—have much in common with the heuristic and functional textbook series, but distinguish themselves through, for example, offering more rich problems and often choosing a thematic approach and a problem-oriented way of teaching. The realistic textbook series also contain new components such as calculations with the aid of a calculator and spatial geometry related to the world around us. A third difference involves the comprehensive use of contexts and models, such as the arithmetic rack, the (empty) number line, the (percentage) bar, and all kinds of diagrams, schemas and tables.

6.1.3 Dual Textbook Series

Dual textbooks series form an intermediate category of textbooks series. They cannot be considered as purely procedural textbook series nor as conceptual, but contain elements of both. This can, for example, imply that a textbook series on the one hand gives no attention to flexible mental calculation, and shows little interest in including applications, but on the other hand deals with algorithmic calculation in an insightful manner. The reverse can as well be the case. Then, conceptual elements of flexible (mental) arithmetic, estimation and their applications go together with the mechanistic approach of a procedural textbook series. Dual textbooks series can also have different approaches for the lower and the higher grades.

6.1.4 Textbooks Series in Use Over Five Time Periods

When describing the most important textbook series that were in use from 1800 to 2010 we make a distinction in five time periods. For each of them we highlight the textbook series that were characteristic for that time (see Fig. 6.1). The first period, labelled as 'Procedural didactics and semi-textbook use', running from 1800 to 1875, is a kind of pre-stage of textbook use. From 1875 to 1900 was the period in which the use of complete textbooks started. Then, the first type of conceptual textbook series was used, namely those with a heuristic orientation. In the next half century, from 1900 to 1950, the dual textbook series were in use. The subsequent time period, from 1950 to 1985, is both characterised by the use of procedural textbook series as well as by the use of conceptual textbook series with a functional orientation. The time between 1985 and 1990 is taken up by the conceptualisation of a new curriculum for mathematics education in primary school, published in the *Proeve*[1] (Treffers, De Moor, & Feijs, 1989). This programme was meant to get more coherence in what and how mathematics is taught. In the time period from 1990 to 2010, the divide in procedural and conceptual textbook series was basically over. The large majority of the textbook series which are now in use belong to the conceptual textbook series and have a 'realistic' signature or have at least a number of RME characteristics (Treffers, 2015, pp. 130–135; Van den Heuvel-Panhuizen & Drijvers, 2014).

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[1] The full title is *Proeve van een Nationaal Programma voor het Reken-wiskundeonderwijs op de Basisschool.*

1800	**Procedural didactics** & **Semi-textbook use**	*Lower grades* No textbooks; mathematics taught on the blackboard *Upper grades* textbook series for individual use	1835: Hemkes 1850: Boeser
1875	**CONCEPTUAL** **Heuristic** textbook series		1875: Versluys 1878: Van Pelt
1900	**DUAL** textbook series		1918: Bouman & Van Zelm 1935: Diels & Nauta
1950	**PROCEDURAL** textbook series & **CONCEPTUAL** **Functional** textbook series		*Procedural textbook series* 1950: Naar Zelfstandig Rekenen 1970: Niveaucursus Rekenen *Functional textbook series* 1949: Geef Acht! 1958: Functioneel Rekenen 1969: Nieuw Rekenen
1985-1990	*Proeve*: Towards a National Programme for mathematics education in primary school		
1990	**CONCEPTUAL** **Realistic** textbook series		1981: De Wereld in Getallen 1983: Rekenen en Wiskunde 2001: Pluspunt 2001: Rekenrijk 2002: Alles Telt 2009: Wizwijs
2010	**PROCEDURAL** textbook series		2009 Reken Zeker

Fig. 6.1 Overview of two centuries of textbook series for primary school mathematics in the Netherlands

6.2 The Period 1800–1875: Procedural Didactics and Semi-textbook Use

6.2.1 Teaching Mathematics on the Blackboard and No Complete Textbook Series Available

In Article 4 of the first Dutch Education Law launched in 1806, mathematics was considered a compulsory school subject alongside reading and writing. In 1810 the government compiled an official list of recommended mathematics textbooks series. These textbooks were not used for whole-class instruction, but to teach students individually. Moreover, they were only meant for the upper grades of primary school. Therefore, the period 1800–1875 can be referred to as semi-textbook use, since no complete textbook series for all grades of primary school were available.

In the lower grades mathematics was taught without textbooks. This involved mechanistic teaching of mathematics on the blackboard. The approach was clearly procedural. An assistant teacher wrote the problems on the blackboard and the students then had to calculate them on their slate. Beforehand, drill-and-practise activities were done with the whole class to prepare carrying out algorithmic calculations. The approach was purely procedural—there was no space for flexible calculations. Once students had mastered the standard algorithms, then in the upper grades they were given a textbook for the first time and had to work through the book individually under their teacher's guidance.

6.2.2 The Textbook Series by Hemkes

The so-called 'ten cent' textbook series by Hemkes (1846) were the most used textbooks during the period 1800–1875. In 1835 Hemkes published the textbook *De Kleine Rekenaar* (The little calculator), meant for beginners. In this textbook each of the basic operations starts with problems with bare numbers, followed by word problems with named numbers. Then follow assignments for practising with mixed problems for all operations that have been dealt with so far, addition, multiplication, subtraction and division respectively. This is followed by a repetition chapter with these problems—all in all nearly 400 problems. What is remarkable, is the large variation in difficulty between the problems within one category. Take, for example, the difference in difficulty level between the first two of the following problems and also notice in the third problem the often occurring elaborated and informal style of the word problems.

1. Count together: 3, 2, 8 and 9.
2. How much does the sum of all numbers that you can make with the numbers 5, 6 and 9 amount to?

3. Hans could not work out how much money one would have to pay for 5 cows if one cow cost 85 guilders. Oh, do give the fool a hand.

The other 'ten cent' textbooks, consecutively dealing with decimal fractions, money, measurements and weights, the 'rule of three' and proper fractions, follow the same structure. Each time the textbook begins with bare number problems, followed by word problems with named numbers for each operation in the previously mentioned order.

The Hemkes textbooks were rather popular. Only in the second half of the 19th century they were gradually replaced by Boeser's mathematics textbooks.

6.2.3 Boeser's Mathematics Textbooks

The textbook series by Boeser (1850) address the same content topics as Hemkes' textbooks, but Boeser's word problems, contrary to those of Hemkes, were quite factual and brief in their formulation. In the foreword of Boeser's *Eerste Rekenboekje*, which he published in 1850, he wrote:

> Yet, according to me, some of the books for beginning reckoners are too extensive, too tedious; others contain too few imaginable situations derived from the children's world and daily life, whereas most of these books (I give my own opinion here) suffer from the evil of the mechanistic approach. Not only do these books contain a series of dry problems which children have already done while working on the blackboard, the books also offer a number of problems with the assignment 'add together', 'subtract', and so on, which can be done without developing or practising the children's ability to reason.

Here Boeser has a point; when application problems have been placed within a particular category of problems, for example in a section about multiplication problems, the students automatically know that they can find the solution by doing a multiplication. And a further point is: why should a mathematics textbook contain bare number problems if the students previously practised these problems through doing whole-class calculations on the blackboard? Hence, it is no surprise that Boeser's textbooks particularly contain mixed application problems. An example follows now.

> A man worked daily for 12 hours; he spent 2 hours on eating and 3 hours on leisure activities. The remaining time he spent on sleeping. How many weeks did he spend sleeping in a year?[2]

Boeser's books with the mixed applications were particularly popular and they were reprinted until the 20th century.

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[2] All quotations from textbooks are translated from Dutch by the authors.

6.3 The Period 1875–1900: Conceptual Textbook Series of a Heuristic Orientation

6.3.1 Influence from Germany

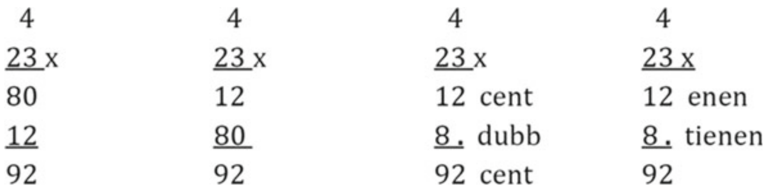
At the end of the 19th century the conceptual approach to teaching mathematics made its appearance. This was also the first time that a complete textbook series was used. This new approach was launched with the publication in 1865 of Rijkens' translation of a handbook of the well-known German mathematics didactician Hentschel (Hentschel & Rijkens, 1865), who is called the 'father of the new arithmetic in public schools'. Partly due to this publication, a renewed interest arose in the work on early mathematics education done by Grube, who was another important German mathematics didactician, whose *Leitfaden für das Rechnen* (Guide for arithmetic) was published in 1842. In the Netherlands, Grube's publication became known through the slightly adapted version that was published by Brugsma in 1847 (see also Brugsma, 1872).

6.3.2 Versluys

Versluys, the founding father of the Dutch didactics of mathematics, was heavily inspired by the German mathematics didacticians Hentschel and Grube when writing his textbook series *Rekenonderwijs ten Dienste van de Lagere School* which was published in 1875 (see Versluys, 1875). He characterised his conceptual textbook series as 'heuristic', which means that the focus was on insightful, self-inquiry-based learning of mathematics within a whole-class setting guided by the teacher.

Yet, in Versluys' textbook for the first year of primary school not much can be recognised of what is usual in today's mathematics education. The main reason for this is the monographic method that Versluys applied. Here, he followed Grube. Typical for this method is the 'operational' way the numbers 1–20 are handled. Every time a number is built up, split up, and 'measured' with preceding numbers. This means that from the very beginning the four basic operations are involved in the teaching of mathematics, but first through oral assignments and word problems and not in formal symbolic notations. In the teacher guide, for each number under ten about one hundred problems are provided for making calculations with present or absent objects and amounts, and with unnamed numbers. For the numbers from ten to twenty the quantity of provided problems has been reduced to less than fifty.

However, the way Versluys treats calculations up to one hundred has a lot in common with how we do it today. Additions and subtractions are not done in a prescribed manner, but flexibly. At the same time the multiplication (and division) tables are memorised more or less automatically, following a heuristic approach, structured



4	4	4	4
23 x	23 x	23 x	23 x
80	12	12 cent	12 enen
12	80	8. dubb	8. tienen
92	92	92 cent	92

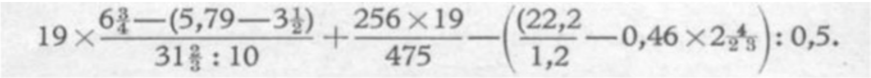
Fig. 6.2 Structure of teaching multiplication by Versluys (1875) (cent = cent; dubb = dime; enen = ones; tienen = tens)

through doubling and reversing, and by means of products with five and ten. The properties of the multiplication operation which are at stake here can be used later for mental arithmetic. Here, too, there is again a lot of attention for applications. With problems such as 9 × 13 and 13 × 9 the impetus is given for flexible (mental) calculation, whole-number-based written calculation and insightful algorithmic (digit-based) written calculation, plus the relationships between them. For all these, a rectangular model serves as a visual basis.

Algorithmic multiplication is structured ingeniously. On the one hand, the problem 4 × 23 (“4 children each get 23 cent”) makes a connection with repeated addition, and thus with the shortened algorithmic method the students learned earlier for addition. On the other hand, through 23 × 4 (“23 children get each 4 cent”) students learn the zero rule of algorithmic calculation that is applied in the case of multipliers with multi-digit numbers (Fig. 6.2).

The digit-based algorithm of long division is prepared through a whole-number repeated subtraction approach. Again, an elementary word problem serves as a concrete basis: “How many times can 4 guilders be taken from 936 guilders?” First a chunk of 4 guilders is taken away 200 times, then from the remainder 30 times a chunk of 4 and finally 4 times a chunk of 4 guilders; altogether this makes 234 times.

Versluys assigns as much value to flexible mental arithmetic as to algorithmic calculation. For both mathematical domains, he starts with numbers up to one hundred. What is noticeable, is the large amount of word problems and the rather small number of bare number problems—for Versluys arithmetic is in the first place applied arithmetic. Only in the upper grades of primary school does this change: then complex algorithmic calculations appear which are in Dutch called ‘vormsommen’ (form problems). Figure 6.3 shows an example of such a problem.



$$19 \times \frac{6\frac{3}{4} — (5,79 — 3\frac{1}{2})}{31\frac{2}{3} : 10} + \frac{256 \times 19}{475} — \left(\frac{(22,2}{1,2} — 0,46 \times 2\frac{4}{23} \right) : 0,5.$$

Fig. 6.3 Example of a ‘vormsom’

6.3.3 Van Pelt

Van Pelt, who published *De Nieuwe Rekencursus* in 1878 (Van Pelt, 1878, 1896, 1903), also makes use of the monographic method for calculations up to twenty. However, for calculations up to hundred and thousand, his approach differs from that of Versluys. In Van Pelt's approach, there is no room for algorithmic calculation in the first three years of learning. It is mental arithmetic all the way. Furthermore, Van Pelt makes it very clear that for him the final goal is that the multiplication tables are known by heart, and that on the way to that goal the students have to learn to understand a variety of properties that they will be able to utilise in mental arithmetic in future. Van Pelt (1903, p. 15–17) states the approach in his textbook series as follows:

> No sensible teacher will ask here whether this is the fastest way for students to learn the tables. He will understand that the main goal of our teaching has to be development, development by doing and searching on your own. He will quickly notice that this way of working has such an influence on the students that later on they will choose this approach themselves, especially in mental arithmetic. (…). The author of the arithmetic course expresses his annoyance that at this time so many still learn their tables by memory work: yet, every teacher knows that arithmetic education should be purely *heuristic*.

And a bit further on, when it is about calculation up to one thousand, next to the smart calculations, Van Pelt places the stylised, whole-number-based written calculation, which is the prelude to insightful algorithmic written arithmetic in the fourth year of primary school. If there is ample attention in mental arithmetic for insightful solutions for problems such as 40 × 55—via 40 ells[3] at 55 cents—then learning the calculation algorithm for 43 × 55 should not be a problem. According to Van Pelt, you first calculate 3 × 55 and 40 × 55 separately and then combine the results from both.

As far as practical applications are concerned, there is no essential difference between Versluys and Van Pelt. They both give word problems a central place in their textbook series, and utilise them from the beginning as a concrete starting point for learning formal calculation procedures.

6.3.4 The Adage of the Conceptual Mathematics Textbook Series with a Heuristic Orientation

The didactic adage of the conceptual mathematics textbook series with a heuristic orientation is "First know, then do, first think, then do"—exactly the reverse of the procedural motto!

However, this adage can be easily misunderstood. Because knowing and thinking do not only refer to understanding of the calculation rules and procedures, but

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[3] An 'ell' is an old length measure. In the Netherlands, an ell was 69.4 cm.

also to comprehending their applicability. And according to the heuristic point of view, this applicability can be guaranteed only when applications are part of the teaching trajectories from the very start. This even goes so far that initially the calculation procedures are adapted for the sake of applicability! So, it may happen that students learn two versions of the division algorithm: whole-number-based division with repeated subtraction involving quotative division ("How many weeks are there in 364 days?") and digit-based long division for problems that imply partitive division ("Fairly divide the amount of 364 guilders among 7 people. How much does each of them receive?"). Only after some time, these two forms of division are brought together and combined into the common form of long division. Also, the gradual, whole-number-based start of the algorithmic calculations for the other basic operations is inspired partly by the criterion of applicability.

From the above it can be concluded that 'conceptual' is considered as 'first knowing why' and only then 'knowing how' is not correct. In fact, the why and the how are intertwined here. The teaching of the multiplication tables, as Versluys and Van Pelt advocate, is an eloquent example of this.

6.4 The Period 1900–1950: Dual Textbook Series

In the first two decades after 1900, new textbook series were published, which can be called 'dual', since they combine characteristics of both procedural and conceptual textbook series. They are characterised by the fact that they:

- Abandon the monographic treatment of the numbers up to twenty, and the teaching of multiplication and division in the first year of primary school
- Restore the prominent place of counting in the initial phase of education
- Do no longer give priority to word problems, but start with bare number problems
- Assign a larger relevance to algorithmic calculations than to mental arithmetic
- Put more emphasis on skills than on insightful calculation.

The textbook series by Bouman and Van Zelm (1918) was one of these new dual textbook series. This textbook series gained greatly in popularity from around 1920. Considered from the viewpoint of the very beginning of learning arithmetic, Bouman and Van Zelm's (1918) textbook has a one-sided focus on counting—one-sided because this textbook series rejects number images and all kinds of visual means that elicit grouping procedures. Numbers should remain unnamed; they are mathematical conceptions, and as such cannot be related to objects and figures.

In the teacher guide it says: "Mathematics education should put the emphasis on calculation with unnamed numbers." Yet, the authors permit that the students can start by drawing dots and circles as long as they have no specific meaning.

Aside from elementary mental calculation, the textbook series of Bouman and Van Zelm does not give attention to flexible mental calculation outside the area of numbers up to one hundred. Only in 1934 do the authors revise their deviating approach regarding flexible mental calculation.

Algorithmic calculation is taught in an insightful manner through whole-number-based calculations, but the authors remain true to their beliefs: the use of concrete objects and named numbers such as cents, ten cents and guilders to provide insight into the calculation procedures within the decimal system, is discouraged strongly. For example, considering the ten as 'equal' to ten cents does not make the understanding and applicability of the concept simpler, according to Bouman and Van Zelm.

Furthermore, it is curious how little interest they have in the issue of applicability. Take for example the topic of division. In the Booklets 5 and 6 (meant for Grade 3), there are 2500 bare number problems and only fifty application problems for quotative and partitive division. Applications only appear after Booklet 9 (meant for Grade 5), so around the point that students who will not go to secondary education are about to stop.

Because of neglecting flexible (mental) calculation and disregarding the inclusion of application problems in algorithmic calculation in the lower and middle primary school grades, the original version of the textbook series by Bouman and Van Zelm is closer to procedural textbooks than to conceptual textbooks. The great attention that this textbook series devotes to complex thinking problems does not affect this conclusion, since learning to solve these problems is mostly an issue of training to recognise the type of problems for which the solution methods have been practised. If you have not been trained to solve these problems, it will be impossible or at best very hard to solve them correctly—consider, for example, the following percentage problem from Bouman and Van Zelm.

> A pays *ƒ*160,38 for a bale of coffee. If he enjoyed 1% for cash payment and had 10% tare and had to pay 45 cents per half kilogram, what was the gross weight of the bale?

In 1935, Diels and Nauta published their textbook series *Fundamenteel Rekenen* (see Diels & Nauta, 1939, 1944), which can also be characterised as a dual textbook series. Diels and Nauta were strongly against this type of thinking problems. Nevertheless, despite their critique and resistance from primary school teachers and thinking psychologists, they had to include them in their textbook series, because these problems remained part of the secondary school entry examinations. In addition, the textbook series by Diels and Nauta differs not only at the end, but also at the start of mathematics education from (the first version of) the textbook series by Bouman and Van Zelm. In arithmetic up to twenty, in Niels and Nauta's textbook, together with counting, the use of number images is involved. In addition, these authors do not abandon problems with named numbers, although these problems only appear little by little at the end of the teaching trajectory for arithmetic up to twenty.

From the second school year on, Diels and Nauta reserve ample time in each lesson for mental calculation with word problems and oral calculation exercises with bare numbers. Some examples of the word problems from Booklet 9 (meant for Grade 5) of *Fundamenteel Rekenen* published in 1944.

(1) An airplane is at a height of 3600 m and suddenly descends 735 m. How many m is it still above the earth?

(2) A troop of soldiers covers 23 km in 4 h. Now they still need to cover 11½ km. How long does that journey take, if they rest for two hours along the way?

(3) Michiel de Ruyter was born in 1607 and died 1676. How old did he become? How many years ago did he die?

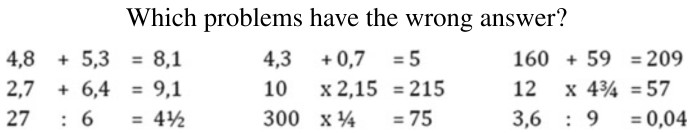
(4) A gentleman loses a wallet with *ƒ*4000. He will give 2½% as a reward to the finder. How much reward will the finder receive?

The oral calculation exercises (according to the teacher guide, these exercises mean that "the teacher reads out the problem, the students write down the answer") usually consist of bare number problems (Fig. 6.4).

With this approach, Diels and Nauta firmly followed the 1936 school inspection guideline that states: "Mental arithmetic should take up a prominent position; when it is done with small numbers, it does not merely have practical value, but also fosters understanding."

The textbook series *Fundamenteel Rekenenl* is the first one that has put estimation on the educational programme. This is motivated as follows: "It was attempted to further discourage the purely mechanical work by repeatedly having the result estimated before calculating it."

Considered so far, this textbook series by Diels and Nauta is a purely conceptual textbook series. However, this changes when we consider algorithmic calculation. As can already be seen from the quotation on estimation, the authors are critical, not to say negative, about algorithmic calculation. In this context, they speak about training and mechanisation which lead to thoughtlessness and would obscure understanding of the number system. The amount of algorithmic problems is therefore considerably smaller than with Bouman and Van Zelm—for some parts it is only a quarter of it. Another noteworthy point is that in the Diels and Nauta textbook series the algorithmic procedures are not taught in a comprehensive way, not even for relatively small numbers such as 364 ÷ 7, where whole-number-based division by means of repeated subtraction would be an obvious approach. Also, the applications of algorithmic calculation with large numbers including decimal numbers are only treated in the higher grades. In this respect, this textbook series shows all in all a typical procedural approach. In other words, along with the innovative, conceptual elements of flexible (mental) arithmetic, estimation and their application, the textbook series of Diels and Nauta does also contain elements that are in line with the procedural textbook series. Therefore, this often-used textbook series, just as the textbook series


Which problems have the wrong answer?

4,8 + 5,3 = 8,1
2,7 + 6,4 = 9,1
27 : 6 = 4½

4,3 + 0,7 = 5
10 x 2,15 = 215
300 x ¼ = 75

160 + 59 = 209
12 x 4¾ = 57
3,6 : 9 = 0,04

Fig. 6.4 Oral calculation exercise from Diels and Nauta (1944) of Bouman and Van Zelm, has to be characterised as dual, even if the Diels and Nauta textbook series is closer to the conceptual approach than the latter.

It was only in the course of the 1950s that these two textbook series lost their leading position in the market.

6.5 The Period 1950–1985: Procedural Textbook Series and Conceptual Textbook Series with a Functional Orientation

Considered quantitatively, the procedural textbook series dominate in the period from 1950 to 1985. Five of such textbook series managed to acquire a substantial market share, with *Naar Zelfstandig Rekenen* (Zandvoort, Venekamp & Kuipers, 1955/1970) for many years the largest. With respect to content, this textbook series fully meets the general characterisation of the procedural approach. Nevertheless, what makes the series special is the practical organisation of independent, individual work in a looser classroom setting, which is made possible by the particular setup of the booklets. Alongside every page with problems there is a page with suggestions and worked examples that can, if necessary, be further explained by the teacher. The learning content is systematically divided into small units which have been organised on the basis of increasing complexity. Based on this information about the textbook series, it can only be concluded that this procedural textbook series is proficiently assembled: the structure of the learning content is well thought out, and the instructions for the students are adequate. Partly because of this, the textbook series is fairly easy to use in everyday teaching. The other procedural textbook series with a large market share, *Niveaucursus Rekenen* (Vossen et al., 1970), is also based on a more loosely organised classroom setting and the approach and content of this textbook series largely corresponds with *Naar Zelfstandig Rekenen*.

Halfway through the 20th century, parallel to the procedural textbook series, a new type of conceptual textbook series emerged, which can be called ‘functional’. The ideas about mathematics education laid down in these textbook series were as innovative as the approaches reflected in the conceptual textbook series with a heuristic orientation that were in use in the 1870s. For the new conceptual textbook series with a functional orientation, new educational avenues were followed inspired by the German Psychology of Thought and its didactical application by the Amsterdam school of Kohnstamm (1952).

The best known functional textbook series is *Functioneel Rekenen* written by Reijnders and Snijders (1958). However, this textbook series was not the first of its type. The honour for being the first to develop a new conceptual textbook series goes to Rombouts and his textbook series *Geef acht!* (Rombouts, 1948). This textbook series is the counterpoint to the pure recipe-based procedural textbooks series. Finally, the third functional textbook series was *Nieuw Rekenen* (Bruinsma, 1969), that advertised itself as a functional textbook series and was, based on its market share, the most successful one of these three series.

The textbook series *Functioneel Rekenen* (Reijnders & Snijders, 1958) emphasises:

- The importance of a whole-class discussion about the various solution methods for bare number problems and context problems
- Own productions of problems in various forms, for example, making up problems that go with a given answer, or finding a suitable word problem that fits to a given bare number problem, or formulating a question for a problem without a question
- The power of visual models such as the number line, tables and bars, in solving problems
- A lot of attention for flexible mental calculation
- The value of estimation for roughly determining or checking the outcome of a calculation, but especially as a didactical tool for teaching precise calculation.

The latter point is an entirely new idea that is explained further using the example of 52 + 29. First the outcome is given a lower (70) and an upper (90) limit, and then this outcome is made more precise in various ways using questions such as "How much more than 70?" and "How much less than 90?" The explanation provided in *Functioneel Rekenen* finishes with: "These kinds of solution methods should emerge constantly in whole-class or group discussions. The numbers are handled in an entirely different way than in algorithmic calculation." About algorithmic calculation the following is said: "Throughout the whole textbook series the goal has been to avoid mechanical work with numbers—that is, applying tricks that have been learned by heart—as much as possible."

Although the title of the textbook series emphasises the functional aspect, this principle is not applied to the teaching of algorithmic multiplication. As with Diels and Nauta the standard procedures for whole numbers, decimal numbers and fractions are taught in a purely recipe approach. Yet, for the upper grades of primary school the textbook series does contain applications of various forms: problems without questions, free productions, closed problems in which the students have to find the right operations, and also elementary mental calculation and estimation problems.

The textbook series *Nieuw Rekenen* assigns an important role to mental calculation. This mainly involves calculations done while using your head, rather than done in the head. The general introduction of the textbook series *Nieuw Rekenen* (Bruinsma, 1969, p. 12) states this as follows:

> Mental calculation is always arithmetic that is functional and based on understanding. What it is not: algorithmic calculation done in the head or applying tricks. Mental calculation always demands understanding of the structure of the numbers. Mental calculation certainly does not imply that paper may never be used. It is often advisable to give especially those children with a less good memory the opportunity to write down intermediate answers; this puts the child at ease and can strengthen confidence. Real life demands the ability to perform simple mental calculations quickly; the child must be able to quickly understand relationships and know which calculations to perform; mental calculation increases understanding of the number system and encourages discovering the many possibilities that lead to the same answer.

In short, flexible (mental) calculations are not just a goal in themselves, but also function as a didactical tool to foster number sense, insightful arithmetic and applicability. Five examples from Booklet 4b for Grade 4 on multiplication show how this basic concept is made concrete.

1. Calculate:
   $5 \times 98 = 5 \times 90 + 5 \times 8 =$
   but also $5 \times 100 - 5 \times 2 =$
   and half of $10 \times 98 =$
   Calculate in different ways.
   7×98 4×98 12×25

2. Calculate in the simplest way.
   6×94 8×97 28×29

3. Make your own problems.
   There is a fence around a meadow.
   The meadow is long 120 m, wide 80 m.

4. Make 5 multiplications.
   The outcome is always 450.

5. Estimate first!
   $9 \times f3.75 =$ $85 \times f0.97 =$

Problem 1 puts the students on the trail of smart calculation that can be applied in Problem 2. In Problem 3, algorithmic calculation comes into view with 28 × 29.

Moreover, the textbook series *Nieuw Rekenen* makes an insightful transition from calculation by splitting and whole-number-based calculation to algorithmic calculation. For multiplication, this transition could take place in a few lessons and could go as follows (Fig. 6.5).

In (a) and (b) the whole-number-based calculation is shown. In (c) the transition to calculating with digits is made. Earlier, this approach is also followed for addition

7 x 98 = 7 x 90 + 7 x 8 = 630 + 56 = 686.

		5
98	98	98
7 x	7 x	7 x
630	56	686
56	630	
686	686	
(a)	**(b)**	**(c)**

Fig. 6.5 Structure of multiplication in *Nieuw Rekenen*

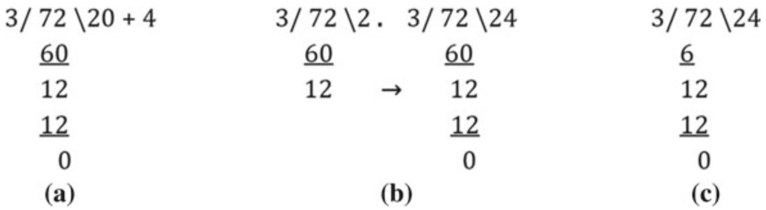


Fig. 6.6 Structure of long division in *Nieuw Rekenen*

and subtraction, after first showing the position values of the ones, tens and hundreds using coins (cents, dimes, and guilders).

Long division (Fig. 6.6) is introduced using a problem such as 72 ÷ 3 with the question: “Into how many groups of 3 can I divide 72?”

The students can start with (a) for a few problems, but soon they will have to switch to (b). Using the shorter notation, from ‘20’ to ‘2.’ to ‘2’ takes place in one lesson. The transition from (b) to (c) is made in more difficult problems.

This phased approach can also be found in the conceptual textbook series with a heuristic orientation by Versluys and Van Pelt and in the dual textbook series by Bouman and Van Zelm. The main difference with these previous textbook series is that in *Nieuw Rekenen* the transition from (a) via (b) to (c) happens in a few lessons and the description in the textbook is apparently mostly aimed at the teacher who can use this to give an insightful explanation of the shortened standard procedure. In *Nieuw Rekenen*, fractions, ratios and percentages are also taught in an insightful manner in the sense that the teacher, following the textbook series, can explain the concepts and operations as clearly as possible on the basis of models and schemes.

Moreover, *Nieuw Rekenen* contains many and varied application problems, that are sometimes placed together in a thematic series, for example, a series about shopping, sales receipts, foreign money, train journeys and distances in Europa. The closed, half-open and open assignments match well with what is being taught at the time; they are applications of what students have previously learned in a purely numerical way.

In this respect, this textbook series distinguishes itself from the textbook series *Geef Acht!* In the teacher guide Rombouts (1959) says:

> The problem, the genuine mathematics problem, is both the start and the end. It is used to give context to the calculations, and it is returned to over and over again, since everything has to have a purpose for the students as well. Not start with ‘mathematics’ and later on ‘applied mathematics’, but together, connected.

This textbook series is further characterised by the emphasis on flexible mental calculation, own productions and a reduction of the learning content. The latter was, with respect to the thinking problems and problems about the metric system, was only made possible when at the beginning of the 1970s, the comprehensive entry examination for secondary education was replaced by the less broad Cito End of Primary School Test.[4]

6.6 The Period 1985–1990: Towards a National Programme for Primary School Mathematics

The time period between 1985 and 1990 is characterised by the many efforts that have been made to get a new programme for mathematics education in primary school. The start of working on this programme goes back to the beginning of the 1970s when the large-scale Wiskobas[5] project was launched. From 1971 to 1981 the Wiskobas team developed, together with the educational field and in school practice, a large collection of rich problems and themes for various topics within arithmetic, measurement and geometry, and they developed initiatives for new textbooks. In this period, also the foundation was laid for what later became known as Realistic Mathematics Education (RME).

The Wiskobas publications functioned as a source of inspiration for the new, realistic textbook series, that is, RME-oriented textbook series, that hesitantly appeared on the educational market around 1980—hesitantly, because their content was not a seamless match for the then prevailing mathematics teaching practice and the Cito End of Primary School Test. The effect of all this was a (too) large diversity in textbook series. The need for a (new) national mathematics curriculum with explicitly stated end goals made itself felt. On the initiative of the Nederlandse Vereniging tot Ontwikkeling van het Rekenwiskundeonderwijs (NVORWO; Netherlands association for the development of mathematics education), hundreds of people involved in the field of education were consulted about the future of mathematics education in primary education. In 1987 this resulted in the *Proeve* (Treffers et al., 1989), the first design for a national programme for mathematics education in primary school. A few years later, the mathematics end goals for primary school described in this publication were officially given approval by the government (OCW, 1993) and served as beacons for textbook authors and test developers.

The concrete learning goals in the *Proeve* involve the domains of basic skills, algorithmic calculation, ratio, fractions, percentage, and measurement and geometry. These core goals can be typified as follows.

- For the domain of basic skills much emphasis is put on the understanding of the decimal place value system, mental calculation, estimation, applications, as well as appropriate use of calculators.
- Algorithmic calculation provides room to learn variants of the conventional procedures, such as the ones earlier described in this chapter for long division.

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[4] The Cito End of Primary School Test is developed by Cito, the Netherlands national institute for educational measurement.

[5] Wiskunde op de Basisschool (Mathematics in Primary School).

- Because of its broad applicability the domain of ratio is interpreted much more widely than the formal ratios of traditional arithmetic. Practical calculations with percentages are also given a lot of attention—with the focus firmly on understanding the concept of percentage.
- Fractions and decimal numbers, and the relations between them, are given meaning in various ways. The students should be able to compare, order, add, subtract, multiply and divide fractions and decimal numbers in simple application situations. Directly leading students to mastery of the mathematical rules for these four basic operations is rejected.
- More attention than in the past is given to measurement, calculations with common measures and representing measurement data in schemes and graphs. However, there is less attention for practicing the metric system.
- Teaching geometry starts with focussing more on observing (peep dioramas, photos, localising, light and shadow, building block constructions) than on making calculations. Students come across a great diversity of aspects of mathematics from starting with observable reality, such as visualising, using geometrical models, spatial orientation and reasoning, reflecting on one's own actions, applying geometrical knowledge and insights to practical and puzzle-like problems, and all this in relation to topics from the field of arithmetic and measurement.

The *Proeve* does not limit itself to describing the end goals of mathematics education, but also makes statements about the didactics. Because it is a good habit within RME to use examples from teaching practice or from a textbook series for explaining particular approaches, we also will do this now. To illustrate what this didactics means, we have chosen a problem where different core aspects of 'numbers and operations' can be seen. RME-oriented mathematics education sometimes uses newspaper clipping to ask mathematical questions. In this case this is done as well.

> Hard work in the bulb fields
>
> Every year in spring in the Netherlands there is a lot of work to be done in the bulb-growing industry. This is the fourth year that Johan has been doing this work; he has worked both in the fields and in greenhouses. Now he works in the transport department of a company in the auction halls. "I usually load the trucks, that is heavy work. I usually work 220 hours a week. That's good, because you earn money that way", according to Johan.

How would students in Grade 3 and 4 react to this newspaper clipping? It depends on one's expertise and sensitivity regarding the thinking of students whether one can judge students' solutions in advance. But even if one has the experience, children will still come up with surprises. To begin with, there are students who do not (purely) calculate. Here are some examples:

- No, that is impossible because people work 36 or 40 h a week and this is way too much.
- Yes, it is possible because it's very heavy work and that often takes long to do.
- No, because my mother already works 180 h a week. If he worked harder, he'd be working the whole day, that's a bit much.

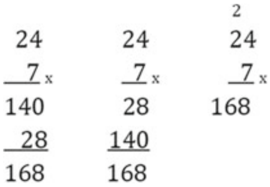

Fig. 6.7 RME structure of algorithmic multiplication

- Yes, because you can load trucks and grow bulb fields at the same time.
- It's not possible, because a week only has 168 h.

Moreover, the calculations that lead to the last conclusion are very diverse:s

- repeated addition: 24, 48, 72 … 168
- repeated doubling: 24, 48, 96, 192; 192 – 24 = 168
- multiplication by splitting: 7 × 24 = 140 + 28 = 168
- algorithmic multiplication: 7 × 24 written 'underneath each other'
- smart multiplication: 7 × 24 = 7 × 25 – 7 = 175 – 7 = 168
- idem, but wrong: 7 × 24 = 7 × 25 – 1 = 175 – 1 = 174 (!)
- estimation: 10 full working days is 240 h, so 220 h would be well over 9 days and a week only has 7 days.

The newspaper clipping about Johan is an interesting problem, because it is located on the crossroads of different content strands. Calculation by splitting, smart calculations, algorithms and estimation can all be included in it. It is clear that the students can learn much from each other as a result of discussing differences in reasoning and calculations. The initial focus will be on the various calculations of 7 × 24, starting with the fairly laborious approach of repeated addition. Next, the approaches of multiplication by splitting and algorithmic multiplication provide an opportunity to show once again that the algorithmic approach to multiplication is in fact a shortened procedure of the splitting approach (see Fig. 6.7).

The smart calculating of 7 × 24 via 7 × 25 requires further explanation. We know that 7 × 25 − 1 = 174 results in an incorrect answer. How can we make the right approach insightful?

The teacher has a model at hand to visualise the correct solution method: a stack of 7 boxes (days), filled with 25 units (hours), from which one in each box, so a total of 7, has to be removed: 175 − 7 = 168. How will the students in fact try to explain this calculation to each other?

Furthermore, extensive attention can be given to the magnificent 'suppose that' reasoning. You can start from 220 h per week and show that you end up with an impossible number of days per week, that is, 220 − 24 or in other words well over 9 days. Another possibility when faced with that kind of reasoning is to divide 220 by 7, ending up with a day having over 31 h. These last-mentioned argumentations will automatically lead to estimation. At the end of the lesson the question arises: "Is Johan that dumb, or was it simply a slip of the tongue?"

After the students have discussed the problem in groups of two or three for a few moments, they will collectively conclude that Johan probably made a mistake and meant 220 h a month. This would mean that he works about 55 h per week, that is, 11 h per day in a five-day working week, or 9 h per day in a six-day working week, and that is working hard.

In the procedural approach to arithmetic, which is the dominant approach within traditional mathematics education, the problem 7×24 appears in three forms:

- As a multiplication problem in horizontal notation; meaning the problem has to be calculated as $7 \times 24 = 140 + 28 = 168$
- As a multiplication problem in vertical notation; meaning the student has to use algorithmic calculation
- As a dressed-up version with the question “How many hours are there in a week?” and a free choice of how to calculate.

In this procedural approach, no attention at all is given to smart calculation of 7×24 via $7 \times 25 - 7$.

The lesson about the newspaper clipping shows where a conceptual approach with a realistic orientation distinguishes itself from a procedural approach. Moreover, this lesson also makes it clear that the lesson also differs from a conceptual approach with a functional orientation where, in general, such rich problems are not used.

6.7 The Period 1990–2010: Realistic Textbook Series

6.7.1 An Abundance of Textbook Series

The revision of the mathematical content and the end goals, along with the didactic repositioning as described in the *Proeve*, did not fail to have an influence on textbook series. Although the realistic textbook series resemble the functional ones, there are differences as well. What both conceptual textbook series have in common is that they in addition to algorithmic calculation with not too large numbers, give attention to insight into numbers, flexible (mental) calculation and estimation, and that they work only with often used fractions and metric measures. New topics in the realistic textbook series involve applied arithmetic using a calculator and geometrical knowledge of the world. The didactic approach of the new, realistic textbook series is broadly along the lines of the functional approach—broadly, since the functional approach falls short for the second half of primary school. Here, there is a difference with the realistic textbook series, in which in fact the entire mathematics curriculum is modernised. Another difference can be found in the more problem–oriented and (often) thematic character of the realistic approach. Yet another difference is the extensive use of contexts and models, such as the arithmetic rack, the (empty) number line, the (percentage) bar and all kinds of diagrams, schemes and tables.

The two most commonly used realistic textbook series after 1990 are *De Wereld in Getallen* (Van de Molengraaf et al., 1981; Huitema et al., 1991, 2001, 2010) and *Pluspunt* (Groen et al., 2001; Van Beusekom et al., 2010). Since 2000, the total market share of these two textbook series is 70%. Other realistic textbook series with a substantial distribution rate are *Rekenrijk* (Bokhove et al., 2001, 2010), *Alles Telt* (Boerema et al., 2002) and to a lesser degree *Wis en Reken* (Buijs et al., 1999; Bergmans et al., 2001), which is the successor of one of the first realistic textbook series *Rekenen en Wiskunde* (Gravemeijer et al., 1983). All these realistic textbooks series are based on the principles of RME, which in general means that students give productive input in an interactive, (semi-)whole-class setting.

In the case of *Pluspunt* students even have so much input that in some parts of this textbook series the leading role of the teacher is marginalised. *Pluspunt* sets three of the five lessons each week for having students to work independently. Partly for this reason, this textbook series is referred to as semi-instructive, as opposed to the instructive *De Wereld in Getallen*. This latter textbook series not only organises four out of the five weekly lessons in a whole-class setting, but also has an approach in which more teacher guidance is provided. The difference between these two realistic textbook series is specifically reflected in the topic of algorithmic calculation (for wich *De Wereld in Getallen* offers a better structure) and the topic of percentage (for wich *De Wereld in Getallen* offers more starting points for teacher guidance). Both its organisation with offering students opportunities to work on their own and its thematic approach make *Pluspunt* an appealing textbook series that manages to acquire a 45% market share in the period 2000–2010, against 25% for *De Wereld in Getallen*. This choice was clearly not inspired by student results, as the first three national evaluations of the progress in educational achievement by Cito, the PPON[6] studies (Wijnstra, 1988; Bokhove, Van der Schoot, & Eggen, 1996; Janssen, Van der Schoot, Hemker, & Verhelst, 1999) that were carried out from 1987 to 1997, were in favour of *De Wereld in Getallen*.

6.7.2 The Results from the Cito PPON Studies

If one compares in the first three PPON studies (Wijnstra, 1988; Bokhove et al., 1996; Janssen et al., 1999) the mathematics achievements Cito found for the various textbook series in use, it is clear that the conceptual textbook series, both the functional and realistic ones, perform much better than the procedural textbook series.

In the third study, it was found (see Janssen et al., 1999) that on the 24 mathematical sub-domains that have been investigated, both *Nieuw Rekenen* and *Pluspunt* score on average more than 5% points higher than the procedural textbook series *Naar Zelfstandig Rekenen* and *Niveaucursus Rekenen*. The textbook series *De Wereld in Getallen* finishes on average 10% points higher than these procedural textbook series. For the basic operations with numbers, for estimating, for applied arithmetic with

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[6] Periodieke Peiling van het Onderwijsniveau (Periodic Assessment of the Education Level).

the use of a calculator, and for calculations with percentages this difference even increases to around 20% points!

The findings of this Cito study into the effects of various types of textbook series is of lasting relevance. Students perform better when conceptual textbook series are used than when procedural ones are used.

With respect to the sub-domain of operations (mostly algorithmic calculations), which is the main focus of the procedural textbook series, the first three PPON studies show that the differences between conceptual and procedural textbook series are only marginal. Nevertheless, from the results of the 1997 PPON study (Janssen et al., 1999) it is clear that the scores in this mathematical sub-domain are decreasing in comparison to the first study in 1987—a trend that continues in the fourth study carried out in 2004 (Janssen, Van der Schoot & Hemker, 2005), but stops in 2011 (Scheltens, Hemker, & Vermeulen, 2013). Conversely, improved performances are found on other topics, that is, insight into numbers, mental calculation (addition and subtraction), estimation, applied arithmetic with the use of a calculator, calculations with percentages and relations in the contexts of graphs. The performances on all these topics increase on average 15% points—about the same increase as the decrease for algorithmic calculations.

Taken together, these are spectacular research outcomes!

6.8 The Future Landscape of Textbook Series in the Netherlands

In 2015, the market shares of the largest textbook series had undergone a radical shift. That of *De Wereld in Getallen* has increased by 25% points to 50%, making the oldest realistic textbook series by Huitema and his collaborators one of the most successful and influential textbook series in the history of Dutch mathematics education. The most recent edition (Huitema et al., 2010) has been revised for both organisational and didactical structure as content. Students receive a week task for independent working which they can do in the second part of each lesson, after the whole-class instruction. These tasks are available at three difficulty levels, minimum, basic and advantaged, which makes it possible that the students can work on a level that is in tune with their ability. Further revisions in the textbook series are that some of the teaching sequences for measurement have been adapted; less time is spent on digit-based algorithmic calculation; and addition, subtraction and multiplication are more geared towards whole-number-based written calculation.

The market share of *Pluspunt* has fallen by 25% points to 20%. Compared to the 2001 edition, more time is spent on algorithmic calculation, much more than in *De Wereld in Getallen*. The organisational structure of the semi-whole-class system with three lessons a week for students to work independently and two lessons for whole-class instruction has not changed. New is that, as in *De Wereld in Getallen*, for differentiation three difficulty levels are distinguished.

Also, most of the other textbook series published a revised version, but there were only a few changes in their already not large market share. *Alles Telt* became a bit larger and the market share of *Rekenrijk* has decreased. For *Wis en Rekenen* no new, revised version was published. The new textbook series *Wizwijs* (Van Groenestijn et al., 2009) did not acquire a substantial market share. The same is true for the textbook series *Reken Zeker* (Terpstra & De Vries, 2009), explicitly published to implement (again) a procedural approach, which has not obtained a market share of any significance.

Taking into account that replacing a textbook series in a school takes about ten years, the current situation means that for the time being, realistic textbook series are used most frequently. So, we may conclude that two centuries of working on mathematics education have been decided in favour of the conceptual approach. This choice for a conceptual approach rather than a procedural one, is exactly what Freudenthal argued for around forty years ago (Freudenthal & Oort, 1977, p. 337).

> When a child finishes primary school, it has processed between ten and twenty thousand arithmetic problems – the degree to which it succeeded with them will determine its further education and its road in life, following a type of lower vocational education or […] general secondary education. But foremost this fact of learning to calculate (and the achieved or not achieved success in this learning) will determine the mathematical (or rather anti-mathematical) attitude of the student – and, what is even worse – of the teacher who has to teach mathematics […]. [This learning to calculate reflects][7] a view of a human being as an efficiently to be programmed computer, while the performance typical of a computer will never be approached. The education that we develop has been determined by another image of a human being, and by another view of mathematics – not as subject matter, but as a human activity.
>
> I have previously given this the triple characterisation of
>
> - Linked to reality
> - Near to the children
> - And socially relevant.
>
> And I will now sum up these characteristics in one that encompasses all: human worthy, the human being as a learner, as a teacher, as a counsellor and as a creator of education. (translated from Dutch by the authors)

These words that were spoken by Freudenthal on accepting an honorary doctorate at the University of Amsterdam, where he stated his ‘realistic’ vision of mathematics education against the sharply contrasting background of procedural mathematics that dominated education at the time, have lost none of their relevance today.

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[7] Added by the authors.

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[8] The references indicated with * are mathematics textbook series.

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Chapter 7
Sixteenth Century Reckoners Versus Twenty-First Century Problem Solvers

Marjolein Kool

Abstract In this chapter, the focus is on arithmetic which for the Netherlands as a trading nation is a crucial part of the mathematics curriculum. The chapter goes back to the roots of arithmetic education in the sixteenth century and compares it with the current approach to teaching arithmetic. In the sixteenth century, in the Netherlands, the traditional arithmetic method using coins on a counting board was replaced by written arithmetic with Hindu–Arabic numbers. Many manuscripts and books written in the vernacular teach this new method to future merchants, moneychangers, bankers, bookkeepers, etcetera. These students wanted to learn recipes to solve the arithmetical problems of their future profession. The books offer standard algorithms and many practical exercises. Much attention was paid to memorising rules and recipes, tables of multiplication and other number relations. It seems likely that the sixteenth century craftsmen became skilful reckoners within their profession and that was sufficient. They did not need mathematical insight to solve new problems. Five centuries later we want to teach our students mathematical skills to survive in a computerised and globalised society. They also need knowledge about number relations and arithmetical rules, but they have to learn to apply this knowledge flexibly and meaningfully to solve new problems, to mathematise situations, and to evaluate, interpret and check output of computers and calculators. The twenty-first century needs problem solvers, but to acquire the skills of a good problem solver a firm knowledge base—comparable with that of the sixteenth century reckoner—is still necessary.

7.1 Introduction

Over many centuries teaching arithmetic has played an important part in Dutch education. Interest in this subject started to grow in the sixteenth century when the Netherlands began to develop into an important trade nation and arithmetic finally got

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its own Dutch name: ‘*Rekenen*’. At that time the aims, content, organisation, teachers and students differed a lot from what is going on in Dutch arithmetic education of the twenty-first century. This chapter makes a comparison between now and then. The differences are large and plentiful, but there are also some remarkable similarities that we can perhaps learn from.

7.2 Arithmetic in the Sixteenth Century

7.2.1 Merchants, the New Rich of the Sixteenth Century

An early medieval Dutch merchant’s life was not very complicated. He wandered around, visiting towns and villages, trying to barter his goods. He was not schooled in bookkeeping and commercial arithmetic, but that was not a problem. Over time, in the fifteenth and sixteenth century, when the Netherlands grew more prosperous and more goods were produced, merchants were no longer simple wandering adventurers. They stayed in their offices and sent out their traveling salesmen. Business journeys became longer, merchants travelled to different countries, they had to pay salaries, customs rights, costs of transport, assurances of goods, etcetera. They needed to change money in many different ways, because each city had its own money system. They visited exchange banks where moneychangers took care of their affairs. Bankers and bookkeepers were needed. Many merchants earned a lot of money and they spent it on building houses and filling these with luxury goods; so, they needed carpenters, bricklayers, gold and silversmiths and other craftsmen. As trading methods grew more complex, a more advanced arithmetical method was needed, and written records of all commercial transactions and calculations (Swetz, 1989).

7.2.2 Traditional Arithmetic on the Counting Board

In the Netherlands of the early Middle Ages, arithmetic was traditionally done on a counting board with horizontal lines. Each line has a certain value and by placing coins on or between the lines people could express numbers and do calculations. This counting board is a variation of the ancient Greek and Roman abacus with vertical lines and counters of ivory, bones or glass (Mazur, 2014).

Traveling merchants did not always drag along their counting board. Instead, they left it at home and drew chalk lines on a table to do their calculations. Some merchants even omitted the lines. Figure 7.1a shows a picture from the French arithmetic book *Le Livre de Chiffres et de Getz* (Anonymous, 1501). Three merchants are calculating with coins without using lines. In this method (Fig. 7.1b), a decimal system is created by placing coins on a vertical line. These are the so-called ‘layers’. The first layer indicates the ones, the second layer indicates the tens, the next one the hundreds,

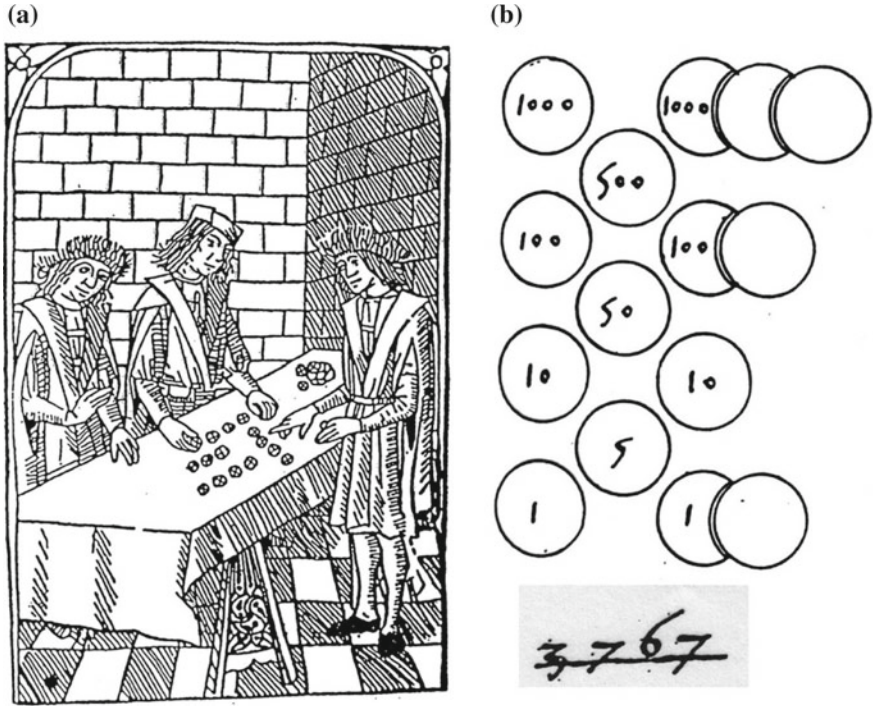


Fig. 7.1 **a** In the French arithmetic book *Le Livre de Chiffres et de Getz* (Anonymous, 1501; picture retrieved from Menninger, 1969, p. 367) the method of calculating with coins without using lines is explained. **b** Calculating 3 × 1000 + 500 + 2×100 + 50 + 1×10 + 5+2 × 1 = 3767 by using coins (this picture is from a book of Van Varenbraken, 1532; Ghent, University Library, ms. 2141. fol. 187r.; picture retrieved from Kool, 1988, p. 170)

etcetera. The value of the fields between the layers increases from 5 to 50, 500, etcetera. The number 3767 is expressed in coins. After you have learned to represent numbers with coins the next step is doing calculations. It is quite easy to add and subtract because you only have to add or remove coins, then rearrange the coins and read the result. Doing multiplication and division is a little bit more complicated, but it is doable. So, this traditional way of doing arithmetic sufficed for quite a long time.

7.2.3 A New Written Arithmetic Method with Hindu–Arabic Numbers

At the end of the 12th and the beginning of the thirteenth century a new arithmetic method appeared in Southern Europe. This method was spread in Arabic manuscripts that reached Spain and Sicily via trade routes. The best-known manuscript is the ninth century arithmetic manuscript of al-Khwarizmi (ca. 780–850), a Muslim mathematician, astronomer, geographer at the court of al-Mansur in Baghdad. His arithmetic manuscript has been lost, but Latin translations still exist. In his work, he describes the Hindu system of numeration and a method to do written calculations using this number system. Several Latin translations and adaptations were made of this manuscript. Inspired by these works, thirteenth-century European scholars like John of Sacrobosco and Alexander of Villa Dei wrote their own arithmetic books. These academic Latin treatises may have been intended for a learned audience (Folkerts & Kunitzsch, 1997).

The Italian Leonardo of Pisa (also called Fibonacci, ca. 1170–1240) learned the new arithmetic method during business journeys with his father in North Africa. In 1202, he wrote the *Liber Abaci*. In this book, he applied the new arithmetic method on a great many commercial problems. This practical part of his work was copied by the authors of dozens of Italian arithmetic books. Translations and adaptations of these books in several languages were made and the new method became popular in many other European countries including the Netherlands. The audience of these practical books was not academic.

7.2.4 The Rise of the New Arithmetic Method in the Netherlands

As far as we know now, the oldest arithmetic manuscript in the Dutch language teaching the new arithmetic method appeared in 1445. Two other Dutch arithmetic manuscripts were written in the fifteenth century. From the sixteenth century, 9 Dutch manuscripts and 24 Dutch printed books on written arithmetic with Hindu–Arabic numbers are in existence. If you take into account that arithmetic books were consumables used by teachers and traveling merchants, many more books and manuscripts must have been published at that time (Kool, 1999).

In some of these books both arithmetic methods are explained, the traditional one with the coins as well as the new written Hindu–Arabic one. Both arithmetic methods stayed in use over a long time. In Fig. 7.2 you see the two methods being practised together at the same table, on the left the modern method and in the middle the traditional one. This picture is from the title page of the arithmetic book written by the German Ries (1533).

Ries explains that learning the traditional arithmetic method with coins is a good preparation for learning the new method with pen and paper. In his book, he describes both methods. It seems that quite a few people in sixteenth century Europe could use both methods. The mathematician Peter Ramus used the new arithmetical method in his *Arithmeticae Libri Tres* (1555), but in private, he said, he preferred the traditional way with coins (Verdonk, 1966). There was no competition between the two methods, as is sometimes wrongly suggested (Boyer, 1968; Swetz, 1989).

Fig. 7.2 The traditional arithmetic method with coins (middle) and the new one with Hindu–Arabic numbers (left) on the same table; title page of the arithmetic book written by Ries (1533) (picture retrieved from Swetz, 1989, p. 32)

In the end, the modern way of calculating with a pen was preferred to the old manner. But this happened only after a rather long period of time. In 1689 calculation coins were still struck in the Southern Netherlands (Barnard, 1916).

Why did it take such a long time before the new method was accepted everywhere? For us it is obvious that it has many advantages as compared with the old one. For example, you can easily check your written calculation afterwards. In arithmetic with coins, the numbers you start with disappear from your counting board during your calculation. Of course, people could check their final result by using the ‘check of nines’, but it is impossible to read over the process afterwards. In the new method, you can. This new method has more advantages. Using Hindu–Arabic numbers extends your mathematical options. It is easy to write big numbers, to extract roots and to calculate with fractions. Using the traditional method people did their calculations with coins and then used a pen to write down their result in Roman numerals. In the new method, the same instrument—the pen—and the same number system—Hindu–Arabic—are used for both calculating and recording the result.

Yet, in spite of these advantages it is understandable why the traditional method with the coins survived for such a long time. Most of the people at that time could not write. Around 1600 in the Netherlands only 40% of women and 60% of men were able to sign their marriage certificate (Dodde, 1997). Perhaps more people could read, because in sixteenth century Dutch education reading was taught before writing and many students left school at the time that writing education started, because they had to work and earn money. The Dutch arithmetician Christianus van Varenbraken explained in his arithmetic manuscript of 1532 that he describes the traditional method with coins for people who cannot write. Another advantage of calculating with coins is that one visualises calculations with concrete objects. And finally, you do not need a zero. It is easy to understand that an empty place on your counting board means nothing. In the new written number system, you need a zero to indicate an empty space. You have to write a sign, although this sign means 'nothing'. And at the same time this magical sign can changes the value of a number when it is added to it. 4 does not mean the same as 40! People found this difficult to understand. Authors gave long explanations about the function of zero. Van Varenbraken (1532, cited in Kool, 1988) wrote about the zero:

> *This 0 means nothing, he has no value of his own, but 0 gives a value to the other 9 number symbols. And he makes their value ten times more than the value they have of their own.*

Some people were even opposed to the new number system because of the zero. In Florence, the Arte del Cambio, the guild of money changers, forbade its members to use the new numbers in their cash books for fear of fraud (Pullan, 1968).

Arithmetic books in the Dutch language, that had been available since the fifteenth century, were not used in the traditional Latin schools, because in these schools all teaching was done in Latin and arithmetic hardly played a part. During the sixteenth century, so-called 'French schools', in whose curriculum the town government did not have a say, were founded by private initiative. Merchants, bankers and other financial and administrative practitioners sent their sons to these schools to study subjects like French, bookkeeping and arithmetic. French was the most important business language at the time. The other subjects at the French schools were taught in the vernacular. It is clear that these schools were good 'nurseries' for future merchants, bankers and money changers. The arithmetic books in Dutch were used in these schools. Some teachers wrote and used their own arithmetic book.

7.2.5 The Content of the Dutch Arithmetic Books from the Sixteenth Century

The authors generally teach the basics of arithmetic, which means that they deal with the reading and writing of Hindu–Arabic numerals including zero, and the arithmetical operations: addition, subtraction, multiplication and division. Some authors also dealt with halving and doubling, which they considered as separate operations. The arithmetic algorithms they teach largely correspond to those in use nowadays. Only the division algorithm shows some differences. First calculating with whole numbers is taught, followed by fractions. To practise the algorithms many examples, worked out in detail, are presented. Most of these examples deal with money, weights and measures. In the sixteenth century, each city had its own system of money and

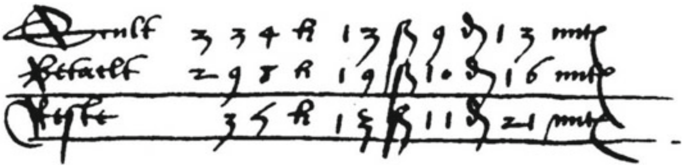


Fig. 7.3 Subtraction (including a mistake) with two amounts of money from the Dutch arithmetic manuscript of Christianus van Varenbraken (1532); Ghent, University Library, ms. 2141. fol. 135r (picture retrieved from Kool, 1999, p. 72)

measures which could make calculations rather complicated. In Fig. 7.3 you see a subtraction with two amounts of money from the arithmetic manuscript of Van Varenbraken (1532) from Ghent: 298 lb, 19 shillings, 10 pennies and 16 mites are subtracted from 334 lb, 13 shillings, 9 pennies and 13 mites. You have to know the Ghent system in which: 1 lb equals 20 shillings, 1 shilling equals 12 pennies and 1 penny equals 24 mites. It is clear that this complicated calculations and many mistakes were made, as you see in the final result of the example: 11 pennies ought to be 10 pennies.

Authors teach their readers to check their calculations, especially the check of nines appears often, but apparently this example was not checked. In the first part of the books sometimes extracting roots is dealt with also, and as said before, calculating on a counting board.

In the second part of the books elementary arithmetic is applied to solve all kinds of practical problems, on buying, selling or exchanging of goods, partnerships, changing money, calculating interest, insurance, profit, loss, etcetera. It is clear that it is useful for future merchants and technical, administrative or financial practitioners to learn to solve these. The most important rule to solve these practical arithmetical problems is the rule of three. This rule is used to find the fourth number in proportion to three given numbers. Because of its importance some authors introduce this rule in a richly decorated frame, see Fig. 7.4. This picture is from the arithmetic book by Van Halle (1568). The text says: 'The rule of three, how you can find the fourth number out of three numbers'. The other arithmetical rules are mostly variants of the rule of three.

If you want to solve a problem with the rule of three, you have to place the three given numbers on a line, multiply the last two numbers and divide the product by the first one. In Fig. 7.5 you see one of the many problems that is solved by the rule of three from the arithmetic book by Van Halle (1568). The problem is: "*If nine seamstresses can make fifteen shirts within one day, how many shirts can six seamstresses make*?" Van Halle places 9, 15, and 6 on a line and calculates $(15 \times 6) \div 9 = 10$ shirts.

Of course, there is a more appropriate way to find the solution of this problem. You can even solve this by doing mental calculations: if nine seamstresses can make fifteen shirts, three seamstresses can make five shirts and six seamstresses can make ten shirts. This is much easier, but this kind of clever alternative solution methods is hard to find in the old arithmetic books. The authors give only one solution method for each problem. They present standard algorithms that always work in the same

Fig. 7.4 The exuberant introduction of the important rule of three in the arithmetic manuscript of Van Halle (1568); Brussels, Royal Library, ms. 3552. fol. 60v (picture retrieved from Kool, 1999, p. 133)

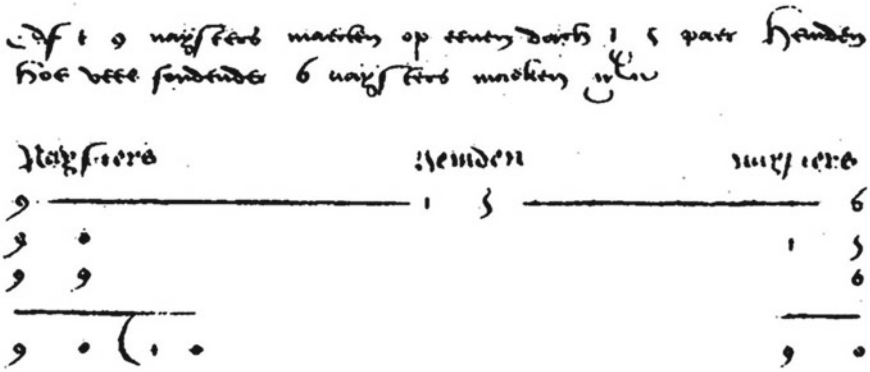


Fig. 7.5 One of the problems that is solved by the rule of three in the arithmetic manuscript by Van Halle (1568); Brussels, Royal Library, ms. 3552. fol. 70v (picture retrieved from Kool, 1999, p. 134)

way, followed by many problems to practise these fixed recipes. There are a few exceptions, which I will discuss later on.

The problem of the seamstresses is quite simple, but the books contain many problems that are (much) more complicated. As you can see in the following example from the arithmetic book by Van Halle (1568):

> *Three merchants are at sea and suddenly a violent storm arises. They have to throw overboard a part of their cargo. The value of this part is 100 guilders. In the end, they come home safely where they have to divide the loss. The first merchant had 300 guilders worth of cargo on the ship, the second had 400 guilders worth of cargo on the ship and the third one had 500 guilders worth of cargo in the ship. The cargo had a total value of 1200 guilders, of which 100 guilders was thrown overboard. What is the loss of each individual merchant?* (Fig. 7.6).

Money changers had to solve problems like the one in Fig. 7.7, from the arithmetic book by Van der Gucht (1569):

> *A merchant from Florence went to the exchange bank in London in order to change* $120\frac{1}{2}$ *ducats at* $42\frac{1}{4}$ *pennies each into angelots at* $66\frac{1}{2}$ *pennies each. The question is: How many angelots will he get in London? The calculation here is:* $(120\frac{1}{2} \times 42\frac{1}{4}) \div 66\frac{1}{2} = 76$ *angelots and the remainder is 594.*

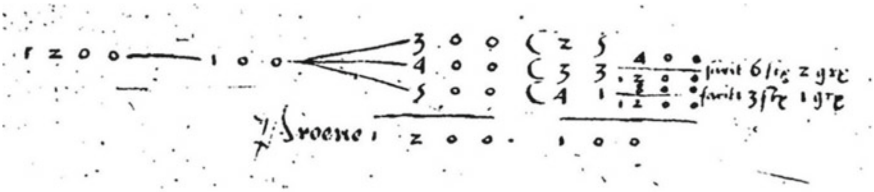
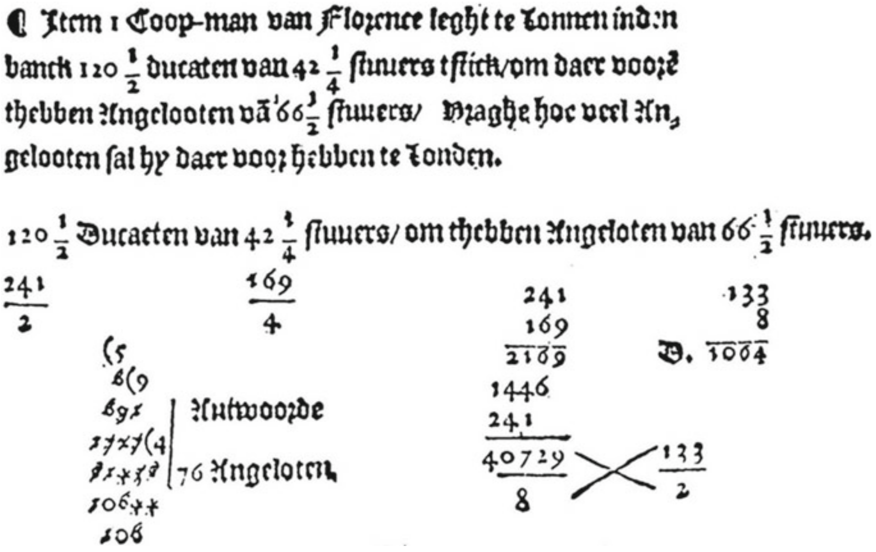


Fig. 7.6 Solution of the problem about three merchants who share the loss they had in a violent storm at sea; this problem is from the arithmetic manuscript of Van Halle (1568); Brussels: Royal Library, ms. 3552. fol. 97r (picture retrieved from microfilm)



¶ Item 1 Coop-man van Florence leght te Londen in den banck 120 $\frac{1}{2}$ ducaten van 42 $\frac{1}{4}$ ſtuuers tſtick/om daer voor thebben Angelooten vã 66 $\frac{1}{2}$ ſtuuers/ Vraghe hoe veel Angelooten ſal hy daer voor hebben te Londen.

120 $\frac{1}{2}$ Ducaeten van 42 $\frac{1}{4}$ ſtuuers/ om thebben Angeloten van 66 $\frac{1}{2}$ ſtuuers.

Antwoorde 76 Angeloten

Fig. 7.7 A problem about changing money, from the arithmetic book by Van der Gucht (1569); Ghent, University Library, Acc. 1463. fol. 96r (picture retrieved from microfilm)

The authors of the sixteenth century arithmetic books only use words and numbers to describe problems and solution methods. In the first parts of these books the solution descriptions are very long and cumbersome, but further on in the books, as you can see in the Figs. 7.5, 7.6 and 7.7, authors use more concise, symbolic notations and try to limit the number of words. They use lines, crosses and other graphical means, and signal words with a special meaning, for example, the word 'proeve', which means 'check'. These schematic presentations increase readability, are easier to learn by heart and reduce the risk of making mistakes. These efforts to shorten the presentation of calculations prepare the way for the later symbolic mathematical notation.

7.2.6 Didactic Principles in Dutch Arithmetic Books from the Sixteenth Century

If you study sixteenth century Dutch arithmetic books you can derive some didactic principles. Arithmetic skills are needed by merchants and financial, administrative and technical practitioners. To develop these skills the authors offer a limited number of standard algorithms to do arithmetic and rules to solve the practical problems they come across in their professions. They present one solution method for each problem type and to practise this method they give many similar problems that differ only in the numbers used. Repetition may help the pupil to remember the solution method. In some situations, alternative and more convenient solution methods are possible, but these are rarely shown. Probably the authors want to achieve that their students can use this method more or less 'blindly'. They must become skilful reckoners. Repetition, practise and drill were the main principles of this education. You can recognise these principles, for example, when studying the tables of multiplication in the books. In the arithmetic manuscript of Christianus van Varenbraken of 1532, you see a 12 times 12 table with the exhortation to learn these tables "as well as your 'Ave Maria' without missing anything". It shows that learning these tables was a serious matter, as important as learning prayers. An anonymous arithmetic manuscript of 1594 contains tables of multiplication even up to 17×27. The author of this manuscript likewise ordered his students to learn these tables by heart. And they probably did, because in a time without pocket calculators, in a society with very complicated systems of money, weights and measures, it will be useful to have many multiplications in your head, especially when you realise that paper was expensive at the time. Calculations were made on a slate. Arithmetic books were used by the teacher and mostly not available for students.

When considering the standard rules in sixteenth century arithmetic books, the practical problems, the many exercises to apply algorithms and fixed recipes, you can imagine that sixteenth century craftsmen became well trained reckoners within their profession. If they came across a new mathematical problem they probably did not know what to do, but that did not matter because they hardly came across new mathematical problems. They wanted to know the arithmetic of their profession and they had no need for learning mathematics.

7.2.7 Interesting Exceptions

In some of the sixteenth century arithmetic books there are problems that do not fit the previously sketched situation. These problems are not practical at all. They contain unrealistic stories and have nothing to do with money and commerce. For example, in the book written by Van der Gucht (1569) there is the following problem:

> *A man walks 11 miles during the day and at night he walks back for 3 miles. The question is in how many days he will reach Rome, if the distance to Rome is 500 miles.*

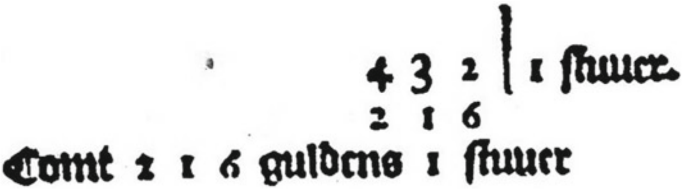
It is quite unlikely that a traveller to Rome would walk back three miles each night. How could this problem end up in a book with practical exercises? Tropfke (1980) discovered that variations of this problem already appeared in India in the ninth century, and also in the Arabic manuscript of al-Karagi (late tenth and early eleventh century), and you can find it in several European arithmetic books, including the *Liber Abaci* of Leonardo of Pisa from 1202. It turns out that most of the unrealistic problems in the sixteenth century arithmetic books are very old and appeared in different historical mathematical manuscripts. Their function in the sixteenth century books is not clear. Perhaps it is a matter of tradition, a kind of cultural heritage. Van Egmond (1980) and Tropfke (1980) think that these problems had a recreational function in the serious practical books, to break the routine. That seems plausible, because authors like Van Varenbraken (1532) and Stockmans (1595) call these problems ‘problems for pleasure’ and ‘entertaining problems’. Van den Dijcke (1591) collected all these curious problems in a special chapter at the end of his book. He introduces this collection with: “Here you will find many different beautiful problems to sharpen and enjoy your mind.”

Only a few of the arithmetic books have some of these unrealistic traditional problems. It is clear that sharpening and enjoying the mind of the readers was not a common or important purpose of the authors. These problems originally belonged to the old sources of the academic mathematical tradition and arrived perhaps more or less ‘accidentally’ in some of the commercial arithmetic books. You can imagine that an author saw a source with these entertaining problems and added a few to his own book to bring some variation, but it is clear that these problems may not distract the students too much from the main aim to learn practical and useful arithmetic.

There is a second unexpected phenomenon in some of the arithmetic books. The authors call it French or Italian practice. This is a collection of alternative arithmetic methods with which the arithmetician can speed up and simplify his calculations. But these methods only work in particular cases and with specific numbers. You cannot use them blindly and you need arithmetical insight to judge if it is possible and efficient to use these special strategies.

For example, in Fig. 7.8 you see a problem from the arithmetic book by Van der Gucht (1569): “*How many guilders can you have for 4321 nickels?*” To solve this problem you have to divide 4321 by 20. Van der Gucht advises to put aside the last cipher of the number and halve the remaining number.

Van Halle (1568) deals with problems like, “*If 16 m of cloth cost 99 guilders, what is the price of 128 m?*” Instead of the standard calculation with the rule of



4 3 2 | 1 ſtuuer.

2 1 6

Comt 2 1 6 guldens 1 ſtuuer

Fig. 7.8 A fast way to change guilders into nickels, from the arithmetic book by Van der Gucht (1569); Ghent, University Library, Acc. 1463, fol. 39v (picture retrieved from Kool, 1999, p. 162)

three (99 × 128) ÷ 16 = 792, he advises to divide 128 and 16 by 8 first. Because then you have to calculate (99 × 16) ÷ 2 = 792, which is much easier. He probably did not realise that he could simplify the problem even more by dividing 16 and 128 by 16, because then the remaining calculation is even more easier (99 × 8) ÷ 1 = 792.

This type of insightful efficient calculation only plays a minor part in some of the arithmetic books. It is conceivable that experienced merchants used many strategies from the French or Italian practice in their daily work, but in the arithmetic books you hardly see them. The core business of the teachers was to practise and drill standard rules and fixed recipes, flexibility was learned during work.

7.3 Arithmetic in the Twenty-First Century

7.3.1 *Comparing Sixteenth and Twenty-First Century Education*

Let us make a giant leap to education in Dutch schools of the twenty-first century. It is not surprising that the differences with the sixteenth century business schools are huge! In our time, all children, including of course girls, go to school; this is not a privilege for sons of merchants and bankers. All students learn arithmetic as part of mathematics for at least ten years, with books of their own, pen and paper, tablets, laptops, smartboards, computers and calculators. The differences between sixteenth and twenty-first century education are huge and numerous, but there is one similarity between the teachers of the sixteenth and their twenty-first century colleagues, they both want to teach their students the arithmetic they need in daily life, in society and in their future profession. It seems that the teachers of the sixteenth century French schools were quite successful in reaching this aim. But concerning twenty-first century education, there is much discussion about the skills that our students need to acquire and the way that modern education can contribute to them.

7.3.2 Twenty-First Century Skills in General

Wagner (2008) speaks of an achievement gap between what schools (in the United States) are teaching and what is necessary for students to succeed in the current knowledge society. He argues that students have simply not been taught the competences that are most important for the twenty-first century. The skills that current and future professions require, differ significantly from what current education offers. Wagner gives the following list of what he calls, "the new survival skills": (1) critical thinking and problem solving, (2) collaborating and leading by influence, (3) agility and adaptability, (4) initiative and entrepreneurism, (5) effective oral and written communication, (6) accessing and analysing information, (7) curiosity and imagination. Wagner is not the only one who discussed this issue. The twenty-first century skills that we need to survive in our rapidly changing computerised and globalised society are discussed by many experts from inside and outside education, and they give lists comparable to that of Wagner.

7.3.3 Twenty-First Century Skills in Mathematics Education

Making a list of necessary twenty-first century skills is a good thing to start with, but the next question is what such a list means for the organisation and content of education, especially mathematics education. Gravemeijer (2012) concluded that critical thinking, problem solving, collaborating and communicating fit very well with problem-centred, interactive, mathematics education. These are also the aims of Realistic Mathematics Education (Van den Heuvel-Panhuizen & Drijvers, 2014) in which students get the opportunity to work in groups on meaningful problems guided and supported by their teacher. In this way, students try to reinvent parts of mathematics. Interaction, discussion, reasoning, asking questions and understanding are important features of this kind of education. In practice, it turns out that it is quite challenging to stimulate students to join actively in interactive problem solving and reasoning, because they are not familiar with it. Students need time to adopt new classroom social norms and to develop enough self-confidence to explain and justify their solutions, to try and understand other students' reasoning, and to ask questions when they do not understand something, and challenge arguments they do not agree with. It takes time to change the classroom into a research, annex learning, community (Cobb & Yackel, 1996). At the same time, it places high demands on teachers. They can no longer give ready-made solution methods, but have to develop students' reasoning to higher levels of understanding by fostering discussions and asking questions like: What is the general principle here? Why does this work? Does it always work? Can we prove that? Can we describe it in a more precise manner? Can we do this in another way? Etcetera. Creating a classroom atmosphere where students construct knowledge by learning from and with each other demands special qualities, competences and efforts of the teacher, but it is worth it.

7.3.4 The Content of the Mathematics Curriculum

Now we know what requirements the classroom culture must meet to develop twenty-first century skills, the next question is: What should be the content of the mathematics curriculum when computers take over mathematical routine tasks? Focusing on standard algorithms seems less important. The rise of computers in society and education places new mathematical demands on students. They have to learn to recognise the mathematical structure of situations and problems, they need to translate these problems into tasks that a computer or calculator can execute; this means quantifying and mathematising reality. So, students must have some idea of what quantifying (measuring) reality entails. Besides that, they have to understand what a variable is, and how to reason about interdependencies between variables, and finally they have to interpret and evaluate the output of the computer. This asks for mathematical topics such as measuring, tables, graphs, variables, models of relationships between variables, and elementary statistics (Gravemeijer, 2012).

The more we leave mathematical work to the computer, the more important it becomes that we control the computer output in a more or less approximate way. This asks for arithmetical skills to estimate calculations, based on networks of number relations and flexible and meaningful use of features of arithmetical operations. For example, if you want to check calculations such as: $4 \times 26 = 104$ and 13% of $888 = 115.44$ it is useful to know number relations like $4 \times 25 = 100$ and 12, 5% equals 1/8. And if you want to check $7 \times 99 = 693$ it is good to know the distributive law $7 \times 99 = 7 \times 100 - 7 \times 1$.

You can check $8 \times 1.76 = 14.08$ by calculating 8×1.75. You know $8 \times 2 = 16$ and $8 \times 0.25 = \ldots$? You may think $8 \times 25 = 200$, so $8 \times 0.25 = 2$ or $8 \times ¼=2$, you will find $8 \times 1.75 = 16 - 2= 14$. But you can also use the arithmetical rule of halving and doubling, like $8\times1.75=4\times3.5=2\times7=14$. It is clear that $8\times1.76=14.08$. These are just a few examples to show how you can use number relations and arithmetical rules in many different ways to check calculations. It is obvious that there is still much work to do in the arithmetic education of the twenty-first century, as it will be a big effort to equip students with sufficient flexible and meaningful arithmetic skills.

The contrast with the educational aims of the sixteenth century arithmetic books is enormous. Instead of recognising the type of problem and choosing the standard recipe to solve it, twenty-first century students have to mathematise a given problem situation, solve it with or without a computer or calculator and interpret and evaluate the output by checking it in an approximate way using flexible knowledge of number relations and arithmetical rules. Instead of recognising a well-known situation, our students need to recognise the mathematical structure of a new situation. Instead of using a ready-made solution method, our students need to construct a new solution method using the arithmetical knowledge and tools they possess.

We may not underestimate the arithmetical skills of the sixteenth century practitioners. They had to deal with complicated money, weight and measure systems. They learned fixed arithmetic recipes at school, and it is plausible that they became experienced in the flexible arithmetical tricks of the French and Italian practice during their working life. They were not taught to deal with new arithmetic problems, but they were experienced, flexible reckoners within the borders of their profession. Learning arithmetic to solve applied problems is part of the Dutch didactic tradition until today, but the nature of the applied problems changed during the years and that asked for different knowledge and skills.

The twenty-first century asks for problem solvers, people who can apply their arithmetical knowledge to unknown problems in new situations. At first glance, the computerisation of society makes life easier and more comfortable compared to the sixteenth century. We no longer have to use long and cumbersome arithmetic algorithms. But when you realise what our society asks from its members it is clear that the aims of arithmetic education are much more challenging than they were in the sixteenth century.

In spite of that, we can learn two important things from the sixteenth century. School is not the only place where you can learn things. After school, in your professional life, learning is still going on. In recent years the lifelong learning concept has gained adherents because people realise that it is impossible to reach all goals at the required level in school. That means that we have to make choices in our arithmetic education. The arithmetic books of the sixteenth century make a suggestion. Equipping students with a solid basis of arithmetical knowledge seems a valuable starting point.

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Chapter 8
Integration of Mathematics and Didactics in Primary School Teacher Education in the Netherlands

Wil Oonk, Ronald Keijzer and Marc van Zanten

Abstract From the 1970s, curricula of primary mathematics teacher education in the Netherlands drastically changed. This occurred simultaneously with the changes in primary mathematics education. Teacher educators systematically discussed mathematics teacher education and implemented new content and new approaches in primary teacher education. This chapter provides a chronological overview of how Dutch primary mathematics teacher education developed from the 1970s until the present. We describe ideas about learning to teach mathematics and ideas about the relationship between the development of mathematical literacy and didactical proficiency of student teachers. Furthermore, the influence of national measures such as the introduction of nationwide tests for primary mathematics teacher education is discussed. The chapter ends with an impression of recent learning materials for student teachers and a reflection on new perspectives for integrating theory and practice, emphasising the continuous search for a well-balanced way to interconnect mathematics and didactics.

8.1 Introduction

In 1858 criticism on the education of teachers in the Netherlands led to the establishment of the first teacher education institutions by law (Van Essen, 2006). The curricula of these institutions were almost the same as those for the higher grades of secondary school, with pedagogy and half a day a week for teaching practice

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added. The mathematics curriculum included, for example, algebra, planimetry and stereometry (Van Beek & Van Heek, 1924, 1925, 1926).

This situation did not change fundamentally for decades. Even in 1952, when curricula changed as a result of the Nieuwe Kweekschoolwet (New Teacher Education Act), little changed in daily practice because most teacher educators were the same people as before the Act, and they saw no reason to change their teaching approaches.

In 1968 teacher education institutions were renamed into Pedagogische Academies (Pedagogical Academies). In 1985, after years of discussion (see Innovatie Commissie Opleidingen Basisonderwijs, 1980) kindergarten schools and primary schools, were integrated into primary schools for children from 4 to 12. The respective teacher education institutions were also integrated into the Pedagogische Academies voor het Basisonderwijs (Pedagogical Academies for Primary Education). After a few years, criticism from the Onderwijsraad, the Dutch national advisory council for education (Onderwijsraad, 1988) and OCW, the Ministry of Education (OCW, 1990), mainly on the insufficient academic level and the lack of a clear education concept at the teacher education institutions, again led to a revision of the teacher education curriculum. This time the curriculum had to be based on a well-defined education concept and should fit within the framework of a specific teacher education didactics. Problem-based learning and thematic education were adhered to, and teachers from all disciplines were expected to develop their own educational materials according to these two concepts (Goffree & Oonk, 1999).

From the beginning in 1991, external quality monitoring of teacher education institutions mostly considered general characteristics of the institutions' curriculum and hardly focused on school subjects. Over the years less and less attention was paid to the development and enforcement of mathematics and other subjects in teacher education, which resulted in a decrease in attention for subject-specific content knowledge in the teacher education institutions (Van Mulken, 2002; Onderwijsraad, 2005). However, from time to time the HBO-raad, the Council for Higher Professional Education, took measures to secure the mathematical proficiency of student teachers, mainly through including mandatory mathematics tests (Keijzer & Hendrikse, 2013).

In this chapter, we describe primary school teacher education in the Netherlands, taking the year 1971 as a starting point, which is the year that the IOWO[1] (the first predecessor of the Freudenthal Institute) was founded. In the final section of this chapter we provide a new perspective on learning to teach mathematics, the relationship between the development of mathematical literacy and didactical proficiency and recent influences on primary school mathematics teacher education.

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[1] Instituut voor de Ontwikkeling van het Wiskunde Onderwijs (Institute for the Development of Mathematics Education).

8.2 Mathematising and Didacticising

8.2.1 *The Influence of Freudenthal on Mathematics Teacher Education*

In 1971 Hans Freudenthal became the first director of the IOWO, where two years earlier the Wiskobas[2] project had started (see Freudenthal, 1978; Treffers, 1978; 1987). This Wiskobas project for mathematics in primary school was intended to develop a new approach to mathematics education together with teachers and student teachers, with the intention to teach them at the same time how to implement this new approach. A strong belief of Freudenthal and his team was that the curriculum for primary school mathematics teacher education should be developed in close connection with the primary school mathematics curriculum. Shortly after the first Wiskobas curriculum publications were published, an educational experiment was started at a teacher education institution in Gorinchem (Goffree, 1977; Goffree & Oonk, 1999). Every week, Freudenthal and two members of the Wiskobas team went to this small town and attended the lectures given to the student teachers and visited the school where the student teachers acquired practical experience. Freudenthal worked with the children to show the student teachers how it is possible to initiate and observe mathematical learning processes. His observations and analyses were intended to convey to the student teachers the idea of the teacher as a researcher and give them the feeling that there was much that could be learned from the children themselves. Freudenthal also introduced a narrative element into the teaching with stories such as "Walking with Bastiaan" (Freudenthal, 1977). Materials for the student teachers were developed and tested. Freudenthal made theoretical contributions to these materials as evidenced, for example, by his work on the analysis of fractions and ratio as mathematical objects (Freudenthal, 1983). Freudenthal's approach had a great impact on the teacher education institution in Gorinchem (see, e.g., Goffree, 1979).

8.2.2 *A Model for Learning to Teach Mathematics*

The experiences in Gorinchem were discussed regularly in a national group of mathematics teacher educators. Materials for student teachers were tried out nationwide and rewritten by the staff of the Wiskobas project. All this led to a model for learning to teach mathematics (Fig. 8.1).

The model shows, starting from the left, that mathematics education, both for student teachers and children, takes meaningful mathematical situations as its starting point. For children, it implies the activation of subjective, informal structures, which allows the mathematical learning process to start with children's intuitive

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[2] Wiskunde op de Basisschool (Mathematics in Primary School).

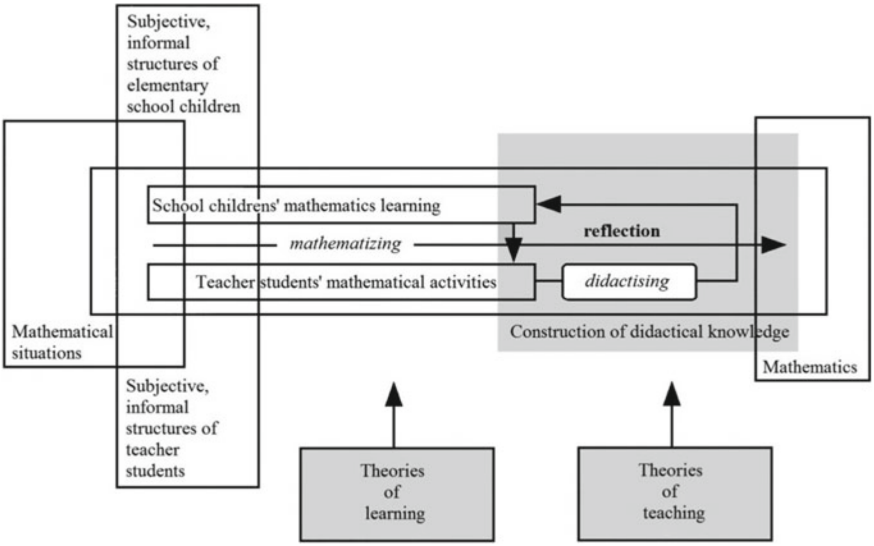

Fig. 8.1 Model for learning to teach mathematics (Goffree, 1979, p. 313; Goffree & Oonk, 1999, p. 209)

notions and informal procedures. Under the guidance of the teacher who knows the objective structures of mathematics (formal mathematics), they get the opportunity to rediscover mathematics (Freudenthal, 1983), experiencing a process of guided reinvention (Freudenthal, 1991).

For the student teacher, whose subjective, informal structures are affected by his or her earlier experiences with learning and teaching mathematics, learning to teach is considered a process of both mathematising and didacticising (Freudenthal, 1991). Keijzer (1994, p. 4) expresses this as follows:

> Reflecting on learning experiences as a starting point for learning to teach mathematics. In teacher education, student teachers' experiences with learning mathematics are discussed from time to time, especially shortly after student teachers enter teacher education. One teacher educator decides to use this focus in the very first meeting with first-year prospective teachers. One student teacher recalls learning the algorithm for long division and another tells how she masked her struggle with mental arithmetic by finger calculations hidden from the teacher. The teacher educator concludes: "Our talk on early experiences with learning mathematics showed that reflecting on one's own mathematical acting forms a fruitful starting point for exploring didactical content knowledge." (translated from Dutch by the authors)

The view on learning to teach mathematics as represented by the model in Fig. 8.1 is that learning to teach mathematics should start by student teachers carrying out mathematical activities at their own level. Reflections on children's learning processes combined with the student teachers' own experiences in learning mathematics contributes to the creation of an educational basis for teaching mathematics. Big ideas from general educational theory, rooted in either didactics or formal mathematics, can also contribute. It is assumed that in this way student teachers will get into a cyclical process of solving mathematical problems, mathematisation, reflective problem solving, and mastering teaching approaches. Meanwhile, student teachers work with children and study their learning processes while continually referring to their own learning processes. While doing so the student teachers integrate their subject matter knowledge and didactical content knowledge,[3] in other words, they coherently develop both mathematical and didactical knowledge.

8.3 New Developments in Primary School Mathematics Teacher Education

8.3.1 Mathematics & Didactics as a New Subject for Student Teachers

In the 1980s, the model described in Fig. 8.1 was elaborated into the new subject Mathematics & Didactics[4] in primary school mathematics teacher education, based on the book series *Wiskunde & Didactiek* (Goffree, 1982/1994, 1983/1992, 1985, 1993/2005, 2000). This series of books was used in the 1980 s and 1990 s in more than eighty percent of the Dutch teacher education institutions. Goffree (1982, p. 7) formulated the approach to primary school mathematics teacher education as follows:

> Learning to teach mathematics
>
> We think that you will best learn to teach by first working on mathematical problems yourself. Of course, these problems have to refer to the subject matter you are going to teach. Therefore, most chapters of this book start with simple mathematics problems. Thinking through these problems together helps you to look back at your own and your peer students' solutions from a different point of view. We call this reflection: thinking deeply on finding new ways, using a sketch or material, getting another explanation, a state of still not understanding or suddenly grasping it […]. We think this is important. It builds up the beginning of your didactical thinking, because you can expect similar situations if you are going to teach mathematics to children. (translated from Dutch by the authors)

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[3] Although in international publications this is often referred to as 'pedagogical content knowledge', in the Netherlands, the term 'didactical content knowledge' is preferred. By using 'didactical', it is made clear that here knowledge is meant that is related to the teaching of mathematics and not knowledge about interacting with children and creating an environment at school that socially, psychologically and physically supports their development.

[4] The &-sign between mathematics and didactics symbolised the ambition of the author of the textbooks to optimize the student teachers' integration of their subject matter knowledge and didactical content knowledge.

8.3.2 The Influence of Quality Monitoring

In 1985 the Dutch Ministry of Education proposed a new system of quality monitoring in higher education, which, in the same year, was adopted by the Dutch parliament. In this system, the institutions for higher education themselves became responsible for quality monitoring, while the government would follow this process from a distance (OCW, 1985; Van Bemmel, 2014). Following this advice, the HBO-raad[5] (Council for higher professional education) arranged inspections in primary teacher education; the first one took place in 1991. Focal points in both this first inspection and in the second one included: curriculum, prospective teachers' level, assessment systems, and student teachers' educational process. These points potentially offered chances to evaluate the mathematics curriculum within teacher education. There was a need for such an evaluation, as primary student teachers' mathematics proficiency was a concern of many (Brandt, Feijs, Groen, & De Moor, 1987). Such an evaluation, however, did not happen. The inspection focused only on general aspects of teacher education and was not equipped to look at more domain specific issues (Keijzer, 1993).

8.3.3 Growing Attention to Student Teachers' Mathematical Literacy

Alongside the development of a programme for primary school mathematics teacher education, there were growing concerns about the mathematical proficiency of student teachers (Jacobs, 1986; Brandt et al., 1987) As a reaction, the *Mathematics & Didactics* series was extended with a book especially aimed at the development of student teachers' mathematical proficiency (Goffree, Faes, & Oonk, 1988, 1994; Goffree & Oonk, 2004). This book contained mathematics problems selected from primary school mathematics textbooks, each problem was provided with reflective solutions at the level of student teachers. The view of the authors was that comparing one's own solutions with expected solutions of children and with expert solutions and discussing them, would raise the student teachers' mathematical literacy, and also strengthen their didactical proficiency.

Cooperating mathematics teacher educators also encouraged the Council for Higher Professional Education, to support the development of materials to tackle the problem of student teachers' low mathematics abilities. This led to the series *Wiskunde Verplicht* (Mathematics Required) (Faes & Oonk, 1989a, b, c, 1990) with which student teachers could refresh their mathematical skills. In addition, there was the series *Gecijferdheid* (Mathematical literacy) (Faes, Van den Bergh, & Olofsen, 1992) with example problems on mathematical literacy, which teacher educators could use to develop a test for student teachers, that could be administered parallel

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[5] Currently 'Vereniging Hogescholen'.

- Calculate using an efficient strategy: 412 – 97.98 =
- What scale should be used to fit a map of the Netherlands on an A4-sheet? Make sure the space on the sheet is maximally covered with the country. Write your answer as: 1 : _____.
- Design a situation that leads to the number sentence ‘2.25 ÷ 0.75’.

Fig. 8.2 Problems from the 1992 Mathematical Literacy Test

to the other first-year test. The problems provided for the mathematical literacy test assessed, for example, student teachers’ ability in using efficient strategies in tackling number problems or showing insight in measurement. Figure 8.2 shows some typical problems from this test.

From 1992 on, most teacher education institutions for primary education developed and used local adaptations of this test to assess their first-year student teachers’ mathematics skills. In addition, as this test asked for mathematical knowledge of a specific nature, the first-year curriculum gradually incorporated this knowledge. This curriculum started to focus more on efficient strategies to solve number problems, various meanings of (rational) numbers and number relations, meaning in measurement (including personal references to measures), estimation, and geometry. However, although the tests used were inspired by a series of prototypical examples, the (adapted) tests differed significantly between institutions, with respect to the topic in the test, the difficulty level of the problems, and the pass mark.

8.4 Standards for Primary School Mathematics Teacher Education: Adapting the View on Learning to Teach Mathematics

8.4.1 Towards Standards for Primary School Mathematics Teacher Education

After discussions with all the experts involved, the new approach to primary mathematics education in the Netherlands that had been stimulated by Hans Freudenthal, and that was now known as Realistic Mathematics Education (RME), led to the *Proeve van een Nationaal Programma voor het Reken-wiskundeonderwijs op de Basisschool* (Treffers, De Moor, & Feijs, 1989), a first design for setting up a national programme for mathematics education in primary school. Following this programme and the standards for mathematics evaluation and teaching of the National Council of

Teachers of Mathematics (NCTM, 1989, 1992), a group named PUIK,[6] consisting of ten mathematics educators, started in 1990 to develop standards for primary school mathematics teacher education. For example, on the student teachers' insight into children's learning processes these standards state the following:

- Student teachers acquire insight into children's learning processes in the area of mathematics.
- Student teachers analyse data from children's mathematical activities (written or oral data, or data on videotape) from various perspectives.
- Student teachers develop activities themselves to acquire insight into children's learning processes.
- The student teachers regularly talk with individual children (in clinical interviews) about specific problems and their solutions.
- Student teachers study material (such as from Kwantiwijzer[7]) about carrying out diagnostic interviews with children, and then hold interviews in accordance with it.
- Learning processes in the area of mathematics are a frequent topic of lectures, small group work and reading assignments.
- How to increase the level of understanding of both children and students is a topic of mathematical educational research.
- Children's own mathematical productions provide study material for small group work on mathematics education and also serve as illustrations of knowledge transfer (Goffree & Dolk, 1995, p. 74).[8]

8.4.2 *Constructive, Reflective, Narrative*

The philosophy of teacher education elaborated in the handbook of Goffree and Dolk (1995) was founded on three pillars: primary school mathematics teacher education should be constructive, reflective, and narrative. This approach to teacher education is an adaptation of the socio-constructivist vision of knowledge acquisition, reflection as the main driving force of the professionalisation of teachers, and the interpretation of practical knowledge as a way of narrative knowing. According to Oonk, Goffree, and Verloop (2004, p. 137), "Real teaching practice has to be the starting point of teacher education." In the attempt to have these pillars into new curriculum materials for primary school mathematics teacher education, the PUIK group faced essential questions: What represents 'real teaching practice'? How can curriculum designers give a learning environment a 'natural' aura? Moreover, what is meant by 'natural'? Fieldwork practice is natural by definition, but when student teachers discuss this

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[6] Programmering, Uitlijning, Invulling en Kwaliteit (Programming, outlining, filling-in and quality).

[7] The Kwantiwijzer project developed diagnostic instruments based on the ideas of Wiskobas (Van Eerde, 2005).

[8] See for all eighteen standards: http://www.mtedu.utaipei.edu.tw/mathweb/opendata/%E8%8D%B7%E8%98%ADstandards.pdf

practice, they often stick to a superficial interchange of ideas and opinions (Verloop, 2001; NCTM, 2000). Rarely do these discussions reach a level of theoretical reflections. How could the PUIK group solve this problem?

Oonk et al. (2004) mention three important issues that are central to the discussion about these problems. First, learning in practice is mostly a solo task because student teachers do rarely have the opportunity to discuss common experiences and observations, necessary to acquire deep rooted knowledge. Second, student teachers usually focus on fulfilling responsibilities and on survival issues, so their reflections on their profession are dominated by talking about actions. Third, as a result, student teachers do not acquire practical knowledge that can be generalised across situations or organise their narratives of teaching into a broader framework.

The PUIK group got a new perspective on these problems when they visited the School of Education of the University of Michigan where they were introduced to the *Student Learning Environment* (SLE) designed by Lampert and Loewenberg Ball (1998). The SLE became a source of inspiration for the making of the *Multimedia Interactive Learning Environment* (MILE) for primary mathematics student teachers in the Netherlands (Dolk, Faes, Goffree, Hermsen, & Oonk, 1996).

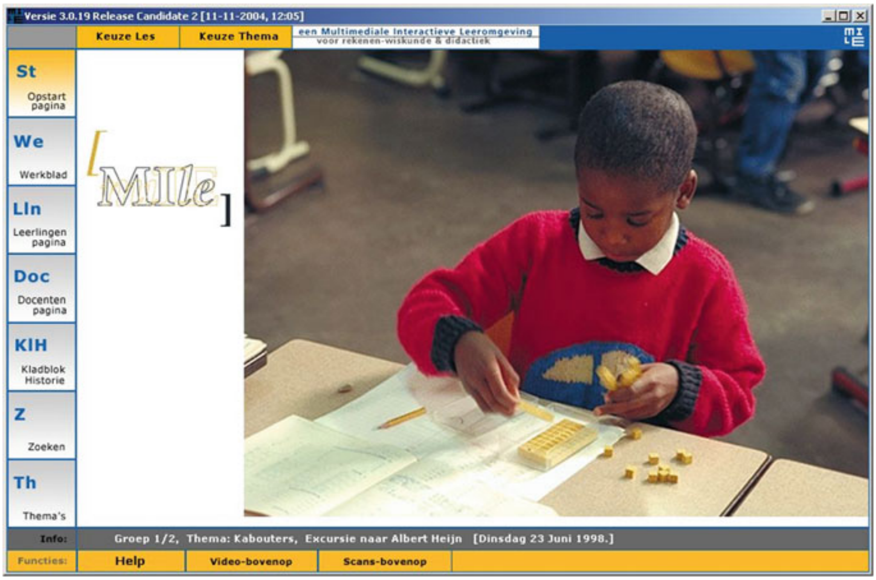
8.4.3 Mile

All Dutch teacher education institutions participated in the MILE project. The goal of MILE was to enable student teachers to investigate good practice in teaching primary mathematics. ‘Good practice’ here meant practice being in line with the Dutch standards for primary mathematics education and with those for primary school mathematics teacher education. Other characteristics of the good practice offered by MILE were:

- Showing authenticity of real practice in school.
- Representing the complexity of real teaching practice, exemplary for the programme of primary education.
- Taking into account learning strands and of students’ learning processes: education in the vein of RME.
- Providing researchable reflective practice of expert teachers and some theoretical input by the designers (Oonk et al., 2004, p. 145).

The MILE database included materials on mathematics education from Kindergarten through Grade 6, involving recorded, connected lessons, discussions with teachers and supervisors, and textbooks and other materials. It was possible to study each lesson as a whole or in short fragments. Keyword searches of the fragment descriptions and lesson dialogues could be done using a search engine. Every lesson fragment reproduced a teaching instance and a short description that provided further clarification (Fig. 8.3).

Research showed that student teachers were often not only focused on the actual teaching of mathematics when watching the fragments in MILE, but also on general



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Info: Groep 1/2, Thema: Kabouters, Excursie naar Albert Heijn [Dinsdag 23 Juni 1998.]

Fig. 8.3 The MILE start-up screen

didactical and pedagogical issues (Oonk, 2001, 2009; Goffree & Oonk, 2001; Oonk et al., 2004). MILE thus offered the possibility to use the school subject mathematics as an arena for theoretical reflections that connect with larger didactical and pedagogical ideas. Furthermore, working with MILE, four levels of student teachers' knowledge construction were distinguished (Oonk, 2009, pp. 74–75; Oonk et al., 2004, p. 152):

Four levels of student teachers' knowledge construction:

- Knowledge can be taken from the expert teachers in MILE; student teachers expand their own didactical repertoires through assimilation of the practice knowledge contained in MILE.
- Adaptation and accommodation of practice knowledge can modify the repertoires of the MILE teacher to suit student teachers' own purposes.
- Establishing (new) links between the events in MILE and events from student teachers own trainee practice and related theory; this is the level of integrating theory, in which they might (re)consider didactical insights and points of view.
- The level of theorising manifests itself when student teachers designed their own local theories; they built up ideas about causes and consequences through the observation and interpretation of fragments they themselves found in MILE.

The results of research on the activities of fifty second-year student teachers (Oonk, 2001) revealed that they used theory as a means to understand and explain practical situations. The majority of the student teachers themselves believed that working with MILE enabled them to apply and further explore the knowledge that they already had. The following transcript of a discussion shows how two second-year student teachers working in MILE were searching for appropriate theory when they compared, faced and considered which material or model is (or is not) appropriate and why.

> Denise and Marieke are watching and analysing a video clip about Fadoua, a Grade 2 student and her teacher at the instruction table. We see how the teacher identifies in a diagnostic discussion the way of thinking behind Fadoua's mistake (18-6=11). It appears that Fadoua counts backwards starting from 18 ('initial error') and while counting backwards also skips two numbers (12 and 14).
>
> The two student teachers discuss the most appropriate way to assist Fadoua.
>
> Denise I think solving the problem using 18 blocks (units) may help.
>
> Marieke I don't think this will help, because it doesn't solve Fadoua's counting problem.
>
> Denise Maybe the number line?… eh…
>
> Marieke That will not help for the same reason.
>
> Denise I suddenly think that the fives structure of the arithmetic rack with twenty beads can help Fadoua either by directly subtracting 6 or by splitting to yield 8-6 or 18-6. And that doesn't involve counting anymore.
>
> Marieke I agree, I can well remember from earlier clips that Fadoua has most likely mastered splitting the numbers to ten (…). In this case we can probably indeed use the arithmetic rack teaching method. (Oonk, 2001, p. 21)

A number of the student teachers demonstrated a budding appreciation for theory. However, others lost their way in MILE and rarely reached beyond a superficial level of reflection. The frame of reference of these student teachers appeared somewhat diffuse and fragmented. An important side-product of the MILE project was the accompanying professional development for mathematics teacher educators at one-day conferences.

8.5 New Ideas About Learning to Teach Mathematics

After 2000, the earlier ideas about primary school mathematics teacher education remained as a kind of natural fundament within the community of mathematics teacher educators in the Netherlands. These included the ideas of RME, the pillars constructive, reflective and narrative, and the integration of mathematics and didactics.

However, important questions crossed the boundary of the centuries and remained to influence the discussions about the way to go: How could the integration of theory and practice for student teachers really be shaped? And how could student teachers' reflections on (their) teaching practice be brought to a higher level? For example, most teacher educators were convinced that video recordings served the purpose of reflection on real practice, but so far, the student teachers' experiences remained to be proved against this assumption. Developments at different levels of mathematics education in the Netherlands brought new perspectives to answer these questions.

First, the Freudenthal Institute, partly in cooperation with the Netherlands Institute for Curriculum Development (SLO) and the CED-Group,[9] developed in the TAL[10] project the so-called 'Teaching-learning trajectories' for most domains of primary mathematics education (Van den Heuvel-Panhuizen, 2008[11]; Van den Heuvel-Panhuizen & Buys, 2008[12]; Van Galen et al., 2008[13]; Gravemeijer et al., 2016[14]). These extended descriptions of the learning pathways in mathematics provided teacher educators and authors of mathematics textbooks series with well thought out ideas about mathematical learning processes of primary school students.

Second, a large-scale research project for mathematics teacher education was set up. The purpose of this Theorie In Praktijk (TIP; theory into practice) project (Oonk, 2009) was to gain insight in the student teachers' way of integrating theory and practice, and particularly to find out how they relate theory and practice and to what extent they are competent to use theoretical knowledge in multimedia educational situations. The study was performed at eleven teacher education institutions. A learning environment was designed to optimise the opportunity for theory use. Theory was recognisable in a multifunctional set of concepts,[15] covering a local instruction theory (Gravemeijer, 2004). The set was multifunctional in the sense that it became manifest in expert reflections on video clips (Goffree, Markusse, Munk, & Olofsen, 2003), through a link that provided extra information, and in a list of concepts to use during the course. The latter functioned as a tool to check one's own progress in understanding the concepts.

The TIP study showed among other things that the learning environment was a catalyst for the development of the student teachers' so-called 'theory-enriched practical knowledge'. The student teachers' use of theory could be identified univocally and described at different levels. These levels turned out to have a positive correlation with the student teachers' level of mathematical literacy and the level of their previous education. A rise in the level of theory use took place especially in interaction led by the teacher educator (Oonk, 2009).

A third development, that occurred simultaneously with the two previously described developments, is that the Panama Kerngroep (Panama core group)[16] of

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[9] The CED-Group trains and advises professionals in education and child care.

[10] Tussendoelen Annex Leerlijnen (Intermediate attainment targets and learning lines).

[11] Published in Dutch in 1999 and 2001 (Treffers, Van den Heuvel-Panhuizen, & Buys, 1999; Van den Heuvel-Panhuizen, Buys, & Treffers, 2001) and in English in 2001 (Van den Heuvel-Panhuizen, 2001).

[12] Published in Dutch in 2004.

[13] Published in Dutch in 2005.

[14] Published in Dutch in 2007.

[15] The idea for such a set originated from a research project on MILE (Oonk, 2001), where a list of concepts was used (Bos, 1999) to inform the student teachers about keywords for the course at hand.

[16] Panama stands for: Pabo Nascholing Mathematische Activiteiten (Pedagogical Academy Training Mathematical Activities). Panama is the Dutch network of mathematics teacher educators for primary education. One of the activities of Panama is organising the annual Panama Conference.

mathematics teacher educators organised a discourse about the curriculum for primary school mathematics teacher education. This discourse inspired the members of this group to write a book about their experiences, considerations, and dilemmas when teaching primary mathematics student teachers (Van Zanten & Van Gool, 2007).

Reflections on the stories in this book were used to arrive at six quality landmarks of good practice in primary school mathematics teacher education:

1. Mathematics-specific coaching when student teachers practice their teaching in school
2. Enough opportunity to develop a mathematical and didactical repertoire
3. Including student teachers' mathematical literacy in the binding study advice
4. Developing student teachers' mathematical literacy
5. Opportunity for reflection and further professionalisation of mathematics teacher educators
6. Ample attention for mathematics specific development of student teachers, including mathematical attitude. (Van Zanten & Van Gool, 2007, pp. 111–115). (translated from Dutch by the authors)

Some of these landmarks came into being in the form of mandatory mathematics tests and the development of a knowledge base for mathematics for prospective teachers.

8.6 A Mathematics Entrance Test for Student Teachers

Teacher education institutions adapted the 1992 Mathematical Literacy Test in various ways. As a result, this test did not secure a fixed mathematics ability level for the prospective primary school teachers (Straetmans & Eggen, 2005). The inspection of teacher education institutions already brought to the fore in 2002 that this situation was problematic, and also found that not all institutions took the level of mathematical proficiency into consideration in the binding study advice that was given to the students at the end of their first year. In school year 2006–2007, this led to the nationwide introduction of a mandatory entrance test, the Wiscat Test. From this year on, all prospective teachers needed to pass this test in the first year of their study. This meant that the third landmark of quality that the Panama Core Group formulated already had become reality, though not in the way they intended or had wished. The Wiscat Test was designed in such a way that the scores of the student teachers could be compared with the mathematics proficiency of primary school students. Student teachers had to show a better mathematics proficiency than eighty percent of students at the end of primary school. So, it can be concluded that the pass mark of the Wiscat Test is low. This pass mark requires less mathematical ability than was originally intended in the Mathematical Literacy Test, and for several teacher education institutions it meant a lowering of the required level of mathematical competence in the first year of teacher education (Van Zanten & Van den Brom-Snijders, 2007). Therefore, they decided to set additional requirements for student teachers.

Fig. 8.4 Two problems from the 2006 Wiscat Test

2.5 % of an amount is €9.-
What is the full amount?
€[ ].[ ] (blanks should be filled in)

On a map with scale 1 : 12,500,000
the distance between Den Bosch and Prague is 7 cm.
The original distance is?
[ ] km (blank should be filled in)

However, a significant number of teacher education institutions chose not to do so (Keijzer, 2010). Figure 8.4 shows typical problems from the Wiscat Test.

The Wiscat Test is a computer-based test and asks student teachers to provide only an answer. Open questions, as in the 1992 Mathematical Literacy Test, are not included. In a sense, this approach is also reflected in the curriculum for primary school mathematics teacher education. Teaching for mathematical literacy was replaced by teaching how teachers and students could produce answers. Several teacher education institutions offered many hours of support for student teachers who needed practise for the test (Keijzer, 2010). Student teachers in their turn often developed or maintained an instrumental way of learning and practicing mathematics, as this appeared to be appropriate to pass the test. Moreover, many institutions chose to not assess mathematics skills, other than those in the Wiscat Test (Van Zanten & Van den Brom-Snijders, 2007). Consequently, many prospective teachers considered the entrance level of this test as a sufficient end level of mathematical proficiency for teaching mathematics in primary education.

8.7 The Knowledge Base for Primary Mathematics Teacher Education

8.7.1 Background

Mathematics teacher educators kept articulating their concerns about the mathematical proficiency of their students (Van Zanten, 2006) and the inability of the Wiscat Test to address this problem. In the first decade of this century, there were also growing concerns about the declining amount of attention that teacher education institutions paid to mathematics (as well as to other subjects) (Onderwijsinspectie, 2008; Onderwijsraad, 2005).

After 2005, primary school mathematics teacher education in the Netherlands became one of the issues in a debate that followed the somewhat disappointing results for Dutch students in TIMSS and PISA (OECD, 2004; KNAW, 2009; Mullis, Martin, Foy, & Arora, 2012). Especially, the study load in teacher education was discussed. In 2008, over the four years in teacher education, student teachers spent on average about 350 hours on studying mathematics, which many experts in the field consider a very low study load to cover the whole range of mathematical and didactical content knowledge. Moreover, there were enormous differences in the mathematics study load between teacher education institutions. In some institutions, the study load did not even exceed the amount of 120 hours in four years, i.e., an average of 30 hours per year (Keijzer, 2010).

The concerns about the proficiency of primary school teachers led to the decision to make a knowledge base for mathematics and language (HBO-raad, 2008; OCW, 2008). The HBO-raad assigned the development of a knowledge base for mathematics to a group of mathematics teacher educators under the name of ELWIeR[17]/Panama. This resulted in the publication of *Kennisbasis Rekenen-wiskunde Voor de Lerarenopleiding Basisonderwijs* (Van Zanten, Barth, Faarts, Van Gool, & Keijzer, 2009), or in short, the Knowledge Base. The ELWIeR/Panama group developed the Knowledge Base in close collaboration with mathematics teacher educators from all Dutch teacher education institutions (Van Zanten, 2010).

8.7.2 *Defining Professional Mathematics Literacy*

The Knowledge Base was meant to provide a description of mathematical knowledge for teaching. Therefore, it had to include both subject matter knowledge and didactical content knowledge. In line with the idea that mathematising and didacticising are interconnected, the developers of the Knowledge Base saw these types of knowledge as two sides of the same coin. Their basic assumption was that teachers' own mathematical literacy provides the foundation of their didactical repertoire. Mathematical content knowledge and knowledge of didactics for mathematics were seen as highly interrelated and inextricably connected. Following an earlier study on mathematical literacy for student teachers (Oonk, Van Zanten, & Keijzer, 2007), the ELWIeR/Panama group defined professional mathematical literacy as a subject-specific competency for student teachers including: having a sufficient level of mathematical literacy of one's own, being able to give meaning to mathematics for students, being able to stimulate calculation methods and raise students' competency, and being able to stimulate students' mathematical reasoning (Van Zanten et al., 2009).

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[17] Expertisecentrum Lerarenopleidingen Wiskunde en Rekenen (Expertise Centre Teacher Training Mathematics and Arithmetic).

8.7.3 Content of the Knowledge Base

The final version of the Knowledge Base consists of two parts: general theory and domain descriptions. The first part broadly describes general information about mathematics education. This includes several kinds of goals of mathematics, varying from the legally established core goals to underlying values of mathematics education, such as preparing students for participation in society. Second, the general theory describes the varying learning processes that occur, for example mathematical reasoning, verbalising solution processes and memorisation, to name a few. In connection with this, didactical insights are given, with a prominent place for the notions of RME.

The second part includes descriptions of five mathematical domains: whole numbers; proportions, percentages, fractions and decimal numbers; measurement; geometry; and relations. These descriptions consist for the largest part of descriptions of domain specific mathematical and didactical content knowledge. Furthermore, appearances and relevance of the domain in reality are specified, as well as the intertwinement and connectedness with other domains and with other school subjects.

8.8 The Knowledge Base Test

8.8.1 Content of the Knowledge Base Test

After the Knowledge Base for mathematics was established, a test was developed for assessing student teachers' knowledge. This new nationwide Knowledge Base Test for third-year student teachers had to guarantee that the prospective teachers master the knowledge described in the Knowledge Base. However, this test only includes the mathematical content knowledge, and not the didactical content knowledge described in the Knowledge Base (Keijzer, Garssen, & Peijnenburg, 2012).

With the introduction of the Knowledge Base Test in 2013, prospective teachers' mathematics proficiency is not only assessed at a basic level in an entrance test (through the Wiscat Test) but also at a far higher level in the third year. This situation demanded specific investments in mathematics content matter knowledge in primary school mathematics teacher education (Keijzer, 2015a).

The Knowledge Base Test is computer-based. Student teachers' knowledge is assessed in all five domains described in the Knowledge Base. The test items are related to what Ball, Thames, and Phelps (2008) refer to as mathematics on the horizon, common content knowledge, and specialised content knowledge. Figure 8.5 shows some typical problems from this test.

- Estimate the number of minutes you have lived the day you celebrate your 18th birthday.
- Which numeral is in the ten-position in the answer of 877651 × 76523?
- What is the decimal number 25 written as a binary number?

Fig. 8.5 Problems from the 2013 Knowledge Base Test

8.8.2 Influence of the Knowledge Base Test on the Curriculum for Primary School Mathematics Teacher Education

Between 2009 and 2015 the average study load for primary school mathematics teacher education rose from about 350 hours to about 485 hours, again with huge differences between the teacher education institutions. The emphasis on the curriculum for primary school mathematics teacher education shifts from didactical content knowledge to all aspects of mathematical knowledge for teaching, including mathematics content matter knowledge. Furthermore, to support student teachers, there was a shift from preparing prospective teachers for the 2006 Wiscat Test to preparing them for the Knowledge Base Test (Keijzer, 2015b). But, when doing so, teacher educators struggled a bit with the intentions of the Knowledge Base Test. The Knowledge Base was introduced to guarantee mathematical knowledge for teaching as a connected body of knowledge for prospective teachers. Teacher educators expressed concerns that prospective teachers might be unable to pass the Knowledge Base Test and will drop out of teacher education, while shortly before these prospective teachers would have graduated and functioned well in teaching practice (Lit, 2011). Another concern was that teacher educators with insufficient background in mathematics foresaw that they might have difficulties when preparing their student teachers' for the Knowledge Base Test (Keijzer et al., 2012). Moreover, the nature of this test might force many mathematics teacher educators to spend a significant portion of teaching time to test preparation at the expense of paying attention to connecting the mathematical content to didactical content knowledge.

8.9 Recent Learning Materials for Student Teachers

Previous developments led to new learning materials for primary school mathematics teacher education. Currently, the market for this educational domain in the Netherlands is mainly determined by the book series *Reken-wiskundedidactiek* (Didactics of mathematics)[18] and *Rekenen-wiskunde in de Praktijk* (Mathematics in practice).[19] These two book series that cover the complete curriculum for primary mathematics teacher education show many similarities in how they combine the implementation of the Knowledge Base and the Knowledge Base Test with the approach of integrating mathematical and didactical content knowledge. Both series encompass both the mathematical and didactical content knowledge described in the Knowledge Base and explicitly connect these two types of knowledge with each other. Both provide an overview of mathematical and didactical concepts in doing so. Also, the contents of both series are inspired by RME, and in both series the pillars constructive, reflective and narrative are recognisable. Each of the series pay attention to raising the mathematical literacy of student teachers towards the level required for the Knowledge Base Test. Further, both series stress student teachers' competences in learning to teach mathematics meaningfully, in stimulating students' mathematical reasoning. Other similarities include the attention paid to carrying out assignments, and the attention paid to differences between student teachers' in learning mathematics. However, the two series differ in their approach to student teachers learning to teach. The structure of *Reken-wiskundedidactiek* follows the mathematical content of the primary school curriculum. For example, the first book about whole numbers, starts with a chapter about the history and the properties of our number system, followed by a chapter about the teaching-learning trajectory of counting and number sense. The structure of the series *Rekenen-wiskunde in de praktijk* is built up along themes and big ideas of practice. For example, the first book of this series starts with an orientation on teaching practice in kindergarten followed by a chapter with a self-assessment to let student teachers figure out their knowledge of meaningful concepts.

In addition to these book series, there are also two books that focus on one part of the curriculum. *Rekenen met Hele Getallen op de Basisschool* (Veltman & Van den Heuvel-Panhuizen, 2010/2015), is a teacher education version of the teaching-learning trajectory for calculation with whole numbers developed in the TAL project (Van den Heuvel-Panhuizen, 2008). The book describes learning trajectories and intermediate attainment targets, and provides examples of activities for student teachers selected from primary mathematics textbooks. All eight chapters of the book start with a practical activity at the student teachers' own level followed by reflections. Then, each chapter discusses the relevant learning trajectory in detail, including examples of activities described in primary mathematics textbooks. In between, there are didactical tasks and ideas for student teachers' activities in school practice. In summary, it can be said that this book is in the first place a book about the didactics of whole numbers.

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[18] This book series consists of the following publications: (Van den Bergh, Van den Brom-Snijders, Hutten, & Van Zanten, 2005/2012; Van den Brom-Snijders, Van den Bergh, Hutten, & Van Zanten, 2007/2014; Van Zanten, Van den Bergh, Van den Brom-Snijders, & Hutten, 2006/2008/2014; Hutten, Van den Bergh, Van den Brom-Snijders, & Van Zanten, 2010/2014).

[19] This book series consists of the following publications: (Oonk, Keijzer, Lit, & Barth, 2013; Oonk et al., 2011/2015; Oonk, Keijzer, Lit, & Figueiredo, 2016).

Rekenen+Wiskunde Uitgelegd (Ale & Van Schaik, 2011/2014) explains the primary mathematics subject matter at the level of the student teachers. These explanations are exemplified with contexts and models (e.g., bar, ratio table and number line). The text contains tips for student teachers, for example the suggestion to use special strategies for operations and examples for activities with students. In summary, it can be said that this book may be characterised as a book to strengthen student teachers' mathematical literacy before taking the Knowledge Base Test.

Although it might seem that these latter two books mean a break with the approach in which the development of mathematical proficiency and didactical proficiency are integrated, in practice it does not work out in this way. Teacher education institutions that use one of these books combine them with the book series mentioned previously.

8.10 Perspective: Searching for a Balance

Looking back over the past decades, we can observe a continuous search of in the field for a well-balanced way to interconnect the didactical education of student teachers and their development of mathematical literacy. That was—and is—not an easy enterprise. It occurred more than once that student teachers' mathematical proficiency was judged insufficient. As a consequence, many teacher educators struggled to adapt the curricula in the sense that less time could be spend on the didactical development of the student teachers in favour of more time for supporting student teachers' preparation for nationwide obliged tests.

Actually, two external forces influenced the development of curricula for primary school mathematics teacher education simultaneously. On the one hand, quality monitoring focused on the full curriculum in primary teacher education and did not signal the need for investments in the mathematics curriculum. On the other hand, concerns about primary student teachers' mathematics proficiency, also elicited by the TIMSS and PISA reports showing that mathematics results in the Netherlands went down in comparison to other countries, gave reason for introducing nationwide tests, namely two entrances tests (the Mathematical Literacy Test in 1992 and the Wiscat Test in 2006) and a third-year (the Knowledge Base Test in 2013). These nationwide tests did influence the mathematics curriculum in a sense that more focus was put on student teachers' development of mathematical literacy and less on their development of didactical proficiency. Teacher education institutions now offer a relatively large number of hours for preparing student teachers for the Knowledge Base Test (Keijzer, 2015b). Nevertheless, there are still big differences between curricula, which could be caused, in combination with the effects of the external forces, by the freedom that teacher education institutions have to organise their own curriculum.

Although teacher educators are concerned about the previously sketched situation, the majority keeps searching for new possibilities to maintain and increase the achievements of student teachers, especially their competence to integrate their mathematical and didactical knowledge.

First, the ELWIeR group of primary mathematics teacher educators have focussed their practice-based research on improving mathematics teacher education. Questions addressed in this group include:

- What study load do primary teacher education institutions reserve for mathematics and how does this develop over time? (Keijzer, 2015b)
- How could horizontal content knowledge be included in primary mathematics teacher education? (Duman, 2015)
- How can high performers in mathematics be adequately supported in primary teacher education? (Kool & Keijzer, 2015)
- What are characteristics of low performers in mathematics in primary teacher education? (Keijzer & Boersma, 2017)
- How can strategies for connecting subject matter knowledge and didactics be schematised? (Keijzer & De Goeij, 2014; Keijzer, 2013).

Second, researchers on primary school mathematics teacher education presented a new approach of integrating theory and practice, called 'enriching practical knowledge',[20] as well as a way to assess and describe this knowledge in a systematic manner (Oonk, Verloop, & Gravemeijer, 2015). By analysing and discussing real teaching practice and describing their own reflections on that practice, student teachers show they are using theoretical ideas and terminology of mathematics and of teaching mathematics in a meaningful manner. In this way, practical knowledge can develop in 'theory-enriched practical knowledge'. This approach is less or more recognisable in the current two mainly used book series for primary school mathematics teacher education.

Last but not least, there is an increasing interest in problem-oriented education in Dutch primary education (Platform Onderwijs 2032, 2016) as well in primary school mathematics teacher education. In line with this, the study group Wiskunde voor Morgen (Mathematics for tomorrow) (Gravemeijer, 2015) consisting of twenty experts in the area of mathematics teaching from primary to higher vocational education, is searching for an answer to the question how to adapt mathematics teaching to prepare the present generation of students for tomorrow's society. The discussions about the development of 21st century skills in this group may also influence the goals, the content and the instructional formats of primary mathematics teacher education, not least because of the changing role of computers as an extension of human possibilities, and the increasing role of mathematics in other sciences. The latter argument reminds us of Freudenthal's statement that mathematics (in the future—Freudenthal was talking about the year 2000) should not be taught as a separate subject, but as

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[20] The word knowledge in practical knowledge is used as an overarching, inclusive concept that summarises a variety of cognitions from conscious and well-balanced opinions to subconscious and unreflected intuitions (Verloop, Van Driel, & Meijer, 2001). Mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008) is considered as the core of practical knowledge for mathematics teaching.

a part of integrated education (Freudenthal, 1976). It requires an adapted view on learning to teach mathematics! We like to accept this challenge.

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Chapter 9
Secondary School Mathematics Teacher Education in the Netherlands

Joke Daemen, Ton Konings and Theo van den Bogaart

Abstract In this chapter, we discuss the education of secondary school mathematics teachers in the Netherlands. There are different routes for qualifying as a secondary school mathematics teacher. These routes target different student teacher populations, ranging from those who have just graduated from high school to those who have already pursued a career outside education or working teachers who want to qualify for teaching in higher grades. After discussing the complex structure this leads to, we focus on the aspects that these different routes have in common. We point out typical characteristics of Dutch school mathematics and discuss the aims and challenges in teacher education that result from this. We give examples of different approaches used in Dutch teacher education, which we link to a particular model for designing vocational and professional learning environments. We end the chapter with a reflection on the current situation.

9.1 The Dutch Educational System

As a start of this chapter we first give an overview of the Dutch educational system. For the sake of clarity, we will focus on the main stream of the system and not go into all exceptions. In other words, we will describe how education is organised for 90% of the Dutch students. For example, education for students with special needs will

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be left out. After describing the system, we will discuss the different forms of initial teacher education and the statutory framework of required teacher competences that they take as a basis. We end this first section about the Dutch educational system with a discussion of continuous professional development for teachers.

9.1.1 The School System

Figure 9.1 shows how the Dutch educational system is organised. By law, education in the Netherlands is compulsory up to the age of eighteen. Primary education lasts eight years from the age of four to twelve. Mother tongue and elementary mathematics are the core subjects. At the end of primary school all students have to take a test; mostly the so-called ‘Cito Test’[1] is used for this. Based on the students’ test score and the opinion of the teacher the students are allocated to a particular level of secondary education. Consequently, at the end of primary school the track of students’ secondary

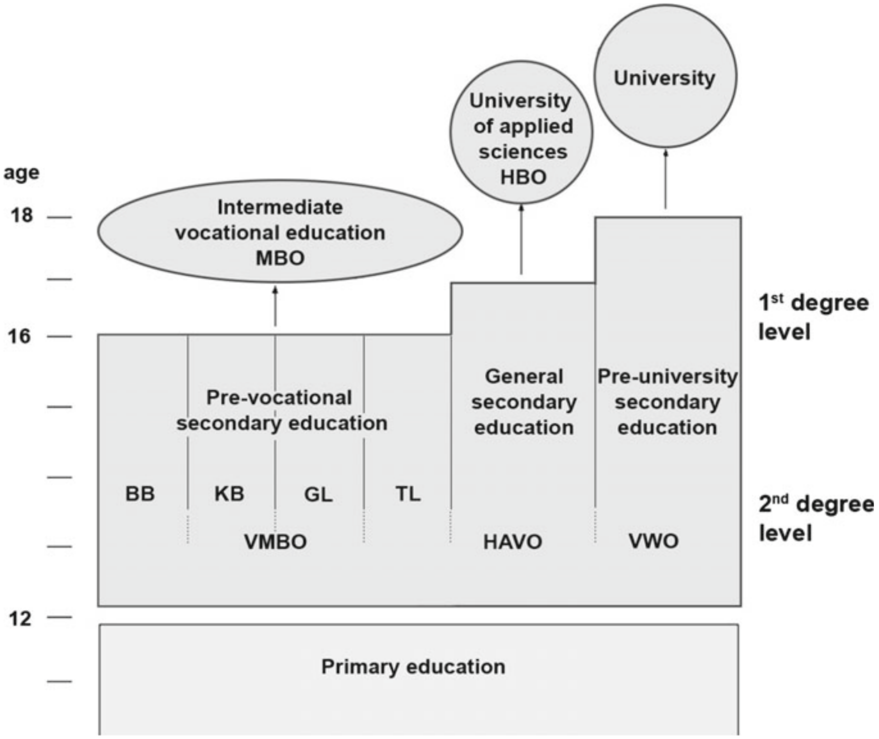

Fig. 9.1 The Dutch school system

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[1] The Cito End of Primary School Test is developed by Cito, which is the Netherlands national institute for educational measurement.

education is largely determined. The Dutch educational system can therefore be labelled as ‘early tracking’—a fact sometimes criticised for its negative consequences on students with a low socio-economic background (OECD, 2007; Onderwijsraad, 2010).

Secondary education is divided in several tracks:

- VMBO: Pre-vocational secondary education, which contains in itself four tracks ranging from practical tot more theoretical, continuing in MBO, intermediate vocational education and training
- HAVO: General secondary education, which qualifies for higher professional education (HBO, also called ‘universities of applied sciences’) where students can study for primary school teacher and second-degree mathematics teacher, who are qualified for VMBO and the first years of HAVO and VWO
- VWO: Pre-university secondary education, which qualifies for studying at a university where students can study for first-degree mathematics teacher, who are qualified for VMBO and all grades of HAVO and VWO.

After finishing one of these tracks, it is possible to proceed to a higher track. For example, after finishing VMBO, a student can go to HAVO.

Within each track students can choose different directions. All tracks start with the so-called ‘basic education’, where a broad range of obligatory subjects is offered, including mathematics. For example, when studying at a VMBO school, a student can select one of four profiles: Technology, Care and Welfare, Economics, or Agriculture. In MBO, for which VMBO prepares, mathematics is in general no longer a separate subject, although in recent years government policy has led MBO schools to introduce courses in basic arithmetic.

In HAVO and VWO students have to choose a profile in year four. Here, the profiles are: Culture and Society, Economics and Society, Nature and Health, or Nature and Technology. In VWO mathematics is obligatory in all profiles. In HAVO and VWO, mathematics is distinguished in four kinds of mathematics, ranging from more applied mathematics to more formal and abstract mathematics (see Table 9.1).

At the end of VMBO, HAVO and VWO, there is a national examination in mathematics. The content of the examination is prescribed by the Dutch government. As a result, the curriculum is largely fixed and teachers are largely limited in the topics they teach (Webbink et al., 2009; Snoek, 2011). In contrast with the prescribed

Table 9.1 Mathematics in upper HAVO and VWO

Kind of mathematics	Content
Mathematics A	Applied analysis in economics and health contexts, statistics
Mathematics B	Formal analysis, analysis applied in technical and scientific contexts, analytic geometry
Mathematics C	(VWO only) Topics aimed at liberal arts, e.g., logic, or symmetry and perspective in art
Mathematics D	(complementary to B) Euclidean geometry, more analysis, choice topics about applications in technical and scientific contexts

content, the teachers are free to make their own choice regarding the didactics they use.

9.1.2 Secondary School Teacher Education

There are several routes to obtain a qualification to teach mathematics. Although in most countries teacher education involves a study at university level (Deinum, Maandag, Hofman, & Buitink, 2005), in the Netherlands it is also possible to follow a study at an HBO school. This HBO route to become a mathematics teacher is currently by far the most popular route (source www.stamos.nl) (see Table 9.2).

The main difference between the university route and the HBO route are:

- Admission to a HBO school is less strict than to a university; students from lower secondary education tracks can enrol in a HBO school but not in a university
- The route via university provides a much stronger grounding in formal mathematics
- Universities have a separate curriculum for mathematics and didactics, while HBO schools have an integrated curriculum of mathematics and didactics.

9.1.2.1 Secondary School Teacher Education at a HBO School

There are twelve HBO schools in the Netherlands that provide mathematics teacher education. They offer a four-year bachelor's programme leading to a second-degree qualification. Although the HBO schools were initially meant for students coming directly from secondary school, the current student population is highly heterogeneous: about 65% (an estimation based on personal communication with staff from several institutions) consist of students between 25 and 65 years old who have decided to make a career move. These older students often already have a job as mathematics teacher. A Dutch law allows them to do this as long as they obtain their qualification within a certain amount of time. The competences acquired by these students before they start their teacher education differ greatly. Some are already qualified for teaching a particular secondary school subject or for teaching primary education, others

Table 9.2 Total number of students in types of secondary school teacher education in 2014

Type of secondary school teacher education	Number of students
HBO teacher education for second-degree teachers	2,456
HBO teacher education for first-degree teachers	322
University teacher education for first-degree teachers	40

have a financial or technological background, while there are also students who have not yet done anything related to mathematics or teaching.

The regular (full-time) bachelor's programme at HBO is intended for students who have just completed secondary education. They start with this programme at about the age of seventeen. The older students follow a part-time programme, enabling them to combine their study with a job and family life. The adjective 'part-time' is a bit misleading however. The programme has roughly the same content as the full-time programme and it also lasts four years.

Six of the twelve HBO schools also offer a master's programme for obtaining a first-grade teacher qualification. To be admitted to this programme, two conditions must be fulfilled: one must already have a second-degree qualification and one must work as a teacher in secondary education. This means that the master's programme provides a promotion opportunity for second-degree teachers.

9.1.2.2 Secondary School Teacher Education at a University

Only a small part of the Dutch secondary school mathematics teachers is educated at one of the nine research universities that offer teacher education. Here, three routes are possible.

- Traditionally, teacher education at a university is a post-master's course. Having obtained a master's degree in mathematics or in a related subject, a student has to follow a one-year programme to become a teacher with a first-degree qualification. This route is strongly consecutive: at the start of the post-master's course, a student is already fluent in mathematics.
- Doing the Master's of Science Education and Communication is another option that is offered at five universities. This two-year programme entails, besides mathematical subject knowledge, also theoretical insights and practical skills in both formal and informal educational practices. In this master's course, mathematics is put in a broader societal perspective, relevant to secondary education and to the public at large. The master's course prepares for a first-degree teacher qualification as well as for a career in communication or research and development.
- By following an education minor as part of the Bachelor's of Mathematics, one can obtain a second-degree teacher qualification. The minor takes up to half a year of the bachelor's study, which lasts three years.

The latter two routes have only existed since 2011. The main reason for these additional routes is the shortage of qualified teachers in secondary education.

9.1.2.3 Quality Assurance in Teacher Education

Institutions for teacher education have autonomy in designing their curriculum (Snoek, 2011). Therefore, quality assurance is organised through an accreditation system. The accreditation takes place every six years. There is no formal governmental regulation for ensuring the quality of teacher educators, who almost all have at least an appropriate master's degree. Also, there is a voluntary professional register of teacher educators, which is maintained by VELON, the Dutch Association for Teacher Educators.

9.1.3 Continuous Professional Development

The types of teacher education described in the previous section can be seen as the starting point for the teacher's professional career. After students have completed the teacher education programme they are fully qualified to teach. Although it is without doubt that a teacher at the start of his or her career as a teacher still has a lot to learn, currently there is no formal policy in the Netherlands, like in other countries (see Deinum et al., 2005), for an induction phase. Nevertheless, life-long learning skills are an important perspective in the curriculum. Therefore, institutions for teacher education tend to focus more and more on 'on the job' education of in-service teachers.

Education in the Netherlands is qualified as good, but compared to other countries the performance of Dutch students in secondary education has started to go a little backwards (SLO, 2015). Since the teacher largely determines the quality of education, the government is taking steps to improve the quality of teachers (Snoek, 2011). The school, as an employer, has the duty to maintain the quality of teachers and give them the opportunity to professionalise continuously. Therefore, schools maintain competence dossiers for every teacher. The Inspection of Education has the task to check the quality of schools and report to the government.

All qualified teachers in the Netherlands have to join the national teacher register. Until August 2017 this was a voluntary registration. Currently, by law, teachers have to participate in professional development activities throughout their working life. They have to spend 40 h of professional development each year. For mathematics teachers, ten of these hours are labelled as 'involving mathematics'. The Nederlandse Vereniging van Wiskundeleraren (NVvW; Dutch association of mathematics teachers) and universities provide professional development courses for teachers in order to meet the future requirements for registration in the register of teachers.

Another governmental measure is that teachers can acquire a grant to pursue a bachelor's or master's degree in education, or even a PhD degree. One reason for this measure is the growing shortage of highly qualified teachers. Another reason is that it gives teachers further career possibilities.

9.2 Aims of Teacher Education

Of course, the aim of initial teacher education is to train people to become a capable mathematics teacher in secondary education in the Netherlands. In this section, we will elaborate on this aim. First, we will discuss the teacher's professional competence from a nationwide, governmental perspective. This involves the question: What makes someone a 'capable' teacher? Then we will highlight the aims with respect to the teacher's knowledge of the didactics of mathematics, mathematical subject knowledge, and research skills.

9.2.1 Professional Competence a Teacher Must Have

In the Wet op de Beroepen in het Onderwijs (law on professions in teaching), implemented in 2013, the Dutch government describes the competences a teacher must have. In the resulting statutory framework of competences, the term 'competences' comprises knowledge, skills and attitudes which are specific for the educational domain and sustainable over a longer period of time (Van Merriënboer, Van der Klink, & Hendriks, 2002). These competences are based on the different roles and situations that teachers can face. The statutory framework of teacher competences is the most important guideline for institutions for teacher education. The framework describes the following competences (Onderwijscoöperatie, 2014):

- Interpersonal competence—the ability to create a learning climate
- Pedagogical competence—aimed at the personal development of students
- Subject knowledge and didactical competence
- Organisational competence
- Competence for collaboration with colleagues
- Competence for collaboration with the environment—e.g., parents, organisations
- Competence for reflection and personal development.

While the previously described statutory framework is meant for teachers in all subjects, the NVvW has presented a professional profile specifically for mathematics teachers (Jonker, Lambriex, Van der Veen, & Wijers, 2008). The mathematics-related competences are divided in four categories. These are shown in Table 9.3 together with examples of standards.

Based on the statutory framework, the teacher education institutions must indicate what competence level a student must have attained in order to be sufficiently apt to start as a teacher and when a level is reached equal to a bachelor's or master's degree, as prescribed by the international standards for these degrees (see Bologna Working Group on Qualifications Frameworks, 2005). To do so, institutions have to formulate the required 'behavioural indicators' for each competence. Helpful for this are the so-called 'knowledge bases' that, as a result of government policy to raise the quality of teachers (Ministry of Education, 2008), have been developed recently

Table 9.3 Domain-specific competences for secondary school mathematics teachers (Jonker et al., 2008)

Mathematics-related competence	Example of standard
Subject knowledge	A teacher oversees the internal coherence of a mathematical subject area and links different areas
Environmental factors	A teacher knows of the different mathematics related contents and methodologies in primary education
Learning processes and didactics	A teacher encourages mathematical activity in the students
Assessing, judging and evaluating	A teacher can analyse students' mistakes and provide them with adequate feedback

specifically for domains such as general didactics and pedagogy, educational use of ICT and mathematical subject knowledge at HBO level.

9.2.2 A Broad Range of Teacher Competences is Required

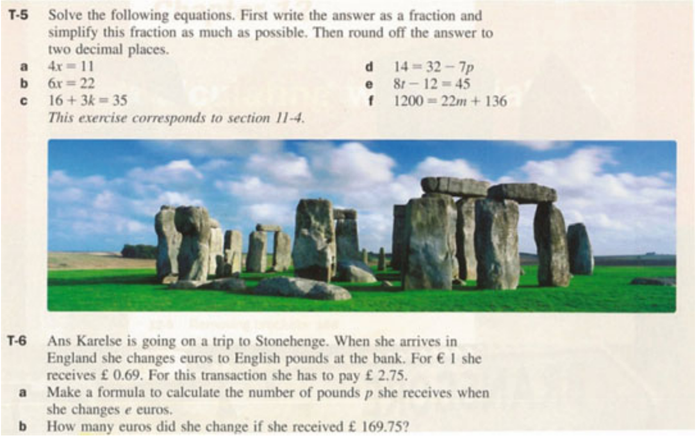
Mathematics teachers in secondary education need to be able to teach a broad range of tracks and topics. A second-degree teacher is expected to teach in the lowest track in VMBO up to the third year of VWO. This means that a teacher has to cope with students with quite different levels of understanding. Figure 9.2 exemplifies how the cognitive demands of the tasks in the first year of secondary education can differ in different tracks.

Furthermore, a teacher must also be able to operate in MBO, intermediate vocational education, which places special demands on both subject knowledge (the application of mathematics in the students' future profession and the workplace) and didactics (taking into account the practice-oriented learning style of the students) (see, e.g., Schaap, Baartman, & De Bruijn, 2012). On top of that, in the last few years it has also become expected of mathematics teachers to teach basic arithmetic in secondary education. This new policy of the Dutch government arose from discontent about secondary school students' arithmetic skills.

The mathematics teacher has to cover a broad range of topics, as the following examples of tasks (Fig. 9.3) may illustrate. Both tasks are from a VWO examination. The first task is from Mathematics C and the second one from Mathematics B (see Table 9.1).

9.2.3 The Approach to Mathematics Education

The present approach to mathematics education in the Netherlands is influenced by Realistic Mathematics Education (RME) as formulated by Freudenthal (1991). One of the main principles of RME is that mathematics has to be relevant for students.

<table>
<tr><td>T-4 Frida rents a bike at ‘Budget’.
Budget calculates the price via the formula:
$\textit{number of days} \times 7.50 + 5 = \textit{rental price}$
The rental price is in Euros.</td><td></td></tr>
<tr><td>a. Copy the chain of arrows and place the numbers 5 and 7.50 in the right place.
b. Calculate the rental price if you want to rent a bike for two days.
c. Make the reverse chain of arrows.
d. Frida has to pay € 35, how many days did she rent a bike?</td><td>$\textit{number of days} \xrightarrow{\times\cdots} \cdots \xrightarrow{+\cdots} \textit{rental price}$</td></tr>
<tr><td colspan="2">From: Textbook <i>Moderne Wiskunde 1B, VMBO-basis, 2008</i>
<i>(Modern Mathematics 1B, VMBO-basic)</i>
(translated from Dutch by the authors)</td></tr>
<tr><td colspan="2">

T-5 Solve the following equations. First write the answer as a fraction and simplify this fraction as much as possible. Then round off the answer to two decimal places.

a $4x = 11$
b $6x = 22$
c $16 + 3k = 35$
d $14 = 32 - 7p$
e $8t - 12 = 45$
f $1200 = 22m + 136$

This exercise corresponds to section 11-4.

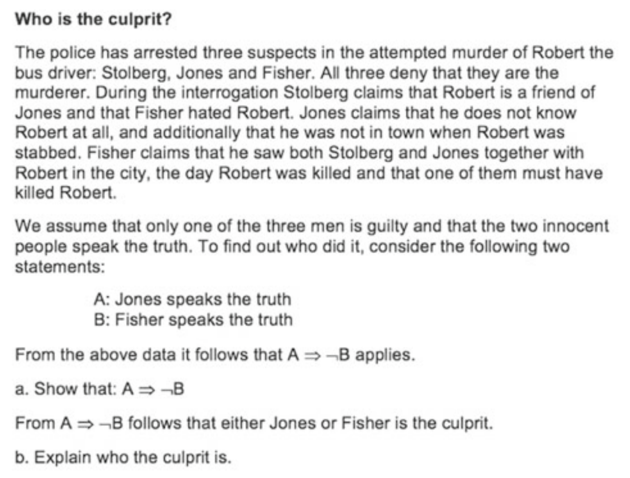
T-6 Ans Karelse is going on a trip to Stonehenge. When she arrives in England she changes euros to English pounds at the bank. For € 1 she receives £ 0.69. For this transaction she has to pay £ 2.75.

a Make a formula to calculate the number of pounds p she receives when she changes e euros.
b How many euros did she change if she received £ 169.75?</td></tr>
<tr><td colspan="2">From: Textbook <i>Modern Mathematics 1B, VWO,</i> 2008 (bilingual track)</td></tr>
</table>

Fig. 9.2 Two tasks from the first year of secondary school: one from the lowest track in VMBO (top) and one from VWO (bottom)

According to Freudenthal, mathematics must not be approached as fixed knowledge to be transmitted, but it should be seen as a human activity. Education in mathematics should therefore give students the opportunity to, albeit guided, ‘re-invent’ mathematics by doing it. The focus should be on the process of mathematisation: starting with experiences with contexts or problem situations, from which the student constructs relations between mathematical objects.

In the Netherlands, more than in other countries, school practice seems to be dictated by textbooks (Drijvers, Van Streun, & Zwaneveld, 2012). The textbooks offer extensive and complete materials for lessons and assessments. Teachers hardly deviate from their textbook (SLO, 2015). The popular textbooks are not unambiguous regarding RME. They incorporate many contexts, but do not use them in mathematisation activities, relying instead on formal definitions and algorithms (Van Stiphout,



Who is the culprit?

The police has arrested three suspects in the attempted murder of Robert the bus driver: Stolberg, Jones and Fisher. All three deny that they are the murderer. During the interrogation Stolberg claims that Robert is a friend of Jones and that Fisher hated Robert. Jones claims that he does not know Robert at all, and additionally that he was not in town when Robert was stabbed. Fisher claims that he saw both Stolberg and Jones together with Robert in the city, the day Robert was killed and that one of them must have killed Robert.

We assume that only one of the three men is guilty and that the two innocent people speak the truth. To find out who did it, consider the following two statements:

A: Jones speaks the truth
B: Fisher speaks the truth

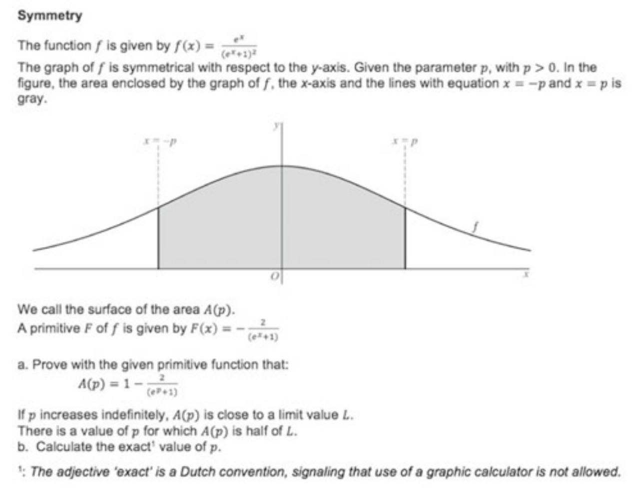
From the above data it follows that $A \Rightarrow \neg B$ applies.

a. Show that: $A \Rightarrow \neg B$

From $A \Rightarrow \neg B$ follows that either Jones or Fisher is the culprit.

b. Explain who the culprit is.

From: VWO Mathematics C examination in 2013 (translated from Dutch by the authors)



Symmetry

The function f is given by $f(x) = \frac{e^x}{(e^x+1)^2}$

The graph of f is symmetrical with respect to the y-axis. Given the parameter p, with $p > 0$. In the figure, the area enclosed by the graph of f, the x-axis and the lines with equation $x = -p$ and $x = p$ is gray.

We call the surface of the area $A(p)$.
A primitive F of f is given by $F(x) = -\frac{2}{(e^x+1)}$

a. Prove with the given primitive function that:

$$A(p) = 1 - \frac{2}{(e^p+1)}$$

If p increases indefinitely, $A(p)$ is close to a limit value L.
There is a value of p for which $A(p)$ is half of L.
b. Calculate the exact[1] value of p.

[1]: *The adjective 'exact' is a Dutch convention, signaling that use of a graphic calculator is not allowed.*

From: VWO Mathematics B examination in 2015 (translated from Dutch by the authors)

Fig. 9.3 Tasks in the first-degree area: Mathematics C (top) and Mathematics B (bottom)

2013). Also, policy makers think there is too much weight on procedural knowledge. Therefore, at the moment, there is a strong emphasis on balanced mathematical proficiency (Kilpatrick, Swafford, & Findell, 2001; cTWO, 2013) in the classroom, incorporating procedural, conceptual and metacognitive knowledge.

Teachers are expected to employ new technology in their lessons. This includes the use of software as a didactical tool, but also of hardware such as graphic calculators and smartboards and of modern instructional approaches like ‘flipping the classroom’.

All the above puts heavy demands on the didactical competence of a mathematics teacher. Being an effective teacher involves much more than just lecturing and helping students with individual problems. It involves asking good questions, engaging students, connecting new and existing knowledge, formative assessments, active learning, use of materials, and the like.

9.2.4 Mathematical Subject Knowledge for Secondary School Teachers

9.2.4.1 Mathematical Subject Knowledge Taught at HBO Schools

In the HBO bachelor’s programme that qualifies for the second-degree level, about 30% of the curriculum is devoted to mathematics. The subject knowledge that is required of a second-degree mathematics teacher is described in the nationally agreed Knowledge Base (HBO-raad, 2009). Table 9.4 shows which mathematical domains are included.

In addition to knowledge of these mathematical domains teachers should also have proficiency in skills such as being able to communicate about mathematics, use ICT to explore mathematical situations, being able to model real-life problems. The rationale behind the knowledge base seems to be that a second-degree mathematics teacher needs to be fluent in his subject at a level that surpasses the level of secondary education. This means, in practice, a level that is a little bit more than that of VWO. Also, a second-degree mathematics teacher needs to have knowledge of domains that are related to the application of mathematics in various professions, such as the domain of complex numbers that is necessary for work in electronics.

Table 9.4 Mathematical domains in the Knowledge Base for second-degree mathematics teachers

Mathematical domain	
Calculus	Mainly real valued function in one variable
Geometry	Mainly planar analytic geometry, basic Euclidean geometry
Algebra	Basic algebraic skills and elementary set theory
Stochastics	Probability theory and descriptive and predictive statistics
Other domains	Graph theory, linear optimisation and history of mathematics

At the moment, a revision of the Knowledge Base for mathematics is taking place. More than was previously the case, the central question is now what kind of mathematics a mathematics teacher needs to know. Furthermore, a distinction is made between knowledge and skills that must be ready at hand when teaching in pre-vocational secondary education and the first years of general secondary education, and knowledge a teacher must have heard somewhere in his or her educational career. This discussion is now taking place.

In the HBO master's programme that qualifies for the first-degree level about 60% of the curriculum is devoted to mathematics. This programme is also based on the Knowledge Base (HBO-raad, 2011), but goes beyond the bachelor's programme, which is extended by, for example, incorporating analysis of several variables and number theory.

9.2.4.2 Mathematical Subject Knowledge Taught at Universities

Teacher education at university implies that students first follow the regular mathematics curriculum and choose the teacher education track when they have already studied mathematics for some time. This means that teachers who graduated from university in general are more proficient in mathematics than their HBO counterparts. In particular, they have a much stronger education in abstract thinking and deductive reasoning.

Students who did not study mathematics at university, but studied a subject in the natural sciences related to mathematics, can also enrol in the teacher education track if they first repair their mathematical deficiencies by following a small subset of courses from the regular mathematics curriculum. Since september 2015, seven special courses can be offered to this category of students: Analysis, Foundations of Mathematics, History of Mathematics, Applied Mathematics, Geometry, Algebra, and Stochastics.

9.2.5 Research Skills for Secondary School Teachers

Since about ten years, learning to conduct educational research has become one of the key goals of teacher education. Institutions differ in the research activities that they demand their students do. In several institutions, the focus is on design-based research in which the design can be a sequence of lessons or another instructional design, which the students have to use in practice to see how it works. Sometimes the training school of the students commissions them to do a particular design-based research. Other institutions allow for other types of research. In any case, the research is always aimed at the innovation of a particular practice, not at gaining scientific knowledge. The latter is, as an exception, only included in the two-year teacher education programmes of some universities, which have a research project lasting at least a full semester.

9.3 The Curricula for Secondary School Teacher Education

In this section, we will focus on the part of the curriculum that aims at the third teacher competence of the statutory framework of teacher competences: subject knowledge and didactical competence. We will ignore here the subject knowledge, which is implemented in a more traditional, academic way. In contrast, regarding the didactical competence, the teacher education institutions seem to have adopted a ‘practice-oriented approach’ of which the key idea is to have the relevant theory closely related to the practical concerns of students who begin to practise teaching (Hammerness, Van Tartwijk, & Snoek, 2012; Korthagen, Kessels, Koster, Lagerwerf, & Wubbels, 2001). We will illustrate this approach by giving examples of various learning activities that Dutch institutions for teacher education have incorporated in their curricula. To organise these examples a model (see Fig. 9.4) is used that has been developed for designing vocational and professional learning environments (Zitter, 2010; Zitter, Hoeve, & De Bruijn, 2016).

The vertical axis is the process dimension, referring to the two kinds of learning processes that the learning activities want to trigger: acquisition versus participation

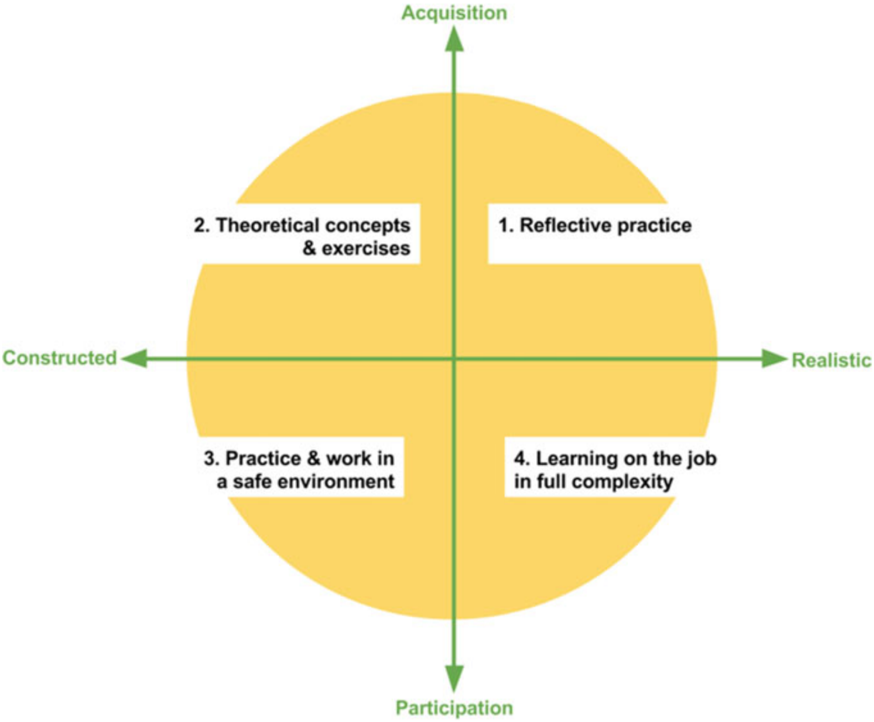


Fig. 9.4 Model to design vocational and professional learning environments (Zitter, 2010; Zitter, Hoeve, & De Bruijn, 2016)

(Sfard, 1998). The horizontal axis depicts the condition dimension: from constructed to realistic. Here, there is also a gradual transition in the amount of control by the curriculum and the amount of guidance a teacher educator offers.

9.3.1 Quadrant 1: Reflective Practice

Reflection on practice plays an important role in the curriculum. Reflection is the instrument by which experiences are translated into dynamic knowledge (Korthagen et al., 2001). There are several ways, such as showing teacher students video clips or student work, to offer teacher students situations on which they are elicited to reflect. In Example 1 the teacher students have to analyse questions asked of an experienced teacher.

Example 1. Analysis of asked questions

From: Van Helden, Krabbendam, & Konings (2011)

(translated from Dutch by the authors)

Assignment: Observe an experienced teacher when he deals with a somewhat large and difficult task. Analyse his behaviour with respect to the questions that are asked related to learning content and problem solving.

In Example 2, the teacher educator does not only have an important role in guiding the reflection, but also in linking mistakes to the underlying didactical theory.

Example 2. From mistakes to didactical theory behind mistakes

From: Utrecht University's teacher education

(translated from Dutch by the authors)

Assignment: In a group meeting of student teachers, the teacher educator asks about mathematical mistakes the student teachers have observed in their students. The teacher educator collects the mistakes on the blackboard, and in a discussion tries to structure the different cases that are brought in. Together, a case is selected and treated in more detail, for example, a mistake such as

$$2\frac{1}{2} - \frac{1}{2} = 2, \text{ therefore } 2a - a = 2.$$

Students discuss the source of this mistake and possible interventions to remedy it. The teacher educator links this to theory, such as the didactics of the use of variables and operations. Then, a new cycle starts.

9.3.2 Quadrant 2: Theoretical Concepts and Exercises

Having teacher students develop a theoretical framework is an important aspect of teacher education. However, if this happens in isolation, students find it difficult to make a transfer to practice. Therefore, finding applications and linking theory to practice are important. To support theory development by teacher students and link this to practice, it is very helpful that currently several textbooks are available that provide the theoretical basis for the courses in didactics of mathematics. Table 9.5 gives an overview of these textbooks and their content.

In the following two slightly paraphrased examples are given from these textbooks used in teacher education. Example 3 illustrates how in Faes et al. (2011), a textbook meant for teacher education at the second-degree level, the teacher students become acquainted with the different levels on which tasks can be presented to students.

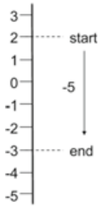
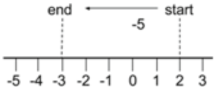
Table 9.5 Textbooks for mathematics teacher education

Second-degree level	First-degree level
Serie Wiskunde voor Leerlingen van 12–16, voor de Lerarenopleiding (Series Mathematics for Students 12–16, for Teacher Education) (see Dutch version at http://www.fisme.science.uu.nl/wiki/index.php/Samenwerkingsgroep_Lerarenopleiding_Wiskunde_2e_graads) – Elementary arithmetic – Geometry – Algebra	*Handboek Wiskundedidactiek* (Handbook Didactics of Mathematics) (Drijvers et al., 2012) – Learning and teaching mathematics – Variables and equations – Functions – The derivative – Geometry – Probability – Statistics – Modelling – Technology in mathematics education – Assessing mathematics – Mathematical proficiency
Serie Leren effectief lesgeven, voor de lerarenopleiding wiskunde (Series Learning to teach effectively, for teacher education) (see http://www.fisme.science.uu.nl/wiki/index.php/Samenwerkingsgroep_Lerarenopleiding_Wiskunde_2e_graads) – Preparing and developing mathematics education – Learning of mathematics – Problem solving and mathematics Assessing mathematics	

Through the assignment to analyse a secondary school textbook the teacher students can build experience with recognising this well-known theoretical three-phase model.

Example 3. From concrete to abstract

From: Faes et al. (2011) (translated from Dutch by the authors)

Subtraction from positive numbers to negative ones	
CONCRETE	Temperature exercise: The initial temperature is 2° it falls by 5° then it is $-3°$
SCHEMATIC	Number line:  Then: 
ABSTRACT	$2 - 5 = -3$

The phases ‘concrete-schematic-abstract’ are sometimes called the ‘context-model-formal’ phases.

- Context phase: Students need to familiarise themselves with a context. What is the framework? What does the action look like? Often, you must do this several times.
- Schematic/model phase: In this phase the context is gradually released […]

Assignment: Check in a secondary school textbook on the pages devoted to addition and subtraction of negative numbers what you notice concerning the transition from concrete to abstract.

Example 4 is taken from Drijvers et al. (2012), the handbook that is used in teacher education for the first-degree level. Here, again the teacher students become acquainted with theoretical didactical knowledge. This time they learn about the concept of function, in particular about how secondary school students can express their understanding of this concept and how this informal understanding by the students can be understood by the teacher at a more abstract level in which several hierarchical categories of understanding can be distinguished.

Example 4. Concept definitions and images

From: Drijvers et al. (2012)

(translated from Dutch by the authors)

Previously it has been described that concept images of the concept of a function can vary greatly between students. These differences became apparent when Vinner and Dreyfus asked students what they thought a function was. The answers they obtained fell in six categories:

a. The formal definition from the book: each x in the set A is associated to exactly one element from the set B;
b. A dependence relation: y depends on x;
c. A rule with a certain amount of regularity;

[…]

When we look at these images a little bit closer, we can make the following hierarchical distinction:

1. The function as an *input-output process*: this matches the image of a machine. The function f with $f(x) = 2x + 3$ for example, is considered as a local computational process: "take twice the input x and add 3, which gives the output y."
2. The function as *covariance*: […]
3. The function as *object*: […]

9.3.3 Quadrant 3: Practice and Work in a Safe Environment

A teacher is required to perform many tasks in a complex, hectic environment, where it is not always feasible to reflect, to discuss mistakes and to receive feedback on your teaching. For this reason, the teaching of teacher students is practised in a more controlled, simulated context where there is ample opportunity for analysis and where, moreover, the more complex task of teaching can be subdivided into subtasks (scaffolding).

Example 5 shows an assignment in which the teacher students have to practise whole-class teaching. The example is taken from Faes et al. (2011), one of the courses of the teacher education for the second-degree level. Although the setting is artificial, it creates an opportunity to assess and discuss the didactical choices teacher students have to make.

Example 5. Introducing new content

From: Faes et al. (2011) (translated from Dutch by the authors)

Assignment: Deliver a presentation in which you give an introduction to new content for Grade 1. This introduction has to last between 10 and 15 min. Your fellow teacher students will simulate to be your secondary school students.

To prepare your presentation, we recommend to make all relevant tasks from the book yourself and to analyse aspects of the content that are worthwhile to be discussed.

Also, make a written preparation of your presentation in which you address the following questions:

- What is the goal for the students?
- Which questions are you going to ask them?
- Which instructions will you give?
- What materials do you need?

You will find an example of such a preparation in the appendix […]

Example 6 shows an assignment taken from the first-degree teacher education programme at Utrecht University. The assignment is on curriculum design. Through this assignment, students can become aware of their own conceptions about how a lesson can or should be built up. Moreover, they can also experience that their peers might have other ideas that may have equal merit. It will turn out that most of the choices teacher students make are implicit and are based on the way they have learned the subject themselves when they were secondary school students. When they now have to design lessons themselves, they will learn that choices have to be made.

Example 6. Curriculum Planning

From: Utrecht University's teacher education

(translated from Dutch by the authors)

Assignment: In […] you will find a series of problems and fragments copied from Chap. 4 of the secondary school textbook *Modern Mathematics*, *2 VWO*, […]. The chapter is on Pythagoras's theorem. Only some parts of the chapter are copied and are placed in random order.

a. Think about how you want to deal with this subject, especially in which order. Replace the parts in the order that you prefer. You can leave problems or information which you do not find relevant aside. If this is easier you can cut out the fragments and put them in order.

   If it is important you can add missing information in keywords. Note it does not have to cover the complete subject. The idea is to think through the order of teaching.

b. Discuss in pairs where large differences in your order occur. Compare the arguments on which you have based this order. Also discuss the problems you do not use. Write down the arguments.

c. Compare your sequence with the sequence from the original book.

d. If you have time, check out […]. Discuss what current educational practice requires from students and teachers with regard to the actual curriculum in mathematics education.

e. Plenary exchange: differences on the basis of the arguments.

9.3.4 Quadrant 4: Learning on the Job

Workplace learning, or immersion in the workplace curriculum (Billet, 2006), is a major component in teacher education curricula. In the bachelor's programme, this amounts to approximately 25% of the programme, culminating in a trajectory where a teacher student has sole responsibility for several classes (see Example 7). In the university master‘s course (see Example 8), as much as 50% of the programme takes place in secondary school (Snoek, 2011). Workplace learning is not only arranged through the classic way of an internship. Teacher education institutions seek close cooperation and partnerships with secondary education schools, resulting in a tendency towards 'training-for, training-with-and-in, and training-by' the school for secondary education (Deinum et al., 2005). This has benefit for both. From the perspective of teacher education, a motivating learning environment is created where teacher students can integrate theory and practice and where there is opportunity to become part of the professional community (Schaap et al., 2012). From the school's perspective, having the opportunity for professional development for their personnel and having the occasion to work on innovation and research through this collaboration with a teacher education institution, can serve as an incentive (Snoek, 2011).

Example 7. An internship at the end of the bachelor's programme
From: Instituut Archimedes (2014) (translated from Dutch by the authors)

Assignment: For a year, deliver, on your own, 6 to 8 lessons weekly. […] As well as delivering lessons, also perform the following tasks:

- Prepare and grade assessments
- Participate in staff meetings
- Attend report meetings
- Have contact with parents

[Depending on the context,] perform several other activities such as:

- Have contact with support staff
- Design educational products

[…]

Example 8. Scaffolding
From: Utrecht University's teacher education
(translated from Dutch by the authors)

Assignment: Make a video recording of yourself in which you give support to one or more secondary school students. […] Then, select two video fragments that show interaction. Chose one fragment with interaction where you are satisfied about the way you are giving help to your students and one where you are less satisfied about the help you are giving.

Regarding the next questions, try to interpret as little as possible and try to reason as much as possible based on what you see. Name as many concepts as possible.

(a) In which respect is scaffolding used in the video fragments? What elements of scaffolding are included? What observations make you say this?

(b) Which elements of scaffolding do you miss? What observations make you say this?

(c) Watch each video fragment according to two principles, namely the ‘contingent shift principle’ and the ‘principle of adaptivity in term of cognitive complexity’; does this lead to the same results?

9.3.5 Merging All Activities: Exhibiting and Assessing Competence

The plethora of learning activities in the different quadrants could lead to fragmented learning. Moreover, this could be amplified by the fact that these activities are divided over different courses in the curriculum, which are not all specific for mathematics and are taught by different teacher educators. To prevent this, so-called ‘profession-related tasks’, consisting of large, central and targeted assignments, play an important role. This approach is called a ‘whole-task’ model (Van Merriënboer & Kester, 2008). Such a task could be, for example, designing a lesson or a test, or designing a lesson series that one has to carry out. Examples 9 and 10 give an impression of such tasks.

Almost all teacher education institutions have as part of their curriculum an extensive assessment in which a teacher student must present a portfolio that gives insights into his or her competences. An interview with a teacher educator and a senior teacher at the training school is often part of this assessment.

Almost always, a teacher educator attends a lesson by a teacher student as part of the teacher education curriculum. This visit sometimes has the form of a summative assessment, although it is frequently used to give constructive feedback and to plan the further developmental needs of the teacher student.

Example 9. Designing a lesson series

From: Van den Bogaart & Konings (2015) (translated from Dutch by the authors)

[This text] addresses the preparation of several lessons when treating a chapter in a secondary school textbook, but also in case one has a theme for which one wants to produce materials on one’s own.

In a lesson series, there are a great many points of attention:

- How can I analyse the learning content and the textbook? What do my students need to be able to do and know? How can I assess this?
- Which didactical interventions enable me to support the learning of my students? […]

[During the chapter, a student teacher designs and carries out a series of lessons. Here is part of the rubric used to grade the teacher student:]

	Starting	Developing	Competent	Exemplary
Content choice	Choosing a chapter based on the schedule of the schoo	…and takes his/her own didactical interest into account	…and takes the wishes of the teacher team of the school with respect to special innovations into account	…and takes the further development of his/her competences into account
Learning goals	"Knowing that…": Describes the goals in detail and links this to representative tasks	"Knowing why…": …and how the content can become meaningful for the students […]	"Knowing how…": …also long-term goals are taken into account like problem solving, reasoning abilities, […]	"Knowing about knowing…": …and study skills are taken into account

Example 10. 'Master proof'

From: Utrecht University's teacher education (translated from Dutch by the authors)

Assignment: To make your 'Master proof' you should develop all the materials and the didactical instructions necessary to outline a lesson unit of subsequent lessons for senior levels in secondary school […]. This means you should think about the aims and objectives (learning goals), the study guide, work sheets, assessment activities or tests, students' surveys, suggestions for improvements and so forth. It also includes collecting and development of appropriate software, presentations, slides, video's, CD-ROMs and websites. Furthermore, there needs to be a clear distinction between the materials intended for the secondary school students and the texts meant for you and other teachers, that should be in the teacher guidelines. Finally, the justification of specific parts of the lesson unit needs to be made explicit, recognisable and verifiable.

The lesson unit: In agreement with your mentor at your secondary education training school you will make a particular choice regarding class/grade, topic, duration, timing etc., and you will design a lesson unit of at least 4 and at most 10 lesson hours. When you have discussed the design of the lesson unit with your supervisors (at school and at university) and you have made the appropriate adjustments, you will carry out your lessons and evaluate the unit. Using the feedback from the evaluation you will most likely need to make a few more changes in the design.

The product: The final product of your ‘Master proof”, needs to contain the following components: (i) a justification of your approach; (ii) the students’ materials; (iii) a summary of the evaluation and the changes made as a consequence; (iv) a reflection on the process of design and performance of the lesson unit; (v) the teacher guidelines to enable you, or a colleague, to carry out the lesson unit another time.

The assessment: The criteria to assess the quality of the ‘Master proof’ are derived from an ‘advertisement text’ in which the requirements for a school subject teacher are described. This means that from your product it should become apparent that:

- You are able to articulate a vision on your school subject and its place in society; […]
- You show understanding of the construction of the subject curriculum including the performance objectives and assessment goals; […]
- You understand how students acquire knowledge and understanding of your subject area; […]
- You can plan, give and evaluate lessons (in authentic activities) and you are able to translate your vision, insights and knowledge into effective and entertaining lessons: […]
- You dared to experiment; […]

9.4 Reflections on the Current Situation

In this section, we look at the current state of affairs in secondary school teacher education in the Netherlands and discuss some important merits, challenges and points for improvement. We will follow the ordering used in structuring this chapter. First, we reflect on the educational system, then we focus on the aims of teacher education, and finally we conclude with a reflection on the curricula for secondary school teacher education.

9.4.1 Reflection on the Dutch Educational System

In an analysis of mathematics education in the Netherlands, Van Streun (2001) recognises as the most important development during the last century that nowadays almost 100% of students at age 16 have had mathematics education, while previously this was the prerogative of a small elite. We see two additional merits. First, the quality of mathematics education is high. International comparison in PISA (OECD, 2014) shows that Dutch students do well, especially when compared to other European countries. Second, the content of the mathematics curriculum is tailored to the needs of different student populations. There are many levels, several profiles and several kinds of mathematics with their own specific subjects. This diversity in mathematics education that is offered is more apparent than in neighbouring countries (Kaper, 2013). This broad range makes the teacher's job both varied and challenging. Also, it gives ample opportunities for making connections with other school subjects.

Besides these merits, we see several concerns. First, the aforementioned Pisa-rankings show a score that gradually declines in time. This might indicate a slow but steady decreasing level in relation to other countries. Second, although generally the amount of mathematics lessons is comparable to that of neighbouring countries, this is not the case for students at the highest levels of secondary education that prepare for further study in engineering or the natural sciences—in this respect, the Netherlands is far behind (Kaper, 2013) and this situation has worsened in the last eight years.

Another point that is an issue of concern is that in the Netherlands, there is a shortage of mathematics teachers. The expectation is that this shortage will increase (Fontein, Adriaens, Den Uijl, & De Vos, 2015). In 2015, 2.2% of the full-time positions for mathematics teachers could not be filled and this will grow to 5.1% in ten years. Moreover, a significant part of the jobs is filled by teachers who do not have the required qualification (18.5% of the lessons in 2013, according to Fontein, De Vos, & Vloet, 2015).

In the light of this shortage, Dutch mathematics teacher education has two quantity-related problems (Amerom & Drijvers, 2013). First, the efficiency of teacher education is low and decreasing. Although it is difficult to get precise measures, roughly one third of the teacher students have not completed their study one year after the nominal duration of teacher education. Second, a significant part of those who graduated at a teacher education institution do not become teachers—for example, this is the case for 34% of second-degree teachers who graduated between 1996 and 2005. Additionally, there are concerns about the very small number of teachers graduating at universities (Commissie Deltaplan Wiskunde.NL, 2015).

A further issue to be worried about is that Dutch teachers have little time for professional development, lesson planning and evaluation, which in part explains the great dependency on textbooks. This is especially a problem for starting teachers, who are often overloaded with work. This can lead to their quitting their job as a teacher.

A striking characteristic of the Dutch system for secondary school teacher education is that there are many routes by which one can acquire a teacher qualification, that there are many teacher education institutions and that these institutions are quite autonomous. This leads to a diverse population of teachers, which is enriching. However, the many routes are not enough. Schools for secondary education demand still more flexibility. On the one hand this is a major point of concern at the moment, but on the other hand there are limits to the flexibility that can be realised due to quality assurance and the small scale of Dutch teacher education (Dekker, 2016). In view of the foregoing, one can raise the question whether the large number of routes and institutions does not lead to too much fragmentation. Especially in teacher education at universities where student population is very low, this is a major problem. Therefore, there is cooperation between institutions.

Continuous professional development of teachers is an important aspect of current educational policy. At the moment, too many teachers regard receiving their degree as the end point of their development. The professional register of teachers aims to change this, but we fear that it will take some time before this change in culture is realised. Another important aspect is the high dropout rate of beginning teachers. An extension of the duration of teacher education could remedy this. In addition, it would also be helpful for keeping more teachers in their job if teachers had more career possibilities. Currently, Dutch teachers can make progress through getting a higher salary based on length of service as well as by being promoted to higher-ranked positions at their school or at another school. However, compared to other jobs the career perspective for teachers is still small. Also, the teaching profession has a lower status than many other professions (Sikkes, 2015). All these circumstances damage the attractiveness of a teaching job.

Regarding the quality of teacher education, recent accreditations of institutions for teacher education show that mathematics teacher educators as a rule are highly qualified and ambitious. Nevertheless, certain types of professional development for teacher educators could be stimulated more, such as developing knowledge about what it means to combine teacher education with having a teaching job in secondary education. Furthermore, for secondary school teachers who are responsible for the part of the teacher education that takes place in practice in secondary school, more schooling is necessary, especially in the didactics of mathematics.

A particular concern is also the payment of teacher educators: their starting salaries as a rule are lower than those of experienced teachers in secondary educations, which makes it difficult to convince ‘the best teachers’ to work in teacher education.

Finally, the state of affairs concerning research in the didactics of mathematics is worrisome (Verhoef, Drijvers, Bakker, & Konings, 2014). The same holds for research on the education of starting mathematics teachers. Although the research output is respectable, there is little budget for further research, although the need of teacher education institutions for validated insights is growing. A positive development is the creation of budgets for teachers to do a PhD study.

9.4.2 Reflection on the Aims of Dutch Secondary School Mathematics Teacher Education

Dutch teacher education can be characterised as rather focused on the teaching profession: the learning of teacher students takes place in close contact with practice. There is a rich tradition in cooperation between teacher education institutions and schools for secondary education. However, this cooperation is often on education in general and does not always has a strong link to mathematics. Although competence in the didactics of mathematics is highly valued, it is often subordinated to general educational competences. Recent research (Inspectie van het Onderwijs, 2015), focussing on the second-degree level, pinpoints several strong points in the quality of teacher education: reaching a high level of subject knowledge and basic pedagogical and didactical skills in teacher students and realising a high level of competence in teaching and assessment. Nevertheless, there is also room for improvement in focus on curriculum planning, designing assignments, applying differentiation and making links to practice.

As remarked earlier, mathematics teachers in secondary school are strongly guided by their textbooks. The limited time available to teachers for preparation and professional development leads to a conservative way of teaching, which may result in a discrepancy between the aims of teacher education and school practice in secondary education. Thus, there can be a gap between the demands of schools in secondary education and the aims of the teacher education institutions. Governmental policy therefore emphasises having more interchange between secondary education schools and teacher education institutions (Dekker, 2016).

For five years, the mathematical subject knowledge for secondary school teacher education at HBO schools has been captured in the Knowledge Base. This has led to a levelling of the mathematical goals, yet often resulting into a rise of these goals. However, there is discontent among teacher educators with the large amount of detail in the Knowledge Base and putting too much focus on reproductive skills.

Since the requirements on subject knowledge needed to start with teacher education are more and more relaxed, teacher students at universities are often overqualified in this respect. Therefore, there is an ongoing discussion within universities on the balance between academic and practical training necessary to become a teacher. Three years ago, students in secondary school teacher education qualified the academic focus as insufficient (VSNU, 2013). However, at the moment one tends to put more emphasis on good practice preparation, yet, after all, these students have completed an academic study.

Currently, in secondary school teacher education there is a growing emphasis on educational research. Yet it is not evident how to deal with research in teacher education. This especially applies to teacher education at university, where most students have already done extensive research activities outside education, but where there is little opportunity to really let students experience research in an educational context.

A final point of reflection in this section is about the vision on mathematics education. Although, to a large degree, there is a shared vision on mathematics education among secondary school teacher educators, in a broader societal perspective this is not the case. As a result, this is leading to a debate on the main function of mathematics education, namely teaching formal skills versus skills for functional use of mathematics, and this debate impedes curriculum development (SLO, 2015).

9.4.3 Reflection on the Curricula for Secondary School Teacher Education

In the curricula for secondary school teacher education one can distinguish several types of programme components such as theoretical courses aimed at mathematics, the didactics of mathematics, or general educational competences, and more practice-oriented activities like internships and research projects. Parts of these activities take place at the teacher education institutions and other parts in schools for secondary education. Such a curriculum can easily lead to fragmentation. This danger is also mentioned in accreditation reports on the teacher education institutions. Similarly, there is also the danger of having no coherence in the curriculum because activities are divided over different courses in the curriculum, taught by different teacher educators. The profession-related tasks based on the whole-task model we discussed earlier, could be used to obtain coherence between the theoretical courses and practice-oriented activities. This approach should get more emphasis in the secondary school teacher education curricula.

A further aspect where coherence falls short concerns research skills. Achieving a research attitude in future teachers is regarded as a major goal of teacher education. However, doing research often has an isolated place in the curriculum. Linking research to the profession-related tasks enhances the coherence, as this linking is happening more and more.

Another complain that is sometimes heard from secondary school teacher educators is that teacher students have a lack in applying didactical knowledge when teaching their students. A problem is that experts in the didactics of mathematics do not always go to the training school and observe the teacher students' teaching systematically. Coaching in the training school is often limited to giving practical directions. In a one-year teaching education programme at a university, linking theory and practice is difficult to realise. We see room for improvement here with respect to the following points: (1) the role played by the didactical expert in coaching and assessing teacher students' learning in the training school, (2) the establishment of communities of training school coaches and didactical experts, (3) professional courses for training school coaches and (4) an alumni policy to make the realised level of the teacher students more transparent.

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Chapter 10
Digital Tools in Dutch Mathematics Education: A Dialectic Relationship

Paul Drijvers

Abstract Nowadays, digital tools for mathematics education are sophisticated and widely available. These tools offer important opportunities, but also come with constraints. Some tools are hard to tailor by teachers, educational designers and researchers; their functionality has to be taken for granted. Other tools offer many possible educational applications, which require didactical choices. In both cases, one may experience a tension between a teacher's didactical goals and the tool's affordances. From the perspective of Realistic Mathematics Education (RME), this challenge concerns both guided reinvention and didactical phenomenology. In this chapter, this dialectic relationship will be addressed through the description of two particular cases of using digital tools in Dutch mathematics education: the introduction of the graphing calculator (GC), and the evolution of the online Digital Mathematics Environment (DME). From these two case descriptions, my conclusion is that students need to develop new techniques for using digital tools; techniques that interact with conceptual understanding. For teachers, it is important to be able to tailor the digital tool to their didactical intentions. From the perspective of RME, I conclude that its match with using digital technology is not self-evident. Guided reinvention may be challenged by the rigid character of the tools, and the phenomena that form the point of departure of the learning of mathematics may change in a technology-rich classroom.

10.1 Introduction

Since the origin of mankind, people have developed and used tools to ease their work and to carry out tasks. In the case of mathematical tasks, tools such as the abacus, the ruler and the compass have been used for centuries. More recent is the development of a fascinating category of tools, namely digital tools. This new generation of tools includes software for algebra and calculus (e.g., computer algebra systems or CAS),

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for 2D and 3D geometry (dynamic geometry systems or DSG), and for statistics (e.g., the Dutch software VuStat, see Van Streun & Van de Giessen, 2007). Such powerful tools, in which an impressive amount of mathematical expertise is incorporated, may not only be used for 'getting the mathematical job' done, but may also affect mathematics teaching and learning. In addition, dedicated tools such as applets have been designed for specific educational purposes. These educational roles are central in this chapter.

Among mathematicians and mathematics educators, mathematics is considered as more than a set of algorithms which can be applied to solve routine problems. No matter how powerful these standard solution procedures are, and how much of human intelligence was needed to develop them, doing mathematics and, as a consequence learning mathematics, also encompasses working on problems that are new to the person involved, and requires creative problem solving and the development of new methods and knowledge. From this perspective, much attention has been paid to theories on bottom-up learning, (socio-) constructivism, discovery learning, inquiry-based learning. Students should be given ample opportunity to explore, to investigate, to conjecture, and to prove. In this way, they are expected to develop meaningful mathematical insights, to (re)construct their mathematical knowledge and to acquire general skills that go beyond the specific task at stake. In the theory of Realistic Mathematics Education (RME), which is wide-spread in the Netherlands, this idea is captured in the notion of guided reinvention (Freudenthal, 1973; Van den Heuvel-Panhuizen & Drijvers, 2014). According to this principle, students should be given the opportunity to experience a process similar to that by which a given mathematical topic was invented. While doing so, students in the meantime need guidance from the teacher. A second RME concept, didactical phenomenology (Van den Heuvel-Panhuizen, 2014), highlights the relation between the mathematical thought object and the phenomenon from which it emerges. In particular, it addresses the question how mathematical objects can help in organising and structuring real phenomena. The challenge for the designer, of course, is to find such meaningful phenomena that beg to be organised and structured by the targeted mathematical knowledge.

Some decades ago when digital tools for mathematics education became more widespread and increasingly powerful, mathematics educators and researchers both in the Netherlands and worldwide expected that this might provide levers to change mathematics education in the direction of the aforementioned higher-order goals, rather than focusing on the acquisition of basic paper-and-pen techniques. If digital tools would enable students to easily and quickly investigate different situations, to engage in experimentation without time-consuming work by hand, to outsource the basic techniques to digital tools, would this not offer excellent opportunities for the envisaged bottom-up and meaningful learning? Some optimism seemed appropriate.

In the educational reality of students, teachers, classrooms and schools, however, the use of digital tools for higher-order learning goals turned out to be more complex than foreseen. In addition to the sometimes problematic infrastructural demands that the use of digital tools puts on every day teaching, it became clear that each digital tool for mathematics education does not only offer opportunities, but also comes with constraints, which may be the result of either technological limitations or design choices. The flexibility of digital tools, and the ways in which teachers can customise them for their specific purposes, is often limited. As a consequence, their use for developing higher-order skills, which seems the most subtle, is less popular than for practicing basic skills. Overall, research results on the measurable improvement of learning are only modest (Drijvers, 2016).

As a result, there is a somewhat dialectic relationship between the higher-order goals of mathematics education, as highlighted in RME theory among others, and the opportunities and constraints digital tools offer. Can we manage to use such tools for learning goals that go beyond basic skills, or do they tend to push us back into an algorithmic approach of mathematics? How can digital tools be used for bottom-up, meaningful and realistic mathematics education? How can we optimise the design of digital tools on the one hand, and the didactical design of ways to use them in teaching on the other? In short, how can we deal with the tension between sometimes rigid digital tools and flexible teaching? This is the central issue in this chapter. To deal with this issue, I will, after a brief historical flash-back, discuss two particular cases of using digital tools in Dutch mathematics education: the case of the handheld graphing calculator (GC), and that of the Digital Mathematics Environment (DME).

10.2 A Brief Flash-Back

Over the past 45 years, the world-wide development of digital tools for mathematics education and their use in practice has drastically evolved, both with respect to the type of tools and the type of use. After some early applications of Computer Assisted Instruction for mathematics, in the 1970s there was a major focus on programming in Logo and BASIC, for example to make the 'turtle' move in a specific way (Drijvers, Kieran, & Mariotti, 2010). In his book *Mindstorms*, Papert made a plea for programming in so-called micro worlds, claiming that "[t]he computer presence has catalysed the emergence of ideas" (Papert, 1980, p. 186). Programming was considered a means for enhancing students' mathematical problem-solving abilities. The availability of personal computers in the 1980s not only made programming activities more feasible in practice, but also led to the development and dissemination of dedicated software for mathematics (such as computer algebra systems), and for mathematics education (dynamic geometry systems or dedicated software, see Doorman & Van der Kooij, 1992). General tools, such as spreadsheet software, were also used in mathematics lessons (Sutherland & Rojano, 1993).

By the end of the 1990s, handheld technology such as GCs became widespread. The advantage of these digital tools was not only that the use of the technology no longer depended on the classroom infrastructure, but also that the initiative to use this personal device lay primarily with the students: even if guidance from the teacher was needed, in the end it was the student who decided when to use the technology, and for what purpose. The handheld format also raised the question of the use of digital tools in assessment and examinations.

As Internet speed improved, after 2000 the use of small, dedicated applets in mathematics teaching became more popular. Online educational use gradually replaced work with locally installed software. In addition, digital tools allowed for communication, exchange and collaboration between students, and between students and teachers. Video channels offering mathematical instruction became popular, leading to the 'flipping the classroom' paradigm. Online courses started to attract huge numbers of participants.

Nowadays, we see a myriad of digital tools used worldwide, ranging from desktop PCs to laptops, tablets and smartphones. Students bring their own devices, and broadband internet is the gateway to different types of applications. Using digital tools in the mathematics classroom has become natural, and less prominent than it used to be in dedicated 'technology lessons' in the past.

Developments in the Netherlands took place along similar lines. In the 1970s, programming was popular, including work with flow charts and scratch cards to execute programs written in educational programming languages such as Algol and Ecol (Vonk & Doorman, 2000). In the 1980s, the personal computer started to make life easier. However, schools used a diversity of brands of computers and different operating systems. It was only after a national project called 'NIVO' brought some uniformity, that using ICT in education became more common. In the 1990s, ICT also became integrated in subject curricula. Schools were equipped with computer labs. After 1999, GCs became mandatory for pre-university education for students aged 15–18. With the advent of broadband internet, applets were being used more and more, in particular those from the DME developed at Utrecht University. Over the last decade, classrooms were equipped with interactive whiteboards, and wifi in school allows students to access the internet through their own devices. The Geogebra software is quite popular, and students make more and more use of laptops and tablets, in some schools not just in addition to textbooks, but as a replacement. The question, however, is how the type of use and its didactical and theoretical backgrounds have developed over this period.

10.3 The Case of Handheld Graphing Calculators

To investigate how the dialectic relationship between the goals of mathematics education and the opportunities and constraints of digital tools developed over time in the Netherlands, I now describe the case of handheld GCs. I will confront the initial expectations with the developing practice and also address the specific case of symbolic calculators. The case description closes with a short conclusion.

10.3.1 Initial Expectations

In the mid-1990s, the GC entered the Netherlands. This happened to coincide with a curriculum reform for pre-university mathematics for students aged 15–18, which was carried out by the Freudenthal Institute, which was the cradle of the theory of RME. As a consequence, the GC was seen as a means to bypass institutional constraints and to directly equip students with a device that would support dynamic and interactive exploration and reinvention, and this approach was integrated in the curriculum reform process. More in particular, Drijvers and Doorman (1996, p. 425) claimed that

> [o]bservation of the students' behaviour during the experimental lessons supports the premise that the graphics calculator can stimulate the use of realistic contexts, the exploratory and dynamic approach to mathematics, a more integrated view of mathematics, and a more flexible behaviour in problem solving.

As an example of a task that invites such an exploratory approach, students were asked to graph functions f and g defined by $f(x) = a \cdot 2^x$ and $g(x) = 2^{x+c}$ for such values of a and c that the two graphs would coincide. The students knew that the two graphs could be derived from the standard exponential graph with base 2 through a multiplication with a factor a with respect to the x-axis, and a horizontal translation to the left over a distance c respectively. In this way, students were offered an experimentation space for the discovery of relationships such as $2^{x+c} = 2^x \cdot 2^c$, $2^c = a$, or $\log_2 a = c$, depending on the level and age of the students. Speaking in general, expressing functions in terms of other functions, in this example $g(x) = f(x + c)$, is a powerful means to build chain functions (Kindt, 1992a, b). More examples can be found in Doorman, Drijvers and Kindt (1994, 1996).

If exploration using GCs is part of teaching, it should of course also be assessed. Figure 10.1 shows part of an assignment of the national examination for the schools that piloted the new curriculum in 1999. One question was to find the value of n in

$$x(t) = \left(1 + \frac{1}{n}\sin(nt)\right)\cos(t)$$

$$y(t) = \left(1 + \frac{1}{n}\sin(nt)\right)\sin(t)$$

so that the graph of $(x(t), y(t))$ is the 'curved circle'. This task can be solved through different combinations of reasoning and drawing on the GC.

As mentioned earlier, digital tools come with limitations. An obvious limitation of the early GCs was the low-resolution screen and the not very sophisticated ways to graph functions, with sometimes confusing results. Figure 10.2, for example, shows the calculator's inappropriate way to deal with the vertical asymptote of the function f defined by $f(x) = \frac{x^2+x-1}{x-1}$. Indeed, students were unable to correctly copy this graph on paper. Such limitations may challenge the teachers' intentions of explorative, 'real' mathematics. The solution we found to this was not to avoid such

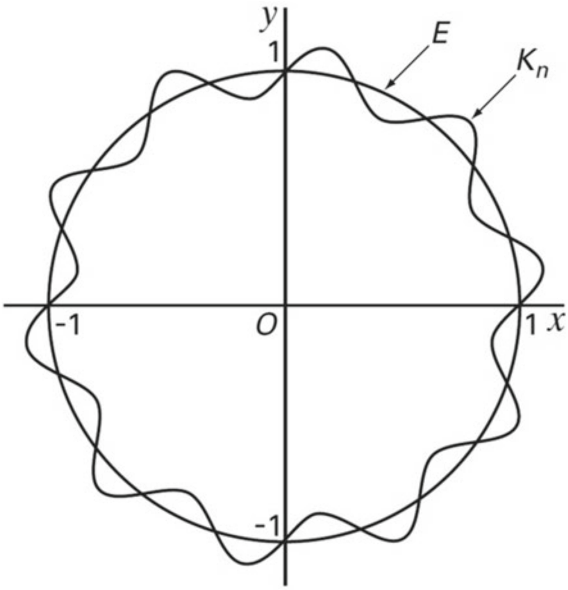


Fig. 10.1 Exploration task in the 1999 national pilot examination

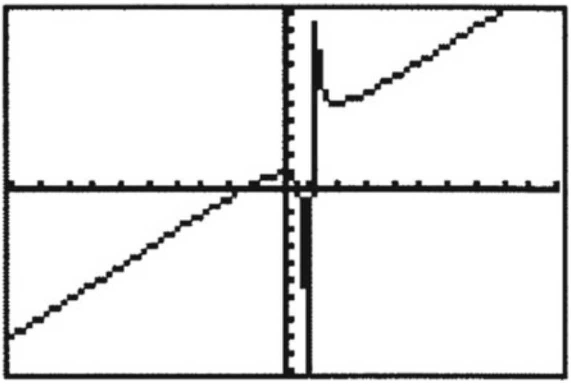


Fig. 10.2 Early GC's misleading graph

constraints, but rather to exploit them by making them explicitly subject to further investigation. In the case of the asymptotes, we invited students to try to come up with as many misleading graphs on the GC screen as they could think of, and, of course, explain why the graph was misleading and how they found it. This approach, inspired by Treffers' (1987) notion of students' own production, proved to be a fruitful one.

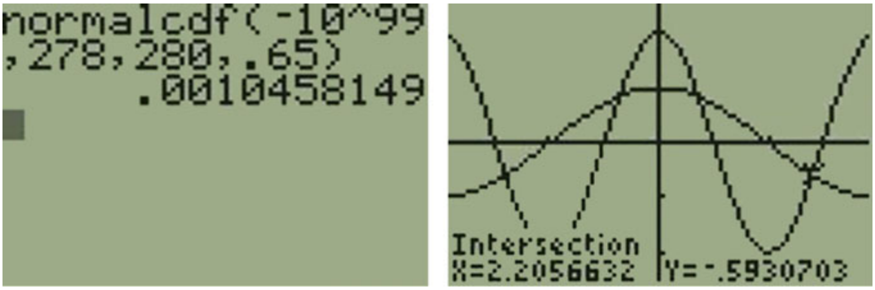
Of course, GCs are 'ready-made' devices that are hard to tailor to specific didactical needs. Through task design and teaching approach, we tried to exploit the tool's potential for the sake of RME. In retrospective, I may say that we were somewhat naive in our initial and optimistic expectations, as we probably neglected the dependency on the task and the teaching, on top of the affordances of the digital tool. From the developing practices during the following years, however, this became quite clear.

10.3.2 Developing Practices

Since 2001, students in the Netherlands were required to bring a GC to the national examination in mathematics for pre-university secondary education. 'Required' does not mean that nobody is allowed to take the exam without a GC; it does mean, however, that assignments may become much harder to do without a GC, and that the 'risk' of not having one, or not being able to use it appropriately, is for the candidate. The idea behind this policy was that a curriculum in which digital tools are recognised as important cannot be assessed in a technology-free manner, and the handheld personal GC would be a feasible way to include this aspect in the national written examination. Also, the national examination was expected to act as a lever to really implement a change in teaching practice in line with the opportunities described in the previous section.

Initially, the GC was also allowed for national examinations in the subjects physics, chemistry, biology and economics. However, this permission was withdrawn once the authorities became aware that students could store information (formulas, applications, texts, even pictures) on their handheld devices, which was not intended and might present candidates with unequal chances. At present (2016), mathematics is the only subject for which GC use is allowed during national examinations.

How did the national examinations change since students have had a GC at their disposal? Different countries have shown different policies to deal with technology in central examinations (Drijvers, 2009). Compared to other countries, initially Dutch policy was relatively far-reaching: the use of the GC was not only allowed, but also indispensable in some assignments, and its appropriate use was credited in some of the tasks. However, some trends need to be mentioned. First, assignments in which the GC plays an essential role in visualising or exploring a mathematical situation, such as the task from the pilot examination in 1999 shown in Fig. 10.1, are quasi non-existing. Apparently, the board that sets the examination assignments considered such tasks as too much depending on GC skills, and to a lesser extent on the mathematical insights to be assessed. Second, the number of credit points that students may get through the use of the GC seems to be decreasing over the years. In this sense, the role of the GC in examinations became smaller over the years. This may be explained by the tendency to re-value exact paper-and-pen procedures from algebra and calculus: assignments nowadays contain phrases such as "Calculate the exact value…" which require algebraic or analytic by-hand procedures, and do not credit GC generated solutions. Third and final, the number of GC techniques that are credited in examination papers became limited and standardised; in fact, students should be familiar with ways to calculate probabilities of normal and binomial distributions in statistic assignments, and with ways to calculate intersection points, zeros,



```
normalcdf(-10^99
,278,280,.65)
      .0010458149
```

```
Intersection
X=2.2056632  Y=-.5930703
```

Fig. 10.3 Standard GC procedures to find intersection points and to calculate probability from Dutch 2015 mathematics examinations

maxima and minima in calculus and algebra tasks. Figure 10.3 shows an example of these two procedures, as asked for in the 2015 national examinations in mathematics in the Netherlands. The left screen shows the calculation of a probability for a normal distribution, and the right one the approximation of a solution of the equation $2 \cdot \cos(2t) = \cos(t)$.

Since GCs became mandatory in mathematics examinations, textbook series of course also included references to these devices. Again, the type of tasks and the ways to use the GC were not as oriented towards exploration, visualisation and reinvention as the exemplary student materials in the curriculum development project had been. Rather, the textbooks focus on the previously mentioned GC procedures for statistics and calculus. As a consequence, teachers also make sure that their students master this small repertoire of standard techniques, rather than exploiting the didactical opportunities of the GC in their lessons.

In short, teaching and assessment practice in Dutch upper secondary mathematics education with respect to the integration of the GC did not have the effect that was hoped for. Compared to the ideas expressed earlier, the role of the GC remained limited to some specific techniques, which students also apply in easy cases. For example, students may use an intersect procedure to solve an equation like $2x + 3 = 7$. On the one hand, this may endanger the maintenance of paper-and-pen skills. On the other hand, this is what technology nowadays offers. The main reasons for this limited use probably lie in the developing opinion in the field of mathematics teachers, educators and mathematicians. On the one hand, innovative and technology-oriented people soon started to consider the GC as ‘old school’ technology, compared to more advanced devices such as tablets, laptops and smartphones. On the other hand, more conservative voices in the field expressed their concern about students’ paper-and-pencil techniques and stress the need to put aside the GC (and other digital tools) to make students master these basic skills. In this way, the GC became tangled between ICT-oriented and back-to-basics protagonists.

10.3.3 Additional Symbolics

In the meantime, symbolic calculators (SCs), i.e., handheld devices that also offer computer algebra on top of GC features, received international attention. Many teachers, educators and researchers were fascinated by the immense mathematical and symbolic power embedded in such small, handheld devices. Still, it was unclear what the consequences of this technological development should be for secondary mathematics education.

In the Netherlands, the use of symbolic calculators in secondary education was investigated in the PhD. study by Drijvers (2003). The study shows both the potential of computer algebra in mathematics education and its constraints. As was the case for the GC, the somewhat rigid character of computer algebra environments may hinder students' expressiveness and teachers' creativity. In the case of computer algebra, the strict syntax for algebraic commands turned out to be one of the most important obstacles. Again, similar to the GC, an interesting didactical approach to deal with these constraints was to explicitly address them and to take computer algebra as an expert system which is subject to the students' investigations: How does the device get its answers? How to explain differences with what would be expected? In this way, obstacles may be turned into opportunities (Drijvers, 2002). For the case of algebraic equivalence, this approach is elaborated in more detail by Kieran and Drijvers (2006).

The SC had a limited impact on teaching practice in the Netherlands. The reasons are to a certain extent similar to those in the case of the GC: as paper-and-pencil algebraic skills are highly valued, equipping students with computer algebra does not seem the right thing to do. Therefore, SCs were banned from national examinations. Also, the limitations of computer algebra and the difficulty to use it were not in the SC's favour. The argument that symbolic calculation tools might free students from calculational drudgery and open horizons for modelling, application, investigation, and reinvention was, once more, not highly valued.

The reason to mention symbolic calculators here in spite of their limited impact is that international research on their use did lead to fruitful theoretical perspectives, which may be applied to the use of digital tools in general. A core point is the bi-directional relationship between tools and their users, in which students' thinking is on the one hand shaped by the digital tool, and on the other hand shapes the way the tool functions (Hoyles & Noss, 2003). This is reflected in the notion of instrumental genesis, the co-emergence of techniques for using digital tools and the mathematical insights involved (Artigue, 2002; Trouche, 2004; Trouche & Drijvers, 2010).

10.3.4 Conclusions on the Graphing Calculator Case

The case of the introduction of the GC in Dutch mathematics education first shows the initial enthusiasm, reflected in the design and use of innovative tasks that exploit the technology's potential for exploration and reinvention. Next, I describe the implementation in both examination and teaching practice, which is to a lesser extent driven by a guided-reinvention view on mathematics, and comes down to equipping students with a limited repertoire of standard techniques for using the GC; techniques which are of course of practical value. This development may be caused by the limited and rigid character of the GC, in combination with public opinion in the Netherlands shifting towards paper-and-pencil basic skills.

Even if the didactical policy of turning constraints into opportunities was in some cases a fruitful one, it became clear that the digital tool's limitations may hinder the creative design of open and engaging tasks. An important criterion for digital tools in mathematics education, therefore, is their expressive power for students, so that they enable students to explore and express mathematical ideas in accessible and natural ways.

10.4 The Case of the Digital Mathematics Environment

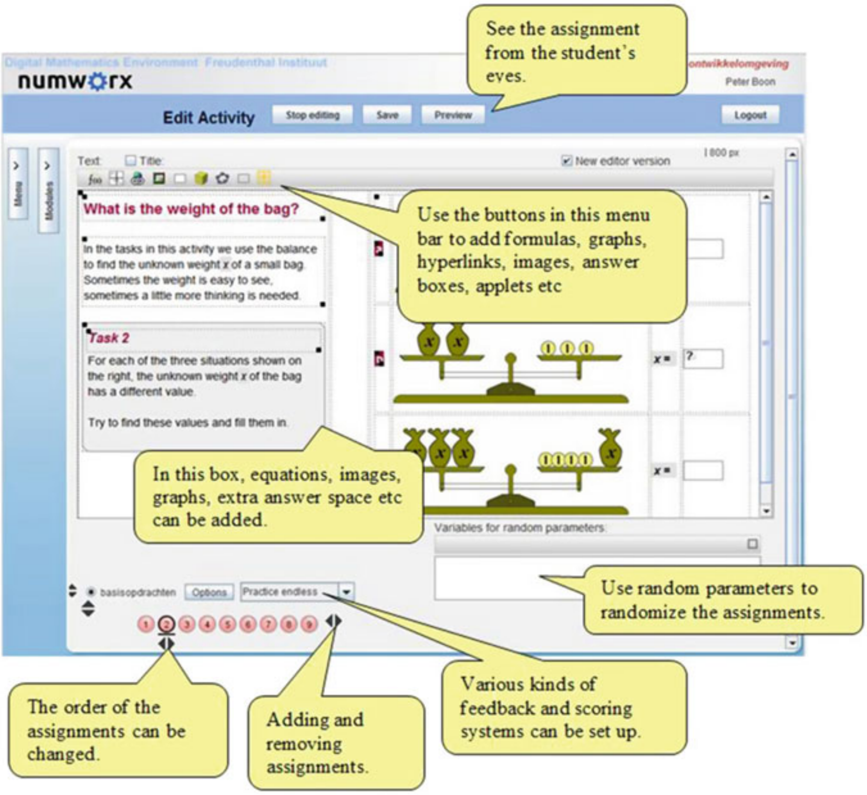
As a second case reflecting the dialectic relationship between the goals of mathematics education and the opportunities and constraints of digital tools, I now consider the development of the DME. The DME is an online environment for mathematics activities developed by Utrecht University's Freudenthal Institute. I will first briefly sketch the DME's technological development. Next, design choices will be discussed, as well as the role of the teacher, which was the topic of adjacent research. The case description closes with a conclusion.

10.4.1 Technological Development

In the late 1990s, the DME (https://www.numworx.nl/en/log-in/) started as an initiative by Peter Boon, who was a mathematics teacher at the time, and an expert in programming. His initial idea was to design Java applets that were available online and that would facilitate students' exploration of mathematical objects and concepts. In collaboration with colleagues at the Freudenthal Institute, applets were designed for several topics, such as 3D geometry (Kindt & Boon, 2001), algebra (Boon, 2004), and on the intuitive notion of functions as chains of operations (Boon & Drijvers, 2006). As these applets were field-tested and soon became popular in schools, and as their number was growing over the years, a content management system was needed to organise the content collection, as well as a player to deliver this content. In addition to programming applets, the architecture of the environment as a whole became a focus.

One of the powerful features of digital tools in general is the option to keep track of student progress, either to inform students and teachers, or to provide automated feedback, or to score student work. For this reason, a learning management system was embedded in the DME, which on the one hand provides students with feedback on their work and on the other hand offers overviews of students' progress to their teachers.

Initially, a core activity while designing the DME was the programming of applets. Gradually, the difference between programming on the one hand, and designing the tasks and activities for students that come with the available applications on the other hand, grew bigger. For this reason, the DME authoring environment was developed. It allows educational designers, such as teachers, educators or text book authors, to design activity sequences for students without engaging in programming the applets that form the basis of these activities. In the authoring environment, authors adapt existing online modules or design new ones, using existing applets and basic tools such as graphing and equation editing facilities as building blocks. Knowledge of the underlying programming language is not required; rather, an intuitive and mathematical interface makes the digital design accessible to a wide audience (Fig. 10.4).



What is the weight of the bag?

In the tasks in this activity we use the balance to find the unknown weight x of a small bag. Sometimes the weight is easy to see, sometimes a little more thinking is needed.

Task 2

For each of the three situations shown on the right, the unknown weight x of the bag has a different value.

Try to find these values and fill them in.

See the assignment from the student's eyes.

Use the buttons in this menu bar to add formulas, graphs, hyperlinks, images, answer boxes, applets etc

In this box, equations, images, graphs, extra answer space etc can be added.

Use random parameters to randomize the assignments.

The order of the assignments can be changed.

Adding and removing assignments.

Various kinds of feedback and scoring systems can be set up.

Fig. 10.4 The DME authoring environment

Nowadays, the DME includes a player, a content management system, a learning management system and an authoring environment. It has moved to html5, provides advanced features for assessment and adaptivity, has a computer algebra engine available, and hand writing recognition.

10.4.2 Design Choices

Even if the DME's design did not follow a fixed road map scheduled in advance, its development has always been guided by a set of (sometimes implicit) design principles. Boon (2009) describes how his points of departure were to make the software flexible and customisable, and to always keep in mind other possible educational applications of a designed piece of software. As a consequence, the DME had a modular character, in which the basic building blocks, the applets, can be re-used and adapted to the specific didactical goals at stake. In this way, the DME became a rich and flexible environment for mathematical activity.

As the DME was developed within Utrecht University's Freudenthal Institute, it is not surprising that the theoretical foundation of its design is rooted in the theory of RME. This theory is reflected in DME characteristics in several ways. With respect to students, many DME applets and activities offer them room for expressing their mathematical ideas, exploring mathematical situations, and reinventing mathematical properties. Also, according to the notion of didactical phenomenology, it is central to engage students in situations that invite the development of mathematical thinking in a natural and mathematically sound way. And finally, students should be productive in the DME activities rather than reproductive. As a consequence of the dedicated design of applets and student activity, instrumental genesis is expected to take place in a more natural way than in the case of more general and less flexible tools, such as the GC.

With respect to teachers, the DME also has some features that can be related to the RME theory. Due to its flexible character, and the availability of the authoring environment and the applet collection as building blocks, the DME offers teachers the opportunity to engage in design, to be productive themselves, and to acquire ownership of their teaching and teaching materials. This ownership is not self-evident in the Netherlands, where teachers usually rely strongly on the regular mathematics textbook series, rather than designing their own materials and lessons. As such, the DME is a less ready-made digital tool than GCs or computer algebra environments, for example.

As the DME is used by a wide variety of users, the RME points of departure do not guarantee educational products (i.e., online modules) that reflect the RME theory. In fact, from the early years on, some applets on practicing solving linear equations became popular, whereas the RME approach in these applications is not very prominent. If the DME starts to 'live' in the mathematics education community, full control will of course be out of the hands of the software architects.

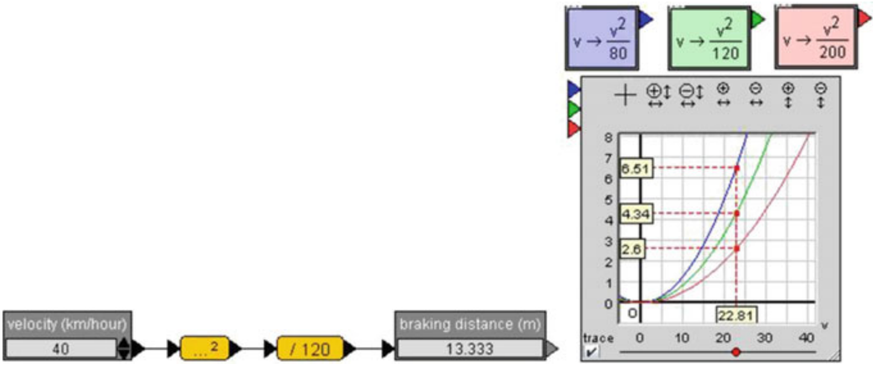


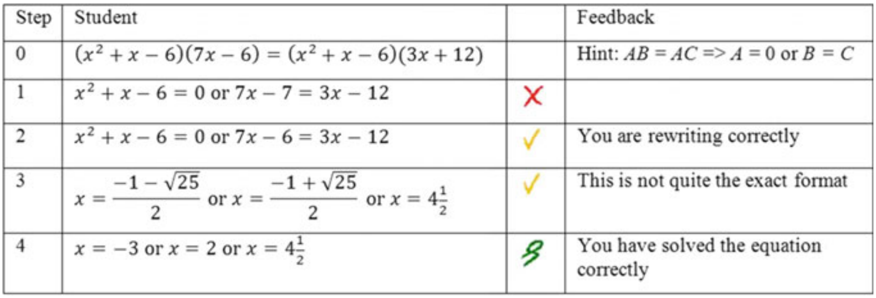
Fig. 10.5 Stepwise arrow chain (left) and collapsed arrow chains (right), from (Drijvers, Boon, Doorman, Bokhove, & Tacoma, 2013)

The development of the DME did not only include technological design, but was part of an iterative process of designing, field-testing, and improving, that is typical for educational design. The development of the DME has always had a strong link to research projects, which in most cases had a cyclic design-based research character. Studies that rely on DME affordances, but also informed its further development, are manifold (e.g., Jupri, Drijvers, & Van den Heuvel-Panhuizen, 2016). Let us briefly consider some examples.

As an example of a study that both made use of the DME and informs its development, Doorman, Drijvers, Gravemeijer, Boon, and Reed (2012) describe how a teaching sequence on functional thinking using an applet called “Algebra Arrows” led students to develop a structural view on function. Based on the notion of emergent modelling (Gravemeijer, 1999), the teaching sequence integrated both paper-and-pen and digital work. As an illustration, Fig. 10.5 shows how student may ‘collapse’ chains of operations into functional objects.

As a second example, Bokhove and Drijvers (2012a, b) investigated feedback design in DME modules for 17- and 18-year-old students on equation solving. Different types of feedback and feedback conditions were compared (see Fig. 10.6). As an overall conclusion, feedback timing and fading seemed crucial for its effects. In an effect study using multilevel models, the feedback-rich intervention indeed turned out to be effective (Bokhove & Drijvers, 2012b). In another study, however, on 13- and 14-year-old students solving linear and quadratic equations in the DME, the intervention was not successful (Drijvers, Doorman, Kirschner, Hoogveld, & Boon, 2014). Apparently, the success of such interventions is not straightforward. The role of the teacher might be a crucial factor here, which is why I consider it in the next section.

In short, the design of the DME is strongly influenced by RME principles and by the interaction between design and adjacent educational research. It is these factors that helped the DME to develop into a rich and flexible environment for mathematics



Step	Student		Feedback
0	$(x^2+x-6)(7x-6)=(x^2+x-6)(3x+12)$		Hint: $AB = AC => A = 0$ or $B = C$
1	$x^2+x-6=0$ or $7x-7=3x-12$	✗	
2	$x^2+x-6=0$ or $7x-6=3x-12$	✓	You are rewriting correctly
3	$x=\frac{-1-\sqrt{25}}{2}$ or $x=\frac{-1+\sqrt{25}}{2}$ or $x=4\frac{1}{2}$	✓	This is not quite the exact format
4	$x=-3$ or $x=2$ or $x=4\frac{1}{2}$	✔	You have solved the equation correctly

Fig. 10.6 Stepwise feedback including hints (Bokhove & Drijvers, 2012a)

education in the way it did, and to reduce the tension between rigid digital tools on the one hand, and flexible didactical approaches on the other.

10.4.3 Role for the Teacher

In addition to the different levels of design in the DME, including programming applets and global environment features, and designing student activities, a third level is of course crucial: the use of the DME in teaching. This includes (1) preparing lessons, in some cases through the adaptation of existing activities to the teaching purpose and target group involved, (2) delivering lessons according to these plans, and (3) dealing with unexpected events while teaching, either from the digital technology or from student behaviour. As for other digital tools, the exploitation of the potential of the DME is not straightforward to teachers. Therefore, research has been carried out at the Freudenthal Institute to investigate the professional development needed by teachers to fully benefit from the opportunities the DME offers.

As a theoretical lens in this research, the notion of instrumental orchestration (Trouche, 2004) is used. An instrumental orchestration is defined as "the teacher's intentional and systematic organisation and use of the various artefacts available in a—in this case computerised—learning environment in a given mathematical task situation, in order to guide students' instrumental genesis" (Drijvers et al., 2010, p. 214–215). Three levels are distinguished: the didactical configuration, the exploitation mode and the didactical performance. A study on three teachers using the "Algebra Arrows" applet shows that teachers have their preferences for specific types of orchestrations, and that these preferences relate to their views on mathematics education (Drijvers et al., 2010; Drijvers, Godino, Font, & Trouche, 2013). A study with twelve mid-adopting teachers using different applets within the DME documents the extension of the teachers' repertoire of orchestrations as a characteristic of their professional development (Drijvers, Tacoma, Besamusca, Doorman, & Boon, 2013).

From the RME perspective of guided reinvention, the DME is challenging. On the one hand, many DME activities are designed to provide means for exploration and reinvention to students. On the other hand, we notice a tendency in teachers, who often are novices in teaching with digital tools and are themselves in the process of instrumental genesis, to step back as soon as students interact with digital tools, and to fall back on ‘old’ teaching strategies such as teacher-driven explanations. As ICT may be a new and complicating element in the didactical configuration, this may hinder the teachers’ flexible attitude that is needed to appropriately support the students’ process of reinvention. To be confident, to identify opportunities and constraints of the digital activities, and to adapt teaching experience and skills to the technology-rich classroom, is a challenge for teachers. Professional development can play an important role in helping teachers to also engage in RME-based teaching in a digital setting.

10.4.4 Conclusion on the Digital Mathematics Environment Case

The case of the DME shows that software design can be more closely related to a theoretical view on the teaching and learning of mathematics than is the case for the GC. Also, the iterative design is underpinned by research, and the interplay between design and research is known to be a powerful one (e.g., see Bakker & Van Eerde, 2015). The fact that educational designers, software designers, teachers and researchers work together guarantees close-to-practice solutions that are also theoretically grounded. This way of working may reduce the tension between a teacher’s didactical goals and the software’s affordances. As a result, much of the DME content offers room for students to explore and to construct, to be productive, and as such may facilitate a guided reinvention approach to mathematics education.

To bridge the gap between task design and software design, that is so manifest in the case of the GC, the DME authoring environment empowers teachers to adapt tasks and applications, and to design new ones. This improves the teachers’ ownership of their teaching materials, which we consider a good thing. In the meantime, we should acknowledge that it is demanding for a mathematics teacher to (re-)design materials in the DME, not only for reasons of time, but also because of the didactical insights and creativity needed. It is here that the need for professional development comes into play. Professional development activities complement the collaboration between designers, teachers and educators mentioned above.

What we do learn from the DME case, finally, is that software for mathematics education should be designed for different educational applications, and should be flexible and customisable. As Boon (2009, p. 10) phrases it, we should design software as “a collection of reusable components and packages”. In this way, activities addressing new phenomena may be easily created based on existing activities.

10.5 Conclusion

In the introduction of this chapter, I mentioned the dialectic relationship between the higher-order goals of mathematics education, and the opportunities and constraints digital tools offer. Indeed, as Hoyles and Noss write: “Tools matter: they stand between the user and the phenomenon to be modelled, and shape activity structures” (Hoyles & Noss, 2003, p. 341). The two cases described in this chapter help us to further specify this dialectic relationship for students and for teachers.

For students, using digital tools for mathematics in addition to paper and pen may lead to new opportunities and constraints. New techniques for using the tool for a type of task need to be developed, and each technique affects the concept image as it emerges in the students’ minds. The interaction between technical mastery and conceptual understanding is a subtle one. Whereas a mismatch between the two may hinder learning, a natural fit between technique and the mathematics at stake may foster mathematical understanding. Whereas the constraints, such as syntactical demands, may frustrate students, room for exploration may foster engagement in rich explorative and productive activities.

The challenge for teachers, therefore, is to exploit the opportunities and to deal with the constraints. Teachers may experience some tension between their didactical aims and goals, and what can really be done in the digital environment. Two levels of educational design come into play: the design of tasks and student activities on the one hand, and the customisation or design of the software on the other. In many cases, the latter type of design is too time-consuming or too difficult to be within the teachers’ scope. Therefore, an important criterion for educational software is the option for teachers to tailor it to their didactical intensions, taking into account instrumental genesis and the subtle relationship between tool use and mathematical thinking. What counts, after all, is not the digital tool itself, but the way it is part of a didactical approach, including tasks, activities, discussions and assessment. To oversee the role of digital tools in this spectrum is not trivial for many teachers; professional development may be useful here.

From the perspective of RME, I conclude that the match between RME and using digital technology is not self-evident. With respect to guided reinvention, the integration of digital technology in mathematics teaching may initially be a complicating factor to teachers, which challenges established teaching techniques. To remain in control, teachers may react to this by focusing on traditional forms of teaching such as demonstrations and explanations. This may lead to more guidance and less room for reinvention. As a consequence, the guided reinvention approach may need extra attention when technology enters the classroom.

As for didactical phenomenology, I conclude that the phenomena may change in a technology-rich classroom. The digital environment itself may be a meaningful phenomenon to the student. The GC’s limitations with respect to graphing asymptotes turned out to be an inspiring phenomenon to elicit algebraic thinking. In the meantime, using digital tools may also create some distance to the phenomena at stake. For example, drawing a circle with a physical compass does require a circular hand movement; in a dynamic geometry software environment, this physical connection between the hand movement and the geometrical object is less evident, as students just have to click on a centre point and a point at the radius distance. It is important that the phenomena explored in a digital environment can be presented and manipulated in a natural way, which corresponds with representations and manipulations in the physical world. Interesting ongoing research investigates how such embodied experiences can be simulated on digital devices (Abrahamson, Shayan, Bakker, & Van der Schaaf, 2016).

Altogether, the challenge for teachers, designers, educators, and researchers is to create digital tools that are flexible and customisable, that offer room for exploration to students, and that teachers can easily adapt to their specific didactical goals. Teaching with technology should not default to traditional techniques because of the increasing complexity of the teaching environment, and attention needs to be paid to presenting phenomena in natural and meaningful ways. These are not straightforward challenges; in the meantime, progress has been made and a joint effort is needed to make the integration of digital tools in mathematics education a widespread success.

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Chapter 11
Ensuring Usability—Reflections on a Dutch Mathematics Reform Project for Students Aged 12–16

Kees Hoogland

> *Change in education is easy to propose, hard to implement, and extraordinarily difficult to sustain.*
> Hargreaves and Fink (2006, p. 1)

Abstract In this chapter, I look back at the implementation of W12-16, a major reform of mathematics education in the lower grades of general secondary education and pre-vocational secondary education in the Netherlands including all students aged 12–16. The nationwide implementation of W12-16 started in 1990 and envisioned a major change in what and how mathematics was taught and learned. The content was broadened from algebra and geometry to algebra, geometry and measurement, numeracy, and data processing and statistics. The learning trajectories and the instruction theory were based on the ideas of Realistic Mathematics Education (RME): the primary processes used in the classroom were to be guided re-invention and problem solving. 'Ensuring usability' in the title of this chapter refers to the aim of the content being useful and understandable for all students, but also to the involvement of all relevant stakeholders in the implementation project, including teachers, students, parents, editors, curriculum and assessment developers, teacher educators, publishers, media and policy makers. Finally, I reflect on the current state of affairs more than 20 years after the nationwide introduction. The main questions to be asked are: Have the goals been reached? Was the implementation successful?

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11.1 Vision

11.1.1 Radical Innovation

The W12-16[1] reform of mathematics education in the lower grades of general secondary education and pre-vocational secondary education in the 1980s and 1990s was widely seen as a radical innovation in mathematics education. The reform affected all elements of mathematics education in secondary schools: a new and broader curriculum, alternative ways to approach students, fostering students to develop more and other skills such as problem solving, and using different assessments such as contextual and open-ended problems.

The realisation of such a change was only possible with broad support. In the 1980s and 1990s, there was in the Netherlands a great deal of agreement between teachers, mathematics education developers from the Freudenthal Institute (the former IOWO[2]), mathematics educators from the SLO,[3] the staff of APS,[4] and teacher educators from various teacher education institutions.

Through and with these leading institutions, publishers, other teacher educators, teacher unions, educational support agencies, researchers and developers in mathematics education worked together to change mathematics education. Furthermore, a great many of these people were involved in writing mathematics textbooks. This broad collaboration also made it possible to offer in-service training on a large scale. And last but not least, there was support, although limited, to this mathematics education reform movement from professional mathematicians; because of the eminent stature of Hans Freudenthal.

This broad engagement was also visible in the two teams that were the driving forces in the development and implementation of W12-16 reform. First, the W12-16 team started as a development and design team with members of various institutions. Later, this team was transformed into the SW12-16[5] team, a broad implementation team with dozens of teachers and mathematics educators as team members, working together to implement the new curriculum.

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[1] Wiskunde 12-16 (Mathematics 12-16).

[2] Instituut voor de Ontwikkeling van het Wiskunde Onderwijs (Institute for the Development of Mathematics Education).

[3] Netherlands Institute for Curriculum Development.

[4] National Centre for School Improvement.

[5] Samenwerkingsgroep Wiskunde 12-16 (Collaborative working group Mathematics 12-16).

11.1.2 Pioneering

For most members of the W12-16 team, innovation in mathematics education began well before their participation in the team. Some members were looking for opportunities to innovate teaching methods, while others had an affinity with at-risk students and acted from a background of special needs education. Other team members were mostly interested in the professional development and empowering of mathematics teachers. Yet some other members were working on promoting mathematics for girls. All team members had one thing in common; they were looking for a setting in which all students could be inspired by mathematics, motivated by the content and the approach, and be actively engaged in mathematics. The team members were the pioneers who advanced the initial developments.

In some sense, the work of the W12-16 team was an extension of the mathematics education development that was already taking place in the Netherlands. At the same time, W12-16 was the focal point through which all the initiatives came together and were moulded in a coherent vision. From the beginning of the 1970s, at the IOWO people had been working on the design of Realistic Mathematics Education (RME) in which students are given practical problems from everyday life or other sources which can be experienced as real by the students. By solving these problems and reflecting on them with mathematics teachers, students construct their own set of mathematical concepts. In RME this is called ‘guided reinvention’ (Gravemeijer, 1994, 2004). In the first instance, the IOWO staff focused on primary education and primary school teacher education. As a follow up, a number of booklets on particular mathematical domains were developed for lower secondary education using the same approach. The influence of this material was limited because the content of the examinations changed little, if at all. Because of this, teachers and mathematics textbook authors were hesitant to use these booklets in their teaching programmes.

More substantial for the development of the W12-16 team’s vision was a change in upper levels of secondary education resulting from the HEWET[6] and HAWEX[7] projects, which introduced a new mathematics curriculum. The influence of the HEWET project (1978–1985) was the most substantial because it concerned, amongst other things, the development and introduction of a new curriculum for pre-university secondary education. Mathematics A was intended for students pursuing a university education in the social sciences; the contents were considered a kind of ‘forerunner’ of mathematical literacy (De Lange, 1987; OECD, 1999), including functional mathematics, contextual problem solving, and statistics and probability. Mathematics B was meant for students pursuing a university education in the natural sciences and contained more technical mathematics with a strong calculus

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[6] Herverkaveling Wiskunde I en II (Re-allotment Mathematics I and II); the HEWET project resulted in Mathematics A and Mathematics B, a new mathematics curriculum for the upper grades (age 16–18) of VWO, the pre-university level of secondary education.

[7] HAVO Wiskunde Experimenten (HAVO mathematics experiments); the HAWEX project resulted in Mathematics A and Mathematics B for the upper grades of HAVO, general secondary education which qualifies for higher professional education.

approach, including functions, graphs, and advanced calculus. In addition to the calculus domain, the curriculum of Mathematics B included a domain of geometry in which mathematical proofs were reintroduced in the curriculum as an example of the scientific mathematical method.

At this stage, the developments in primary education had also progressed. In the beginning of the 1990s about 40% of primary schools used a RME textbook series (Van den Heuvel-Panhuizen, 2010).

So, for both age ranges, the 4–12-year-old students and the 16–18-year-old students, the mathematics curricula were changing. One last gap remained: lower general and pre-vocational secondary education. In 1987 a committee was set up to review the mathematics curriculum for students in these tracks in the age range of 12–16 years.

11.1.3 The Educational and Societal Context of the Change

The experiences with a new curriculum in primary education and the upper levels of general secondary education fed the vision of the W12-16 team. However, it was not only developments in mathematics education which left their mark. While the W12-16 team was at work, more general educational changes took place and influenced the development of the team's vision. The direction and size of the educational change in W12-16 were determined to a large extent by the social context in which the plans were developed. In the 1980s and 1990s, in which the W12-16 and SW12-16 teams operated, there were several developments that affected classroom norms, educational policies and curriculum development. In this particular time frame, there was a focus on equity in education: schools organised students in heterogeneous classes; there was a general need for basic education for all, and consequently for mathematics for all; and last, but certainly not least, to prevent the waste of enormous human potential in mathematics, there was a focus on the mathematical competence of girls. In the Netherlands, compared with surrounding countries, girls were underrepresented in technology sectors of education and were underachieving in secondary education because many were choosing tracks with either no mathematics or easier mathematics.

Moreover, at that time, there was also increasingly widespread use of calculators in society, though this had not yet spread to schools. And, to complete the picture of this period of time, it is important to note that it was prior to the common use of the internet and the World Wide Web.

11.1.4 The Dutch School System

The Dutch school system has a few very distinct features, which also influenced the implementation of the new program (Fig. 11.1).

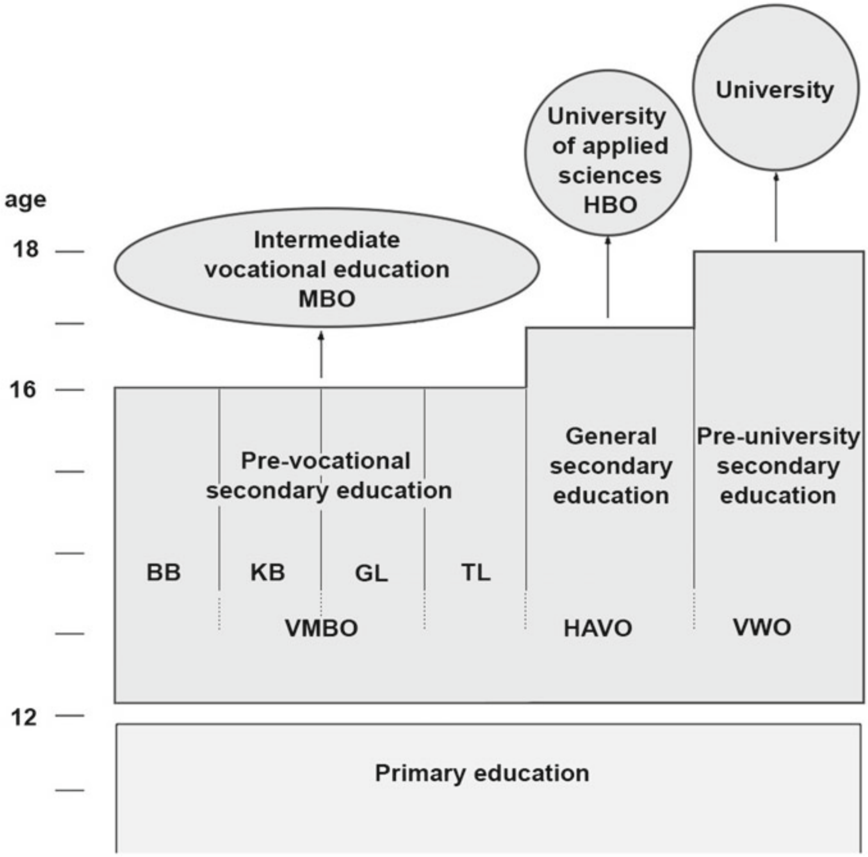


Fig. 11.1 The Dutch school system

Early streaming and a focus on vocational education from a very young age are typical features of the Dutch school system. Although it is internationally recognised that such early streaming limits the full developmental potential of the student population (OECD, 2013), discussing a change to this has been a no-go area in Dutch politics for decades. Reducing streaming is seen in Dutch politics as aiming at egalitarian and uniform education, instead of aiming at differentiated education, that is, dealing with differences in the classroom. The strong and early focus on vocational education could indeed have a benefit for many students, as they can engage in meaningful and job-related activities early in their education. But at the same time, it can lead to a sharp divide between vocational and general education. Preventing a two-tiered education structure, and in the long run a two-tiered society, was and is a serious educational challenge for the Netherlands. The aim of the W12-16 team was to make mathematics education meaningful for all students, regardless of level, gender, ethnicity or educational stream, preferably in an inclusive educational setting.

To summarise, developments in both mathematics education and society worked together to create a vision of mathematics for all students that targeted usability and inspiring and meaningful mathematical content.

11.2 The Content of the New Curriculum

11.2.1 RME—The Vision in a Nutshell

The Dutch approach to mathematics education has become known as 'Realistic Mathematics Education' (Gravemeijer & Terwel, 2000; Van den Heuvel-Panhuizen, 2000; Van den Heuvel-Panhuizen & Drijvers, 2014). The present form of RME has been mostly determined by Freudenthal's (1973) view on mathematics education and was further developed by the staff of the Freudenthal Institute at Utrecht University. Freudenthal viewed mathematics as an educational task that, for it to be of human value, should be connected to reality, remain within children's experience, and be relevant to society. In his view, teaching mathematics is much more than a transfer of knowledge to be absorbed by students. Freudenthal stressed the idea of learning and doing mathematics as a human activity; it should give students a guided opportunity to re-invent mathematics by actively doing it. This means that the focal point of mathematics education should not be on mathematics as a closed system but on the activity and on the process of mathematisation (Freudenthal, 1980).

11.2.2 RME in Secondary Education

In secondary education, mathematical concepts become more sophisticated and formal than in primary education. In many mathematics curricula all over the world (Hodgen, Pepper, Sturman, & Ruddock, 2010a, b) formal mathematics is used as both a goal and an organisational principle for the curriculum, as reflected in names of content domains such as 'algebra' and 'geometry', which are basically domain names from the early eighteenth century. In such mathematics curricula, contextual problems are most commonly used for knowledge application tasks at the end of a learning sequence, as a kind of add on. In the mathematics curriculum for lower secondary education, which is the curriculum being reflected on in this chapter, a broader scope was chosen in this reform: 'algebra' became 'functions, formulas and relations'; 'geometry' became 'geometry and measurement'; 'numeracy' was added with a focus on mathematical literacy; and 'data processing and statistics' were addressed. In this way, the organisational principle for the curriculum shifted towards a categorisation in topics related to the world around us and how mathematics plays a role in it, rather than a categorisation of mathematical concepts.

In RME, context problems also have another function than mere application of mathematics. They are typically used in the exploration and development of new mathematical concepts. In RME, context problems play a role in each new learning trajectory directly from the beginning. Learning trajectories start with the presentation of a problematic situation that is experientially real to the student. The contextual problems are intended to foster a re-invention process that enables students to become involved in problem solving and modelling processes and at the same time provides them with grips for more formal mathematics. In RME, context problems can function as anchoring points for the students to re-invent mathematics themselves. Moreover, guided re-invention and emergent modelling offer ways to address the generally perceived dilemma of how to bridge the gap between informal knowledge and formal mathematics.

11.2.3 Examples from Final Examinations

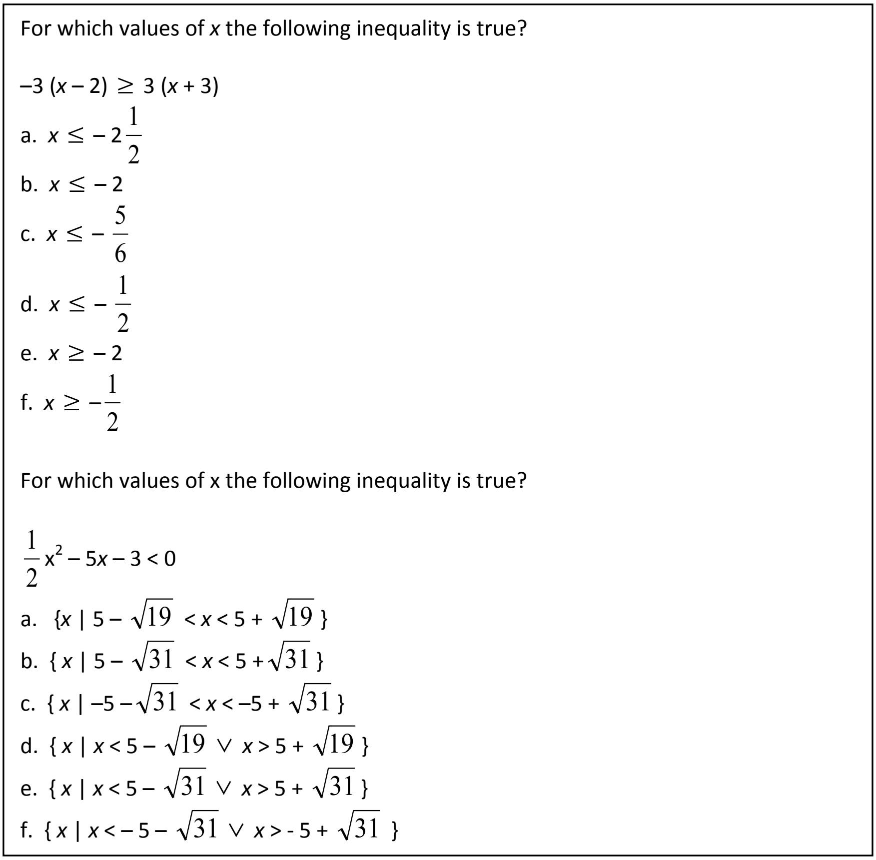
Curriculum changes are documented in formal curricula describing the skills and knowledge goals to be taught. For teachers, however, the most common way to communicate curriculum changes is through discussing exemplary tasks in final examinations and comparing 'old' tasks with 'new' tasks.

The tasks in Figs. 11.2 and 11.3 are from a mathematics final examination for pre-vocational secondary education. Figure 11.2 shows tasks from the examination in 1995, which are typical for the old curriculum, while the tasks in Fig. 11.3 are from the examination in 1996, which are typical for the new curriculum.

The differences between the final examination tasks of the old and the new curriculum are striking. First, the starting point for tasks in the final examination of 1996 involves problems from the real world, sometimes accompanied by pictures or diagrams. This approach contrasts considerably with the formal mathematical approach used previously. Second, the focus has shifted from making calculations at a formal level to mathematical problem solving and modelling. Third, multiple choice questions are abandoned to keep students in a problem-solving mind-set as long as possible. And finally, the mathematics is personalised in the sense that actual people are introduced in the tasks and, while the question may not be directly relatable to the life experience of the students, it is at least imaginable for them.

11.2.4 The Change in Content

Until 1992 the mathematics curriculum for lower secondary education was based on the classical mathematical subjects of geometry and algebra. In algebra, the focus was on algebraic manipulation, solving equations, and linear and quadratic functions and their graphs. In the domain of geometry, the focus was on plane geometry—measuring angles, Pythagoras, and goniometry—with a strong calculational approach. In



For which values of x the following inequality is true?

$-3\ (x-2) \geq 3\ (x+3)$

a. $x \leq -2\frac{1}{2}$

b. $x \leq -2$

c. $x \leq -\frac{5}{6}$

d. $x \leq -\frac{1}{2}$

e. $x \geq -2$

f. $x \geq -\frac{1}{2}$

For which values of x the following inequality is true?

$\frac{1}{2}x^2 - 5x - 3 < 0$

a. $\{x \mid 5-\sqrt{19} < x < 5+\sqrt{19}\}$

b. $\{x \mid 5-\sqrt{31} < x < 5+\sqrt{31}\}$

c. $\{x \mid -5-\sqrt{31} < x < -5+\sqrt{31}\}$

d. $\{x \mid x < 5-\sqrt{19} \vee x > 5+\sqrt{19}\}$

e. $\{x \mid x < 5-\sqrt{31} \vee x > 5+\sqrt{31}\}$

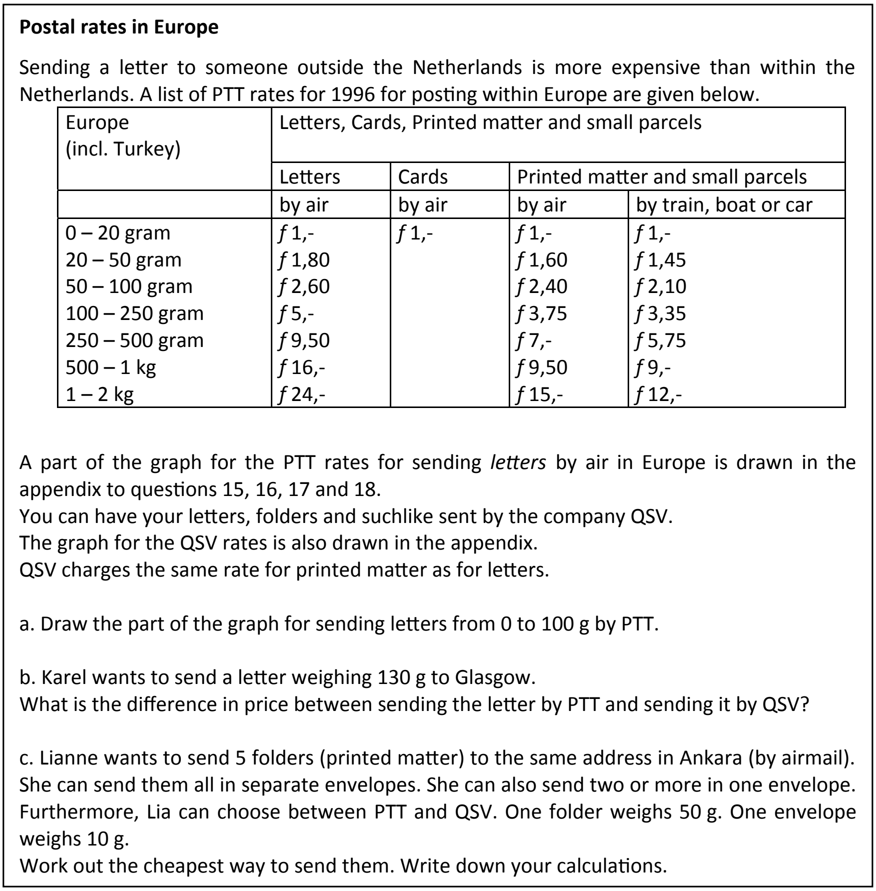
f. $\{x \mid x < -5-\sqrt{31} \vee x > -5+\sqrt{31}\}$

Fig. 11.2 Tasks from the final examination Wiskunde VMBO GT 1995 (pre-vocational secondary education, upper track, mathematics, old curriculum) (translated from Dutch by the author)

the new programme, there was a new approach to algebra and geometry and the scope was broadened to include numeracy and statistics. Furthermore, a new curriculum domain of integrated mathematical activities was added. The aim of this addition was for the students to intertwine the different content strands in a more thematic approach.

11.2.4.1 A New Approach to Algebra

The focus within the algebra domain shifted from algebraic and computational manipulation to reasoning on the relationships between variables and to flexibility in switching between four different types of representations of relations: graphs, tables, verbal representations of situations, and formulas. Other characteristics of the new algebra approach were:



Postal rates in Europe

Sending a letter to someone outside the Netherlands is more expensive than within the Netherlands. A list of PTT rates for 1996 for posting within Europe are given below.

Europe (incl. Turkey)	Letters, Cards, Printed matter and small parcels			
	Letters	Cards	Printed matter and small parcels	
	by air	by air	by air	by train, boat or car
0 – 20 gram	ƒ1,-	ƒ1,-	ƒ1,-	ƒ1,-
20 – 50 gram	ƒ1,80		ƒ1,60	ƒ1,45
50 – 100 gram	ƒ2,60		ƒ2,40	ƒ2,10
100 – 250 gram	ƒ5,-		ƒ3,75	ƒ3,35
250 – 500 gram	ƒ9,50		ƒ7,-	ƒ5,75
500 – 1 kg	ƒ16,-		ƒ9,50	ƒ9,-
1 – 2 kg	ƒ24,-		ƒ15,-	ƒ12,-

A part of the graph for the PTT rates for sending *letters* by air in Europe is drawn in the appendix to questions 15, 16, 17 and 18.
You can have your letters, folders and suchlike sent by the company QSV.
The graph for the QSV rates is also drawn in the appendix.
QSV charges the same rate for printed matter as for letters.

a. Draw the part of the graph for sending letters from 0 to 100 g by PTT.

b. Karel wants to send a letter weighing 130 g to Glasgow.
What is the difference in price between sending the letter by PTT and sending it by QSV?

c. Lianne wants to send 5 folders (printed matter) to the same address in Ankara (by airmail). She can send them all in separate envelopes. She can also send two or more in one envelope. Furthermore, Lia can choose between PTT and QSV. One folder weighs 50 g. One envelope weighs 10 g.
Work out the cheapest way to send them. Write down your calculations.

Fig. 11.3 Tasks from the final examination Wiskunde VMBO GT 1996 (pre-vocational secondary education, upper track, mathematics, new curriculum) (translated from Dutch by the author)

- Dealing with diversity in the representations of relationships between variables instead of focusing on uniformity and formal conventions.
- More focus on interpretation of representations of relationships between variables than on manipulation skills.
- More focus on broad techniques like translating representations of relationships between variables than on specialised techniques like using the abc-formula to solve quadratic equations.
- More focus on a concentric curriculum with a gradual increase in complexity rather than a linear curriculum.

With these characteristics, the new curriculum aimed for a more usable, practical, and meaningful interpretation of algebra. For mathematics in the upper levels of secondary education, it was also seen as possible to design a usable calculus course based on these fundamental principles; see, for example, Gravemeijer and Doorman (1999).

11.2.4.2 A New Approach to Geometry

In the curriculum domain of geometry, the focus shifted away from two-dimensional plane geometry with a strong calculational approach and towards two- and three-dimensional geometry with a focus on so-called 'vision geometry'. This geometry is based on seeing, observing, perceiving, representing and explaining spatial objects and spatial phenomena, in which the idea of vision lines and intervisibility plays an important role.

Geometry for primary schools was developed from similar ideas prevalent in the 1970s and 1980s, primarily informed by everyday geometric phenomena. Emphasis was placed on 'observing, doing, thinking and seeing', as Goffree (1977) described the Wiskobas[8] geometry concept.

The approach could almost be seen as a revival of the ideas of Tatiana Ehrenfest-Afanassjewa, who in 1931 published her *Übungensammlung zu einer Geometrische Propädeuse* (Ehrenfest-Afanassjewa, 1931) in which she substantiated her thinking on geometry based on everyday experiences from a practical point of view. This book contains a collection of problems of an entirely different nature than the traditional geometrical problems around constructions and proofs. It presents everyday geometrical phenomena that could be examined by 10-year-olds or even younger. Accordingly, these problems served to stimulate children's intuitive notions of geometric concepts and properties, thus forming a basis for later formal and systematic work. In the secondary education geometry programme these ideas were continued. So, geometry moved to more usable geometry, with strong links to the surrounding reality. For an extensive overview of specific developments in geometry education, see De Moor (1999).

11.2.4.3 Numeracy as a New Domain for Secondary Education

Within the domain of numeracy, the focus in W12-16 was on mathematical literacy. The functional use of basic mathematics (and arithmetic) was the key element. The aim was to contribute to the basic competences of students in dealing with everyday quantitative situations or problems. The focus on operations with numbers was reduced, and the focus on problem solving and modelling was intensified. This was an approach comparable with the approaches in PISA (mathematical literacy), and PIAAC[9] (numeracy) which were also emerging in the 1990s. Because of the practical nature of the numeracy envisioned, much attention was given to proportional reasoning, estimating, dealing with measurement, and using the calculator.

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[8] Wiskunde op de Basisschool (Mathematics in Primary School).

[9] Programme for the International Assessment of Adult Competencies.

11.2.4.4 Data Handling and Statistics

The new curriculum domain of data handling and statistics was focused on the ways data are collected, visualised and used in decision making. It could be considered as a kind of forerunner of dealing with big data. It contrasted with what was common in this domain. In the pre-1996 programme some statistics was mentioned, again with a strong calculational approach, for example, how to calculate mean and standard deviation and how to produce histograms and circle diagrams.

In the W12-16 curriculum, data handling and statistics was treated as mature and serious components of the mathematics curriculum. They were seen as increasingly important aspects of the mathematical competences students needed in their future lives. The focus also shifted from calculations to interpreting the large amount of numbers and data that is ubiquitous and used more and more in communication between people. This vision was quite new and innovative at the time of the change. As mentioned before, in those years, internet or the World Wide Web were not yet available.

11.2.5 From Mathematics for a Few to Mathematics for All

One of the pedagogical and didactical consequences of seeing mathematics as a human activity (Freudenthal, 1973, 1980) was that it allowed the engagement of every student, not only those who are cognitively privileged or with a strong inclination to mathematical thinking. Mathematics as a human activity is an inclusive philosophy for teaching, learning and doing mathematics.

In W12-16, there was a strong belief that every student should be involved in mathematics on an appropriate level. The change from more specialised topics to a broader view of mathematics and the shift to a broader range of topics was one way of making mathematics more accessible to all. This broader scope followed a worldwide tendency in mathematics education towards more usability. Whereas until the 1970s mathematics curricula were defined as subsets of the mathematical knowledge structure, from the 1980s on there was a global focus on the usability of mathematics, and therefore curriculum elements were sought which had visible applications, including arithmetic, proportions, measurement, data collection and chance. This was a fundamental change in designing curricula, because it made aspects of the real world the basis for the categorisation scheme rather than the logical structure of the mathematical domains (Kilpatrick, 1996; Niss, 1996).

11.3 Implementation

11.3.1 Implementation Theories

From literature on educational change there are many theories and studies that detail the conditions necessary to make curriculum change successful (Hargreaves, Lieberman, Fullan, & Hopkins, 1998). Nevertheless, there are also many reports that describe curriculum changes that have failed. In a most cynical way, implementation of new curricula worldwide is sometimes summarised as the ‘fiasco pattern’. If the target group for the curriculum change is set at 100%, after a few years these outcomes are most common: 70% have heard of it, 50% actually saw it, 30% have read it and have the documentation, 15% use it, 5% use it according to the intention of the change and 0–3% use it and attain the intended effect on the learners.

However, the particular changes in mathematics education described in this chapter have reached maturity over a period of 25 years and have proven to be quite sustainable. The curriculum over this period did not stay completely unaltered, but it still contains some clearly recognisable elements of the original ideas and intended outcomes.

In the following sections this particular implementation of mathematics education is analysed from the perspective of theories of educational change. The purpose of this analysis is to reconstruct which elements of the implementation strategy had a positive effect on the sustainability of the change. A reference is made to the frameworks of Miles and Fullan, which undeniably already in the 1990s inspired and influenced the implementation of the new mathematics curriculum. Fullan (1982), Fullan and Stiegelbauer (1991), and Miles, Ekholm and Vandenberghe (1987) identified three broad phases in the change process: initiation, implementation and continuation. Most of these ideas even go back to the writings of Pierce and Delbecq (1977) on organisational change. The phases can be visualised as in Fig. 11.4, which is based on the work of Miles et al. (1987).

For each phase, the relevant factors from literature are highlighted and it is shown how these factors were addressed in substantial change in the mathematics curriculum in the Netherlands realised through W12-16 and SW12-16.

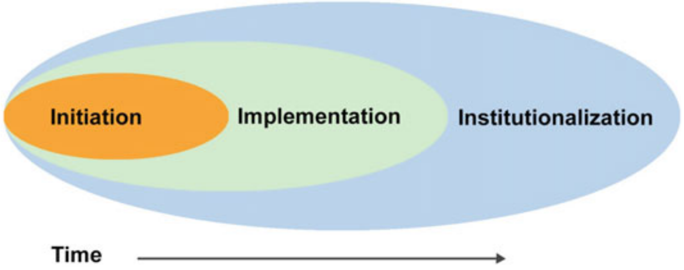

Fig. 11.4 The three overlapping phases of the change process (based on Miles et al., 1987)

11.3.2 Initiation Phase

According to the aforementioned frameworks of Miles and Fullan, the factors that affect the initiation phases include:

- Existence and quality of innovations
- Access to innovations
- Advocacy from central administration
- Teacher advocacy
- External change agents.

The quality of the innovation that resulted from W12-16 was to a great extent positively influenced by the thinking power of Hans Freudenthal. His thoughts about mathematics as an educational task (Freudenthal, 1973) influenced literally all the mathematics educators in the Netherlands and many mathematics educators abroad in the 1980s and 1990s. Access to the innovation for other schools was made possible through the publication of experimental lesson materials, through conducting a large number of information meetings, and through the two major Dutch journals on mathematics education which published monthly on aspects of the new curriculum. The Ministry of Education supported the reform and made funds available for pilot schools and development of experimental teaching materials. For the envisioned change a change in the formal, legislated, curriculum was also necessary. The Ministry of Education made that possible by changing the formal curriculum and the final examinations for pre-vocational secondary education (in the examination year 1996) with broad support from parliament. Moreover, the Ministry of Education commissioned and funded a committee to start with pilot schools. Through the work with pilot schools a group of mathematics teachers was created that acted as advocates for the reform. These so-called ‘advocate teachers’ also had an important role in the in-service teacher education activities. Most pre-service mathematics educators were also involved in the reform movement. Important external ‘agents of change’ were the in-service and pre-service teacher education institutions, the publishers, and the education inspectorate, who all supported the chosen vision.

The elements of this successful initiation were planned and documented in the W12-16 report *Operatie Acceptatie*[10] and involved a series of activities focusing on establishment of acceptance with all key stakeholders. At a conference in October 1989, which was attended by mathematics teachers and mathematics educators this idea arose. Members of the W12-16 team and the largest in-service teacher education institution APS were very much aware of the need to work carefully on creating support for the quite radical innovation. In Table 11.1 the most important activities of the initiation phase of W12-16 and SW12-16 are shown.

At the conference in October 1989, the first plans for an implementation strategy were formulated. In a series of follow-up consultations and meetings with a great number of stakeholders the contours for the implementation strategy were developed

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[10] Operation Acceptance.

Table 11.1 Summary of activities in the initiation phase of W12-16 and SW12-16

School year	Activities by W12-16	Activities by SW12-16
1987–1988	• Start W12-16 • Three development schools developed mathematics booklets	
1988–1989	• Development of mathematics booklets	
1989–1990	• Development of mathematics booklets • First draft of new mathematics curriculum • First pilot final examinations • Start of ‘Operation acceptance’	
1990–1991	• Preliminary version of new mathematics textbooks • Regional information and information meetings • Second draft curriculum • Second pilot final examination	• Start SW12-16 • Start at 10 new pilot schools in the first year of secondary education • Introduction plan • Regional information and publicity meetings
1991–1992	• Regional information and publicity days • New curriculum • New mathematics textbooks • Background book for teachers: *Mathematics 12–16* • Third pilot final examination • End W12-16	• Pilot schools in the second year of secondary education • Regional information and publicity days

further. This strategy was taken over by the Minister of Education as is expressed by his statement:

> I request that special attention be paid to mathematics. The activities of the Commissie Ontwikkeling Wiskundeonderwijs[11] (COW) will be completed in the first half of 1992. On 1 August of that year, recommendations for a new examination syllabus for lower vocational education (LBO) and junior general secondary education (MAVO) shall be available, amongst other things. I intend to transform these recommendations into a definitive syllabus as soon as possible. With this in mind, I ask you now to carry out all the preparations in 1991 and to prepare the way for introduction as far as possible in order to enable a rapid introduction of the new examination syllabus. In order to promote a good connection, with regard to content, between development and support, I would ask that you also set up and carry out the above in consultation with the commission. (quoted by Kok, Meeder, Pouw, & Staal, 1999, p. 22) (translated from Dutch by the author)

The transition of initiation to implementation was marked by the mandate that was given by the Ministry of Education to set up a new commission. It marked the birth of the implementation team SW12-16.

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[11] Committee Developing Mathematics Education.

11.3.3 Implementation Phase

Fullan and Stiegelbauer (1991) identified three major factors which affect implementation: characteristics of change, local characteristics, and external factors. As summed up in Table 11.2, they identified different stakeholders at local, federal and governmental levels. They also identified characterisations of change for each stakeholder and the issues that each stakeholder should consider before committing to a change effort or rejecting it.

Although school leaders as well as policy makers supported the implementation of W12-16, mathematics education was often seen as troublesome with respect to the attained performance levels of the students and their motivation. Moreover, the formal and selective approach of the formal mathematics curricula were seen as opposite to an education that aimed for more equity, better motivation, and providing useful content for all students and not only for mathematically gifted students. To inform all involved as much as possible, the envisioned change was laid out extensively in pilot materials, new examinations and in courses for professional development of teachers. On all levels stakeholders were informed and they were all at least benevolent to the change.

During the implementation process the systematic effort to involve all stakeholders remained one of the key elements. One of the lessons learned from earlier curricular reforms was that forgetting one or more stakeholders will lead to major resistance, not just from the forgotten stakeholders, but from others as well. Strengthening ownership on all levels by involving stakeholders was seen to be of major importance. There were many stakeholders within this implementation, including teachers, school leaders, ministry officials, testing agencies, teacher educators, parents, publishers, policy makers and the media. A continuous and extensive dialogue through a series of meetings was a crucial aspect of involving these stakeholders.

In addition, in the implementation phase a strong need was felt to create and show good practices by ‘regular’ teachers in ‘regular’ schools. After an intensive tour of schools the SW12-16 team and APS agreed to experimentally introduce the programme in ten pilot schools and follow the progress with great care.

Table 11.2 Characteristics of change, local factors, and external factors

Characteristics of change	Local factors	External factors
• Need for change • Clarity about goals and needs • Complexity: the extent of change required for those responsible for implementation • Quality and practicality of the programme	• The school district • Community Board • Principal • Teacher	• Government and other agencies

11.3.4 Continuation and Institutionalisation

Continuation of an innovation is strongly dependent on the institutionalisation of key tenets of the innovation. Continuation depends on whether or not:

- The change is embedded/built into the structure (through policy/budget/timetable)
- The change has generated a critical mass of school leaders and teachers who are skilled and committed to the change
- The change has established procedures for continuing assistance.

In the case of W12-16 the combination of incorporating the changes in final examinations, in the major mathematics textbook series, and the incorporation in the teacher education programmes, both for in-service teacher education and for pre-service teacher education, was key to the implementation and the sustainability of the change. The networks of textbook authors, mathematics educators, and test designers overlapped heavily and made a relatively uniform interpretation of the new curriculum come to blossom. According to the implementation literature mentioned before, these kinds of complex changes in education take at least 20–30 years to come to full crystallisation. After 25 years one can analyse whether the change has reached the stage of institutionalisation.

The next and final section discusses what results can be seen 25 years after the initiation of this mathematics education reform.

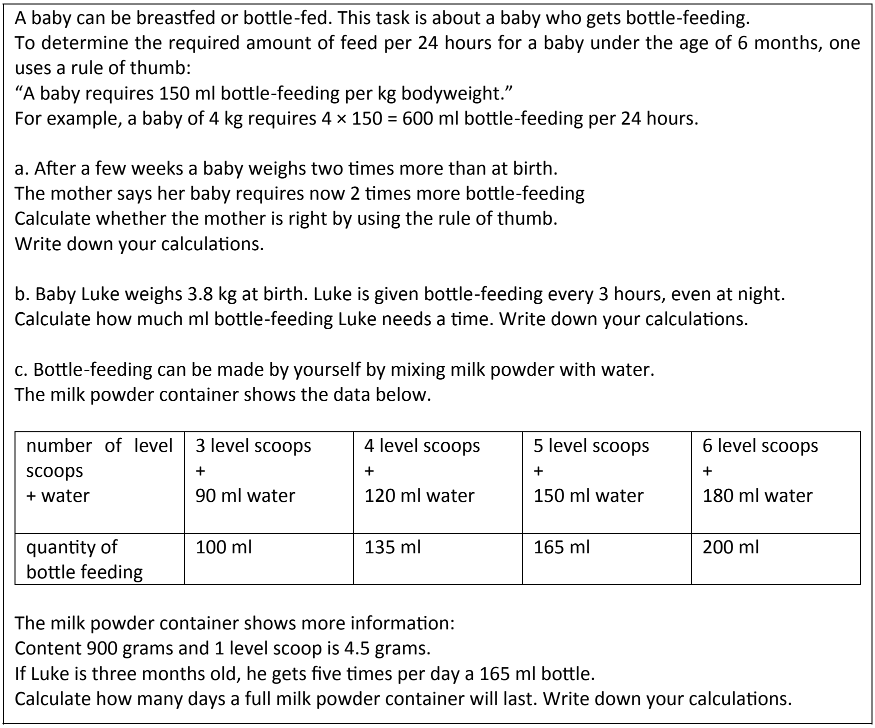
11.4 Reflection

11.4.1 How Sustainable Is the New Situation?

The changes in the mathematics curriculum since 1992 have been most sustainable and successful within the pre-vocational secondary education track of the Dutch education system. The programme has been running in this track for more than twenty years without any problems in classrooms or debates on the content. In the last twenty years, we can say that the students following this pre-vocational track did more mathematics with more usability, with better results, and with higher motivation than students in any other period in the history of mass education.

As a serious indication of sustainability, the mathematics textbook series and the final examinations still reflect the essential tenets of the original vision. Figure 11.5 shows a task from the 2015 final examination for the pre-vocational intermediate track (VMBO-KB).

As is shown in this examination task, most characteristics of the envisioned changes are still visible: based in reality, open-ended questions, and meaningful problems. At the same time, however, the change is still very vulnerable. In the last ten years, with the rise of social media, the persistent idea that children are performing poorly at mathematics and that this can be remedied with simple training and



A baby can be breastfed or bottle-fed. This task is about a baby who gets bottle-feeding.
To determine the required amount of feed per 24 hours for a baby under the age of 6 months, one uses a rule of thumb:
"A baby requires 150 ml bottle-feeding per kg bodyweight."
For example, a baby of 4 kg requires 4 × 150 = 600 ml bottle-feeding per 24 hours.

a. After a few weeks a baby weighs two times more than at birth.
The mother says her baby requires now 2 times more bottle-feeding
Calculate whether the mother is right by using the rule of thumb.
Write down your calculations.

b. Baby Luke weighs 3.8 kg at birth. Luke is given bottle-feeding every 3 hours, even at night.
Calculate how much ml bottle-feeding Luke needs a time. Write down your calculations.

c. Bottle-feeding can be made by yourself by mixing milk powder with water.
The milk powder container shows the data below.

number of level scoops + water	3 level scoops + 90 ml water	4 level scoops + 120 ml water	5 level scoops + 150 ml water	6 level scoops + 180 ml water
quantity of bottle feeding	100 ml	135 ml	165 ml	200 ml

The milk powder container shows more information:
Content 900 grams and 1 level scoop is 4.5 grams.
If Luke is three months old, he gets five times per day a 165 ml bottle.
Calculate how many days a full milk powder container will last. Write down your calculations.

Fig. 11.5 Exemplary task from final examination Wiskunde VMBO KB 2015 (pre-vocational secondary education, middle track, mathematics) (translated from Dutch by the author)

rote learning is still very much alive. See Van den Heuvel-Panhuizen (2010) for a report on this debate. In recent years, in the (social) media a framing can be witnessed that the educational change in the 1990s is to blame for the alleged low level of mathematics of today's students. This is in spite of results provided by international comparison studies like TIMSS, PISA and PIAAC, which consistently find the Netherlands performing quite high. But as is common in (social) media framings of education, assertions om low performance in mathematics are persistent and mainly fact free. This has led to a demand for rote learning, and a ban on the use of ICT-based mathematical tools. In the pre-vocational track of secondary education these demands did not have much effect yet because the curriculum proved itself to be useful for these students and their teachers support the practical and common-sense approach to mathematics.

In the higher tracks of secondary education, however, especially on tracks leading to university, there has been a recurring debate on the alleged low level of mathematics performances in the Netherlands since the late 1990s and the early 2000s, especially in regard to procedural skills. In the debate, we see many references to a 'backward utopia'. It is alleged that sometime in the past all students of all ability levels were able to execute all standardised procedural number and algebraic operations correctly and fluently. This backward utopia, which arguably did never exist, has influenced the current mathematics curricula of lower and upper secondary general education through a re-incorporation of rote learning of algebraic skills. But more importantly, the discussion paralysed to some extent the further development of the didactical ideas introduced in the early 1990s and made it harder for teachers and prospective teachers to develop the skills necessary to make a more inquiry-based and guided re-invention-based mathematics education come to full blossom.

11.4.2 The Way Forwards

We have witnessed that the type of discussion in the media as described above has taken place in many countries over recent decades and will most likely pop up with every new generation of teachers, educators, mathematicians, and policy makers. If, for example, the now popular criteria of Hattie (2009, 2015) had been used to evaluate the work of the SW12-16 team the positive findings described in this chapter would never have been noticed. Luckily, a group of mathematics educators who had a critical mass in the Dutch educational infrastructure believed that mathematics education should be for all and envisioned a mathematics education that ensured usability.

In 2016, in the Netherlands a campaign entitled Onsonderwijs2032[12] is being launched for a curriculum reconsideration in the coming years. Its title reasons that students who enter education today will enter the workforce in 2032. In this campaign three major functions of education are emphasised. The Dutch philosopher and educator Biesta (2010, 2013), amongst others, strongly advocates a renewed focus on these functions: qualification, socialisation and subjectification. Biesta pleads for a greater focus on subjectification as the opposite of socialisation and calls for the uniqueness of each individual human being to be acknowledged. In the advice report for Onsonderwijs2032 (Platform Onderwijs2032, 2016) the goals are presented in a less philosophical way. The report emphasises the relevance of (1) development of knowledge and skills, (2) equipment for future society, and (3) personal development.

In this perspective, RME is more than ever a relevant approach to mathematics education and matches more general developments in education. There is a tendency worldwide to involve all students in mathematics, and RME can offer relevant content as well as an approach that contributes to the aspiration for all students to be involved in mathematics. RME is in that regard still one of the most widely known instruction theories for mathematics education. The broadened scope and the focus on mathematics as a human activity are also relevant in today's world because of the new focus in education on personalised learning and on students creating their own personal learning trajectories.

In the Netherlands, most mathematics educators ignore the current superficial framing in the media and continue designing and developing meaningful mathematics

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[12] Our education 2032.

education alongside international partners that share a vision for ensuring inspiring, meaningful and sensible mathematics education for all students. Good practices in RME can function as examples for the new tendencies in (mathematics) education in the decades to come.

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Chapter 12
A Socio-Constructivist Elaboration of Realistic Mathematics Education

Koeno Gravemeijer

Abstract This chapter describes a socio-constructivist elaboration of Realistic Mathematics Education (RME) that emerged from my collaboration with Paul Cobb and Erna Yackel. It is argued that RME and socio-constructivism are compatible and complement each other. Socio-constructivism points to the critical role of the classroom culture, while RME offers a theory on supporting students in (re-)constructing mathematics. Furthermore, the role of symbols and models is discussed, which was considered problematic in constructivist circles, while being central in RME. The emergent modelling design heuristic is presented as a solution to this puzzle. Together, guided reinvention, didactical phenomenology, and emergent modelling, are combined to delineate RME as an instructional design theory. This is complemented by a discussion of pedagogical content tools as counter parts of the emergent modelling and guided reinvention design heuristics at the level of classroom instruction. Finally, research on student learning and enactment of RME in Dutch classrooms is discussed.

12.1 Introduction

When Freudenthal (1971) coined his adage of mathematics as a human activity, concrete elaborations of what that would mean in practice still had to be worked out. This became one of the main tasks of the IOWO,[1] the predecessor of the current Freudenthal Institute. In the 1980s, Treffers took stock of what had been developed up to then and construed the Realistic Mathematics Education (RME) theory by generalising over the characteristics the prototypical instructional sequences and local instruction theories that were available had in common. This resulted in the publication of a

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framework for a domain-specific instruction theory for RME (Treffers, 1987). The RME approach did fit very well in the broader trend in the international mathematics education community which emerged around that time. This concerned the general recognition of the importance of students' active constructive role in appropriating new mathematics, and an emphasis on applicability. RME did not only share similar starting points, it also offered a fitting theory on how to support students in constructing mathematics; moreover, this was accompanied by various concrete elaborations. In this respect RME arrived at the right time. This undoubtedly contributed to the international interest in this approach—which in turn led to collaborative projects in various countries. This was, however, not a one-way stream; international collaborations influenced RME as well. Collaborative projects in the United States, for instance, brought researchers from the Freudenthal Institute in contact with (socio-)constructivism, which influenced their thinking about RME to a greater or smaller extent. Especially my ten-year collaboration with Paul Cobb, Erna Yackel and colleagues was marked by a mutual influence (see also Cobb, with Gravemeijer & Yackel, 2011). This resulted in a new elaboration of RME, which we will denote as, 'a constructivist elaboration of RME'.

This constructivist version, which emerged next to the original RME theory, will be the theme of this chapter. We will start by looking into the compatibility of the underlying conceptual positions of RME and socio-constructivism. This will be followed by a discussion of the implications of the socio-constructivist collective perspective as elaborated by Cobb and Yackel (1996) for RME. Next, we will discuss the apparent contrariness of the views on the role of symbols and models, and how those positions were reconciled in the emergent modelling design heuristic. Subsequently we will discuss how the socio-constructivist perspective can illuminate the complexity of enacting RME in everyday classrooms. We will complement this with recent investigations of the state of affairs in Dutch classrooms.

12.2 Conceptual Compatibility of (Socio-)Constructivism and Realistic Mathematics Education

In the early stages, some protagonists of RME quickly acknowledged the compatibility of constructivism and RME. Some, however, were more reluctant. Freudenthal (1991) was actually very negative about constructivism. He rejected Von Glasersfeld's critique on the status of objective scientific knowledge by pointing out the achievements of science. He argued that we should not focus on the philosophy of science, but on the philosophy of education, and stated:

> I cannot see any bond between mathematics instruction on the one hand and an alleged or assumed lack of faith in objective mathematical knowledge on the other hand, whether it is called constructivism or anything else (Freudenthal, 1991, pp. 146–147).

At the time that Freudenthal expressed this critique, there were various ideas around on what the radical constructivist position would mean for education. Including the position that teachers should not interfere because of the risk of endangering the students' own constructive activity. Gradually, however, a more pragmatic standpoint won out. In this process, Paul Cobb was very influential. While his theoretical perspective evolved towards a more pragmatic stance; from radical constructivism towards socio-constructivism. He saw great value in what Putnam (1987, cited by Cobb, 2001) denoted 'pragmatic realism' (Cobb, 2001). And he contended that, whereas radical constructivism claims that it is impossible to bridge the gap between one's own knowledge and some pre-given external reality, pragmatic realism questions this focus on the dichotomy between this external reality and our personal knowledge. Instead of focusing on an unknowable pre-given external reality, we should focus on the realities which people experience. This pragmatic realism is clearly compatible with Freudenthal's conception of reality, which he does not link to some pre-existing external reality, but to one's self-constructed experiential reality: "I prefer to apply the term 'reality' to what at a certain stage common sense experiences as real" (Freudenthal, 1991, p. 17).

Cobb (1994a) himself made a nice connection between the two views when pointing out that (socio-)constructivism is not a pedagogy. He argued that if it is true that people always construct their own knowledge, then students will do so in every classroom—even with direct instruction. The issue, he went on to say, is not *whether* they construct, but *how* and *what* they construct. The question therefore is: What do we want mathematics to be for the students? Cobb (1994a) concluded that a potential answer to that question was in Freudenthal's notion of mathematics as a human activity.

12.3 A Socio-Constructivist Perspective on Teaching and Learning

A shared belief in the compatibility of (socio-)constructivism and RME formed the basis for a ten-year collaboration between Cobb cum suis and me, in which we further elaborated RME theory while working on a series of classroom design experiments. The starting point was that RME and socio-constructivism are not only compatible, but also complemented each other. On the one hand, socio-constructivism offers a background theory for RME, and, more importantly, adds a collective perspective. On the other hand, RME offers an instructional design theory that aims to support students in constructing mathematics.

Adopting a socio-constructivist view implies a collectivist perspective on teaching and learning, which situates the students' activity within the classroom community, whereas RME originally tended to a more individual, psychological, perspective, even though the roles of interaction and collaboration between students were of course acknowledged. Socio-constructivism offers an important addition in that it focuses attention on the crucial role of the classroom culture in the enactment of RME in the classroom. To analyse the situated activity of students, Cobb and Yackel

(1996) developed an interpretative framework, denoted ‘emergent perspective’, in which they try to coordinate a social and a psychological perspective. The former involves the norms and practices of the classroom community. The latter focusses on individual students’ reasoning, and concerns beliefs of students and teacher. Cobb and Yackel (1996) discern classroom social norms, socio-mathematical norms, and practices. The social norms describe the expected ways of acting and explaining in a given classroom. They elucidate that the social norms are reflexively related to the students’ and the teacher’s beliefs about their obligations, which are shaped by the classroom’s history. Typical social norms of the traditional mathematics classroom are that students are expected to try to come to grips with the knowledge and procedures presented by the teacher and the textbook. The teacher’s role is to explain and clarify, and the students’ role is to try to figure out what the teacher has in mind and act accordingly. RME asks for different social norms, which in line with Cobb and Yackel (1996) encompass the obligations for students to come up with their own solutions, explain and justify their solutions, to try to understand the explanations and solutions of their peers, to ask for clarification when needed, and eventually to challenge the ways of thinking with which they do not agree. The teacher’s role is not to explain, but to pose tasks, and ask questions that may foster the students’ thinking, and help them in this manner to build on their current understanding and to construe more advanced mathematical insights.

This recognition of the need for fitting social norms has significant consequences for putting RME into practice. It signals the need for changing the social norms, which in turn asks for changing the individual beliefs of the students. It also highlights that students’ beliefs about their role and that of the teacher are formed by experience. In traditional classrooms students are used to being rewarded for reproducing the teacher’s reasoning and procedures, and the belief that this is what is expected from them will not change unless they gain compelling new experiences. This takes some conscious effort (Cobb & Yackel, 1996). To establish new social norms, the teacher has to show that what is valued and what is rewarded has changed.

In addition to the general classroom social norms, Cobb and Yackel (1996) discern socio-mathematical norms and mathematical practices. The socio-mathematical norms refer to what mathematics is and what it means to do mathematics in a given classroom. For example, what counts as a mathematical problem, what counts as a mathematical solution, and what counts as a mathematical argument. We may link those socio-mathematical norms with the notion of ‘mathematical interest’ (Gravemeijer & Cobb, 2013). To engage in the activity of mathematising vertically, students will have to develop an interest in mathematical aspects of their solutions. Teachers may cultivate mathematical interest by asking questions such as: What is the general principle here? Why does this work? Does it always work? Can we describe it in a more precise manner?

With mathematical practices Cobb and Yackel (1996) refer to taken-as-shared ways of acting and reasoning, which may evolve over time. The mathematical practices are reflexively related to individual students’ mathematical conceptions. They speak of an established mathematical practice when certain ways of acting and reasoning are no longer challenged by individual students. This does not necessarily mean that all student's conceptions and actions correspond with that practice. Mathematical practices do, however, offer a means of identifying and describing the progress of the classroom as a whole.

The emergent perspective offers an important addition to RME in that it reveals that a certain classroom culture has to be put in place in order to allow for guided reinvention; an aspect that had not yet been articulated in Treffers' (1987) theory. Moreover, it highlights the reflexive relation between the individual's interpretations and constructions and the norms and practices of the classroom community.

Mark that we may look at the relation between Cobb and Yackel's perspective and RME theory in two ways: We may consider the emergent perspective an integral part of a socio-constructivist take on RME, or conceive of the emergent perspective as describing a necessary requirement—as enacting RME is not possible if the students adhere to traditional school-mathematics social norms. Whichever one chooses, socio-constructivism offers a significant expansion of or addition to RME theory. We may note, however, that conversely RME offers a significant addition to socio-constructivism by offering an instruction theory for supporting students in constructing mathematical knowledge. Further, building on both an RME and a socio-constructivist perspective proved especially fruitful in the domain of symbols and tools, a development we will discuss in the following.

12.4 Symbolising and Modelling

Initially there was a strong wariness among socio-constructivist scholars regarding the use of external representations. This was supported by research in contemporary mathematics classrooms, which had shown that students often could not make sense of the symbolic representations introduced by the teacher (see, e.g., Cobb, 1994b). Broadly speaking, the use of tactile models and visual representations was associated with the transmission model of teaching, in which tacit and visual models were treated as powerful means of supporting learning for understanding. By acting with well-designed concrete models, students were expected to discover the mathematics that was embedded in the models. In relation to the latter, Cobb, Yackel, and Wood (1992) speak of a representational view. They argue that mathematics educators, who use tactile models and visual representations in this manner, implicitly or explicitly hold the view that learning is characterised as a process in which students construct mental representations that mirror the mathematical features of external representations. The problem with this approach, however, is that the meaning of external representations is dependent on the knowledge and understanding of the interpreter. This creates the problem, known as 'the learning paradox' (Bereiter, 1985), that can be captured by the following question: How is it possible to learn the symbolisations, which you need to come to grips with new mathematics, if you have to have mastered this new mathematics to be able to understand those very symbolisations?

The underlying problem, Cobb et al. (1992) argue, is that mathematics educators experience mathematics as an objective body of knowledge, which is mirrored by the external representations they use to make the corresponding mathematics accessible for students. This presupposes an objective body of knowledge that exists independently of some agent. According to socio-constructivist theory, however, knowledge has to be constructed by an actor, and cannot be separated from the knowing individual. Thus, for those who have not yet constructed the more sophisticated mathematical knowledge that has to be learned, this body of more sophisticated mathematical knowledge, literally, does not exist, and thus cannot be conveyed by external representations.

12.4.1 Emergent Modelling

However, while constructivist scholars were wary of symbols and models and pointed to the learning paradox, RME relied heavily on the use of models, model situations, and schemata, as is indicated for instance in Treffers' (1987) characterisation of progressive mathematisation. Consequently, the need arose to reconcile the two conceptions of the role of symbols and models. A beginning of an answer could be found in Treffers' (1987) elucidation that in the RME approach, models etcetera, rather than being offered right away, arise from problem-solving activities. In this manner, Treffers' model characteristics (1987) pointed to a dynamic aspect that could be explicated and elaborated as an explicit design heuristic offering a way to circumvent the learning paradox. We should also refer to Streefland (1985) who argued that by modelling reality you create a model of that reality—which he calls an 'after-image' ('nabeeld' in Dutch). This after image may foster reflection, which in turn may lead to the insight that the model can be used for other problem situations. The model has become a 'pre-image ('voorbeeld' in Dutch) that is used for reasoning about other situations (which he in later publications expands with supporting abstracting and level raising (Streefland, 1992, 1993).

The constructivist concerns about the role of models and the associated learning paradox are eventually addressed by the design heuristic that originated from noticing a shift in the thinking of students using the empty number line (Gravemeijer, 1991). It showed that the students initially used calculations that closely matched the situation in the contextual problem, but later on started to come up with solutions that were based on number relations and were only indirectly connected with the context. This implied that the number line had acquired a new meaning for the students; it started to signify number relations. This insight led to the rationale underlying the emergent modelling heuristic, that the learning paradox dissolves when one adopts a more dynamic view of learning in which mathematical symbols and models are developed in a bottom-up manner. The latter appeared to agree with Meira's (1995) observation that in the history of mathematics, symbols did not suddenly appear in their full-fledged form. Instead, these symbols grew out of informal, situated, forms of symbolising that developed over time in a reflexive process in which symbolisations and meaning co-evolved. Following Meira (1995), we may envision a dynamic process in which symbolisations and meaning co-evolve, and in which the ways that symbols are used and the meanings they come to have, are seen to be mutually constitutive. It showed that a similar pattern could be found in many prototypical RME instructional sequences, such as Van den Brink's (1989) design for addition and subtraction, Streefland's (1990) work on fractions, and the various sequences for the written algorithms (Treffers, 1987). The idea of a dynamic process in which symbolisations and meaning co-evolve has been elaborated in the emergent modelling design heuristic. Here the label 'emergent' refers both to the character of the process by which models emerge, and to the process by which these models support the emergence of more formal mathematical conceptions.

According to the emergent-modelling design heuristic, the model first comes to the fore as a *model of* the students' situated informal strategies. In subsequent activities, the role of the model begins to change. As the students gather more experience with similar problems, their attention may be directed to the mathematical relations involved. In this manner, the students start to develop a network of mathematical relations. This changes what the model signifies for the students. Instead of deriving its meaning from activity in the context in which the problem is situated, the model starts to derive meaning from the mathematical relations involved. Consequently, the model becomes more a base for more formal mathematical reasoning than a way of representing a contextual problem. In other words: the *model of* informal mathematical activity develops into a *model for* more formal mathematical reasoning. We should add that, although we speak of 'the model', the model we are referring to is more an overarching conception, than one specific model. In practice, 'the model' in the emergent-modelling heuristic is actually shaped as a series of consecutive *sub-models* that can be described as a cascade of inscriptions (Latour, 1990) or a chain of signification (Roth & McGinn, 1998). Key here is that acting with each new inscription signifies the earlier activity with the preceding inscriptions for the students. Mark, however, that the series of symbolisations is invented by the instructional designer, not by the students. To adjust for this, one may try to ensure that each new tool/symbolisation emerges as a solution to a problem that has its roots in activity with the earlier symbolisation. In this manner, the history of working with the earlier symbolisation may provide the imagery underlying the new tool. Whether this is the case, may be inferred from whether or not the new symbolisation is used flexibly by the students.

From a more global perspective, the symbolisations can be seen as various manifestations of some overarching model that evolves from a 'model of' situated activity to a 'model for' more formal mathematical reasoning. In relation to this, we may discern four different types or levels of activity (Gravemeijer, 1999):

(1) *Situated activity* in a task setting that is experientially real for the students
(2) *Referential activity*, in which models refer to activity in the task setting
(3) *General activity*, in which models refer to a framework of mathematical relations

(4) *Formal mathematical reasoning* which is no longer dependent on the support of models-for mathematical activity.

These four levels of activity illustrate that the students' understanding of models is grounded in their understandings of paradigmatic, experientially real settings. At the level of referential activity, the models are meaningful to the students because they refer to situated activity in the task setting. General activity begins to emerge when the students start to reason about the mathematical relations that are involved. In this manner the students develop a network of mathematical relations. Consequently, the model starts to lose its dependency on situation-specific imagery, and gradually develops into a model that derives its meaning from the emerging framework of mathematical relations. In this manner the model starts to function as a model for more formal mathematical reasoning.

The transition from model-of to model-for coincides with a progression from informal to more formal mathematical reasoning that is interwoven with the creation of some new mathematical reality—consisting of mathematical objects (Sfard, 1991) within a framework of mathematical relations. Thus, the model-of/model-for transition is not tied to specific manifestations of the model, instead, it relates to the student's thinking, within which 'model-of' refers to an activity in a specific setting or context, and 'model for' to a framework of mathematical relations. Mark that the constitution of a framework of mathematical relations—and thus some new mathematical reality—is an essential element of the emergent modelling design heuristic. In this respect, it differs from a modelling conception in which a model of a contextual problem is generalised in order to function as a model for solving similar problems in other contexts. We may add that model-of/model-for transition in the emergent modelling design heuristic has to be understood in a metaphorical sense. Central is the series of symbolisations or sub-models, which together constitute 'the model', which may or may not be placed under one label—such as the notion of a ruler as the overarching model in the measurement annex number-line sequence (Stephan, Bowers, & Cobb, with Gravemeijer, 2003).

Let us briefly return to the aim of supporting students in developing a framework of mathematical relations and the corresponding mathematical objects, which is experienced as some new mathematical reality. This experienced reality corresponds with the perceived body of mathematical knowledge that we identified as the central problem when discussing the learning paradox. Thus, instead of trying to help students to make connections with a mathematical reality that does not exists for them; the emergent modelling approach helps students in constructing such a mathematical reality by themselves. This focus on the constitution of mathematical objects and a framework of mathematical relations also signifies a deviation from Treffers' conception of RME theory, since he tends to characterise students' mathematical development in terms of students' development of increasingly sophisticated solution methods (see, e.g., Treffers, 1991).

12.5 RME in Terms of Instructional Design Heuristics

The conception of emergent modelling as an instructional design heuristic allowed for an alternative description of RME theory in terms of instructional design heuristics by combining it with guided reinvention and didactical phenomenology (Gravemeijer, 2004).

12.5.1 Emergent Modelling Heuristic

We already discussed the emergent-modelling design heuristic above, but we may add that this heuristic has been used in design research projects on a variety of topics, such as addition and subtraction up to 20 (Gravemeijer, Cobb, Bowers, & Whitenack, 2000), addition and subtraction up to 100 (Stephan et al., 2003), written algorithms for addition and subtraction (Bowers, 1995), integers (Stephan, & Akyuz, 2012), data analysis (Gravemeijer & Cobb, 2013), algebraic functions (Doorman, Drijvers, Gravemeijer, Boon, & Reed, 2012), calculus (Doorman, 2005), and differential equations (Rasmussen, 1999).

12.5.2 Guided Reinvention Heuristic

When elucidating the principle of guided reinvention, Freudenthal (1973) suggested the instructional designer should look at the history of mathematics to see how certain mathematical practices developed over time. The designer is advised to especially look for potential conceptual barriers, dead ends, and breakthroughs. These may be taken into account when designing a potential reinvention route. Streefland (1990) developed a second guideline, which suggests that the informal interpretations and solutions of students who do not know the applicable mathematics might ‘anticipate’ more formal mathematical practices. If so, students’ initial informal reasoning can be used as a starting point for the reinvention process. In summary, the designer may take both the history of mathematics and students’ informal interpretations as sources of inspiration for delineating a tentative, potential route along which reinvention might evolve.

As a special point of attention, we may note that reinvention has both an individual and a collective aspect; it is especially the interaction between students that is to function as a catalyst. The designer has to develop instructional activities that are bound to give rise to a variety of student responses. What is aimed for is a variety in responses that to some extent mirrors the reinvention route. When some students come up with more advanced forms of reasoning than others, teachers can exploit these differences. They can try to frame the mathematical issue that underlies those differences as a topic for discussion (Cobb, 1997). In orchestrating such a discussion, they can then advance the reinvention process. Mark that without such differences, the teacher will not have a basis for organising a productive classroom discussion, and will have to refer to soliciting preferred responses by asking leading questions. We may further observe that in line with the emergent modelling heuristic, the end points of a guided reinvention process are typically cast in terms of mathematical objects and frameworks of mathematical relations in the context of a constructivist elaboration of RME.

12.5.3 Didactical Phenomenology Heuristic

The third RME design heuristic concerns the didactical phenomenological analysis, or didactical phenomenology for short (Freudenthal, 1983). Here the word 'phenomenological' refers to a phenomenology of mathematics. In this phenomenology, the focus is on how mathematical 'thought-things' (which may be concepts, procedures, or tools) organise—as Freudenthal (1983) puts it—certain phenomena. Knowing how certain phenomena are organised by the thought thing under consideration, one can envision how a task setting in which students are to mathematise those phenomena may create the need to develop the intended thought thing. In this manner, problem situations may be identified, which may be used as starting points for a reinvention process. Note that such starting-point-situations may also be used to explore students' informal strategies as Streefland (1990) suggests. To find the phenomena that may constitute starting-point-situations, we may look at applications of the concept, procedure or tool under consideration. Assuming that mathematics has emerged as a result of solving practical problems, we may presume that the present-day applications encompass the phenomena which originally had to be organised. Consequently, the designer is advised to analyse present-day applications in order to find starting points for a reinvention route. Mark, however, that, as the students progress further in mathematics, applications may concern mathematics itself. Essential for valuable starting points is that they are experientially real for the students, that they concern situations in which the students know how to act and reason sensibly. An additional function of a phenomenological analysis is that it allows for construing a broad phenomenological base, which may both strengthen and enrich the experiential real foundation and foster the applicability of the concepts, procedures, or tools under consideration.

12.6 Pedagogical Content Tools

We may complete this exposition on instructional design heuristics with a discussion of the 'pedagogical content tools' (PCTs) that have been put forwards by Rasmussen and Marongelle (2006) as instructional counter parts of the design heuristics of emergent modelling and guided reinvention. They define a pedagogical content tool as, "a device, such as a graph, diagram, equation, or verbal statement that a teacher intentionally uses to connect student thinking while moving the mathematical agenda forward" (Rasmussen & Marongelle, 2006, p. 388). They describe two PCTs, 'transformational records' and 'generative alternatives', which in their view address the problem of how teachers can proactively support their students' learning. Sometimes the design heuristics are too general in their view. Transformational records, which are seen as the instructional counter part of the emergent modelling heuristic, are defined as graphical representations that emerge as ways to record student thinking, which are later used by students to solve new problems. As an example, they discuss an episode of a classroom on differential equations, which starts with the task of making predictions about the shape of a population versus time (P versus t) graph for a single species that reproduces continuously and has unlimited resources. The teacher started the discussion by asking whether the initial slope at $P = 10$ and $t = 0$ should be zero or positive. During this discussion—in which the students adhered to the classroom social norms that they were expected to explain and justify their solutions, and try to understand their peers—most of the students began to realise that the slope had to be positive. Thereupon the teacher drew a tangent line vector with a positive slope as "a notational record of the taken-as-shared reasoning of the classroom community" (Rasmussen & Marongelle, 2006, p. 396). In a similar manner, the notational record was supplemented with some more vectors. This extended notational record was used as a means of support by the classroom community, when sorting out whether the rate of change function would depend only on the size of the population, or also on the time. The teacher, in short, took a proactive role in reshaping the initial record, while supporting the students in developing a line of reasoning that corresponded with what an expert in the subject would recognise as an emerging tangent-vector field. He did so in such a manner that he at the same time cultivated the social norms of an inquiry classroom by initiating, and building on, whole class discussions.

The generative alternatives are linked to the notion that guided reinvention tries to find a position between too much and too little guidance. Here one of the examples concerns a problem about salt water—containing 1 lb salt per gallon—that is pumped into a tank at a rate of 2 gallons a minute. The students came up with two different ideas about the rate of change, which should be 2, according to some, or $2t$, according to others. By framing the justification of one of both as topic for a whole-class discussion, the teacher fostered the social norms, as the students were expected to explain and justify their reasoning and try to make sense of others' reasoning. When the students started to lean towards $2t$, the teacher realised that the students were not making a conceptual distinction between rate of change in the amount of salt and the amount of salt. He then assumed more responsibility for the content and the direction of the discussion by pointing out that after t minutes $2t$ pounds of salt are flowing into the tank, and asking: Is that the rate of change? In the then unfolding discussion the students start to realise that the *amount* of salt after 2 min is $2t$ (pounds), whereas the *rate of change* is 2 (pounds per minute). The authors point out that what makes this an example of a generative alternative is not just that two alternatives, t and $2t$, were discussed. Key here is that this discussion advanced the mathematical idea of the explicit distinction between the rate of change in a quantity and the quantity itself.

With the pedagogical content tools, we have moved from RME theory to RME in the classroom. We will discuss the latter more extensively in the following.

12.7 RME and Classroom Practice

As is already noted above, the socio-constructivist perspective reveals the complexity of enacting RME in everyday classrooms. And we may conclude that this is more difficult than the initiators in the Netherlands were aware of. One of the hurdles concerns classroom culture. There was, and to a large extent still is, a lack of awareness of the need to invest in changing the classroom social norms. Another difficulty concerns the need to anticipate and build on the students' thinking. Following Simon's (1995) line of reasoning, teachers have to ascertain the students' level of reasoning and design or choose instructional activities that support students in expanding, and building on, their current ways of thinking. They have to develop hypothetical learning trajectories (HLTs), which involve anticipating the mental activities of the students when they engage in the envisioned tasks, and considering how these relate to the learning goals. This requires teachers to have a sound understanding of the rationale that underlies the instructional sequences they are working with. Usually, however, teachers are insufficiently informed about the local instructional theories that underpin the instructional sequences. Moreover, they are not schooled in thinking about the mental activities of students.

We may add that the students have to play their part as well. We already mentioned the classroom norms, but knowing that they are expected to think for themselves, explicate their thinking, etc., does not necessarily mean that they are willing to do so. An inhibiting factor may be the ego-orientation (Nicholls, Cobb, Wood, Yackel, & Patashnick, 1990) of some students. This includes being more concerned about how one looks in the eyes of one's peers, than about solving the task at hand. Fear of failure may keep those students from starting to work at a challenging task. Teachers, therefore have to invest in fostering a task-orientation (ibid.), the willingness to work on mathematical tasks. Important in this respect is that teachers refrain from judging students by external standards, or comparing them with their classmates, and instead promote that students take their personal progress as an evaluation criterion. This is of course hard to achieve with the current emphasis on testing.

12.8 Recent Research on Instructional Practice in the Netherlands

Following on the discussion of theory on enacting RME we may ask ourselves how RME actually works out in Dutch classrooms. This question is actually in line with a discussion that is going on in the Netherlands about the quality of mathematics education. This discussion was evoked by the results of the national assessments, known as PPON.[2] The PPON survey of 2004 showed a significant decline in the mastery of (procedures for handling) whole number addition, subtraction, multiplication, and division. However, the results did not decline across the board; the results on various other topics showed improvement instead. A comparison of consecutive PPON surveys (Janssen, Van der Schoot, & Hemker, 2005) showed a positive effect on a number of topics that RME innovators deem important (Van den Heuvel-Panhuizen, 2010). We may further argue that national assessments, and also international assessments—on which the Netherlands were, and are, still doing very well—are too crude instruments to come to grips with what is going on in mathematics education. This kind of considerations gave rise to three independent Ph.D. studies, which investigated the proficiency of Dutch students on specific topics, respectively addition and subtraction up to 100 (Kraemer, 2011), fractions (Bruin-Muurling, 2010), and algebra (Van Stiphout, 2011). Analysing the results of those three Ph.D. studies, Gravemeijer, Bruin-Muurling, Kraemer and Van Stiphout (2016) found that Dutch students' proficiency fell short of what might be expected of reform in mathematics education that targets conceptual understanding. In each of those three cases this appeared to be caused by a deviation from the original intentions of the reform. Firstly, the textbooks capitalised on procedures that can quickly generate correct answers, instead of investing in the underlying mathematics while accepting that fluency may come later. In relation to this, the authors speak of "task propensity", "the tendency to think of instruction in terms of individual tasks that have to be mastered by students" (ibid., p. 26). Secondly, there was an overall lack of attention for more advanced conceptual mathematical understandings in Dutch textbooks. Instructional sequences in the textbooks end too early, before the more advanced conceptual goals are reached. What is missing from the instructional sequences is the phase that Sfard (1991) denotes as reification. The students are not supported in constructing mathematical objects. The other reason they bring to the fore is that more advanced conceptual mathematical understandings are not formulated as instructional goals, not in the textbooks, nor in official curriculum documents. They plead for changing the usual goal descriptions in curriculum documents by identifying more advanced conceptual mathematical understandings as key curriculum goals.

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[2] Periodieke Peiling van het Onderwijsniveau (Periodic Assessment of the Education Level).

12.9 Conclusion

We started our discussion of the socio-constructivist elaboration of RME with the question of the compatibility of RME and socio-constructivism, and we concluded that on a meta level both positions are well compatible. We showed that adopting the collectivist perspective that is inherent to socio-constructivism especially has consequences for how we think about enacting RME in the classroom. Establishing social norms that encompass student responsibility for coming up with their own solutions and discussing these and those of other students, is a prerequisite for enacting RME. We further found that a potential irreconcilable difference concerned the different views on the role of symbolising and modelling, which created the need to reconcile the two positions. In relation to this we discussed the emergent modelling design heuristic, which is designed to circumvent the so-called learning paradox. The emergent modelling heuristic tackles the concern of socio-constructivists that symbols do not come with an inherent meaning, by ensuring that symbolisations and meaning co-evolve in a reflexive process, while at the same time supporting the construction of some new mathematical reality, which may be thought of as consisting of mathematical objects that derive their meaning from a network of mathematical relations. The heuristic may be characterised as a transition from a model of the students' situated informal strategies to a model for more formal mathematical reasoning. But the pith of the matter concerns (a) the sequence of sub-models that together form a chain of signification, in which activity with each new sub-model signifies activity with the earlier sub-model, and (b) the construction of a framework of mathematical relations by the students. We observed that by acknowledging that guided reinvention and didactical phenomenology also can be seen as instructional design heuristics, allows an alternative manner of describing RME theory—in which RME theory is described in terms of instructional design heuristics.

When turning to the classroom practice we of course reiterated the importance of the classroom culture. We also highlighted that the constructivist elaboration of RME entails a shift in attention from the instructional sequence with a rationale or local instruction theory that underpins it, to the local instruction theory with a series of instructional activities that can be used a resource. For the constructivist position that students construct their own knowledge implies that teachers have to adjust their teaching to the students' thinking. This means that teachers have less use for ready-made instructional sequences, but instead need at their disposal knowledge about the intended learning process and the possible means of supporting that learning process, or about local instruction theories. On the basis of this, teachers may develop hypothetical learning theories (Simon, 1995), which put the mental activities of the students at the centre of the teachers thinking. Given the results we reported in the last section, we may argue that the local instruction theories the teachers are to be provided with have to target more advanced conceptual mathematical understandings. The latter should also be worked as goals in national curriculum documents.

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Chapter 13
Eighteenth Century Land Surveying as a Context for Learning Similar Triangles and Measurement

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Abstract In our study, we investigated the value and applicability of the history of mathematics as a didactical tool for teaching mathematics. Recent literature has disclosed conceptual, cultural, and motivational arguments for including historical mathematical texts and methods in the mathematics curriculum. We explored how these theoretical assumptions worked out when designing historically-based instructional material and when using this material for teaching. The focus was on teaching measurement skills and the application of similar triangles to eight- and ninth-grade students. The profession of the Dutch land surveyor in the 18th century served as a historical context. Analyses of the data indicated that several aspects of this historical context were helpful for teaching these subjects. The practical activities along the 18th century lines appeared to have a positive effect on the students' motivation and on their conceptual understanding. The ninth-graders reacted more positively to the historically inspired text than the eighth-graders. The integration of historical elements, especially the need to read the old language, was generally not applauded, nor did we observe a positive cognitive effect. Even so, the practical activities inspired by the context appeared significantly effective and were judged positively by the students.

13.1 Introduction

Interest in the application of the history of mathematics as a didactical tool can already be seen in the early 20th century, for example in the work of Otto Toeplitz

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(1927). The new element in the current chapter is the research aspect. Not only do we present a geometry text at secondary school level that is based on historical problems and methods, we also survey how this text was used and evaluated by a large number of students. The survey was part of a project called 'Reinvention of geometry', and followed on the project Reinvention of Early Algebra (Van Maanen, 2002). Both names reveal the connection with the educational philosophy of Freudenthal (1973). In his view, the mathematics teacher should guide his or her students to re-invent mathematics (Freudenthal wrote re-invention with a hyphen). This approach to mathematics education also criticised the antididactic inversion, the phenomenon that in general mathematics is taught in reversed order: not the problem that led to a certain theorem is presented first, but in teaching one starts with proving the theorem and only then comes the problem, as an example or an exercise for the students.

Although Freudenthal was very interested in the history of mathematics, a discipline in which he had great expertise, he doubted whether the reinvention process should follow the historical line. We will argue in favour of the historical line for the reinvention of geometry. The student text will also make it clear that historical reinvention fits seamlessly in realistic mathematics education. We use the profession of the Dutch surveyor in the 18th century as a historical context for teaching measurement skills and the application of triangles. From a cultural perspective, this context is relevant for Dutch students. The profession of surveying was already developed in the Netherlands in the 17th century. The growing need for surveyors arose from the expansion of the nation, from designing maps, from the construction of buildings and fortifications. Since then, surveying served as an educational context and the first surveying books in Dutch were published.

13.2 Surveying and the Teaching and Learning of Measurement by Using Similar Triangles

Measurement is generally done by comparing the object that has to be measured against an instrument. One determines the size of a book by keeping it next to a graded ruler and by reading off the lengths of its three dimensions. The focus of this study is on teaching the mathematics needed when it is impossible or unpractical to compare the object to be measured. For example, think of the situation depicted in Fig. 13.1, where one wants to measure the width of a river. Nowadays it is common in such situations to use electronic tools, either based on optics or on satellite signals (GPS). More attractive for discussion in mathematical education, since more transparent and well accessible at the theoretical level, is the classical method of applying similar triangles.

Until GPS fundamentally changed surveying, the mathematical tool used in these cases was the combination of triangulation and trigonometry with the theory of similar triangles. Much simplified, the surveyor divides the surface to be measured in a net of triangles. One side of one triangle is then measured accurately in the

To determine the width of a river, the 18th century Dutch surveyor Morgenster uses similar triangles. In this way he can stay on one shore and does not need to cross the water. When he has done the fieldwork, his sketch looks like this.

Triangle *BCE* is similar to *BAF*. And since *BC, BA* and *CE* can be measured, *AF* can be calculated. So, the width *DF* of the river is known
($DF = AF - CE$).

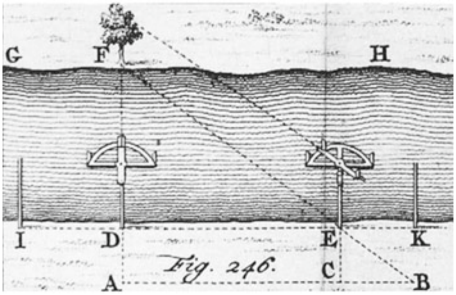

Fig. 13.1 An 18th century application of similar triangles for measuring the width of a river (Morgenster, 1744, figure 246)

classical tactile manner (this segment is the base line), and after that only angles are measured. All other lengths are calculated with trigonometric methods, especially the sine rule.

In our study, trigonometry was not yet in the curriculum of the classes that we wanted to observe, so the students worked with similar triangles and with direct proportionality of two pairs of homologous sides. More generally, we intended to explore the appeal and power of land measurement as an authentic practice, which stimulates students to learn (in our case) the theory of similar triangles. And since the current surveying techniques go beyond the scope of students in eighth and ninth grade, we proposed to the students in our experiment to imagine that they were 18th century apprentice surveyors.

This sets the scene: land measurement as an authentic practice in which the students work with similar triangles, and the historical setting which keeps the mathematics within the reach of the students.

13.3 History of Mathematics as a Context for Mathematics Education

Our study is inspired by the movement in the United Kingdom and France which proposed to teach mathematical subjects in relation with their historical development and, if possible, with the help of original problems and sources. The idea was strongly promoted in the 1990s in France by Evelyne Barbin, who coordinated joint work on the history and teaching of mathematics between teachers and academics within the French IREMs (Instituts de Récherche pour l'Enseignement des Mathématiques), institutes for the research on the teaching of mathematics. In England, this idea was propagated by John Fauvel, who edited a collection of IREM papers (Fauvel, 1990) and published a special issue of the journal *For the Learning of Mathematics* on history in mathematics education (Fauvel, 1991). Fauvel was the driving force behind a series of seminal HIMED (History In Mathematics EDucation) conferences, which started in Leicester in 1990. The idea was fostered by developers and researchers within the International Study Group into the Relations Between History and Pedagogy of Mathematics (also known as HPM), an affiliated study group of ICMI. The key survey of the developments by the end of the 20th century is published in the report *History In Mathematics Education* of the respective ICMI Study (Fauvel & Van Maanen, 2000). In the decade 2001–2010, educational research in the HIMED domain grew more important. Good historical information about mathematical development and suggestions about how this could be used in teaching was still current. But more than before, studies were done to find out in a more structured and less anecdotal manner how teaching went with resources in which historical elements were integrated. Our research, which was part of the broader PhD project Reinvention of Geometry, belongs to this stream of classroom studies. Teaching similar triangles in a context in which students acted as 17th century surveyors was part of the project, discussed in Gulikers (2003). The report of the complete study appeared in Van Gulik-Gulikers (2005). More recent studies in the same vein are Glaubitz (2011) and several articles by Jankvist, especially Jankvist (2011), with a firm theoretical digestion of his earlier work.

The arguments that are mentioned in the 20th century literature on this subject were analysed by Gulikers and Blom (2001) and can be subdivided into conceptual, (multi-)cultural and motivational arguments. From a conceptual point of view, knowledge of the history of mathematics can result in an enrichment of the didactic repertoire of the teacher and in a more conscious use of the teaching methods involved. Students develop a better understanding by familiarising themselves with the way in which mathematical concepts developed. From a cultural perspective mathematics is an activity in which solutions are sought and found for problems from daily practice. During this process overlaps with other disciplines become visible. In this respect, it is not unimportant that a major part of the origin of mathematics is rooted in non-western cultures. This may enhance mutual respect and tolerance in multi-cultural classes. Using historical formulations of a problem in the curriculum can have a positive effect upon the motivation of students, because the historical source material sometimes contains amazing examples that enliven the lessons and challenge the students.

13.4 Research Questions

13.4.1 Role of History for Motivation

An argument in favour of using the history of mathematics as an educational tool that some designers and researchers put forward is that it motivates students. The unusual problems and the practical orientation of historical resources may serve to create a vivid lesson climate, and break through the common chalk-and-talk lessons.

History appeals to different fields of interest and to different skills of the students, and this may enhance their motivation to do mathematics. At least, so it is claimed. This claim is precisely our main question: To what extent is this claim confirmed when we propose to students an ordinary mathematical textbook chapter in 'historical disguise'? This disguise implies that we replaced the usual exercises about similarity by historical methods and problems from the practice of a surveyor.

To answer this question, we compared the motivation observed in the students before the experimental history-based lessons, with that observed after the experiment. We also tried to find out whether changes in motivation are caused by the historical elements, by the practical character of the exercises and the actual execution of measurements, or more in general by the change in the layout of the lessons.

13.4.2 Influence on the Learning Process

A surplus of motivation may have the effect that students understand mathematics better. Since it is difficult to observe how history affects a student, we chose to investigate what students think about their mathematical ability: Did the history make mathematics more accessible to the students, or did they experience more difficulties because of the historical elements? Our expectation was that history would make the content more concrete and that students would therefore find the mathematics easier to understand.

13.4.3 Students' View on the Role of Mathematics in Society

Another, different argument in favour of integrating history is that it enables students to see and almost feel some important applications of history in society. Students often ask, "Why do we have to do mathematics?" Our lesson design includes a series of applications that relate to an important period in Dutch history, when measuring and similar triangles were tools in the hands of builders and architects, the 'Golden Age', in which many Dutch cities were first built or grew larger, in which ports and fortifications were constructed. Also, students can identify with a mathematical practitioner, since no measurement exists without someone who performs that measurement.

13.4.4 Two Questions in the Margin: Is History Essential, and What Is the Role of Old Language?

Although not crucial when the focus is on measurement, two marginal questions that arise naturally within the design of our study still deserve attention. The first concerns causality: If we notice that motivation improved within the experimental history-based lessons, can we be sure that this happened as a result of the history, or is it just the difference in outlook which appeals to the students?

A closely related question is about the authenticity of the historical resources. Written Dutch of the 17th and 18th centuries closely resembles current written Dutch, but especially the fonts used for printing books do cause reading difficulties. This raises the question of how students react to these difficulties, and how they judge the extra energy needed to decipher the texts.

13.5 Method

13.5.1 Background

The idea that measurement, and more particularly surveying of land, can be taught in a historical context and with historical resources, was tested in a design experiment in two cycles. In both cycles the topic of calculations in connection with similar triangles was taught using student texts that were based on an 18th century course for surveyors. The authentic practice of surveying played an important role in the developed teaching material.

The first cycle of the experiment was piloted in five classes in the course year 2001–2002. In 2002–2003 a revised version of the material was tested. In this chapter, we report about this second cycle. The design for this cycle differed in some respects from the design used in the first cycle. A new introduction about calculations based on similar triangles was added to better connect the historical problems with the mathematical knowledge that the students had at hand. Also, the revised teaching material gave better support to the students with respect to tasks which required understanding 18th century Dutch. The Dutch language has not changed dramatically since 1750, but not everything is immediately obvious either.

13.5.2 Participants and Data Collection

The second cycle of the experiment was carried out at 16 schools in various regions of the Netherlands. All in all, almost 1100 students from 46 classes (24 eighth-grade classes and 22 ninth-grade classes) taught by 32 teachers participated. Data were collected through:

- questionnaires for both students and teachers, administered before and after the experimental lessons
- classroom observations of 19 lessons at 9 schools
- evaluations of the students' work, including worksheets and posters.

Due to organisational problems, far less than 1100 students filled in both questionnaires. Only the questionnaires of students who participated both before and after the experimental lessons were analysed.

13.5.3 Teaching Material

The mathematical subject addressed by the teaching material was the application of similar triangles in the determination of distances and lengths. The material replaced the chapter on similarity from two Dutch mathematics textbooks (*Moderne Wiskunde*[1] and *Getal en Ruimte*[2]). The historically oriented assignments contained an introduction on calculating with similarity. Similarity was applied in 17th century land surveying to calculate the height of buildings and the width of rivers. The students were taken back a few centuries and were asked to put themselves in the position of the surveyor. They first did a historical theoretical assignment on the basis of old mathematical texts and bits of historical background information (Fig. 13.2). This was followed by a practical assignment (see below) and finally they made a poster.

The text for the students was in Dutch, but one of the schools had a bilingual programme and taught mathematics in English, therefore here an English translation was used. Figure 13.2 displays the English version of the experimental text as it was used by the students. The text was written so that it fully replaced the textbook chapter on similar triangles. In this way, the participating schools did not 'lose time', and the results of experimental classes could be compared to those of classes that used the textbook.

The practical assignment is a regular element in Dutch education, during which the students, often in couples or small groups, work on a more complex, practical problem which requires good mastery of the subject taught, as well as other knowledge and skills. In this case the students carried out measurements in pairs with the aim to calculate, with the help of the principle of similarity, the height of a building or the width of a ditch. The students could choose from the various methods they had seen in the teaching material. Following the choice of the method they had to design a solution strategy: planning how to do the measurements, followed by how to incorporate all the findings in a poster.

Teachers were supported by a short teacher manual which gave suggestions for the distribution of the material over the lessons, and for didactical arrangements. This teacher guide also contained advice about certain exercises, a scheme for evaluating the practical assignment and the worked-out exercises. Next to this, there was information about the historical background, which included an overview of the history of surveying in the Netherlands. The teacher guide also contained advice and materials for multidisciplinary activities together with the Dutch language teachers.

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[1] Modern Mathematics.

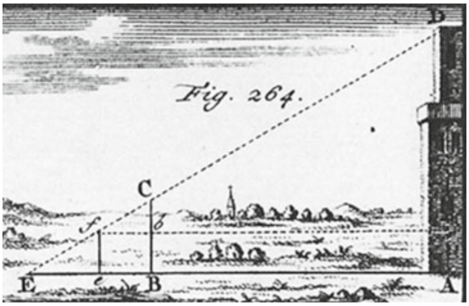
[2] Number and Space.

Historical theoretical assignment: Determining the height of a tower using a stick

The first problem that you will study comes from a mathematics book that was written specifically for land measurers and engineers.
The book was published, in Dutch, in 1744 and it was called *Werkdadige Meetkonst*.
It was written by Johannes Morgenster especially for his students.
The language of the book is old Dutch, which has been literally translated into English.

Fieldwork.

To find the height of the Tower AD[a] without using any Angle measurements; one places a Stick CB Perpendicular to the ground at a point where one has a good and complete view of the Tower, and one sees where the line of sight DC comes to the ground in E, that is, one looks for the place E at which one sees the top of the tower just over the top of the Stick; where this happens, one places a pin: then measure BE, AE and also the height of the stick BC; having written these measurements down, the fieldwork is finished.



(a) Read the description of the 'fieldwork' above. Make the text simpler by making shorter sentences. What belongs together? You may mark in the text. Then write a modern, shorter version of this text.
(b) What is very uncomfortable when doing this fieldwork?
(c) Look at the figure Morgenster used and make a sketch of it on your worksheet. Label all vertices clearly.
(d) Why are the triangles *AED* and *BEC* similar?

After the fieldwork has been done, the measurements are written down as follows:

> We take BE. 9, the stick BC. 6 and the line on the ground AE. 150 Feet.

In the 17th century, the land measurers did not work with metres and centimetres, but in feet (which in England, for instance, they kept until quite recently!). The 'foot' was the standard measurement.
(e) About how long do you think a foot would be in centimetres?

The calculation is written down by Morgenster in the following way:

> *The Work.*
> BE BC AE
> 9 - - - - - - - - - - 6 - 150?
> comes AD. the height of the tower to 100 Feet, if the line on the ground AE is horizontal

(f) Show with a calculation (and explanation!) how Morgenster got his answer.

Fig. 13.2 Historical theoretical assignment, text for students ([a]As Morgenster did not use italics to indicate line segments, we will not use italics when literally translating his text)

13.6 Findings

13.6.1 An Observation: Two Students as Surveyor

In this section, we describe our findings gathered from observing two students when they were engaged in a so-called ‘practical exercise’, which included the preparations, the practical measurements and the calculations afterwards. The school where the observation took place was a bilingual Dutch school (Maartenscollege, Haren), which teaches mathematics in English. Before the students started on the actual measurements, they had to design a solution strategy (“plan van aanpak”), in which they showed the methods they intended to use and calculated the expected results (Fig. 13.3). The practical exercise was independent student work, about which the students reported on a poster. When Stefan and Marco started this exercise, they had already worked on the booklet with application problems about similar triangles in the context of surveying. So they already knew several classical methods. In the exercise they were asked to measure the height of the school gym. Their solution strategy was based on two methods, taught in the booklet:

- Using a mirror which is placed on the ground.
- Looking over the top of a stick.

The first method is based on the work (Fig. 13.5) of the Dutch reckonmaster Cardinael (1620). See also Fig. 13.3. See for the second method and the elaboration on the poster Figs. 13.4 and 13.9.

13.6.1.1 Solution Strategy

Stefan and Marco used the two methods mentioned above to measure the height of the gym of the school. Figure 13.3 shows their worksheet describing Method 1. They first designed a solution strategy using fictitious numbers. Method 1 is the mirror method, illustrated in Cardinael (Fig. 13.5). However, in the following observation it will become clear that they did not quite grasp the strength of this method, which is based on the assumption that one cannot know the distance between the mirror and the object to be measured.

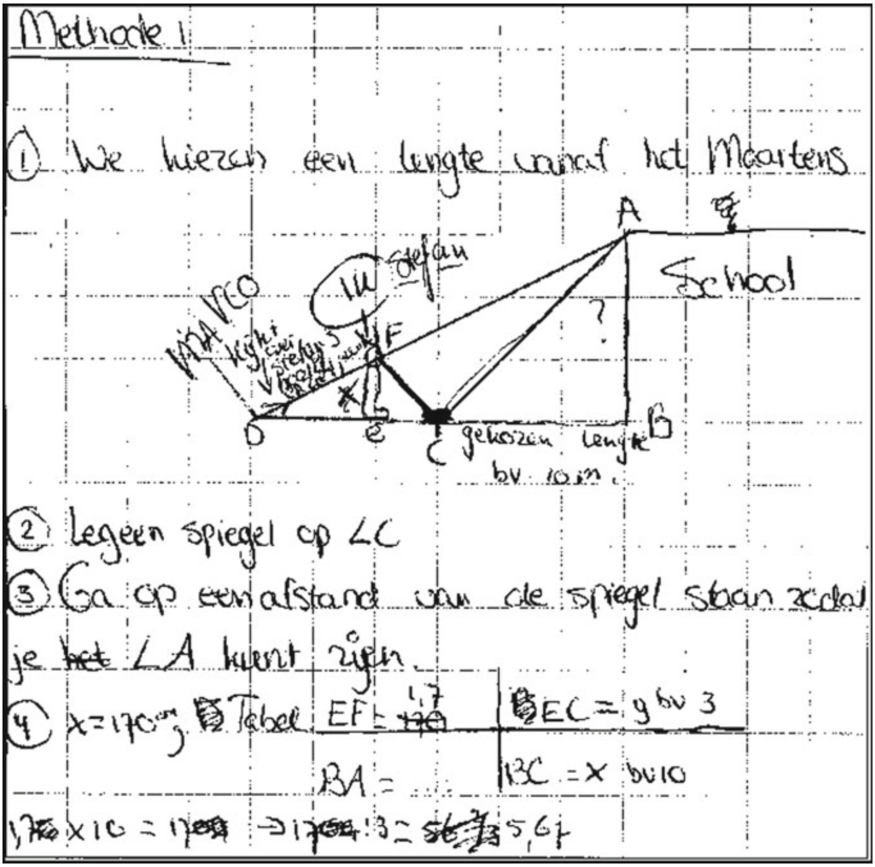
Their text on the worksheet reads as follows:

1. We choose a length from the Maartens College building

   [Below the drawing is written:] chosen length e.g., 10 m
2. Place a mirror at $\angle C$
3. Position yourself at such a distance from the mirror that [looking into the mirror] you can see $\angle A$.

   [The drawing indicates that Stefan, drawn as a standing figure *EF* with length *x* and called ‘me’, looks into the mirror. On the left side of the drawing is written:] Marco looks over Stefan’s head and sees point *A*.

   [The calculation then reads:]



Methode 1

① We kiezen een lengte vanaf het Maartens

② legeen spiegel op ∠C

③ Ga op een afstand van de spiegel staan zodat je ~~het~~ ∠A kunt zien.

④ x = 170 cm, Tabel

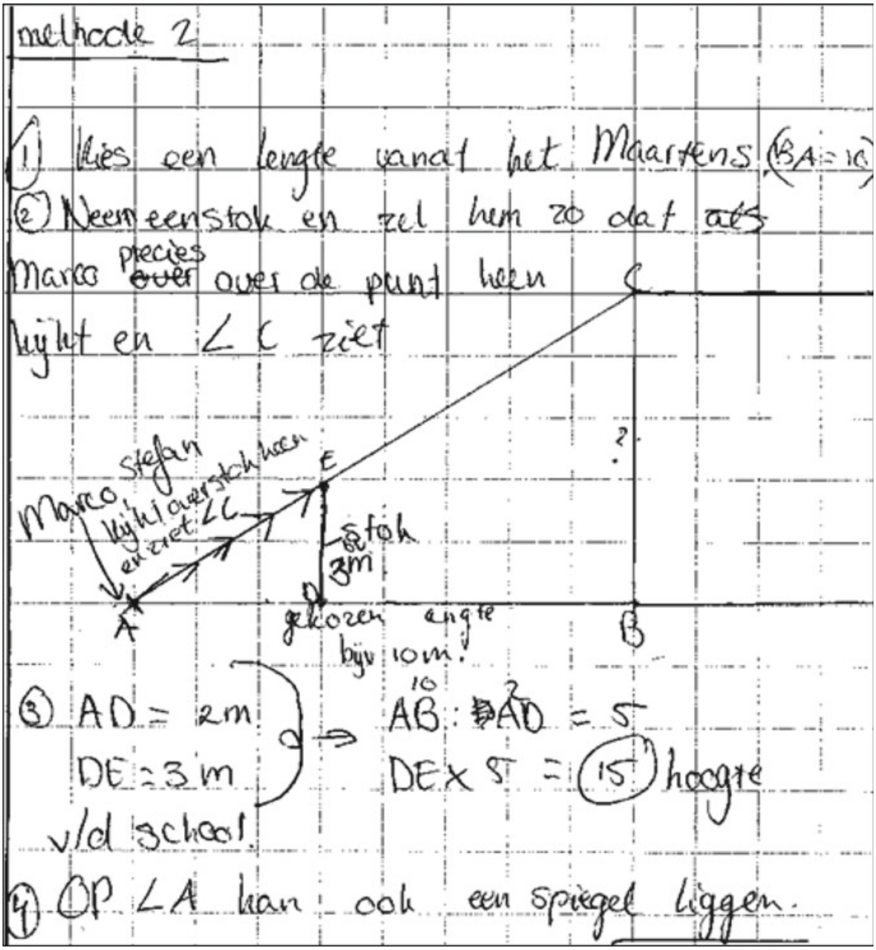
EF = 1,7 ~~170~~	EC = y bv 3
BA =	BC = x bv 10

1,7 x 10 = 17 → 17 : 3 = ~~5,6~~ 5,67

Fig. 13.3 Work by Stefan and Marco, Method 1

4. $x = 170$ cm [*BA* follows from the equal proportions $EF : AB = EC : BC$ represented in the table, EC is supposed to be y, for example 3, and *BC* was chosen to be *x*, for example 10; the cross product produces for *BA* is] $(1.7 \times 10) \div 3 = 5.67$.

Figure 13.4 shows the worksheet of Stefan and Marco describing Method 2. Here they work in a similar manner: *ED* is now a ‘stick’ (or Marco can serve as a stick; see below); Stefan looks over the stick and sees $\angle C$. Again, the lengths *AD*, *DE* and *EB* are fictitious (2, 3, and 10 respectively). Subsequently, Stefan and Marco collect all tools that they need for the actual measurements. They borrow a tape-measure from their teacher. At the physics lab, they acquire a mirror and the porters help them with a broom, which they will use as a stick.



methode 2

① Kies een lengte vanaf het Maartens (BA=1

② Neem een stok en zet hem zo dat ~~als~~ Marco precies ~~over~~ over de punt heen kijkt en ∠C ziet

③ AD = 2m
DE = 3m
v/d school
⇒ $AB : AD = 5$
DE × 5 = 15 hoogte

④ Op ∠A kan ook een spiegel liggen.

Fig. 13.4 Work by Stefan and Marco, Method 2

13.6.1.2 The Actual Measurement

Before we continue the observations of our two surveyors, it is worthwhile studying Cardinael's diagram (Fig. 13.5) in its own right.

The method proposed by Cardinael does not involve the length *BC*. The only lengths that are to be measured are *CD*, *DE* and *DF* (the mirror is at *C*, the stick is at *D*, so over its top E one sees the top of the tower *A* in the mirror, and *F* is the point on the ground from which one sees *A* and *E* in one line). If triangle *CDE* is then reflected in *DE*, the angles $\angle ACB$ and $\angle EGD$ are equal, since they are both equal to $\angle ECD$. Therefore, lines *AC* and *EG* are parallel and therefore triangles *ACF* and *EGF* are similar. So, $AB : CF = ED : GF$, and in this proportionality *AB* is the only

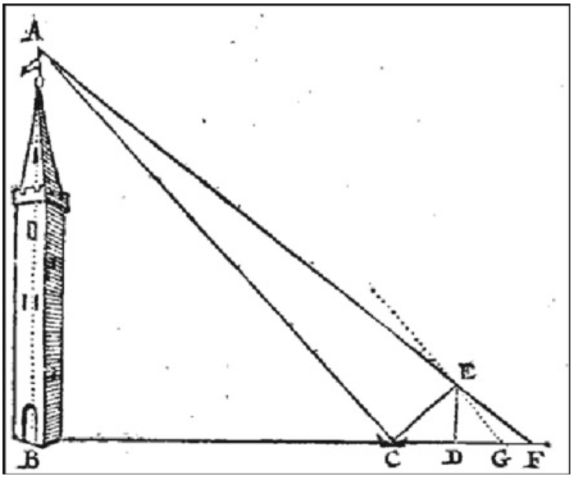


Fig. 13.5 Cardinael's diagram measuring a height

unknown. That, however, was too advanced for Stefan and Marco (they might have worked like this, but our observation shows that they did not).

Another way of solving this problem, provided one can measure the distance between the foot of the tower and the mirror, is as follows. Stand where you can see the top of the tower in the mirror, which will be at point *D*. Measure the height of your eyes from the ground (*DE*), the distance between you and the mirror (*CD*) and the distance of the mirror from the tower (*BC*). It is this second method that Stefan and Marco use, placing the mirror at a point *C* (Fig. 13.6). First, they set a fixed distance for the distance *BC*, in their case the distance between the mirror and the wall of the gym. They take $BC = 10$ m. Stefan searches from where on the extension of *BC* he sees the edge of the gym's roof in the mirror (Fig. 13.6). Marco measures Stefan's height (1.64 m) and the distance from Stefan's feet to the mirror (1.67 m) and concludes: "Now we have all measures and we can start our calculations." Stefan makes a note of the measures, and they decide to continue the calculations inside.

For the second method, where they use only a stick, Marco tries to find a point on the ground from which he can see the edge of the roof and the top of Stefan's head in one line (Fig. 13.7). Next, Stefan measures his distance to Marco (4.12 m), and comments: "The measures are not nice, but [that is not really a problem, because] we can use a calculator." Then, Marco, speaking to Stefan: "We do not even use the broom, since you are the stick."

13.6.1.3 The Calculations and the Poster

While Stefan is still engaged in drawing, Marco also elaborates on the method with the stick (Fig. 13.8). For the height of the roof he finds 6.29 m ($\frac{15.79 \times 1.64}{4.12}$, in which 15.79 comes from ($15.79 = 10 + 1.67 + 4.12$). Marco says: "That is a very big difference between both methods." He reconsiders his own work and continues:

Fig. 13.6 Stefan sees the edge of the roof in mirror *C*; Marco measures Stefan's distance from *C*

Stefan and Marco are engaged in determining the height of the roof of their school's gym. With his head on the ground, near the threshold of the school building, Marco tries to find a point from which he can see the roof and the top of Stefan's head in one line. "Measuring is really difficult! Especially, to keep the door open!", is the text by Stefan and Marco on the poster on which they report about their measurements

Fig. 13.7 Marco tries to find a point from where he can see the roof and the top of Stefan's head in one line

"Both methods look alright, but there is quite a difference; 9 m seems rather high to me." In the meantime, a classmate asks whether Stefan and Marco have calculated the height in two ways.

"Yes", Stefan says, "but we have two different answers." The classmate says: "How is that possible?" Stefan answers: "That is what we are trying to find out." Then, Marco proposes to make a scale drawing of the situation for the stick method (Fig. 13.8). First, he draws the distance from the gym to the point on the ground, and

Stefan first makes a careful drawing of the first method (with the stick).
Marco calculates the result from the method with the mirror directly. He writes down the following proportionality table

1.64	1.67
x	10

and then calculates the result with a cross product

$\frac{10 \times 1.64}{1.67} = 9{,}8$ m

Fig. 13.8 Marco and Stefan at work with their data

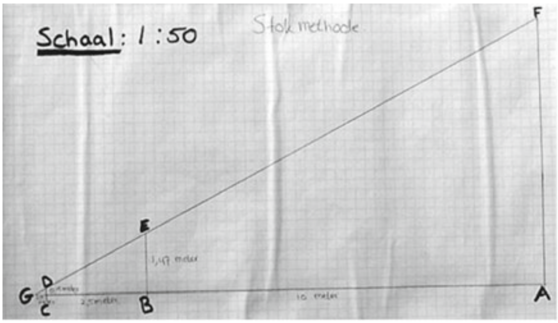
then he inserts the 'stick' in the right position. With the help of the ray from the eye, he then draws the edge of the gym roof. Then he inserts the mirror in the drawing, in the right place.

Then, it is time to measure in the drawing. Marco determines the height of the gym, taking the scale into account. He says: "The method with the stick seems to have given the right answer: 6 m seems logical for the gym." Afterwards, the boys proceed to find out what went wrong with the mirror method. Marco suggests measuring the two angles at C (incidence and reflection) in the drawing. If these angles are not equal in the drawing, the boys must have made a mistake in their measurements. They inspect the angles in the drawing and find them to be unequal. Marco says to Stefan: "So, you did the sighting incorrectly." Stefan answers: "Or you were wrong when you measured the distances." The end of lesson signal ends this collaboration, which lasted a double lesson (100 min). Stefan and Marco decide that they want to redo the measurements at a later moment.

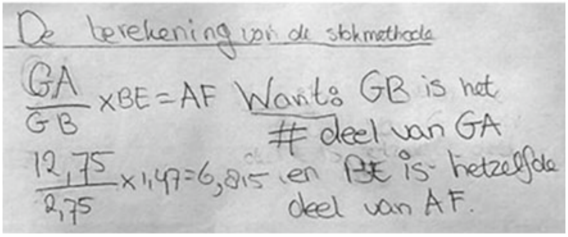
Stefan and Marco reported about this practical exercise in a poster. They clearly had first repeated the measurements. In the poster they illustrated the methods used with drawings. With the new figures they presented their final calculations (Fig. 13.9) and their conclusions (Fig. 13.10). They found different results for the two methods. The stick method led to 6,8 m. With the mirror method their calculations for the height of the building resulted in 6,697 m.

With respect to the technical execution of the performed measurements, two comments can be made, which also throw some light on the deviations between the final results. First, in Fig. 13.6, Stefan looks into the mirror at C. In this case, not Stefan's length (1,64 m) should be used in the subsequent calculations, but the height of Stefan's eye. This will be about 10 cm less than his length, which will lead to a more accurate result in Fig. 13.8, where the calculation is carried out.

Second, in Fig. 13.7, Marco sights the roof and "the top of Stefan's head in one line" (at least, that is what the two are thinking). But the ray from Marco's eye to the roof is tangent to the side of Stefan's head and not to its top. In this case they should not calculate with Stefan's length but with a larger number, which depends on Stefan's length, the radius of his skull and the slope of the ray from Marco's eye



Stick method
Scale: 1:50



De berekening van de stokmethode

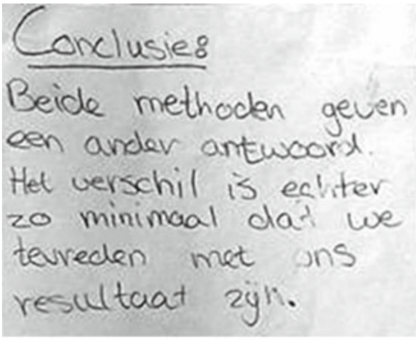
$\frac{GA}{GB} \times BE = AF$ Want: GB is het # deel van GA

$\frac{12,75}{2,75} \times 1,47 = 6,815$ en BE is hetzelfde deel van AF.

The calculation with the stick method

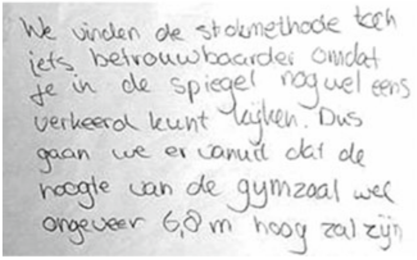
want = because
is hetzelfde deel van = is the same part of

Fig. 13.9 Part of the poster with report on the stick method



Conclusie:
Beide methoden geven een ander antwoord. Het verschil is echter zo minimaal dat we tevreden met ons resultaat zijn.

Conclusion: Both methods give a different answer. The difference, however, is so minimal that we are satisfied with our result.



We vinden de stokmethode toch iets betrouwbaarder omdat je in de spiegel nog wel eens verkeerd kunt kijken. Dus gaan we er vanuit dat de hoogte van de gymzaal wel ongeveer 6,8 m hoog zal zijn

We still think that the stick-method is somewhat more reliable because the sighting in the mirror is regularly wrong. So, we assume that the height of the gym is about 6.8 m.

Fig. 13.10 The part of the poster with conclusions to the roof. Both problems do not arise if a pointed stick is used instead of the human body.

13.6.1.4 Provisional Conclusions Based on the Observations of the Two Students

Apparently, the exercise had a number of features which only come up when students actually carry out the measurements. The well-known but not often experienced fact that people make errors when measuring, is revealed here thanks to the two methods. Quite probably the method with the mirror is less precise than the stick method, although a deeper analysis is necessary to clarify this matter. The decision to make a scale drawing is also very interesting. There is nice intuitive reasoning here. If in reality the angles are equal (they should be equal because of the reflection law), then they must also be equal in a scale drawing. Finally, the attention that the students pay to the (probably unexpected) problem of finding different heights for the same object, their honesty and eagerness in dealing with this problem and the decision to redo the measurements gives insight into how scientific ethics develop in young students. The problems inherent with measurement prompted them, and they took up the challenge.

13.6.2 Findings from All Students Involved in the Data Set

Next to doing observations on a micro-level, we investigated the judgements of the students about the historical and practical aspects of the teaching material. Table 13.1 gives an overview of our findings based on the pre- and post-questionnaires, on interviews with students and teachers, on classroom observations and on evaluations of written material produced by students (exercises and posters). Only the questionnaire data of the students who filled in the questionnaire both before and after the experimental lessons were included in the analysis. The results are marked positive (+) when there is significant positive statistical evidence, indecisive (±) when there is both positive and negative evidence and negative (–) when there is significant counterevidence. In the remaining part of this section these results are further elaborated.

13.6.2.1 Students’ Motivation

Working with the text for students in which elements from the history of mathematics were integrated, had a negative effect on the motivation to learn mathematics. Judgements by students were rendered on a scale from 0 (“history is not useful, not agreeable, …”) to 1 (“history is interesting, nice, useful, …”). The difference in

Table 13.1 Judgement of students on historical and practical aspects

	Hypotheses, and related points of inquiry	Result
1	Thanks to the instruction according to the experimental design, students became more *motivated* for doing mathematics	–
	By which characteristics of the design do students become more motivated? By	
	– the integration of history in the tasks	–
	– the execution of a practical exercise	+
	– the change in working procedures	±
	– the change in the type of tasks	–
2	Thanks to the instruction according to the experimental design, students develop a better *understanding of mathematical concepts and methods*	±
	According to the students, which aspects of the learning process were influenced by the experimental design? Students say about mathematics that they	
	– find it easier because of the integrated history	–
	– find it easier to remember	+
	– have a better understanding thanks to the concrete materials	+
3	Thanks to the instruction according to the experimental design, students have a better *view of the role that mathematics plays in society*	–
	Which elements in the design are responsible for the better view of the role of mathematics in society?	
	– The historical applications	–
	– The possibility to identify with the (life and work of a) 17th century surveyor	+
	– The execution of the practical exercise	+

motivation between before and after working with the experimental text was found to be significant ($p < 0.01$) (Table 13.2).

There were also significant differences between subgroups. This was the case between those students who find mathematics easy and those who find mathematics difficult (see Table 13.3). Students who in the pre-questionnaire indicated finding mathematics easy, were less motivated to learn mathematics after working with the experimental text. For those who found mathematics difficult beforehand, nothing had changed in their motivation after working with the text.

Another relevant difference is that between those students who preferred to do problems with bare numbers, and those who reported to like problems that they

Table 13.2 Motivation to learn mathematics before and after working with the experimental text

	N	*M* (*SD*)	Difference
Motivation before working with the text	429	0.57 (0.21)	−0.03*[a]
Motivation after working with the text	429	0.54 (0.21)	

[a] Here, and in the following tables, one asterisk indicates that the result is significant at the level of $p < 0.01$

Table 13.3 Motivation to learn mathematics before and after working with the experimental text by students who indicated finding mathematics easy and students who indicated finding mathematics difficult

	"Math is…"	*N*	*M* (*SD*)	Difference within subsets (*SD*)	Difference between subsets
Motivation before Motivation after	Easy	304	0.63 (0.19) 0.59 (0.20)	−0.04* (0.15)	0.04*
Motivation before Motivation after	Difficult	106	0.41 (0.20) 0.41 (0.21)	0.00 (0.17)	

Table 13.4 Motivation to learn mathematics before and after working with the text for students who prefer bare number problems and for the students who prefer context problems

	"I prefer…"	*N*	*M* (*SD*)	Difference within subsets (*SD*)	Difference between subsets
Motivation before Motivation after	Bare number problems	225	0.59 (0.20) 0.56 (0.21)	−0.03* (0.17)	0.00
Motivation before Motivation after	Context problems	190	0.56 (0.23) 0.53 (0.22)	−0.03 (0.14)	

have to derive from a story (see Table 13.4). The motivation of the first group had decreased after working through the text, while for the second group the change was not significant.

One explanation for the negative effect on the motivation of the students is that the students appeared to encounter many difficulties with the historical sources presented in the old Dutch language. They were not acquainted with it, and they experienced reading and interpreting these texts as too complicated and time-consuming. Some students remarked that working with these sources "has nothing to do with mathematics". They reported that it had "no use" for them and they also called it "a waste of time". This reaction was not unanimous, though, since other students said that they found it interesting to learn something about the old language. These students considered it a challenge to derive the mathematics from the historical source material.

Another important reason why the motivation of students was not stimulated by the integration of history is that it made the tasks much more complex than they were used to. Working with this material requires other competences, and, as some students said, they do not like that. It makes mathematics less attractive for them. Much also depends on the insight and support of the teacher.

Although the quantitative data were not that positive, classroom observations made by the first author and interviews with students and teachers enabled us also to signal some encouraging effects (though these positive reactions did not dominate). During the observations and the interviews, it came to the fore that the variation in activities was considered as positive; especially the change in perspective. History gives an unexpected turn to mathematics and it makes the lessons livelier. Both teachers and students appreciated learning about how mathematics was applied in the past, for example, the methods to measure the height of a building. And the interviewed students were especially enthusiastic about applying these methods themselves in reality.

In particular, the concluding practical assignment was valued positively. An open question at the end of the lesson series revealed that three quarters of the students liked the practical assignment. Most students found that the practical assignment brought interesting variation and also a challenge, namely, they themselves could make a practical application of the theory that they had studied in the preceding lessons. In doing so, they saw that what they had learned in the text about 17th century surveying "really worked". Students liked that they were allowed to go outside during their mathematics lesson, which is "different from an ordinary lesson in the classroom". Also, the cooperation with fellow students in measuring the height of a building was welcomed. Students were thrilled that, "without modern tools", but "with simple materials", they were able to do the necessary measurements themselves.

Students who did not fully understand the mathematical explanation in the experimental text did not like the practical assignment, because they had no clear clue how they should do the measurements. Some students found the step from the theoretical explanation of plane geometry in their regular textbook to the practical task of measuring in space difficult. Some of them also preferred to do "just abstract mathematics". They called the practical assignment superfluous, because they "know for certain that they will never apply these measuring methods".

13.6.2.2 Influence on the Learning Process

Working with the historically-oriented experimental text did not change the view of students about how well they were able to do their mathematical work (the feasibility). Some students found the exercises easier because the information about the ancient ways to measure made the text less abstract. They commented that they learned much from carrying out the applications. They memorised better what they had to learn, because it fired their imagination. Thanks to the practical assignment—they said—they sooner noticed mistakes that they had made in earlier, formal exercises, since now they "had to really think about the matter". This put a check on their acquired knowledge.

Teachers said that they noticed that students remembered with more ease how calculations on similar triangles had to be performed. They expected that the students in the experimental group would have a better command of the concept of similarity in a later school year than the students who did the regular chapter from the textbook. Our study did not extend so far that we can confirm or reject this claim.

It would certainly be interesting to investigate the effect of such experimental text on memorisation and retention.

Clearly, other students had problems due to the historical elements. One third of the students indicated that the historical practical assignment was one of the most difficult elements of the experimental text. Problems arose from the old language, and also because the mathematics was not as straightforward as it is presented in the textbook. Many students needed to get acquainted with more complex and more extended exercises. They were used to questions subdivided into sub-questions, and often directly related to the mathematical theory and to examples that were just discussed. Now, they had to figure out a strategy themselves, and that was difficult for them. Students who had previously worked on unusual exercises, in projects and small research tasks, were better equipped for these practical exercises.

We found a remarkable significant difference in the judgement about feasibility between two subsets of the students, that is between the students who had answered in the pre-questionnaire that they found mathematics easy and those who had answered that mathematics was difficult for them. In both groups there were many changes in judgement (see Table 13.5). The subset that at first found mathematics easy shifted towards difficult, and in the other group there was a shift from difficult in the pre-questionnaire to easy in the post-questionnaire.

There was also a noticeable difference in shift between the eighth-grade and ninth-grade students who worked with the experimental text (see Table 13.6). For the eighth-graders, the idea that mathematics is feasible shifted significantly to less feasible. For the ninth-graders there was no significant shift.

The posters and reports about the practical assignments revealed that in most cases the students mastered the calculations based on the concept of similarity quite well. This does not hold to the same extent for the historical exercises in the theoretical part of the experimental text. In several lessons, the first author observed that for the practical assignment, in which calculations with similar triangles were the basic ingredient, students first returned to the theoretical sections with the mathematical explanations and retried the unsuccessful calculations. Those students who were again unable to do the calculations, also failed on the practical assignment.

Table 13.5 Students' judgement about the feasibility of mathematics before and after working with the experimental text for students who at first found mathematics easy and who at first found mathematics difficult

"Math is…		*N*	*M* (*SD*)	Difference within subsets (*SD*)	Difference between subsets
… easy", before … easy", afterwards	Subset easy	345	0.74 (0.13) 0.70 (0.15)	−0.04* (0.12)	0.10*
… easy", before … easy", afterwards	Subset difficult	126	0.35 (0.11) 0.41 (0.16)	0.06* (0.14)	

Table 13.6 Eighth- and ninth-grade students' judgement about the feasibility of mathematics before and after working with the experimental text

"Math is…		*N*	*M* (*SD*)	Difference within subset (*SD*)	Difference between subsets
… easy", before … easy", afterwards	Subset Grade 8	205	0.64 (0.22) 0.60 (0.21)	−0.04* (0.14)	0.05*
… easy", before … easy", afterwards	Subset Grade 9	266	0.63 (0.21) 0.64 (0.19)	0.01 (0.13)	

Table 13.7 Students' judgement about the usefulness of mathematics before and after working with the experimental text

"Mathematics is …	*N*	*M* (*SD*)	Difference
… important and useful", before	476	0.67 (0.18)	−0.04*
… important and useful", after	476	0.63 (0.18)	

13.6.2.3 Students' Views on the Role of Mathematics in Society

The importance and usefulness of mathematics is, unexpectedly, noticed less by the students after having worked through the experimental text (see Table 13.7).

For an explanation of this finding, one could argue that students already knew about the importance and usefulness of mathematics before they started to work with the experimental text. The historical context would then not add anything to their view. Moreover, the problems with the old language might even have distracted them from their views and beliefs about mathematics.

However, in the open questions of the post-questionnaire, students answered that the experimental text sheds light on the societal role of mathematics. They noted that they saw how mathematics was applied in the 17th century, especially in the measurements done by the surveyor. This raised enthusiasm, since they now saw what they could do with their mathematical knowledge. Whether or not the historical presentation helped, is still disputed. Some students appreciated the elementary character of the tools, while others said that there was no need to work with these old instruments, since current measuring equipment would let you determine location and height much easier.

13.6.2.4 Students' Opinions About the Integration of History

Table 13.8 summarises what students thought about the integration of history in the experimental text.

Table 13.8 Students' opinions about the integration of history

"Thanks to history…	*N*	*M* (*SD*)
… I was more motivated"	654	0.38 (0.24)
… I found mathematics easier"	604	0.44 (0.22)
… I see that mathematics is useful"	665	0.42 (0.22)

In the open questions in the post-questionnaire 37% of the students indicated that in their view history made the mathematics lessons more attractive. It was important for them to see where mathematics comes from, and "how people worked in the past". This was useful if they wanted to apply mathematics themselves. They also appreciated the experimental approach as a welcome interruption of the usual "throwing around figures" and boring problems "without a story".

Some students emphasised that the experimental text attracted them because they liked the two disciplines, mathematics as well as history, and they said that they liked the combination. Others valued that they now knew more about the development of mathematics, but for them it had rather been "just mathematics" and not "that old Dutch".

When students were asked what they found least attractive about the experimental text, 20% of the students pointed at the exercises with a historical character. Moreover, 30% counted these exercises among the most difficult elements in the experimental text. They reacted by saying "all these stories are difficult" and "are not really about mathematics". Just as some students were in favour of both mathematics and history, there were students who disliked both disciplines. For them the experimental text was doubly unpleasant. A common objection was that the two disciplines should be treated separately. Students who had this objection found that when history is integrated within a mathematical text, "it is not about mathematics anymore". Also, the focus on the past was rejected by some students. They would rather hear about current developments and were of the opinion that "mathematics should rather be directed towards the future".

As expected, regarding the students' opinions about the integration of history several differences were found between different groups of students. The students who liked to derive a problem from a story were more positive about the historical elements than those who preferred to do problems with bare numbers (see Table 13.9).

As is shown in Table 13.10, differences were found between the eighth- and ninth-grade students with respect to motivation, perceived feasibility, usefulness of

Table 13.9 Appreciation of the historical elements in the experimental text by students who prefer bare number problems and by students who prefer context problems

	Subset "I prefer…"	*N*	*M* (*SD*)	Difference between subsets
Appreciation of history in text	Bare number problems	343	0.35 (0.23)	0.06*
	Context problems	237	0.41 (0.26)	

Table 13.10 Eighth- and ninth-grade students' opinions about their motivation, perceived feasibility and usefulness of mathematics, and attractiveness of history-based mathematics lessons

		N	*M* (*SD*)	Difference between subsets
More motivation	Grade 8	317	0.34 (0.24)	0.07*
	Grade 9	337	0.41 (0.24)	
Mathematics is easier now	Grade 8	289	0.40 (0.22)	0.09*
	Grade 9	315	0.49 (0.21)	
Mathematics is useful	Grade 8	319	0.41 (0.22)	0.02
	Grade 9	346	0.43 (0.22)	
Attractiveness of history-based lessons	Grade 8	291	0.31 (0.46)	0.12*
	Grade 9	287	0.43 (0.50)	

Table 13.11 Students' opinions about usefulness of mathematics for students who had joint lessons in Dutch language and mathematics and students who did not have joint lessons

	Joint lessons Dutch language and mathematics	*N*	*M* (*SD*)	Difference between subsets
Mathematics is useful	"No"	579	0.41 (0.22)	0.08*
	"Yes"	86	0.49 (0.22)	

mathematics and the attractiveness of history-based mathematics lessons. The ninth-graders were more positive than the eighth-grade students about their motivation and the perceived feasibility of mathematics. For their opinion about the usefulness of mathematics some difference was found between the eighth- and ninth-grade students, but this difference was not significant. The open question about the attractiveness of mathematics lessons in which history was integrated was answered more positively by the ninth-grade students.

As expected, there was also a significant difference between the students who studied the experimental text in lessons in which the Dutch language teachers and mathematics teachers worked together, and the students who studied the text only in the mathematics lessons. The first group recognised the usefulness of mathematics significantly more clearly (see Table 13.11).

13.6.3 Judgements and Views of the Teachers

A first finding was that experience helps in teaching history-based mathematics lessons. This was clearly concluded by teachers who participated in both cycles of the experiment. They reported that they could deal with the material more easily in the second cycle. The experimental text and the supporting teacher material were designed so that all teachers could work with it, including those who had no prior experience with the integration of historical elements in mathematics lessons. Of the teachers who were not involved in the first cycle, many did not have experience with this. When they were asked in advance what the integration of historical elements meant for them, it appeared to be restricted to giving information about a great name or an important year as a bit of background for a new subject. Despite the fact that they had less experience with using historical material, the overall reaction of the teachers was that they were stimulated by this material. It increased their motivation for mathematics.

Since the commitment and approach of the teachers are important for the execution of such uncommon lessons, it is important to note that all the 32 teachers who taught the experimental text in the second cycle did so on a voluntary basis and that their overall reactions were positive. To illustrate this, we quote here a number of comments by one of the teachers, who sharply analysed some of the central issues discussed in our chapter. According to this teacher,

> [t]he benefit of the project must be found in the integration of your textbook and a part of the history of mathematics. The textbook chapter on similarity can be used straightaway as an introduction to its practical application. That is your main benefit. You do not lose time, whereas you acquire a definite additional advantage.

Another remark was that the experimental text helped to answer questions about the usefulness of mathematics.

> Students often come up with the remark that they do not understand what purpose is served by mathematics. In what way is it enjoyable and what can you use it for? As a teacher, you have of course some examples at your disposal to clarify things a little bit. In the textbook, there are certain contexts that may help you. But still, how do you visualise that mathematics has a certain relevance?

Also, the teacher emphasised that not only students can profit from the practical assignment, the teacher does so as well.

> Especially the practical assignment appeared to be a challenge. Leaving the classroom and then finding an object in the vicinity of the school that you have to measure. Making your personal choice from the methods that have been discussed. You are bound to notice large differences in the ways students try to come with a solution. An excellent opportunity to see what skills are being used. Accompanying several groups gives you the possibility to observe the thought process.

Furthermore, the teacher discussed the problem of working with texts in older Dutch.

> Students must get used to this kind of ‘old language’. However, if you help them along they will usually manage. It gives you the opportunity to have a useful talk about mathematics. In any case, for a part of the students it is a challenge. Students help each other and together they are trying to find a translation for the text. It is a good thing to see them working and trying to solve a problem. Especially working together under the supervision of the teacher gives an extra dimension to the work. Reading, reading again, drawing up a sketch, adding the missing data and classifying them.

13.7 Conclusions and Discussion

Earlier, we already drew some provisional conclusions based on the observations of the two students. Based on the larger group of students and teachers involved, we can conclude now that the domain of measurement is important in mathematics education for several reasons:

- it engages students in practical activities
- it often requires cooperation in order to produce good results
- the measuring errors provoke valuable discussion between students; it could also be taken as the starting point for statistical considerations
- the use of tools, and the discussion about what tools to choose in relation with their possibilities and precision, is a welcome aspect; the discussion needs not to be provoked, since now that the world has GPS, students rightly question why one still needs all these triangles and calculations
- it is an area well within reach of the students, in which they themselves can apply formal mathematical theory.

The analysis of the pre- and post-questionnaires showed that students appreciate practical measurement activities and surveying. Students produced their own results, independently from the teacher. The results mean something, and in doing the measurements and the resulting calculations students could identify themselves with a mathematical practitioner: the surveyor.

Furthermore, in addition to what is said already, particularly in Table 13.1, about our findings on the role of history in teaching mathematics we would like to add here that we observed that quite some students have a view on mathematics that leads them to a quick rejection of the experimental text. For them mathematics is doing short problems according to a clear example, and preferably with the least amount of text possible. When they start working on a problem, they expect that the solution strategy is at hand as soon as they have read the problem. For these students, the experimental text was ‘wrong’ in a number of respects: it required reading, it required time-consuming practical work, and the strategies were not immediately clear. Moreover, the methods were outdated and produced results that can be obtained much quicker and with greater precision. Indeed, they do have a point. However, there is also a point to the wide range of possibilities that the historical aspects offer, namely to connect mathematics teaching to culture, to language, to appreciating the centuries when the Netherlands were an economical power. There is a point in learning to solve complex problems, in learning to do practical work and learning to cooperate. Another finding is that we could identify particular subgroups of students that responded more positively to the history in the experimental text: those who were taught mathematics and the old language jointly, as well as those who already liked to solve ‘word problems’. Also, the ninth-grade students were more positively than eight-graders. Therefore, we may conclude that history is not essential, but that for some students it is beneficial.

The concept of similarity and the applications of similar triangles is a topic that can adequately be taught (and learned!) in the context of practical problems. Measurement is a crucial part of this approach. Integration of historical elements is possible, but probably one will obtain the same or maybe better results without history. The topic is suitable for inviting a visiting practitioner for a guest lesson.

A major reason for students to dislike the experimental text was that it was too complicated and long for them, and that they did not immediately understand the problem or see the solution strategy. This reaction of the students is maybe influenced by the structure of the textbooks. In secondary education, textbooks are meant for students to work on independently, i.e., without the help and sometimes also without the presence of a teacher. The consequence is that tasks that require fundamental thinking, broader exploration and endurance, are skipped or split into a number of small, easy-to-digest parts. Hence, it is no surprise that students noticed the difference between the more complex tasks in the experimental text and the usual problems from the textbook, and that some of them answered that for that reason they disliked the experimental text.

The responses of 32 teachers who volunteered to teach similar triangles in the second cycle of the experiment were stimulating. It might well be that in some schools the teachers learned more from the experiment than their students. The comments of the teacher quoted earlier, showed which aspects of the text were welcomed as they are.

In this study we have not studied how the approach and attitude of the teacher influenced the results and views of the students. This is one important question that remains open for later research. In addition, there were many more questions that came up during our study that need to be answered before we can fully benefit from history as a didactical tool for teaching mathematics. Further research is necessary in several directions, including issues such as

- the age of the students (would this type of text be more suitable for older students, maybe for student teachers?)
- the value of such a text for the teacher (would it be a useful resource for professional development, for a lesson study?)
- the influence of the approach and attitude of the teacher
- the possibly avoidable problems with the old language (Is it possible to maintain the historical character without direct access to the sources?)
- the aspect of memory and retention (what is the effect of a text like the one studied here in the longer term and does it lead to better retention of the concepts and methods?).

Although the work on using the history of mathematics as a didactical tool for teaching mathematics is not yet finished, and many important questions require more research, it is remarkable that, since we carried out our research and reported about it, the history of mathematics has become in many teacher education institutions a standard feature in the education of teachers. In most of the institutions which teach the history of mathematics, designing lessons in which history of mathematics plays a major part, has become one of the assignments.

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Chapter 14
The Development of Calculus in Dutch Secondary Education—Balancing Conceptual Understanding and Algebraic Techniques

Martin Kindt

> Some time ago I read about efforts to improve the teaching of analysis by raising standards of rigour. I believe that analysis at school can be improved only by relating it closer to reality. If more abstraction is not counterbalanced by a closer proximity to reality, it will only yield more unrelated and thus worthless stuff.
> Freudenthal (1973, p. 580).

Abstract Compared to neighbouring countries, in the Netherlands a nationwide introduction of calculus in secondary education took place rather late. This happened in 1958 after a discussion of fifty years between advocates of the teaching of calculus in school and their opponents. From the 1960s on, there has been an acceleration in curriculum changes. First the curriculum was influenced by the New Math movement. This resulted in a rather formal course that became compulsory for almost all students in pre-university secondary education. Many of them experienced serious problems with the acquisition of this topic. Then, in reaction to that, influenced by Realistic Mathematics Education, three calculus courses were developed that tried to build on a meaningful introduction, with applications incorporated, for secondary pre-university education and secondary pre-higher vocational education. Finally, calculus became relevant and within reach of all students of the higher levels in secondary education.

14.1 A Forwards Run of Fifty Years

In 1905 two Dutch mathematics teachers, F. J. Vaes and C. A. Cikot, argued strongly in favour of the introduction of differential and integral calculus in the curriculum of Dutch secondary schools. The principal argument for introducing calculus was that in the subjects physics and mechanics calculus was applied in a disguised way in

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only one context, which limited understanding of the underlying process. The idea of introducing calculus at secondary school was discussed seriously, but in the end rejected by the association of secondary school teachers. Nevertheless, there were some teachers in that period who were experimenting with teaching calculus, and at least two calculus textbooks were issued.

The textbook by Van de Vooren (1919) was titled *Grenswaarden. Eene inleiding tot de differentiaal-en integraalrekening*, or in English: Limits. An introduction to the differential and integral calculus. According to me, this book is one of the best Dutch calculus textbooks for secondary education ever written. It starts with an introductory chapter on constant and variable quantities. This was in fact a short and informal way to teach the concept of function with real-world examples like the weight of a person dependent on his age, or the length of an iron bar as a function of its temperature. These examples were combined with graphical representations. This kind of introduction was very modern at that time, but nevertheless the implementation of so-called 'functional thinking' in mathematics would take almost the whole twentieth century.

In Van de Vooren's book, after the introduction, a paragraph about limits of infinite sequences followed. As an example of a geometric limit the author considers the slope of the tangent in the point $P(1, 1)$ of the parabola with equation $y = x^2$ as a limit of chords. He uses the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ of changes in the x-coordinate of a variable point Q on the parabola tending to P. The slope of the chord PQ in the general case that Q has an x-coordinate $1 + \left(\frac{1}{2}\right)^n$ is equal to $2 + \left(\frac{1}{2}\right)^n$, and this tends to 2 for $n \to \infty$, which may be called the local slope of the curve in P. This is an example of 'the road from discrete to continuous' which in my eyes could be—or should be—an important principle in the teaching of elementary calculus.

Another remarkable point in Van de Vooren's book is the inclusion of many practical applications, like Newton's law of cooling, atmospheric pressure related to altitude, the working of a windlass, the refraction law of Snell, current intensity related to resistance, the harmonic movement. In short, one may conclude that this book was a real treasure for a good teacher. But how many teachers would have used this book or similar teaching resources for calculus? I am quite sure that there was only a very small minority of teachers who would spend a number of lessons on tackling these first principles of calculus.

Anyway, pioneers like Vaes, Cikot and Van de Vooren initiated and promoted thinking about curriculum reform, and, as a result of this, in 1926 a report about a new curriculum was published in *Euclides*, the Dutch magazine for mathematics teachers. The committee responsible for this report consisted of four people of which the president and secretary, H. J. E. Beth and E. J. Dijksterhuis, were highly qualified teachers in mathematics. Both had published a number of scientific books. Beth wrote a history-based book about non-Euclidean geometry. Dijksterhuis wrote a book about the mechanisation of the world view, which is his most famous book and for which he received the prestigious P. C. Hooft Award. A typical quote from the Beth-Dijksterhuis report is:

> … the aim of placing mathematics education in the service of functional thinking with the aid of graphic representation to display a realistic view of how a quantity changes in relation to a variable quantity, leads inevitably to the teaching of the basic elements of calculus. (Beth, Van Andel, Cramer, & Dijksterhuis, 1925, p. 125) (translated from Dutch by the author)

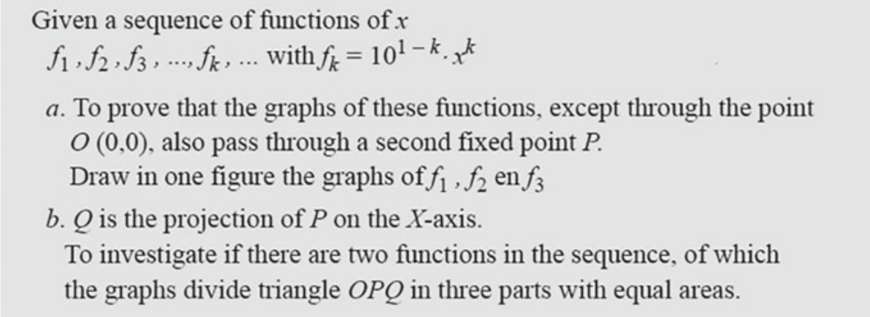
The report proposed explicitly to include elements of differential and integral calculus (with applications in kinematics and in solid geometry) in the new curriculum. This proposal led to a great deal of discussion between supporters and opponents. Some professors at the Technical University in Delft were not in favour of including calculus in the mathematics curriculum in secondary school. One argument was the impossibility of teaching the limit concept in a concise way, and consequently future students in exact sciences would be spoiled in secondary school.

Finally, in 1937 the existing curriculum was adapted and then confirmed by the Ministry of Education. Calculus appeared in the formal curriculum, but the topic was *not* supposed to be a part of the Centraal Schriftelijk Eindexamen (CSE), which is the national written final examination at the end of secondary school. In the Netherlands, this examination has a strong influence on educational practice. Therefore, calculus became an optional subject, which meant that only a few mathematics teachers made time for calculus in their lessons. This situation would continue until 1958!

14.2 After 50 Years of Discussion, Calculus Entered the National Written Final Examination

In November 1948, a historical event took place: the first two-day conference about the didactics of mathematics was organised. One of the initiators was Hans Freudenthal who was also the chairman of the conference. At this conference Van Hiele gave a lecture, titled “An Attempt to Frame Guidelines for Didactics of Mathematics”, in which he quoted a known author of textbooks for physics who had asserted that in Grade 10 the concept of limit, the derivative and the definite integral should be taught for the convenience of the physics teacher. Van Hiele agreed with this assertion and underlined that kinematics, which is taught in physics, is the most adequate entrance to differential calculus.

In January 1954, a committee was appointed to report on the subject of mathematics as a whole and the examination programme, and in February 1955, the committee’s report was published. In the introduction section of the report it was stated that in science education the mathematics component had arrived at an impasse, particularly concerning differential and integral calculus. They posited that it should be possible to teach calculus in a didactically sound way, with no less rigour than for other subjects and yet according to the capacity of the students. To prevent an unwanted development in the future, the committee proposed to stop the calculus curriculum just before the introduction of the number e and the differentiation of exponential and logarithmic functions. The calculus part of the report was received positively.



Given a sequence of functions of x

$f_1, f_2, f_3, \ldots, f_k, \ldots$ with $f_k = 10^{1-k} . x^k$

a. To prove that the graphs of these functions, except through the point O (0,0), also pass through a second fixed point P.
Draw in one figure the graphs of f_1, f_2 en f_3

b. Q is the projection of P on the X-axis.
To investigate if there are two functions in the sequence, of which the graphs divide triangle OPQ in three parts with equal areas.

Fig. 14.1 An example of a calculus problem in the national written final examination of 1963 (translated from Dutch by the author)

The first new national written final examinations in 1961 and 1962 included only one calculus problem. In 1963, two calculus problems were part of the examination; including one very nice one (see Fig. 14.1).

At the same time, early in the 1960s, a new and international movement arose: New Math. In June 1961, the Commissie Modernisering Leerplan Wiskunde (CMLW)[1] was appointed by the Dutch State Secretary of Education. The committee consisted of 18 members—four secondary teachers, twelve university professors, and two school inspectors—, and had a threefold task: (1) to give advice about new subjects in the curriculum which could be first piloted in schools; (2) to produce ideas for in-service courses for senior teachers to update their mathematical knowledge; (3) to investigate whether a special program could be made for mathematically gifted students.

14.3 The Influence of New Math

The first activity of the CMLW was to organise courses for teachers about 'modern mathematics' giving a mathematical background to intended new subjects in the mathematics curriculum for schools. Most of the topics of these courses were influenced by the international New Math movement. The first concern of the CMLW was to raise mathematical competence in line with the demands created by New Math. In the second phase, didactical meetings were organised about how to teach New Math at school. In *Euclides* and during meetings with mathematics teachers, there was much discussion about the pros and cons of New Math. Not everyone was convinced about its blessings. For example, the logician and university professor E. W. Beth[2] was very sceptical about teaching topics like set theory to 12-year-old students. Freudenthal was also sceptical about the New Math movement, but the majority of the CMLW members kept up with the spirit of the times.

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[1] Commission Modernisation Mathematics Curriculum.

[2] E. W. Beth was the son of H. J. E. Beth who was the chairman of the 1926 curriculum committee.

From 1965 on four experiments with New Math were started in a small number of secondary schools. New subjects like parametrised curves (in New Math terms: images of functions from IR to IR^2) and differential equations were proposed. Much attention was given to limit and continuity. In traditional textbooks, the continuity of a function f in a was defined from the concept of limit, but now the main idea was to start with continuous function (for example defined in a topological style with open environments) and then introduce the limit, say of

$$\Delta_a(x) = \frac{f(x) - f(a)}{x - a} \text{ for } x \to a$$

as the value that makes the function Δ_a continuous in a.

A textbook that appeared in the years after the introduction of a new curriculum for calculus in 1968, started with the limit concept for infinite sequences which led to the definition: If for every sequence $(x_0, x_1, x_2, \ldots)$[3] in a reduced open interval around a with $\lim\limits_{n\to\infty} x_n = a$ we have $\lim\limits_{n\to\infty} f(x_n) = b$, then we say $\lim\limits_{x\to a} f(x) = b$. And in this book, this then led to the 'old-fashioned' definition of 'f is continuous in a'.

Fortunately, there were also positive experiences. The most spectacular extension of the calculus program, the teaching of elementary differential equations together with slope fields turned out, beyond all expectations, to be attainable in the classroom.

A didactical point of discussion in those days was the introduction of more-or-less autonomous differentials. The language as introduced by Leibniz was and is still very effective, but did not fit well in the style of New Math. Some purists wanted to skip the use of differentials, but for example Freudenthal posed that teaching calculus without making students familiar with differentials is unacceptable. Another university professor, A. C. M. van Rooij wrote:

> The learning of differentials in school is a tricky question. The calculations with them is easy, but to understand what they really are, the student must have a rather high level of abstraction. (Van Rooij, 1982, p. 81) (translated from Dutch by the author)

The dilemma of using differentials in calculus is that it may lead to a mechanistic style of working.

> It makes no difference what meaning we attach to the differentials, or where we attach any meaning whatever to them. If we define appropriate rules of operation for them and if we employ these rules properly, it is certain that something reasonable and correct will result. (Klein, 2004, p. 215)

If we say that instead of $\frac{\mathrm{d}}{\mathrm{d}x} f(x) = f'(x)$ we may write $\mathrm{d}f(x) = f'(x)\mathrm{d}x$ and reduce the question of differentials to a grammatical one, a real understanding of differentials is not necessary. For example, to treat (in)definite integrals without the use of differentials is needlessly ineffective.

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[3] Undoubtedly the introduction of the limit by sequences fits our intuition best. The difficulty with the step to the 'limit (or continuity) of a function' is the word *every,* because it asks for a proof that shows that no exception is possible.

In the Netherlands, in 1967 the first steps were set towards a reform of mathematics in the higher grades of secondary school. The initial plans included two mathematics strands for Grade 11 and 12 of the pre-university level, which were Mathematics I (calculus and statistics) and Mathematics II (geometry and linear algebra). Mathematics I became compulsory for students who wanted to study exact sciences (including econometrics), agricultural sciences and medical sciences. Mathematics II aimed at students who were interested in a wider scope of mathematics. The calculus part of Mathematics I turned out to be much more extensive than the programme of 1958. Logarithmic, exponential and cyclometric functions were added, as well as parametric curves.

The results in the first new examinations in 1974 were dramatic. The main cause was not the curriculum, but the fact that many more disciplines at university level, like economics, psychology, sociology and even history, required students to follow Mathematics I in secondary school, which was never meant to be a preparation for students in these disciplines. So, in the late 1970s the conclusion was drawn that two new programs were needed for the pre-university level in secondary education: Mathematics A for students who were going to study economic and social sciences and Mathematics B for students who were going to study exact and technical sciences.

14.4 The HEWET Project: A Small Revolution in Pre-University Secondary Education

The Dutch Ministry of Education commissioned the development of two new mathematics programmes for pre-university secondary education and started the HEWET project.[4] In fact, this development was meant to be only a simple reallocation: statistics and probability (which were in Mathematics I), some parts of linear algebra (which were in Mathematics II) and some parts of differential calculus (which were in Mathematics I) would move to Mathematics A. Analytical geometry (which was in Mathematics II) and the more advanced parts of calculus, including integral calculus and differential equations (which were in Mathematics I), would move to Mathematics B. To work on these ideas the HEWET working group was set up. This group existed of two secondary school inspectors, four university professors (two of them representing the exact and technical sciences, the other two representing economic and social sciences), one didactician and two secondary school teachers. The group was advised by three members of the IOWO (Institute for the Development of Mathematics Education), the predecessor of the Freudenthal Institute, M. Kindt, J. de Lange, and G. A. Vonk, the first two of whom would later be responsible for developing new materials for classroom experiments with the new topics and new didactical approaches. A concept version of the HEWET report appeared in 1979 and after a nationwide consultation of the secondary and higher educational sector the final

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[4] Herverkaveling Wiskunde I en II (Re-allotment Mathematics I and II).

report appeared in 1980. With respect to calculus, the report proposed for Mathematics A to include applied differential calculus (no integrals), and for Mathematics B it was proposed to include calculus as it was within Mathematics I.

The need for careful experiments was clear, since the proposed programme for Mathematics A was rather revolutionary because of the applied character preparing students for being able to use mathematics in economic and social sciences. In 1981 two pioneer schools started, with an experimental written final examination in 1983. From that year on, ten new schools joined the project, one year later the next forty schools started and in 1985 the remaining (circa 430) schools followed. The first national written final examination was in 1987.

The following elements of calculus were expected to be meaningful, relevant and within reach of Mathematics A students:

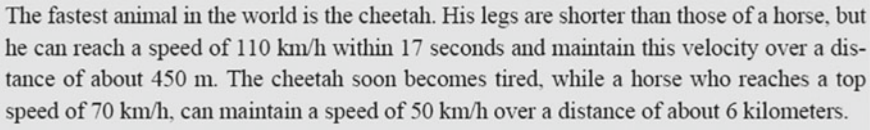
- Trigonometric functions as models for periodic phenomena and trends
- Exponential and logarithmic functions as models for types of growth
- The derivative of a function as a rate of change in a variation of contexts.

Generally spoken, the intended approach was contextual and informal.

As an introduction to this program, three units for Grade 10 were designed at the IOWO, meant as a preparation for both Mathematics A and B. The (translated) titles were respectively: “Logarithms and Exponentials”, “Functions of Two Variables” (both designed by J. de Lange) and “Differentiation 1” (designed by M. Kindt). The last unit was an introduction of the concept of the derivative of a function and had as its subtitle “A Way to Track Changes”.

This last unit consisted of eight chapters: A Changes; B Time, distance, speed; C Measuring slopes; D Slope functions; E Free fall; F Differentiation; G Polynomials; H Maximums and minimums. In one of the meetings for teachers, one participant said that you could skip the first five chapters, because in the F chapter the ‘real stuff’ began. For this teacher differentiation was merely a type of algebraic trick. And so it was for many students at that time. The idea behind the unit was just to give a broadly oriented entrance to differential calculus. The unit was translated in German and after some adaption used in a number of schools in Berlin. The students there, like those in the Netherlands, gave the course a very positive evaluation. A student wrote: “We have never had so much fun with mathematics before.”[5] In the unit (see Kindt, 1982, p. B7), a text about the fastest animal in the world (Fig. 14.2) was used.

This text was followed by only one question:



The fastest animal in the world is the cheetah. His legs are shorter than those of a horse, but he can reach a speed of 110 km/h within 17 seconds and maintain this velocity over a distance of about 450 m. The cheetah soon becomes tired, while a horse who reaches a top speed of 70 km/h, can maintain a speed of 50 km/h over a distance of about 6 kilometers.

Fig. 14.2 The fastest animal

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[5] Translation from German by the author. Literally the student wrote: “Wir haben niemals vorher so viel Spass gehabt mit Mathematik.”

> A cheetah is woken up from his afternoon nap by the sound of horse's hooves. He decides to pursue the horse at the moment that the horse has a lead of 200 meters. Does the cheetah overtake the horse?

In addition, the following hint was given:

> You may assume that the cheetah reaches his top speed after 300 m.

This was followed by a figure of empty graph paper, which suggested tackling the problem by drawing a graph. Most of the students tried to draw a time-distance graph for both animals. The cheetah graph had to be inferred from the brief data in the text and by realising that a variable velocity induces a curved graph. There were also students (and teachers) who started with a time-velocity graph.

In an article that Freudenthal (1979) wrote about this problem, he focused on the hint that the cheetah reaches his top speed after 300 m and compared three models for the initial phase of the cheetah's run (Fig. 14.3).

Each of the areas of the three squares represent a distance of roughly 520 m. The first diagram shows a velocity which grows slowly to then rise rapidly. The start-up distance may be estimated as 65 m (area below the graph is about $\frac{1}{8}$ of the square). The second diagram shows a spectacular acceleration in the beginning, but it takes a relatively long time to reach top speed. The corresponding distance is about $\frac{7}{8}$ of the square, so 455 m. The third model (uniform acceleration) gives a distance of 260 m. The hint of 300 m is not so strange, because the acceleration will not be zero at once in the last second. So, the time-velocity graph will be like the third one, but branch off more smoothly to the highest point. In his article Freudenthal proposed to delete the hint, to make the task even more open.

The examples used in the teaching units developed in the HEWET project (18 in total) inspired the authors of commercial textbooks. So did the cheetah problem. But contrary to the idea of Freudenthal, the authors extended the data and designed many sub-questions or introduced formulas for the time-distance function. One may say that this was a general trend: the textbooks which appeared after the nationwide introduction of Mathematics A were much more structured and less challenging than the units which were successfully used in the experimental phase.

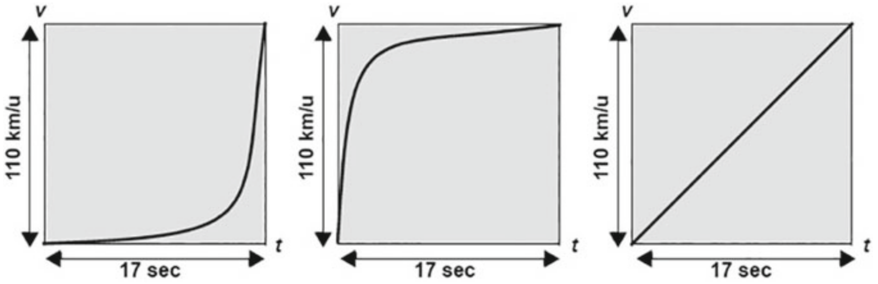


Fig. 14.3 Three graphs of the initial phase of the cheetah's run

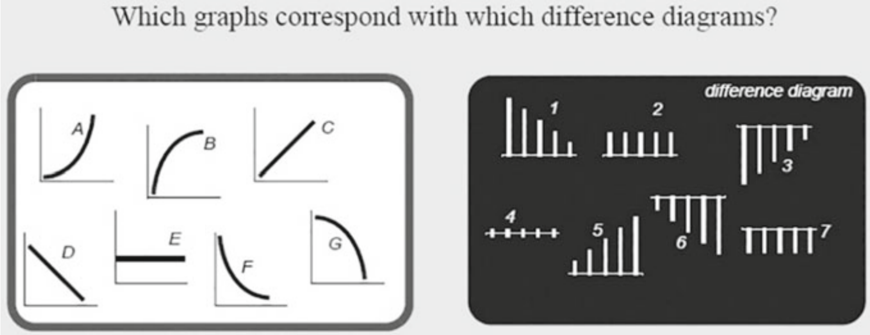
The general conclusion about the content of Mathematics A was positive (De Lange & Kindt, 1984; De Lange, 1987). The strand of applied algebra was judged as the most adequate part, because of the relative simplicity of the mathematical structures and the possibility for the students to model in an autonomous way. Moreover, the main part of this strand had to do with discrete mathematics, which is more concrete—not always easier—than the continuous mathematics. The strand of probability and statistics was undoubtedly very useful in a lot of disciplines and there were more than enough interesting and realistic contexts available to teach. The crown on the course was the strand of testing hypotheses and this strand turned out to be attainable for the majority of the students. The strand of applied calculus was seen as a less successful part. Especially differential calculus with its many rules demanded much more mathematical endurance of the Mathematics A student than the other two strands. Moreover, it seemed to be very difficult to design meaningful contexts for the Mathematics A level in which students could make analytic models by themselves. The common practice in the national written final examination was (and still is) that for the calculus part a mathematical model in the form of an algebraic formula is given, accompanied by a lot of (often simple) questions, but this is done without challenging the students to do a critical investigation of why this formula would be a good idea and without offering students opportunities to construct or adapt a formula.

All in all, one can say that the introduction of Mathematics A thoroughly influenced general ideas about mathematics education. Teachers who thought before that it would not be possible to design written final examinations in more-or-less real-world contexts, were now more-or-less convinced about the feasibility of the RME approach, and in the commercial textbooks for secondary education one tried to implement this approach increasingly.

14.5 Discrete Calculus in Secondary Pre-Higher-Vocational Education

After the introduction of Mathematics A and B in the Dutch mathematics curriculum for secondary pre-university education (VWO), the Ministry of Education decided that there should also be Mathematics A and B in the Grades 10 and 11 of secondary pre-higher-vocational education (HAVO) as well.

Calculus in HAVO Mathematics B would be in the same spirit as calculus in VWO Mathematics A, but in HAVO the applications would be particularly inspired by the exact sciences. For HAVO Mathematics A it was decided not to include differential calculus, but rather to choose for a discrete approach of studying change. The strand was called TGF (tables, graphs and formulas). In this strand, the so-called 'difference diagram' was introduced as a tool to study changes. The teaching experiments with this approach showed no conceptual problems with this idea. The difference diagram



Which graphs correspond with which difference diagrams?

Fig. 14.4 Difference diagrams for studying change (Roodhardt, 1990, p. 23)

turned out to be a surprisingly powerful instrument. One particular good exercise was the following matching problem (Fig. 14.4).

In this exercise students have to relate patterns in graphs to patterns of growth (linear, progressive, degressive). Verbal exercises were also used to make this relation. The exercise was preceded by another activity (Fig. 14.5) that was inspired by a newspaper article.

An illustration of a problem in the experimental written final examination of 1989 for Mathematics A was cast in a realistic context and required conceptual reasoning about growth (Fig. 14.6).

This example appeared in the final examination for the experimenting schools and many students could solve this problem correctly. One student even formulated his answer, using good arguments, in the form of a letter to the farmer! Here, we have to realise that this mathematics programme was not meant for the most gifted students. It appeared to be possible to study the behaviour of functions in connections with local change, with less sophisticated means than differential calculus through a discrete and contextual approach (see also, Doorman, 2005).

The decreasing growth of crime in the Netherlands of the last few years has been turned into an increasing growth.
a. Which of these three pictures fits the text?
b. Write a text for the other two pictures.

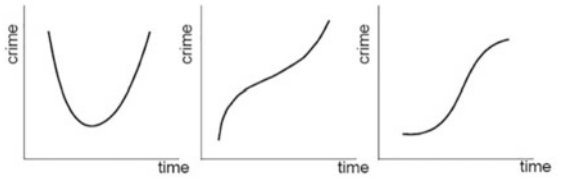
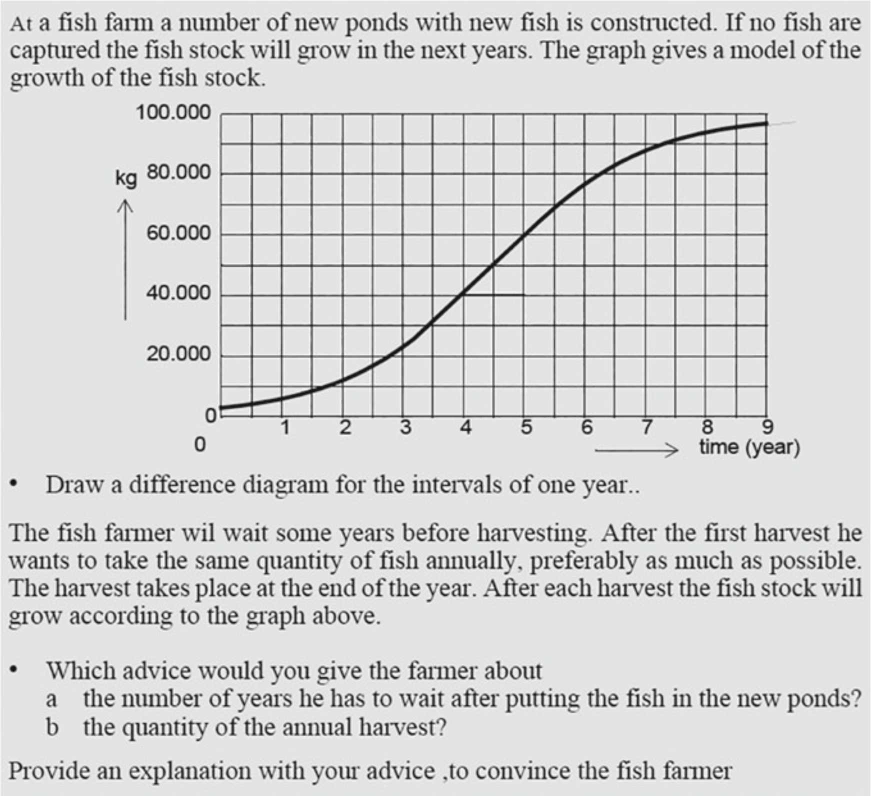


Fig. 14.5 Change in crime (Roodhardt, 1990, p. 17)



At a fish farm a number of new ponds with new fish is constructed. If no fish are captured the fish stock will grow in the next years. The graph gives a model of the growth of the fish stock.

- Draw a difference diagram for the intervals of one year..

The fish farmer wil wait some years before harvesting. After the first harvest he wants to take the same quantity of fish annually, preferably as much as possible. The harvest takes place at the end of the year. After each harvest the fish stock will grow according to the graph above.

- Which advice would you give the farmer about
  - a the number of years he has to wait after putting the fish in the new ponds?
  - b the quantity of the annual harvest?

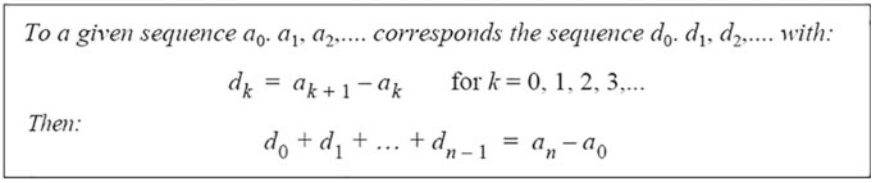
Provide an explanation with your advice ,to convince the fish farmer

Fig. 14.6 A problem in the experimental written HAVO examination for Mathematics A (Translation from Dutch by the author)

14.6 Calculus in Mathematics B for Secondary Pre-university Education

In 1998 upper secondary education was reorganised again. In the last two years of VWO and HAVO the students would now have to choose between four profiles: Culture & Society, Economics & Society, Nature & Health, and Nature & Technology. For the last two profiles the Mathematics B program had to be changed with more emphasis on abstraction, modelling and reasoning. The teaching of calculus was expected to emphasise conceptual understanding with numerical and discrete approaches to differential and integral calculus at the cost of extensive training in procedural fluency (Wiskunde, 1995, p. 70).

In the period 1996–1999, the team of the PROFI project of the Freudenthal Institute designed experimental units for calculus in Mathematics B for secondary pre-higher-vocational education. One of the main ideas learned from the experiences at this school level was to start with the discrete concepts difference (Δ) and sum (Σ) of



To a given sequence $a_0, a_1, a_2, \ldots$ *corresponds the sequence* $d_0, d_1, d_2, \ldots$ *with:*

$$d_k = a_{k+1} - a_k \qquad \text{for } k = 0, 1, 2, 3, \ldots$$

Then:

$$d_0 + d_1 + \ldots + d_{n-1} = a_n - a_0$$

Fig. 14.7 Leibniz's inspiration

functions or sequences. In his retrospective publication *Historia et Origo Calculi Differentialis,* Leibniz looked back to his first work *De Arte Combinatoria* and said that this was his source of inspiration to invent calculus (Edwards, 1979) (Fig. 14.7).

One would call this theorem the fundamental theorem of discrete calculus that can also be visualised graphically (Fig. 14.8).

In the experimental unit (Kindt, 1997), the operators Δ and Σ on sequences were introduced. And the theorem of Leibniz could then be formulated briefly as

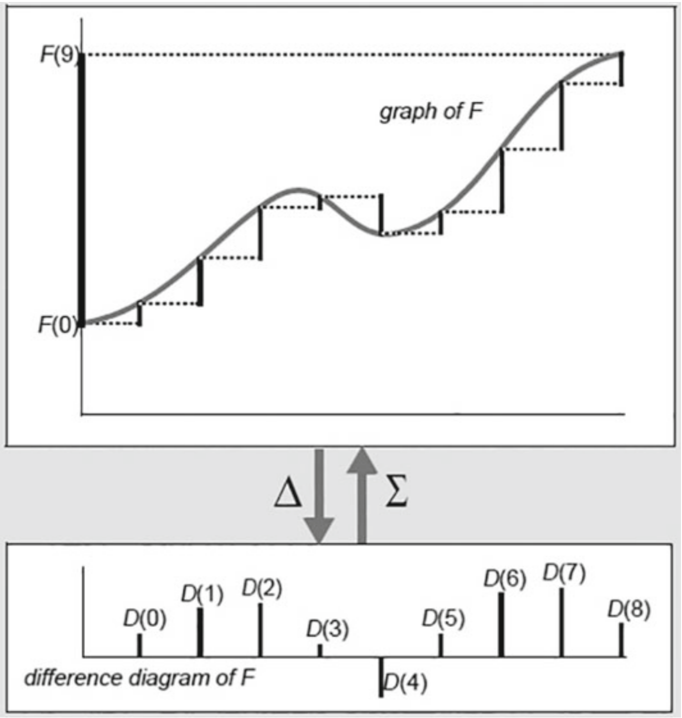


Fig. 14.8 The fundamental theorem of discrete calculus

$$\sum_{k=0}^{n-1} \Delta F(k) = F(n) - F(0)$$

which is the discrete counterpart of

$$\int_0^a \mathrm{d}F(x) = F(a) - F(0)$$

As it is possible to calculate integrals by using one's knowledge about the rules for differentiation, so one can find formulas for the partial sums of a sequence by calculating differences. By expanding $(k+1)^n - k^n$ for $n = 2, 3, 4$ one may deduce

$$\sum_0^n k = \frac{1}{2}n(n+1) \quad \sum_0^n k^2 = \frac{1}{3}n(n+\frac{1}{2})(n+1) \quad \sum_0^n k^3 = \frac{1}{4}n^2(n+1)^2$$

and the last two formulas would then be used for a calculation by Riemann sums of the areas under the curves $y = x^2$ and $y = x^3$.

A special example used in the experiment was the approximation of the area (A) under the graph of $y = 2^x$ on the interval $0 \leq x \leq 1$. The estimation

$$\frac{1}{10}\sum_{k=0}^{9} 2^{\frac{k}{10}} < A < \frac{1}{10}\sum_{k=1}^{10} 2^{\frac{k}{10}}$$

corresponds to a division of the interval in ten equal parts. Using the formula for the partial sum of a geometric sequence, one may write

$$\frac{1/10}{2^{1/10}-1} < A < \frac{1/10 \cdot 2^{1/10}}{2^{1/10}-1}$$

Then by step-by-step refinement—dividing the interval in 100, 1000, etcetera parts—and using a calculator, one gets better and better approximations:

Number of rectangles	Lower estimate	Upper estimate
10	1.393272617	1.493272617
100	1.437700817	1.447700817
1000	1.442195099	1.443195099
10000	1.442645041	1.442745041
100000	1.442690047	1.442700047
1000000	1.442694584	1.442694584

The convergence is visible and this is not an optical illusion. One can simply show that the differences between lower and upper estimate is 0.1, 0.01, 0.001, etcetera! This type of two-sided approximation is probably the best introduction to the mathematical concept of limit. In fact, this is in line with the development of calculus in history which starts with the exhaustion methods of Eudoxus and Archimedes (e.g., Toeplitz, 1963).

Later it was noticed, looking at a chord of the graph of $y = 2^x$ starting from the point (0, 1) and ending in a nearby point, say $(0.1, 2^{0.1})$, that the slope of this chord is equal to the reciprocal of the first underestimate in the column above. This already suggests a sort of inverse relationship between area and slope! Aside, we know from history that Barrow—master of Newton—was the first mathematician who discovered this remarkable relationship, but he used a geometric entrance.

Even later the slope function of $y = 2^x$ was found in a numerical way on the graphic calculator, for example using the input $y_2 = [y_1(x + 0.001) - y_1(x)]/0.001$, with the discovery that the slope function seems proportional with the original function (factor about 0.693). Of course, after this numerical notion, also valid for exponential functions with other bases, came the classical algebraic explanation:

$$\frac{a^{x+r} - a^x}{r} = a^x \cdot \frac{a^r - 1}{r}$$

so, with $\lim\limits_{r \to 0} \frac{a^r - 1}{r} = c_a$

it was found that $\frac{\mathrm{d}}{\mathrm{d}x} a^x = c_a \cdot a^x$.

But what to say about the mysterious constant $c_{a?}$ Using numerical approximations, one may produce a table of values in 9 decimals:

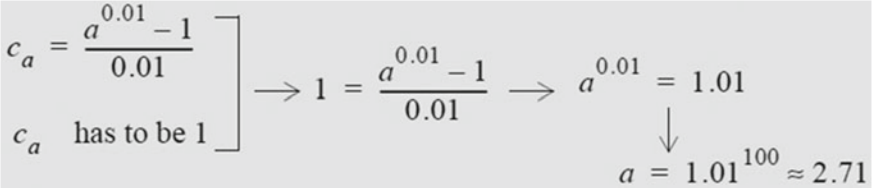
a	c_a
2	0.693147181
3	1.098612289
4	1.386294361
5	1.609437921
6	1.791759469
7	1.945910149
8	2.079441542
9	2.197224577
10	2.302585093

Guided by direct questions, the students could discover a number of relationships such as $c_4 = 2 \cdot c_2$, $c_8 = 3 \cdot c_2$, $c_2 + c_3 = c_6$ and $c_2 + c_5 = c_{10}$. The verification of these equalities was a good exercise in the known rules for derivation, particularly the so-called product rule:

$$\begin{tikzcd}
(ab)^x \arrow[d, "\frac{\mathrm{d}}{\mathrm{d}x}"'] & = & a^x \cdot b^x \arrow[d, "\frac{\mathrm{d}}{\mathrm{d}x}"] \\
c_{ab} \cdot (ab)^x & = & (c_a \cdot a^x) \cdot b^x + a^x \cdot (c_b \cdot b^x) \arrow[d, phantom, "\parallel"] \\
 & & (c_a + c_b)(ab)^x
\end{tikzcd}$$

Looking at the table above one may conjecture that, somewhere between 2 and 3, there must be a base a of an exponential function, for which $c_a = 1$, so a function identical with its derivative. The task for the students—Grade 11—then was to find this number in at least two decimals. Here are the solutions of two students.

Leonie came up with:



$c_a = \frac{a^{0.01} - 1}{0.01}$

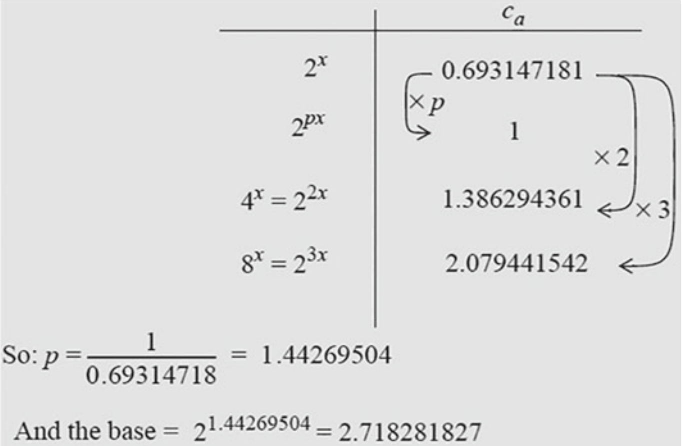
c_a has to be 1

$\longrightarrow 1 = \frac{a^{0.01} - 1}{0.01} \longrightarrow a^{0.01} = 1.01$

$\downarrow$

$a = 1.01^{100} \approx 2.71$

Linda started from discovering $c_4 = 2 \cdot c_2$ and $c_8 = 3 \cdot c_2$. The teacher had asked her for an explanation and had given the hint to use powers of 2. This inspired Linda to come up with:



	c_a
2^x	0.693147181
2^{px}	1
$4^x = 2^{2x}$	1.386294361
$8^x = 2^{3x}$	2.079441542

So: $p = \frac{1}{0.69314718} = 1.44269504$

And the base = $2^{1.44269504} = 2.718281827$

This approximation (of e) was correct in 8 decimals! After these activities and using the discovered property $c_a + c_b = c_{ab}$, the students could discover that $c_a = {}^e\log a$. This is a nice example of the principle of guided reinvention.

The above examples show that the relation between differentiation and integration can be introduced and discussed with discrete and numerical methods. These methods allow teachers and students to calculate and approach specific solutions and characteristics of these solutions. Linda's work showed that students were offered opportunities to approach the value of the number *e* and its role in the derivative of exponential functions. Seven calculus units for the Nature and Health profile at the pre-university level were designed.[6] These units stressed the importance of conceptual understanding and relationships in calculus and illustrated how that can be realised in education with students who are interested in science careers at pre-university level.

14.7 Back to the Future?

In the beginning of the 20th century 'calculus' was sometimes referred to as 'higher mathematics', probably because of its conceptual and technical complexity. Undoubtedly this was the main reason that the implementation of the calculus took so much time. The temptation in teaching calculus is to teach it in a mechanistic way. Interesting concepts are easily overshadowed by algebraic techniques, and the ability to apply calculus in more-or-less realistic situations will then be very low. To illustrate this dominance of algebra I give you now two examples of reactions from classroom.

> Example 1. Teacher: "Last year you learned something about differential calculus, what do you still remember of this?" After a short silence one student reacted: "x squared became two times x."
>
> Example 2. Student: "Differential calculus, I do understand it well, but what does the product rule have to do with it?"

The first example shows that for many students their view on differential calculus is a series of algebraic tricks, not a way to study processes of change. In contrast, the second example shows that after a careful conceptual approach–the teacher was a really good one–the algebraic aspect was experienced as a foreign element. Balancing conceptual understanding and algebraic techniques was, is and will be the dilemma in teaching calculus, perhaps more so than in any other mathematical topic. In an RME approach, such as was realised in the HEWET project, a long conceptual introduction preceded the simplest algebraic rules. Activities involving finding slope functions by measuring slopes using a ruler anticipated the drawing of tangents and calculations with difference quotients. The students were offered the opportunity to discover that the slope function corresponding to a parabola was a straight line. At that time, there was no graphic calculator nor much educational software, but with the help of modern tools there are many more possibilities to build the concept of the derivative in a constructive way (e.g., Drijvers et al., 1996). We have to realise that a numerical approach combined with a geometrical one will give much more insight than the algebraic rules, which of course have their own purpose and utility.

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[6] Six of the units have been adapted and translated in German by a Swiss Mathematics Committee and published by Orell Füssli Verlag Zürich in two books: *Differenzieren–Do it yourself* (2003) and *Integrieren–Do it yourself* (2010).

Leibniz (1684), in his very first article about calculus, propagated what he called the 'Nova Methodus',[7] a means to find maximums and minimums in a great diversity of situations. As an illustration, he gave a new proof of Snell's refraction law. This was a very strong argument to believe in the value of the new calculus. But nowadays, one may object that we have calculators and computers, which after the input of an algebraic model give us the desired result. So, one might say that the main activity in calculus education must be to model, to mathematise change. But on the other hand, algebra is very helpful in proving more general results, for example Snell's law. Of course, one can also use a symbolic calculator–nowadays still not allowed in the Dutch national written final examination–but do we want our students to use rules they do not understand?

At the time of the PROFI project, three main and ideal activities were formulated for Mathematics B: modelling, abstracting and reasoning. Finding a good balance of these activities together with the use of modern technology is the greatest challenge for teaching calculus in a realistic way and as a human activity!

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Chapter 15
The Emergence of Meaningful Geometry

Michiel Doorman, Marja Van den Heuvel-Panhuizen and Aad Goddijn

Abstract This chapter is about a change in geometry education that took place in the last century. We discuss the emergence of meaningful geometry in the Netherlands. Of course, this was not an isolated reform. Worldwide, mathematicians and mathematics educators came up with new ideas as an alternative for the traditional axiomatic approach to teaching geometry. Already at the end of the 19th century, Klein had made a start with this by advocating a transformation geometry, but in this approach the axiomatic structure still played a main role for ordering activities. This was not the case in the work of Fröbel and Montessori who by building on students' intuitions and their attention for students' development of spatial insight were important driving forces towards a meaningful approach to geometry education. In the Netherlands, the pioneers of such a geometry were Tatiana Ehrenfest and Dieke van Hiele–Geldof. Freudenthal was a great promoter of their ideas. For him, geometry is 'grasping space', meaning that geometrical experiences should start with the observation of phenomena in reality. Supported by Freudenthal, from the 1970s on, experiments were carried out in the Netherlands to develop a new intuitive and meaningful approach to geometry education, in which the focus was on spatial orientation. How big the change in geometry education that resulted from these experiments was, is illustrated in this chapter by comparing geometry problems from two Dutch mathematics textbooks: one from 1976 and one from 2002.

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15.1 A World-Wide Change in Geometry Education

The New Math movement which commenced in the United States around the 1960s, and which controlled mathematics education for about two decades in Europe as well, brought about a big change in the mathematics curriculum by giving a central role to mathematical structures based on set theory. This approach corresponded largely with the work of the Bourbaki group that started in the 1930s with reformulating the foundations of mathematics. Although the New Math movement is often referred to as a result of the Sputnik effect requiring the Western world to modernise its education to meet the demands of the upcoming technological innovations, the urge to adapt the content of mathematics education to new developments in science and society had already begun in the 19th century. For example, in 1872, Felix Klein pointed out the importance of attention for structures in mathematics in his Erlanger Programm. He advocated the unification of group theory and transformation geometry (Klein, 1872). According to Botsch (Barbin & Menghini, 2014), Klein inspired most secondary schools in Germany to replace Euclidean geometry with so-called 'motion geometry'. This geometry was a simplified version of transformation geometry.

Klein seemed far ahead of the age of New Math, in which the structural character of mathematics was the central element. Also, by his rejection of Euclidian geometry he was anticipating the ideas of the Bourbaki group. It was almost one century later when during the epoch making Royaumont seminar in 1959, Jean Dieudonné, one of the leading figures of Bourbaki, launched his famous slogan 'A bas Euclide!'. With his slogan Dieudonné drew attention to the outdated content of geometry in secondary schools, still too much based on Euclid, taking geometry as the ideal context for teaching the axiomatic construction of mathematics. This teaching approach did not meet the needs of the new technical society nor the modern language of mathematicians and scientists. In geometry, modern topics were needed, which, to Dieudonné, included vector spaces in finite dimensions. Linear algebra was supposed to provide a 'royal road' to geometry (Choquet, 1964).

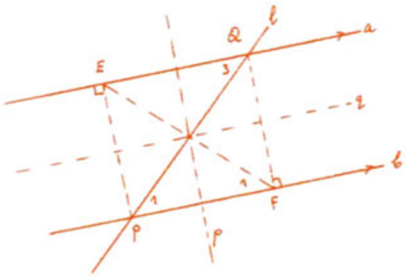
However, these 'modern' views on geometry teaching still had a characteristic of the traditional approach: the structure of a mathematical system was still more or less the main guideline for the learning process. Structuralism dominated mathematics education and resulted in a view on geometry as a means to rush on to analytical geometry, to the world of algebra describing space, and an axiomatic approach of linear algebra. Almost no opportunities were created for students to first develop spatial insight and to become familiar with 'space'.

Yet new developments towards a meaningful approach to geometry education with attention for students' development of spatial insight, had already been proposed early in the 19th century, in particular by German philosophers, pedagogues and psychologists who had been engaged in the content of mathematics education during that era. One of them was the German pedagogue Friedrich Fröbel (1782–1852) who came up with a new programme for geometry education for children aged 4–14, with his blocks, mosaics and other educational toys (Fröbel, 1826). He advocated practical activities for enabling children to get acquainted with characteristics of geometrical shapes in an early phase. The work of Fröbel and also that of Maria Montessori (1870–1952) inspired attention for spatial orientation beginning at the kindergarten level. Choices of activities were based on educational psychological research on spatial insight with young children, and on experiences with playing as a context for geometrical explorations.

15.2 First Steps Towards a New Geometry Education in the Netherlands

Following the international trend, the first steps to reform the then prevailing geometry education were made in the Netherlands as well. Inspired by Klein's Erlanger Programm and influenced by mathematicians like Dieudonne, from the 1960s on, textbooks containing transformation geometry (e.g., Troelstra, Habermann, De Groot, & Bulens, 1962) were also published in the Netherlands. In this approach students were involved in constructing and transforming shapes instead of an emphasis on analysing given angles and triangles and reasoning about congruency. Although the importance of building on students' intuitions was emphasised in the introductions of these new textbooks, the formal axiomatic structure—in this case for reasoning with symmetry—again played an important role for ordering the activities. Figure 15.1 shows an example of a problem that illustrates the logic-deductive reasoning underlying this approach.

The students were expected to draw the two lines *PE* and *QF* perpendicular on *a* and *b*, resulting in the rectangle *PFQE* with diagonals *PQ* and *EF*. This rectangle has two lines of symmetry, *p* and *q*. Reflection in *p* shows that angle P_1 equals F_1 and reflection in *q* shows that angle F_1 equals Q_3. From this it can be concluded that angle P_1 equals angle Q_3.



Given two parallel lines *a* and *b* and line *l*.

Prove that the angles P_1 and Q_3 are equal.

Fig. 15.1 Proving with transformations (from Troelstra et al., 1962, included in Groen, 2004, p. 299)

One of the most systematic investigations into the possibilities of transformation geometry in the early years of secondary school was carried out by the Dutch psychologist A. D. de Groot (De Groot et al., 1968). In this study, the new approach of transformation geometry was compared to the traditional approach through a large-scale teaching experiment involving 12- to 13-years old students who were in their first year of general secondary education. The results showed that between the two approaches in general no difference in performance was found and the students also did not differ in being motivated for geometry.

15.3 Precursors of Meaningful Geometry Education in the Netherlands

Someone who contributed significantly to introducing an approach to geometry education with attention for students' development of spatial insight in the Netherlands, was Tatiana Ehrenfest (1876–1964). She was originally from Russia and lived in the Netherlands for a long time from 1912 on. Ehrenfest had a great interest in teaching and education and gave this interest a practical expression by organising monthly mathematical-didactical colloquia for teachers at her house. Here, spirited discussions were held about the, in her view, fossilised mathematics education in the Netherlands (La Bastide-Van Gemert, 2006, 2015). Among other things, she developed an introductory geometry course with exercises in spatial geometry, titled *Übungensammlung zu einer Geometrische Propädeuse* (Ehrenfest-Afanassjewa, 1931), in which she took geometrical phenomena as a starting point for developing geometrical concepts. With this course, she enriched the domain of geometry with how we experience space. Ehrenfest-Afanassjewa considered activities of looking along two objects, identifying parallel lines in a classroom and lines as light beams and determining angles, basic for an intuitive understanding of the straight line as a mathematical object. In her introduction of the course she motivates the importance of such a phenomenological introduction by contrasting it with the geometrical method that emphasises a logical-deductive approach:

> Den *Weg vom Chaos zum System* und den *Segen*, welchen die *systematische Behandlung* des Stoffes mit sich bringt, zeigen die Logiker nicht, und so erscheint bei ihnen "die Geometrie", als ein von allem Materiellen losgelostes Denkspiel, und anstatt mit *Begriffen* zu operieren – welche ja nur durch eigenen Abstraktionsakt aus eigener lebendigen Erfahrung gewonnen werden können - haben die Schüler mit *Namen* und *Zeichnungen* zu tun, die sie oft an nichts Bekanntes erinnern (Ehrenfest-Afanassjewa, 1931, p. 5, italics in original).
>
> The *road from chaos to system* and the *blessing* resulting from the *systematic dealing* with the learning content, is not shown by the logicians. Therefore, for them "Geometry" becomes a game with thought objects that are isolated of all concreteness, and instead of operating with *concepts* – which can be acquired through the act of abstraction of one's own living experiences – students have to work with *names* and *drawings* which do often not refer to anything they know (Ehrenfest-Afanassjewa, 1931, p. 5; translated from German by the authors).

Halfway through the 20th century, a further impetus to change geometry education in the Netherlands came from the couple Van Hiele (1957) and Van Hiele-Geldof (1957) who proposed introductory activities with concrete materials like folding, cutting, gluing, and paving. As an example, Dieke van Hiele-Geldof started one of her geometry courses with physical cubes. She did not define a cube, but gave the students different kinds of solid cubes and cubes as wire figures of different materials. She discussed with her 12- to 13-years-old students, who were in the first year of the lowest level of secondary vocational education, similarities and differences between these objects, which led to an activity of constructing cardboard cubes. In this process, the students became acquainted with the geometrical objects and with fundamental notions of concepts such as right angle (defined by folding). During subsequent analyses of the objects, other characteristics, patterns and symmetries were identified and relationships were constructed (Van Hiele-Geldof, 1957). This example illustrates a learning process which completely differed from starting with a deductive structure of mathematics. The process that the Van Hieles advocated, passed different levels of understanding, labelled as visualisation (Ground Level 0), analysis (Level 1), informal deduction (Level 2), generalisation and the construction of a formal system of relationships and deduction (Level 3), and rigor (Level 4) (Van Hiele, 1957).

A next step towards a meaningful geometry education was made by Freudenthal who was involved with the work of the Van Hieles. Freudenthal highly appreciated and admired the analysis of classroom observations by Dieke van Hiele and the intuitive approach she promoted in introductory geometry education (La Bastide-Van Gemert, 2006, 2015). Freudenthal was also fond of Ehrenfest's *Übungensammlung*, although he did not agree with the deductive system of teaching geometry that he initially recognised in it. Later however, he understood better what a masterpiece Ehrenfest's publication was (Freudenthal, 1987). For him the relevance of her work was her plea for a resource-based approach to teaching geometry and for the need for an explorative and student-oriented approach to geometry which can be described as 'watching, acting, thinking and seeing'. Geometrical experiences start with the observation of a phenomenon in the surrounding environment. After that you make a model or a drawing to describe the phenomenon with geometrical means. Reasoning about these means will help you to develop mathematics and to understand the modelled phenomena. Freudenthal labelled the research underlying these activities as didactical phenomenology, and he summarised the resulting geometrical experiences and activities more concisely with the term 'grasping space'.

> [G]eometry is grasping space (…) that space in which the child lives, breathes and moves. The space that the child must learn to know, explore, conquer, in order to live, breathe and move better in it (Freudenthal, 1973, p. 403).

By saying this, Freudenthal criticised geometry education as the teaching of structures, an approach that was inspired by New Math and that isolated geometry from reality. He emphasised that apart from some applications of the Pythagorean theorem and some measurement problems about area and volume, the criterion of use entirely failed in geometry (Freudenthal, 1973). As an alternative, he and his colleagues started working on developing a geometry education that later became known as 'Realistic Geometry Education'.

15.4 The Early Experiments: The Focus on Spatial Insight

From the 1970s on, experiments were carried out to develop a new intuitive and meaningful approach to geometry education (De Moor & Groen, 2012; Groen & De Moor, 2013). These experiments were carried out in educational practice through working with teachers and students in real classrooms. Initially, the plan was not to build a learning line for geometry, but to look for themes and problems that result in meaningful mathematical activities.

The designers of this new approach to geometry were focussed on developing the students' understanding of and skills in working with traditionally familiar subjects such as angles, area, symmetry, and the Pythagorean theorem. The intention was to find empirical support for a phenomenological approach to these subjects. To highlight the new character of the geometrical activities, the term 'vision geometry' was used. The experiments in class were aimed at the development of reasoning with vision lines, vision angles, sighting, rays of light, projecting, shadowing and perspective. In particular, this latter subject, perspective, was considered to play a central role in learning basic geometrical concepts and reasoning and the development of a deeper spatial insight. The set-up of the designs was not axiomatic, but based on phenomena and experiences in daily life.

15.4.1 Five Examples of Vision Geometry

The following examples are from tasks designed and tried out during the years 1970–1980. They all reflect the importance of starting with three-dimensional problem situations to evoke and further develop meaningful geometrical reasoning.

15.4.1.1 The Task 'The Singer'

The first example is the task 'The singer' (Fig. 15.2), which was developed, tried out and finally published in a geometry unit for lower secondary education (Schoemaker, 1980). The task is about a singer whose performance is filmed by four cameras. In this task, students can explore the way an object is seen from a certain viewpoint, which is one of the core ideas of the geometry of vision. Because the task fits well within the range of daily available student experiences and intuitions, there was not much need for further explanation. The experiment confirmed that the task is indeed easily accessible for students. Not even a question was needed to put the students to work.

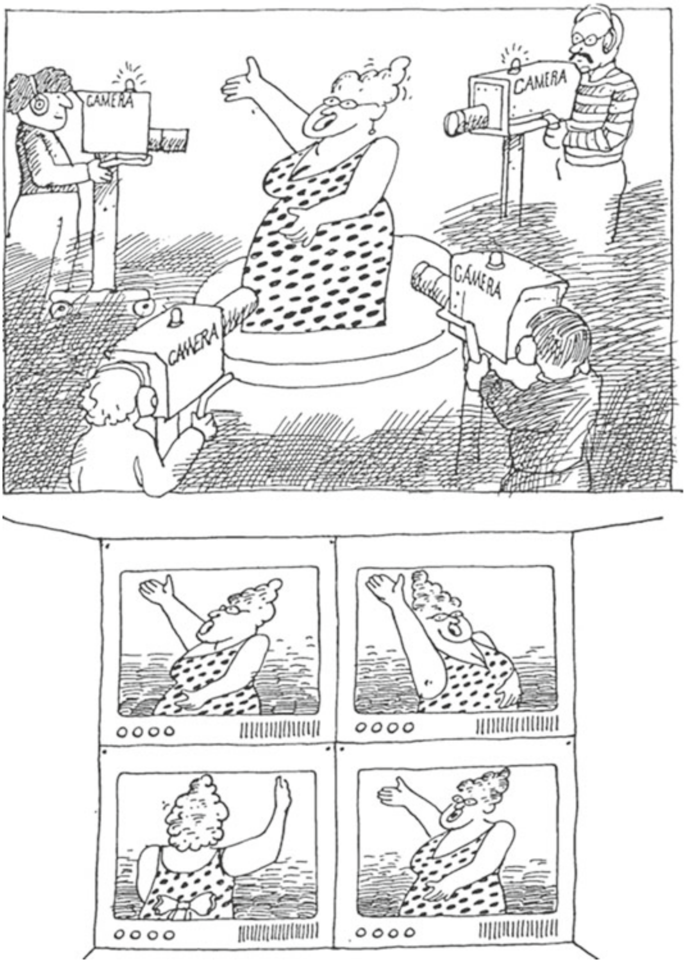


Fig. 15.2 Which image comes from which camera? (Schoemaker, 1980, p. 24)

The students immediately started connecting cameras with the images displayed on the four screens in the control room and they easily determined which camera saw the back of the singer and which camera had the slightly less decent look at the armpit. Deciding which of the four cameras were responsible for the two other images required more advanced reasoning, but the two easy images gave the students a good basis to find which cameras went with the remaining images.

15.4.1.2 The Task ‘Rabbits Behind a Lighthouse’

The second example is the task ‘Rabbits behind a lighthouse’, which is about a boy walking in the dunes (Fig. 15.3). The students were asked whether the boy can see (some of) the rabbits behind the lighthouse, and whether that number changes when he is walking towards the lighthouse. The task evokes the need for drawing lines from the boy to the lighthouse. Students are expected to experience that drawing these lines is difficult in the presented figure and that a top view of the situation would be helpful to be able to decide which rabbits can be seen.

The tasks ‘The singer’ and ‘Rabbits behind a lighthouse’ probe to what extent exploring reality—investigating what we see and how we see things—can be used as a context for geometry. The geometry that emerges is related to becoming aware that space can be projected on a plane, that vision lines and different views of a situation can be used for explanations, and that drawings like top views and side views with vision lines are important tools for reasoning (Goddijn, 1980b).

15.4.1.3 The Task ‘Tower and Bridge’

The third example is the task ‘Tower and bridge’ (Fig. 15.4). This task further elaborates the need for constructing vision lines and reasoning with these lines when a particular situation is shown from another view. This task was used in an experiment for introducing scale and geometrical reasoning in a 3D context (Goddijn, 1979; Schoemaker, 1980). The task was meant to create opportunities for students

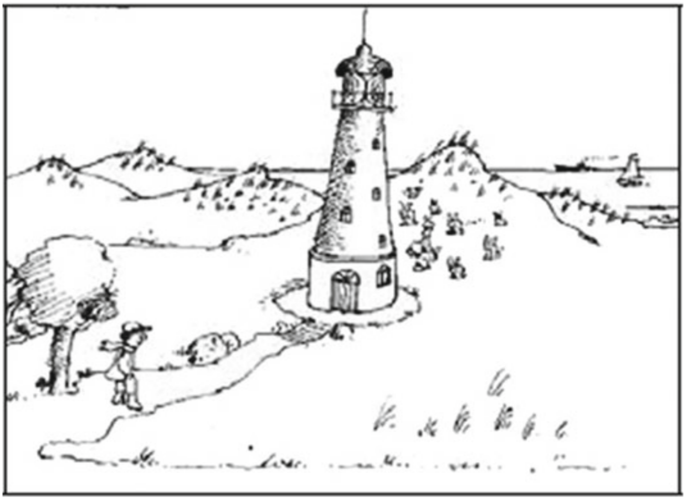


Fig. 15.3 Can he see any rabbits? (Schoemaker, 1980, p. 16)

Fig. 15.4 What is higher, the tower or the bridge? (Goddijn, 1979, p. 2)

to recognise the connection with situations in reality, how they can question them, and how they can use geometry to explain phenomena.

In the left picture, the bridge seems higher than the church, while in the picture on the right the church is higher than the bridge. By constructing a side view of this situation and drawing triangles based on vision lines students can explain this phenomenon and argue that the church must be higher than the bridge. This example shows again how the teaching and learning of geometry can be a constructive and creative activity and that the geometry that focusses on grasping space starts with looking, analysing and creating drawings like top views or side views and the vision lines as tools for explaining phenomena of vision.

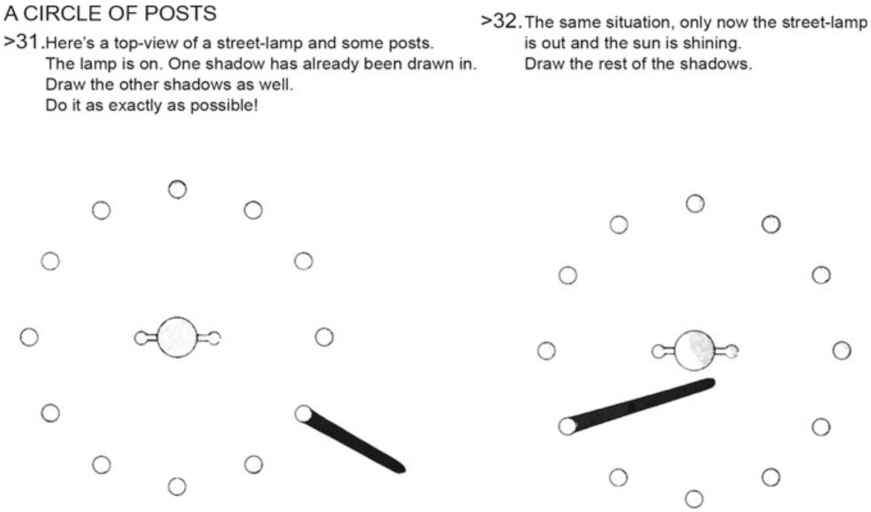
15.4.1.4 The Task ‘Shadows of a Cube’

In the fourth example the geometry also comes with just looking. This ‘Shadows of a cube’ task (Fig. 15.5) is about the polygons that can be created from projections of a cube. The question asked students was, what kind of shadows a cube can have (Goddijn, 1980c). It is obvious that a square must be possible, and a rectangle is also not too difficult. But what other polygons can be created? Can you have a pentagon, hexagon or heptagon as a shadow? Explore and explain.

15.4.1.5 The Task ‘Shadows from the Sun and a Lamp’

The last example is the task ‘Shadows from the sun and a lamp’. This task is also about shadows and addresses different projection methods caused by two different light sources. On the left side of Fig. 15.6 it is night and the street lamp is on. On the right side, it is daytime and the sun is shining. In both cases shadows of posts around the lamp need to be created. Students are asked to describe and explain differences and similarities between the shadows created by the sun and by the lamp.

Fig. 15.5 What kind of shadows can a cube have? (Goddijn, 1980c, p. 20)



A CIRCLE OF POSTS

>31. Here's a top-view of a street-lamp and some posts.
The lamp is on. One shadow has already been drawn in.
Draw the other shadows as well.
Do it as exactly as possible!

>32. The same situation, only now the street-lamp
is out and the sun is shining.
Draw the rest of the shadows.

Fig. 15.6 The different shadows (Goddijn, 1980c, pp. 12–13)

This task illustrates the potential of explorative activities with vision lines, rays of light, projecting, shadows and perspective (see also Goddijn, 1980a). One can also speak of an 'intuitive geometry of the straight line'. The principles of 'parallel perspective' and 'central perspective' and reasoning about what you see and how or why you see it, are the core of this vision geometry. By tasks like these, students are given the opportunity to experience that straight lines of light are the central elements for understanding shadow. Reasoning with these lines in different views, properties of bundles of lines like 'being parallel' or 'all intersecting in one point' come to the fore as natural tools for reasoning in this context. Students are expected to truly experience the characteristics of these situations by experimenting with parallel light beams (sunlight) and a central light source (a lamp). This could intuitively lead to a base for an understanding of invariant characteristics of the two perspective methods.

15.4.2 What These Tasks Have in Common

All foregoing tasks show an approach to geometry education in which fundamental geometrical insights are strongly connected to phenomena that students can experience in everyday life. The tasks that are used for developing these insights are characteristic for Realistic Geometry Education. More specifically this approach to geometry education implies:

- Starting with ‘realistic’ problems
- Considering students as active and creative explainers of problems
- Giving students opportunities for explorative activities through which they can further develop their geometrical intuitions and by which preliminary constructions can emerge
- Eliciting mathematisation in students by focussing on the development of ‘situation models’ like vision lines which bring the students from the informal to the more formal geometry.

We can conclude that these characteristics are in line with the ideas of Ehrenfest-Afanassjewa (1931). In her introductory geometry course with exercises in spatial geometry she also tried to have students develop geometrical concepts from their own living experiences and to prevent that students would work with names and drawings that do not refer to something they know.

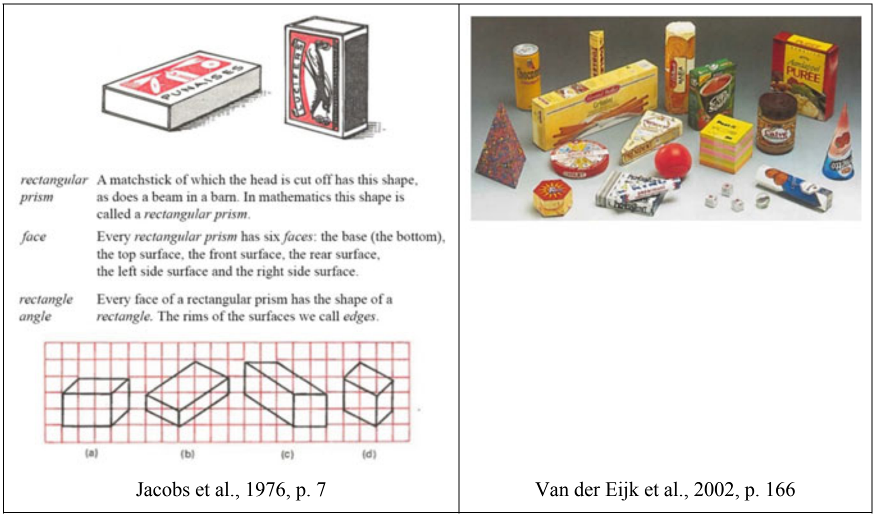
15.5 A Change in Geometry Education: Geometry Problems in 1976 and in 2002

The experiments that have been carried out since the beginning of the 1970s differed hugely from the then prevailing approach to teaching geometry. These experiments brought about a big change in geometry education. Therefore, there is a sharp contrast between geometry as it was offered in textbooks in the 1970s and geometry in current textbooks. Geometry became a discipline that was no longer isolated within the world of mathematics, but connections were made to the daily life situations of students. For example, an important attainment target for students in the lower grades of secondary school was: Students can interpret, describe, spatially imagine and create two-dimensional representations of spatial situations, such as photos, sewing patterns, maps, plans, and blueprints (OC&W, 1997).

What this means for the daily practice of teaching and learning geometry and how this differs from the previous approach to teaching geometry comes clearly to the fore when a textbook series from, for example, 1976 is compared with a more recent one that is published in 2002. The first textbook series is *Moderne Wiskunde voor Voortgezet Onderwijs* written by Jacobs et al. (1976). The second textbook is the series *Moderne Wiskunde* written by Van der Eijk et al. (2002). For the comparison, we took the books for Grade 7, which are meant for the first year of secondary school, and we chose the topics: (a) introduction to 3D shapes, (b) location, in particular the introduction of coordinate systems, and (c) reasoning with lines and angles. Due to space limitations, we can only give a few examples which never will do full justice to the two carefully designed textbook series. Nevertheless, the three examples we provide give a clear impression of the changes that have taken place at the end of the twentieth century in the Netherlands.

The first example is about 3D shapes. As a start for this topic, in the 1976 textbook, the students are shown drawings of two kinds of boxes (Fig. 15.7). The drawings are used to introduce the mathematical terms that describe the elements of 3D shapes (faces, vertices and edges) and characteristics of them. One of the following assignments for the students is to list the edges that are parallel to each other and to learn to draw the mathematical shapes on grid paper. In contrast, the 2002 textbook focusses on providing opportunities to students to explore and analyse shapes that they can see in daily life. Students are stimulated to figure out all kinds of characteristics of the shapes. For example, which objects can roll and what are the similarities and differences between the sides of each of the shapes?

The second example is about the topic of location. Figure 15.8 shows how differently coordinate systems are introduced to students in 1976 and in 2002. In 1976,

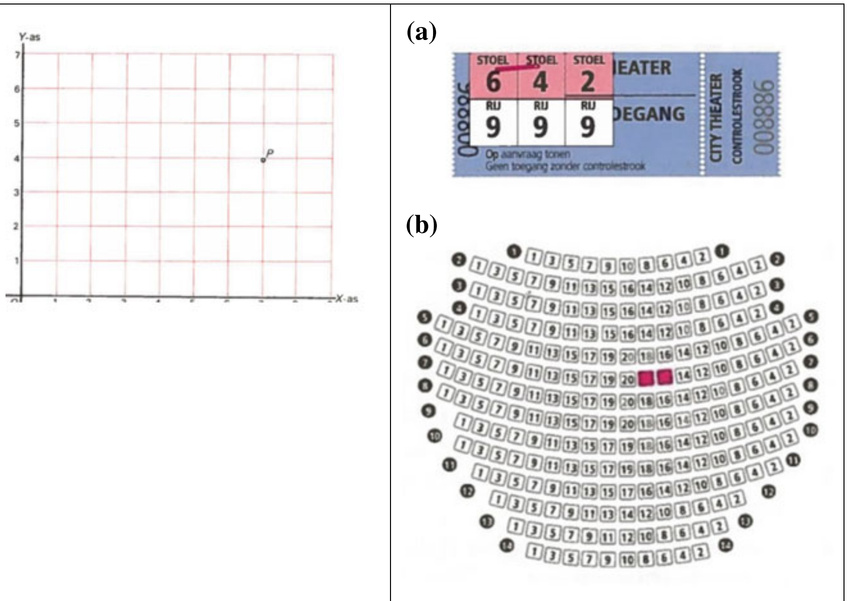


rectangular prism	A matchstick of which the head is cut off has this shape, as does a beam in a barn. In mathematics this shape is called a *rectangular prism*.
face	Every *rectangular prism* has six *faces*: the base (the bottom), the top surface, the front surface, the rear surface, the left side surface and the right side surface.
rectangle *angle*	Every face of a rectangular prism has the shape of a *rectangle*. The rims of the surfaces we call *edges*.

Jacobs et al., 1976, p. 7

Van der Eijk et al., 2002, p. 166

Fig. 15.7 Introduction to 3D shapes



Jacobs et al., 1976, p. 107

Van de Eijk et al., 2002, p. 64

Fig. 15.8 Introduction to coordinate systems

the idea of a coordinate system is posed as a way to organise a plane presented as a grid. The accompanying text in the textbook introduces the students to the language of a coordinate system:

> Start counting from the origin: first seven lines to the right, then four up. We arrive at point *P*. […] The pair of numbers (7, 4) are called the 'coordinates' of *P*.

Next, they have to locate other points following a similar recipe of counting lines to the right and up starting at the origin. In the 2002 textbook, the introduction to coordinate systems is preceded with activities that are connected to the need for such systems. Students are provided with problems in which they can use a coordinate system for reasoning about locations in daily life situations. In the problem from the 2002 textbook, the context of seating people in a theatre is used. The students are asked (a) to figure out where the seats are when you have bought tickets that tell you the chair number and the row number, and (b) to determine what information will be on the tickets when you are seated on the two coloured locations on the floor map of the theatre.

The third example illustrates the differences between the introduction in both textbooks of reasoning with lines and angles. In the 1976 textbook (Fig. 15.9 on the left), the students have to explain that triangles *ABC* and *CDA* are congruent.

In the 2002 textbook (Fig. 15.9 on the right), the topic of reasoning with lines and angles has changed into reasoning about vision lines and angles starting in 3D

Jacobs et al., 1976, p. 126

Van der Eijk et al., 2002, p. 14

Fig. 15.9 Reasoning with lines and angles

contexts. Doing geometry is not limited to reasoning with lines and angles in the plane, but can also start with spatial situations that refer to reality. The students are provided with a picture showing the top view of a room in which a boy and a girl are sitting and showing a garden where there is a cat and two birds are flying around. In the room, there are two windows. The girl who is sitting on a sofa warns that the birds are in danger, but the boy does not understand her. The students are asked to explain this. The purpose of the problem is to introduce students to a situation which they can 'organise' with geometrical means. The students are asked to construct top and side views and to draw vision lines and angles in them that can be used to explain what is seen and how it is seen in reality.

Another remarkable difference between the 1976 and the 2002 textbook is how the topics are ordered. The 1976 textbook starts with teaching the names of 3D shapes on page 7 (see Fig. 15.7). Many pages later, on page 107 (see Fig. 15.8), this is followed with the introduction to coordinate systems and finally, from page 126 (see Fig. 15.9) on, reasoning with lines and angles in the plane is addressed. In contrast, the sequence in the 2002 textbook is the other way around. Here, the introduction to reasoning with vision lines and angles is situated in the beginning of the textbook, on page 14 (see Fig. 15.9). Later, on page 64 (see Fig. 15.8), coordinate systems are introduced with reference to coordinate systems in various real situations. Only in the end, on page 166 (see Fig. 15.7), spatial shapes are explored and geometrical terms for describing these shapes are introduced.

Although in the 2002 examples many of the original ideas for a more meaningful approach to geometry education that were developed in the years 1970–1980 can be recognised, the ideal of geometry as a real constructive activity appeared to be difficult to implement in textbooks. The design of rather closed tasks is more feasible in textbooks than having open tasks that ask for classroom experiments and discussion. Take, for example, a task that deals with the concept of vision angle. Getting a good understanding of this concept requires that it is really experienced through a whole class activity and interactive discussion in which so-called 'why-questions' are asked. However, such questions are often missing in textbooks. Also, in class, attention is seldom paid to reasoning with vision lines and demonstrating their use.

The task that mostly reflects the ideas behind the experiments that started in the 1970s is the task on the right in Fig. 15.9, where the students are provided with a top view of a room and an adjoining garden where birds seemed to be in danger. The power of this task is that the students are offered the opportunity to geometrically organise the situation to understand and know for sure what is going on. According to Freudenthal (1971), this so-called ‘local organisation’ is the way to develop the concepts and reasoning schemes and has the potential to create the need for axioms, definitions and a logic-deductive system. A further example of this idea is presented in the next section.

15.6 An Example of Local Organisation: The Nearest Neighbour Principle

One of the reasons for teaching geometry at secondary school is that the deductive system of definitions, axioms and theorems offers an excellent context for students to experience the mathematics of proof, of being sure and of being creative. However, it requires some maturity of the students to really value and use the very precise definitions of geometrical objects and to understand which constructions are allowed. Therefore, we think it is appropriate to deal with this formal geometry with students who are in the higher grades of pre-university secondary education and who have chosen a science or technology track. Nevertheless, for these students as well, geometry should not start on a formal, abstract level, but with problems that are experienced by the students as real problems and that create the need for further formalisation. The local organisation at the problem level that is necessary for this can be considered as a geometrical activity to re-invent principles of Euclidean geometry. How this works is illustrated by the following example which originates from a unit designed by Goddijn et al. (2014) meant for students in the higher grades of pre-university secondary education (see also Goddijn, 2017).

The task in Fig. 15.10 is part of a series that deals with the topic of the nearest neighbour principle. The so-called ‘Voronoi diagrams’ that can be used to express this principle have many applications that are relevant in reality; for example, in the case of resolving territory conflicts.

To introduce the notion of the nearest neighbour principle the students are shown a map of a desert with five water wells (see Fig. 15.10). The students are asked to colour areas in the desert in such a way that for each possible point (e.g., point J) in a coloured area the corresponding well should be the one that is the closest to that point.

This situation is expected to evoke strategies, like drawing circles and lines. Students are challenged to find the borders between the areas and discover then that these borders seem to be straight lines, which meet each other in one point. After solving

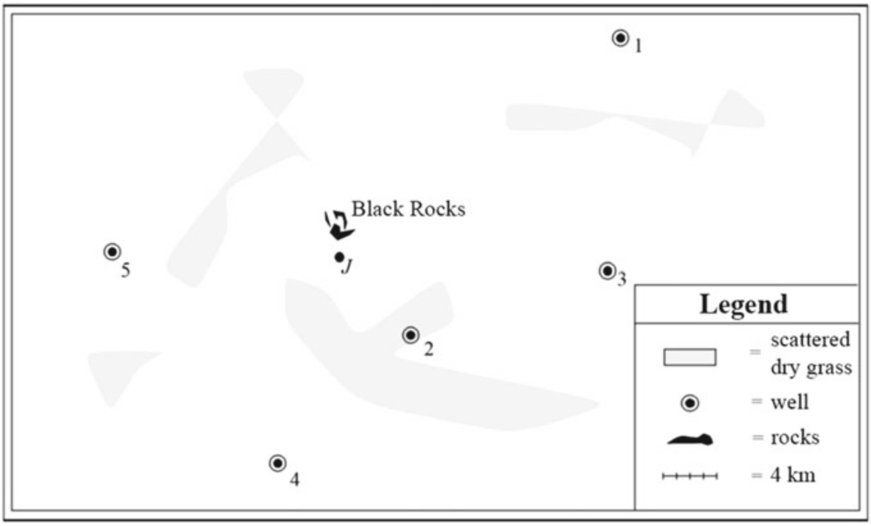


Fig. 15.10 The task ‘Desert map’ (Goddijn et al., 2014, p. 45)

a series of such contextual problems, the focus is changed towards the mathematical characteristics of the diagrams. One of the questions for the students is why these lines, which are called ‘Voronoi edges’, always meet in one point (Fig. 15.11).

Initially this sounds like a useless question, because it is quite obvious for the students that this is the case. Nevertheless, they are invited to look for an answer to this *why* question. To tackle this question, they have to realise that the Voronoi edge between, for example, the centres A and B (the wells in the desert problem) is a set of points P for which the distance to A equals the distance to B. This can be expressed with the distance notation $d(..,..)$. Then, this verbal description is written down as: $d(P, A) = d(P, B)$. This description defines the property of the Voronoi edge and makes the proof that these edges always meet in one point rather straightforward. Assume there is a point M that is the meeting point of the Voronoi edge between A

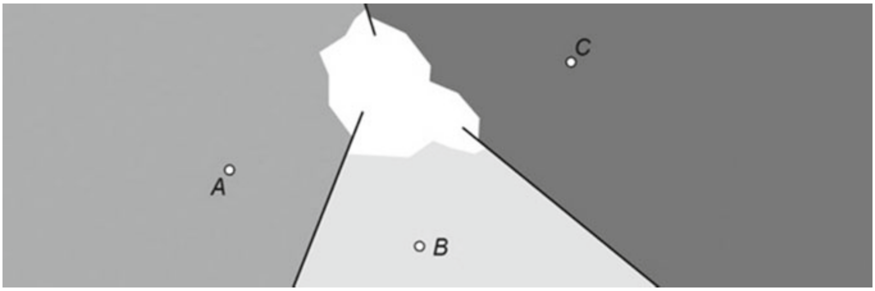


Fig. 15.11 How do three Voronoi edges meet?

and *B* and the Voronoi edge between *B* and *C*, then we have $d(M, A) = d(M, B)$ and $d(M, B) = d(M, C)$. This means that $d(M, A) = d(M, C)$, so *M* is also on the Voronoi edge between *A* and *C*. Consequently, it can be concluded that the three Voronoi edges meet each other in one point *M*.

However, this is not a full proof. Actually, in this proof it is assumed that there is a meeting point of the Voronoi edges which we started with (the edge between *A* and *B* and between *B* and *C*). Yet it might also be possible that this is not the case. So, there is a gap in the argument. Students can detect this, because they already have experienced that when *A*, *B* and *C* are in line, that the Voronoi edges between *A* and *B* and between *B* and *C* are parallel and do not meet. Of course, this can be considered as an exception. Nevertheless, again we can ask whether the proof is complete. Are we sure that the Voronoi edges meet in all other cases?

For example, if there are Voronoi edges that are curved, then it is possible that they do not meet. So, that means that next it should be proved that the Voronoi edge of *A* and *B* is *always* a straight line. This proof is difficult, because it seems so obvious that the Voronoi edge of *A* and *B*, being the collection of all points with equal distance to *A* and *B*, is similar to the perpendicular bisector of *A* and *B*, which is a straight line. But does this mean that if a point is *not* on the perpendicular bisector of *A and B*, that is, not on $pbs(A, B)$, that then this point is also *not* on the Voronoi edge of *A* and *B*.

Suppose, point *Q* is not on the perpendicular bisector of *A* and *B* (see Fig. 15.12). When *Q* is on the left side of the bisector, then the line from *Q* to *B* meets the bisector in *R*. Because *R* is on the bisector it can be concluded that $d(A, R) = d(B, R)$. As soon as it can be established that $d(A, Q) < d(A, R) + d(R, Q)$, it can be inferred that $d(A, Q) < d(B, R) + d(R, Q)$ and consequently that $d(A, Q) < d(B, Q)$. This proves that *Q* does not belong to the Voronoi edge of *A* and *B*. The only thing to be determined is

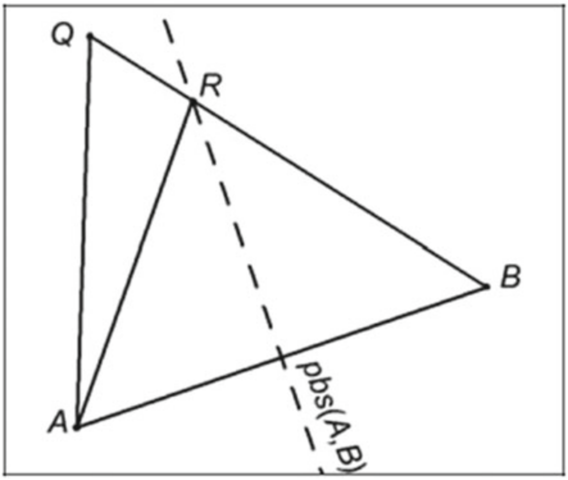


Fig. 15.12 Point *Q* is not on the perpendicular bisector of *A* and *B*

whether $d(A, Q) < d(A, R) + d(R, Q)$. The famous triangle inequality says that this is true if A, Q and R are not on a line.

Students can experience now that there is a bottom in this process of asking *why*-questions towards more fundamental elements, and that this bottom is chosen consciously. In a course which takes distances and 'the nearest neighbour principle' as a topic of departure it is natural to take the triangle inequality as one of the basic truths. However, some protest can be raised against this choice by students who defend the Pythagorean theorem as being a more sure thing. In that case, students can be kindly requested to derive the triangle inequality from the Pythagorean theorem.

The aforementioned example of local organisation around the nearest neighbour principle and the Voronoi diagrams, illustrates the path from exploration to geometry as a logic-deductive system. First, the students have to answer the question about the three meeting Voronoi edges, then they have to explore the character of the Voronoi edge and answering the question whether it is similar to the perpendicular bisector, and finally they arrive at something that is more fundamental and belongs to the geometry as a logic-deductive system: the triangle inequality (Fig. 15.13).

This approach contrasts with a traditional approach starting with the known things at the bottom of the logic-deductive system and building step-by-step theorems with logical arguments. This is the path in which the teaching of geometry stays within the logic-deductive system (Fig. 15.14).

The local organisation described in this section that emerges from the explorative solutions of situational problems results in a reflection on the kinds of definitions that are needed to be able to prove theorems and to establish a strong foundation for (deductive) reasoning. That process guides students from situational problems into the world of geometry and supports them in the development of heuristics for searching for answers to why-questions.

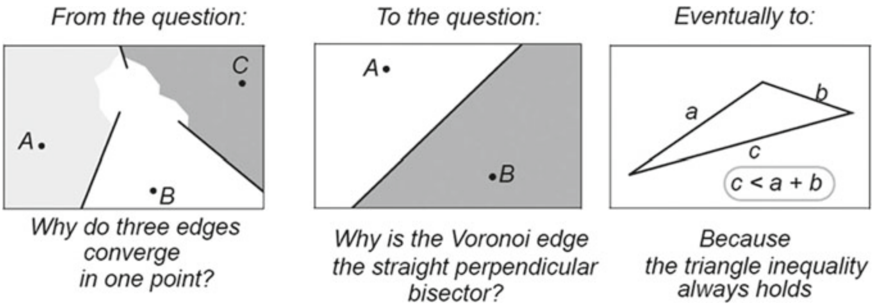
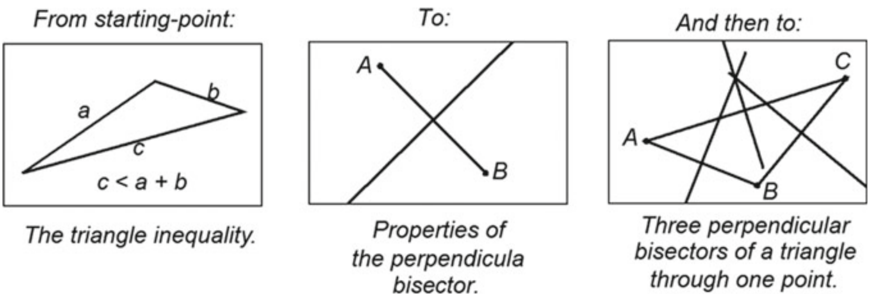


Fig. 15.13 The path from exploration to finding an underpinning argumentation (Goddijn, 2017, p. 30)



The triangle inequality.

Properties of the perpendicula bisector.

Three perpendicular bisectors of a triangle through one point.

Fig. 15.14 The path from axioms to proving statements (Goddijn, 2017, p. 30)

15.7 Final Remarks

In this chapter, we have tried to shed light on the change that took place in the Netherlands in which an axiomatic approach to teaching geometry was gradually superseded by an intuitive and meaningful approach focussed on spatial orientation. Characteristic of the reformed approach, that in the Netherlands later became known under the term ‘Realistic Geometry Education’, is that students are introduced to the world of geometry (the language, the objects and the constructions) by providing them with tasks in 3D contexts that can elicit their intuitive geometrical reasoning. Starting geometry education by developing spatial intuition and ‘grasping space’ was very much supported by Freudenthal (1973) and is exactly at the heart of the ideal of Ehrenfest-Afanassjewa (1931). The result of this reform is that in the Netherlands geometry education nowadays mostly starts with an intuitive introduction (see, e.g., De Lange, 1986; De Moor, 1991; Van den Heuvel-Panhuizen & Buys, 2008), after which it continues in a context-rich course for 12 to 16-year olds (see, e.g., Goddijn, 1991), ending in reflections on definitions and axioms, that is, geometry as a deductive system, by the end of secondary school (see, e.g., Goddijn et al., 2014).

What needs to be stated here is that the reformed approach not only made geometry more meaningful for students, but that this change also widened the scope of the geometry trajectory both in terms of students involved and topics. On the one hand, due to the intuitive introduction some topics, such as vision lines, can now already be dealt with in primary education or even earlier. On the other hand, older students who have reached a certain mathematical maturity can be provided with meaningful imaginable contexts that can be organised locally which gives them access to further learning towards more formal geometry. In this way, at the end of the geometry trajectory, a topic like proofs can become interesting and intriguing for more students.

Furthermore, the change in approach also implies that the structure of the geometry trajectory has changed. Traditionally, structure in a teaching-learning trajectory for geometry was provided by a deductive system starting with formal definitions and basic axioms. This deductive system also dominated the structure of the textbooks, whether they were based on Euclid or on transformation geometry. The traditional trajectory introduced students into a mathematical world without developing their intuitions about this world. Freudenthal and his collaborators criticised this approach to geometry education that is based on geometry as a logic-deductive system. Freudenthal (1973) called this an anti-didactical inversions of learning sequences. This means that this approach takes the final state of the work of mathematicians as a starting point for mathematics education. As an alternative for such an inversion Freudenthal advocated that mathematics education should take its starting point in mathematics as an activity (Freudenthal, 1973, 1991). For him the core mathematical activity was mathematising, that is, organising from a mathematical perspective. Finally, 45 years after Freudenthal wrote his famous paper ‘Geometry between the devil and the deep sea’ (1971), the experiences in the past decades have shown what Realistic Geometry Education can offer students at all educational levels.

Acknowledgements The topic of this chapter is inspired by the work of Ed de Moor, Wim Groen and Sieb Kemme about the development of geometry education in the Netherlands.

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Chapter 16
Testing in Mathematics Education in the Netherlands

Floor Scheltens, Judith Hollenberg-Vos, Ger Limpens and Ruud Stolwijk

Abstract Mathematics testing in the Netherlands focusses on informing schools, teachers, and students about student performance for both formative and summative purposes. The tests are used to monitor whether educational objectives have been achieved and whether content-specific standards have been mastered by the students. In our chapter, we describe the content and objectives of the different national primary and secondary standardised tests. The focus is on the primary function of these tests, but their secondary function where tests are used for accountability is also discussed. In general, the tests are classified into four types: tests to adjust instruction; tests to evaluate proficiency and make decisions about students; tests to evaluate proficiency and make decisions about classes and schools; and tests to evaluate proficiency and make decisions about the quality of the educational system. We show that in practice these types often blend together, as test results are aggregated into class- and school-based indicators at the student level for school evaluation and accountability.

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16.1 Introduction

Dutch primary and secondary schools use a variety of tests, each with its own function. In this chapter, we will focus on the most important mathematics tests, describe the primary function of each of these tests, and explain how the tests are used for accountability. In the concluding discussion section, we will identify the difficulties associated with testing mathematics in the Netherlands.

Testing in the Netherlands focusses on monitoring whether or not educational objectives have been achieved and whether the students have mastered content-specific standards. These standards play an important role in allowing Dutch schools to operate relatively autonomously and design their own programs. The standards for different points in a student's educational career are formulated by SLO (2008), the Netherlands Institute for Curriculum Development, under supervision of the Ministry of Education. Schools need to use these standards as guidelines for setting up the content of their educational programme. So, *what* schools have to teach is determined, but schools can choose *how* they work towards the objectives. A sample of mathematics objectives to be achieved at the end of primary education is shown in Fig. 16.1.

In addition to the relatively broad educational objectives that have been in place for quite some time, more detailed content standards have recently been introduced for basic competencies. As the minimum proficiency level of basic skills in the Dutch language and mathematics[1] in secondary education, and particularly in teacher training programs, were considered too low, the Dutch government introduced content standards: the so-called 'Referentieniveaus' (Reference standards) for Dutch language and arithmetic (Expertgroep Doorlopende Leerlijnen voor Taal en Rekenen/wiskunde, 2007). These standards are described for the main transition points in the Dutch educational system: end of primary education, end of secondary education, and end of vocational education. For each transition point, a foundation level (1F, 2F, and 3F) and an ambition level (1S, 2S, and 3S) are specified. All students in a particular school type or track should be able to master the foundation level, while a substantial percentage of students should also be able to master the more challenging ambition level.

Finally, there is a set of standards in the guiding material and test specifications for the construction of national tests and examinations. In so-called syllabi, more detailed descriptions are given of the objectives. These syllabi are specified by the

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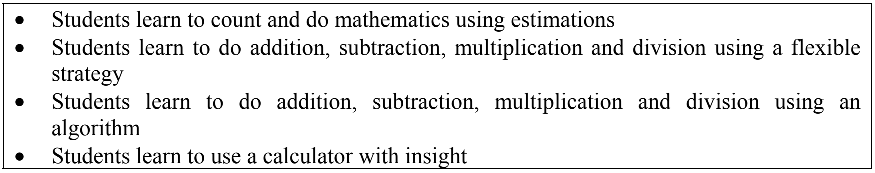
[1] In the Netherlands, a distinction is made between mathematics ('wiskunde') and arithmetic ('rekenen'). In primary school, the term arithmetic is usually used, although this subject also covers other mathematical domains than numbers and operations. In the chapter, we generally use the term 'mathematics', and 'arithmetic' is used to refer specifically to the domain of numbers and operations.

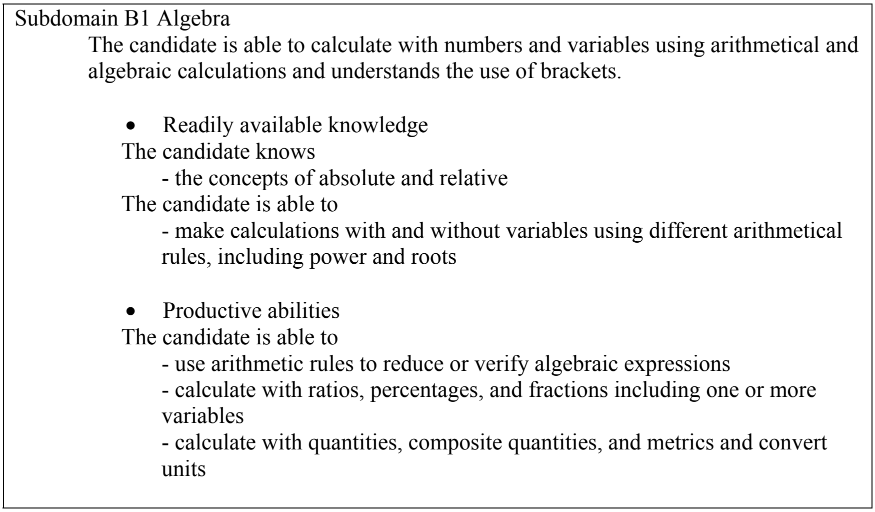


- Students learn to count and do mathematics using estimations
- Students learn to do addition, subtraction, multiplication and division using a flexible strategy
- Students learn to do addition, subtraction, multiplication and division using an algorithm
- Students learn to use a calculator with insight

Fig. 16.1 Sample of objectives of primary mathematics

College voor Toetsen en Examens (CvTE)[2] for all subjects in secondary education and for mathematics and the Dutch language in primary education. They contain examples of potential examination problems to indicate both the difficulty level and the content of the national examinations and tests. Part of the mathematics syllabus for the pre-university level of secondary education in the domain of algebra is shown in Fig. 16.2.

The objectives, syllabi, and content standards together form the base for testing mathematics in the Netherlands. With this framework in mind, we will now describe the different tests used in primary and secondary education.



Subdomain B1 Algebra

The candidate is able to calculate with numbers and variables using arithmetical and algebraic calculations and understands the use of brackets.

- Readily available knowledge

The candidate knows
- the concepts of absolute and relative

The candidate is able to
- make calculations with and without variables using different arithmetical rules, including power and roots

- Productive abilities

The candidate is able to
- use arithmetic rules to reduce or verify algebraic expressions
- calculate with ratios, percentages, and fractions including one or more variables
- calculate with quantities, composite quantities, and metrics and convert units

Fig. 16.2 Part of the mathematics syllabus for the pre-university level of secondary education for the domain of algebra

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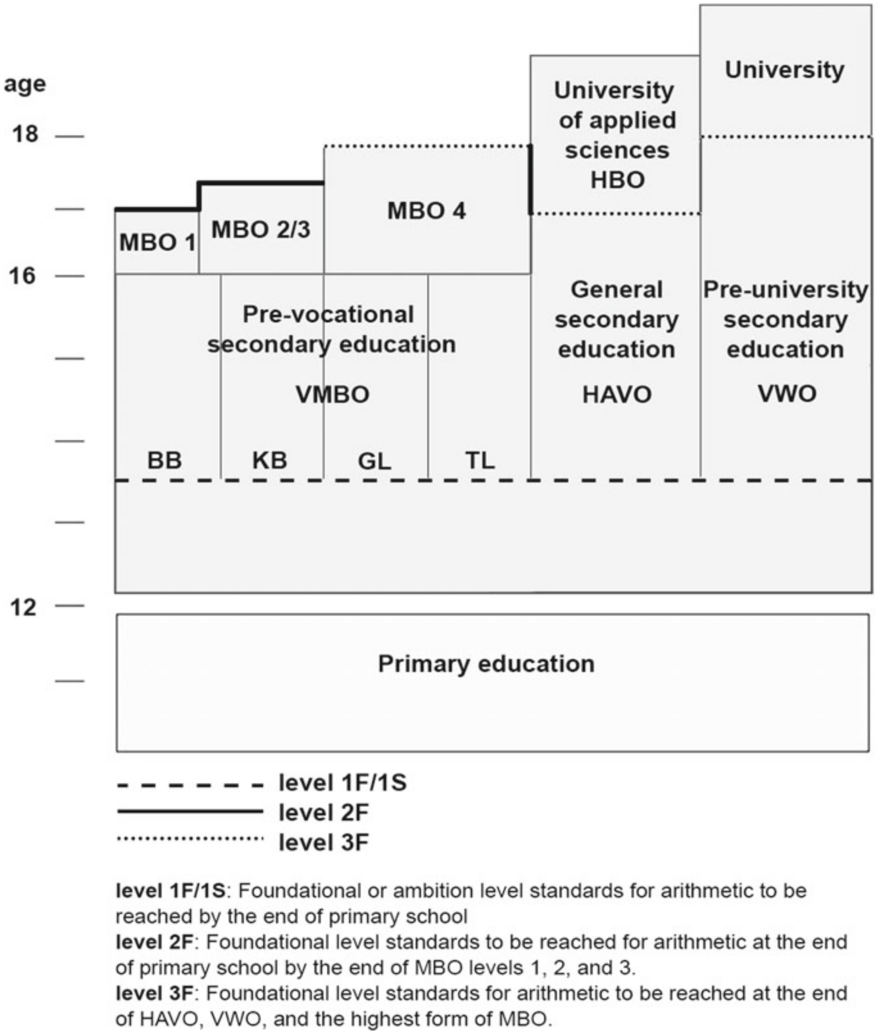
[2] Board of Tests and Examinations.

16.2 Testing Mathematics in the Netherlands

16.2.1 Dutch Education System

Figure 16.3 shows the main elements of the Dutch education system. Primary education includes eight years, starting with two kindergarten years. Children can go to school at the age of four. From the age of five, school is mandatory.



level 1F/1S: Foundational or ambition level standards for arithmetic to be reached by the end of primary school
level 2F: Foundational level standards to be reached for arithmetic at the end of primary school by the end of MBO levels 1, 2, and 3.
level 3F: Foundational level standards for arithmetic to be reached at the end of HAVO, VWO, and the highest form of MBO.

Fig. 16.3 The Dutch school system

Students finish primary education around age twelve and enter secondary education. Secondary education is tracked into three school types:

- VMBO: Pre-vocational secondary education, duration 4 years, subdivided in different levels
- HAVO: General secondary education, duration 5 years
- VWO: Pre-university secondary education, duration 6 years.

Hereafter, students can go to different levels of further education:

- MBO: Intermediate vocational education, duration 1–4 years, subdivided in different levels
- HBO: Higher professional education (also called 'universities of applied sciences')
- University.

At the end of each school level students have to reach particular achievement standards for mathematics/arithmetic (Fig. 16.3).

16.2.2 Primary Education

The main objective in primary education, meant for students aged 4–12 is that students (1) gain, gradually and in meaningful contexts, familiarity with numbers, measures, shapes, structures, and their appropriate relationships and calculations; (2) learn to use the language of mathematics; and (3) are able to deal with various sources of content, including daily life, other courses, and pure mathematics (OCW, 2015).[3]

At the end of primary education, teachers advise students on their secondary education track. To confirm this advice, schools are obliged to administer a test in Dutch language and mathematics. Schools can choose between a number of different tests. When the teacher recommends a lower educational track than that indicated by the test, the teacher's advice can be reconsidered. The end of primary school test also measures whether students have mastered the foundational (1F) or ambition (1S) level standards for mathematics (and Dutch language).

In addition to the test's primary function of indicating a secondary education track or verifying the teacher's recommendation, the aggregated test results for all students can also be used to diagnose areas of improvement for the school (Béguin & Ehren, 2010). For example, they can be used to determine which subjects require more attention and to determine whether measures for improvement have been effective. Most schools use the End Primary School Test, developed by Cito, the Netherlands national institute for educational measurement. The CvTE is mandated by the government of the Netherlands to ensure the quality and proper administration of these national tests and examinations.

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[3] Ministerie van Onderwijs, Cultuur en Wetenschappen (Dutch Ministry of Education).

To monitor the development of primary students in a more formative way, a large number of schools uses a monitoring and evaluation system. One commonly-used system is LOVS[4] developed by Cito. This system contains tests for different subject domains and sub-domains (e.g., Dutch vocabulary and spelling, and mathematics) for Grade 1–6, with assessments twice a year. There is also a system for pre-schoolers (4- and 5-year-old children) for Dutch language and mathematics. The monitoring system for primary school mathematics is a mixture of mostly open-ended items covering different domains. Each assessment results in an ability score.

Because all mathematics tests in the monitoring system are correlated to each other, teachers can compare test results to those of a previously administered test to monitor student growth. The tests are standardised across the country, enabling teachers to compare individual or class test results and growth with the national average. In addition to indicating a student's overall mathematics ability, the tests also provide information for further analysis. For example, the teacher can analyse whether a student scores very poorly or very high in specific areas. Is the result for the sub-domain Numbers and Operations relatively low and for the sub-domain measurement high, then this could indicate that numbers and operations require additional attention.

The Cito Entrance Test for Grades 4–5 with an assessment once a year is an alternative to the student monitoring system. This test uses a multiple-choice format. It provides a complete overview of the student's skills in mathematics as well as in different sub-domains of Dutch language. In Grade 5, the Cito Entrance Test also provides information to indicate the appropriate secondary education track. All the aforementioned tests are also suitable for students with special educational needs.

The Cito LOVS does not assess mathematical fluency (quickly and correctly solving problems). Therefore, schools use several other tests to monitor this aspect of mathematics.

Along with the national standardised tests from Cito and other test providers, schools use other tests for mathematics such as the tests included in textbooks, various exercises, and (digital) test systems.

Appendix A shows some examples of the type of items which are incorporated in the Cito End Primary School Test and the Cito LOVS tests.

16.2.3 Secondary Education

Mathematics is taught in different ways in the different secondary education tracks. In the first few years of VMBO, the lower tracks of secondary education, the focus is on acquiring insight and skills in the sub-domains of numbers and operations, shapes and figures, quantities and measures, patterns, relations, and functions. Because of the vocational focus of this secondary education track, it is important to provide contexts in which mathematics can be applied: contexts related to everyday life,

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[4] Leerling- en Onderwijsvolgsysteem (Student and Education Monitoring System).

other subjects, further education, the workplace, and mathematics itself. In the later years of VMBO, mathematics is only a compulsory subject in the technical sectors; for other students it is an optional subject.

In the higher secondary education tracks, covering HAVO and VWO, of which the highest grades are subdivided into the profiles Nature & Technology, Nature & Health, Economy & Society, and Culture & Society, mathematics is a compulsory subject.[5] There are different mathematics courses targeted at different profiles:

- Mathematics A, targeted at the Society profiles but also permissible for students in the Nature & Health profile; the focus is more on using mathematical methods and on applications of mathematics.
- Mathematics B, targeted at the Nature profiles and compulsory for students in the Nature & Technology profile; the focus is more on the abstract nature of mathematics.
- Mathematics C, exclusively for students in pre-university education in the Culture & Society profile; the course has some overlap with Mathematics A.
- Mathematics D, a supplementary mathematics course in the specialised or optional component of their profile, for students already taking Mathematics B. Schools are not required to offer a Mathematics D course.

Secondary education ends with a final examination in each subject. For most subjects, the final examination comprises a school examination and a national examination; some subjects, such as physical education, only have a school examination. The school examination is prepared by the individual school and is administered in the final school year or years. Tests can be written, oral, and practical. The national final examination is the same for all schools of a certain type and takes place at the same time in all schools. The student's final mark in a subject is the average of the marks in the school and national examinations. In the Appendices C, D, and E, examples of examination items for the various mathematics courses are shown. These items illustrate the significant differences among the mathematics courses.

The national final examinations in the Netherlands are developed by Cito under the supervision of the CvTE. In the lower secondary educational track (VMBO), there are three national final mathematics examinations, differing in level. These examinations exist in both a paper-based and a computer-based version. In the higher secondary educational tracks (HAVO and VWO), there are, as mentioned before, national final mathematics examinations for Mathematics A, B, and C for each school level. Mathematics D has only a school examination. All the examinations are exclusively paper-based.

For secondary education, there are also monitoring and evaluation systems available for mathematics. An example of such a system is the Cito Monitoring System Secondary Education. This system contains four tests which can be administered over the first three years of secondary education. Students can be evaluated on a vertical equated scale (Béguin & Ehren, 2010). Schools have to monitor student progress in a standardised way, but can choose (or develop) their own system of tests. In addition to

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[5] This is not the case for students in HAVO with a Culture & Society profile.

these standardised tests, secondary schools use—similar to primary schools—other tests as well, such as those prepared by the teacher.

16.3 Function of Tests

The previous section of this chapter describes different types of mathematics test. In this section, we will describe the different functions of tests, followed by an outline of functions for the most commonly used tests. Tests can have four different functions: to evaluate and adjust instruction, to evaluate proficiency and make decisions about students, to evaluate proficiency and make decisions about classes and schools, and to evaluate proficiency and make decisions about the quality of the educational system.

16.3.1 Tests to Evaluate and Adjust Instruction

Tests, especially formative tests, ensure that instruction can be adjusted to the students. Tests are designed to provide information not only about the general level of the students but also about student development. Ideally, teachers can use test results to diagnose the specific help or instruction that students need. Examples of tests for evaluating and adjusting instruction are textbook tests and student monitoring systems. The goal of a textbook test is to assess whether students have mastered specific content. When a student answers (almost) all questions correctly, the teacher knows he or she can go on in the textbook. The goal of monitoring systems is to indicate students' current ability levels and growth. These systems contain questions at different levels and in all categories. Teachers can use them to identify specific students who need more instruction or practice and which sub-domains need more attention. In primary education, the student monitoring systems do not aim to classify students. In practice, here the tests are used to identify students who need extra attention or extra challenges. Both in primary and secondary school the tests of the student monitoring systems are also used to choose a secondary education track.

16.3.2 Tests to Evaluate Proficiency and Make Decisions About Students

Tests can also be used to evaluate students' proficiency and make decisions about students. Naturally, these two functions are related. In order to make decisions about a student, the teacher has to figure out whether the student meets the requirements for his or her grade level. This indicates a direction for student's future education.

There are four types of tests for evaluating proficiency and making decisions about students, specifically:

- Tests for selecting students. An example is the examination a student has to pass in order to be admitted to further education, such as succeeding in the national examination for HAVO or VWO, with special requirements regarding the subjects that have been chosen, as a condition for acceptance to higher education.
- Tests for classifying students. Examples are the end of primary school tests. The results of the tests indicate what type of secondary education is best suited for a student.
- Tests for placement. An example is placement in special education. The results of the student monitoring systems are one indicator used to place a student in special education. For special education placements, these results must show that a student's growth is below the growth one might expect for a student at a particular age.
- Tests for certification. The best-known certification test in the Netherlands is the national examination at the end of secondary education.

16.3.3 Tests to Evaluate Proficiency and Make Decisions About Classes and Schools

Tests to evaluate proficiency of students can also be used to evaluate classes and schools. Class growth is central in making decisions about classes. When making these decisions several questions come up. What is the relationship between an increase in ability of a class and the past scores of this class? How is the increase in ability of a class compared to the national increase? But it is also possible to compare the current improvement with previous increases in ability within one school population. How does the improvement of this year's Grade 2 class compare to that of last year's Grade 2? The Cito LOVS incorporates these analyses. Appendix B illustrates and explains a trend analysis at school level.

16.3.4 Tests to Evaluate Proficiency and Make Decisions About the Quality of Education

Schools, school organisations, and also the education inspectorate can evaluate the quality of education. National and international assessments are used to evaluate the quality of education.

An example of a national assessment carried out by Cito is PPON.[6] This assessment is used to evaluate primary school education in detail every five years.

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[6] Periodieke Peiling van het Onderwijsniveau (Periodic Assessment of the Education Level).

Information from this study is used by content experts and decision makers (Béguin & Ehren, 2010). The last PPON for mathematics, carried out in 2011, evaluated 22 different mathematical sub-domains (Scheltens, Vermeulen, & Van Weerden, 2013). In 2014, the responsibility for PPON-like national assessment shifted from the Ministry of Education to the Inspectorate of Education. This change of responsibility will lead to some differences in approach, but the necessity of a national assessment is beyond dispute.

Examples of international assessments are PISA and TIMSS. PISA (the Programme for International Student Assessment) takes place every three years and compares the knowledge and abilities of 15-year-olds in reading, mathematics, and science (Kordes, Bolsinova, Limpens, & Stolwijk, 2013).

The Netherlands also participates in TIMSS (Trends in International Mathematics and Science Study). TIMSS takes place every three years in Grade 4 and 8 and assesses mathematical and science skills. Like in PISA, Dutch students score on average significantly higher than the international average (Meelissen et al., 2012). Table 16.1 summarises the different functions of the most commonly used mathematics tests.

16.4 Use of Tests for Accountability

In the Netherlands, test scores are important for educational accountability. In addition to test evaluations, schools are evaluated by school inspectors who visit the schools. As the Inspectorate of Education is required by law to assess the educational quality that schools offer (including whether the school offers a safe learning environment to students), tests and annual reports are assumed to measure the quality of the school's educational process. The inspectorate uses test scores to identify low-quality schools. Schools that have declining test scores or low test scores over a period of three years are considered to be failing or at risk of failing (Béguin & Ehren, 2010).

Based on the summative or formative function of the tests, it can be assumed that they are valid for measuring the proficiency of an individual student. However, this is not necessarily the case for the aggregated results that are used to indicate educational quality at the school level. Two aspects are important. First, aggregated results as an indicator can misrepresent educational quality if parts of the curriculum are not represented in the tests at hand. For example, the student monitoring tests for primary education do not contain rather open problems in which the student is asked to combine different (mathematical) skills to reach a solution. Nevertheless, a relatively low score in the student monitoring test can still be validly interpreted as a potential lack of quality. Second, one can argue that a test that is a valid measurement of individual students must have different characteristics and content than a test that measures schools (Béguin & Ehren, 2010).

Table 16.1 Outline of functions of the most commonly used mathematics tests

Test	Function						
	Adjusting instruction	Evaluating proficiency and making decisions about students				Evaluating and making decisions on classes and schools	Evaluating and making decisions on quality of education
		Selection	Classification	Placement	Certification		
Testbook test	★						
Student monitoring system mathematics, primary education	★			★		★	
Student monitoring system mathematical fluency, primary education	★					★	
Student monitoring system mathematics, secondary education	★			★		★	
Grade entrance test	★		★	★		★	★
End of primary school test			★				
National final examination					★	★	★
PPON + successors							★
PISA + TIMSS							★

16.4.1 Primary Education

Until recently, the Inspectorate of Education used interim results on the student monitoring system and an end of primary school test as indicators to evaluate the proficiency of primary schools. A new framework for accountability has been available since 2016. This framework focusses on how schools use their test results. The inspectorate no longer sets standards for the interim results of the student monitoring system, but standards are still used for the end of primary school tests (OCW, 2016).

16.4.2 Secondary Education

Since 2016, the Inspectorate of Education has used indicators to judge the quality of a secondary school. First, the inspectorate compares the level of third-year secondary students (Grade 9) to the secondary school track advice that was given at the end of primary school. Next, the inspectorate looks at the percentage of students that pass the first year of secondary school without delay and the percentage of students that pass the last part of secondary school without delay. Finally, the results of the national examination are taken into account. These indicators are compared to a standard established by the inspectorate. The combination of the values achieved on these indicators form a score for the school as a whole. Each of the components contributes to this score and overall it is a balanced system (OCW, 2015). The basic idea of this system of judgement is that schools might do better at one component but worse at another and that this compensates. So if, for example, a school challenges students to achieve a higher level of education than advised, the average scores of these students on the national final examinations can potentially be lower than the scores of students who follow the advised level of secondary school. This will affect the school's indicator for results on the national examinations. Also, it is possible that these students might even need an extra year to finish their secondary education.

16.5 Discussion

16.5.1 Content-Related Issues

16.5.1.1 Testing with or Without Context

Dutch mathematical education has a strong tradition of Realistic Mathematics Education (RME). Mathematics has to be learned in meaningful situations. In the last ten years, a group of experts in mathematics education has advocated for more attention to learning algorithms, teaching fixed procedures for every operation, and teaching mathematics in less meaningful situations. This group has written their own primary education textbooks. As schools have autonomy, they are free to use an RME-based textbook or a mechanistic algorithm-based textbook (or something between the two). This also has consequences for the tests. Today's assessments contain context problems as well as bare number problems. Nevertheless, schools may vary in the attention they pay to bare number problems and context problems. Therefore, it is possible that there are differences in the extent to which the assessments measure what is actually taught in the school.

Another point about RME is that, in problems that relate to real situations, students are faced with more complex situations in which different mathematical competences have to be combined. In tests, however, different competences are tested in isolation. This is partly because tests have to determine whether there are any gaps in mathematical skills. In order to determine this, it is necessary that each question focusses on one particular competence. This is because in more complex computational problems, the outcome is less clear and analysis is more difficult for teachers, making the results less reliable.

16.5.1.2 Should Mathematics be a Compulsory Subject?

In the Dutch educational system, in the lower grades of secondary education all students at each level must do mathematics, but this does not continue through the end of secondary education. In the pre-university secondary school track (VWO), all students are required to do mathematics. For the other levels, mathematics is not obligatory. So, the system requires that pre-university students know about mathematical relations and be able to do some mathematical thinking at a certain level, but for the majority of secondary students, mathematics is an elective. One could ask oneself what this means for society as a whole: will this lead to a social gap (or an increase in an existing gap) between university-educated citizens and others?

16.5.2 Use of Test Scores

Almost all tests, whether monitoring tests, diagnostic tests, or examinations, provide information about student progress towards content standards. In all these cases, mathematical ability is expressed as a value, for example, an ability score. To ascertain whether a student has obtained a content standard, these standards are connected to an ability score. This is a convenient and effective way to access whether a student has attained a particular standard. A disadvantage of this procedure is that mathematical ability is squeezed into one value. If a student scores strongly in one domain, this may compensate for a weakness in another domain. Therefore, students may pass a certain content standard according the test without mastering the specific goals of all the reference standards because they exceed the standards in some other domains. A passing score, therefore, should always be considered into the light of a domain analysis. If a student scores equally (well) in all domains, it may be concluded with reasonable certainty that he or she has mastered the skills described in the reference standards. If the student scores relatively poorly in one or several domains, it is advisable to review the points from the ambition level reference standards in order to establish whether there are gaps to work on with the student.

16.5.3 Use of Tests

16.5.3.1 Autonomy Versus Control

Schools in the Netherlands have the freedom to organise their own teaching programme. As a consequence, they have to account for their choices, for example to the school inspectors. This accountability policy places pressure on schools; they are busy fulfilling all necessary requirements. As a result, autonomy is not what schools experience. By focussing on controlling what schools do, and therefore on collecting test data, there is the risk that the tests partly prescribe the content of the teaching programme. Schools feel that they are judged by the results of the tests, and so they will try to achieve the highest scores. For some schools, this means that the tests determine what they emphasise in their teaching. In these cases, the school does not autonomously decide what they offer their students, but, to put it bluntly, the teaching programme is dictated by the tests.

16.5.3.2 Resistance Against Testing

As mentioned above, schools experience a lot of pressure from testing. Since primary education assessment occurs twice a year for about six subjects, it takes two weeks a year to administer these tests. In addition to that, the use of the student monitoring system is often seen as ‘testing for the school inspectors or the school board’ rather than a monitoring system for students. It is very counterproductive to use these tests for accountability.

Another type of resistance is against tests for pre-schoolers. As most Dutch children enter primary school at the age of 4 (attending school is required from age 5 on), there is a monitoring system for these young students, too. For mathematics, these tests measure some elementary knowledge of numbers, such as the ability to count, adding small collections of objects, and knowledge of mathematics-related terms such as long(er), short(er), first, and last. Like all monitoring tests, these tests are also ability tests. As a result, the test contains questions that the average student can do well, but it also includes questions designed for lower- and higher-than-average-level students. Since the test includes questions for the more proficient students, some of these assignments demand more than strictly required for the average-level goals. Especially when dealing with young children, this calls for a lot of discussions with teachers. On the one hand, they want to give their students an experience of success, thus, they want their students to pass as many questions as possible. On the other hand, teachers often talk about the importance of play for children aged 4 or 5. Some teachers find that, in order to cover all the topics that are in the tests, they cannot let the students play as much as they think is necessary for their students' development. The message that students are also allowed to make mistakes in the monitoring tests is a difficult one and has been insufficiently communicated to schools.

16.5.3.3 Teaching-to-the-Test

The main goal of the (monitoring) tests is to monitor the development of students in order to adjust instruction to their potential and needs. The use of test results to assess the quality of schools is of minor importance. In actual practice, however, it seems that the main purpose of the monitoring tests is for external parties to assess school quality. The result of this is that schools, against all advice, adapt their teaching to the tests and have their students practise for the test. The consequence of this is that the expectations of the Inspectorate of Education rise further, since the average assessment test scores increase. The fact that the average assessment score increases does not mean that mathematics proficiency has automatically increased. In this case, the higher assessment scores are the result of more frequent passes of certain parts of the test, not an indicator that teaching methods in mathematics have improved overall. To fairly determine student ability, it is necessary to update the tests frequently. However, this is (very) expensive. In the future, adaptive testing, in which questions in each test are different for different students, may be one solution. Teaching aimed at specific test problems would then be less feasible for schools. Furthermore, schools should be encouraged to keep in mind the real purpose of the tests.

Teaching-to-the-test is a phenomenon that occurs not just with monitoring tests, but also, for example, with end of school tests and examinations. In these cases, however, it is less 'helpful' for schools because these types of test are updated annually. Therefore, teaching to specific assignments is not possible, and, in fact, it is never advisable.

16.5.3.4 Misuse of Tests

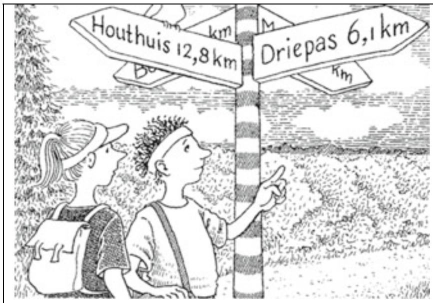
An end of primary school test is administered to ascertain whether a student has successfully completed the curriculum in order to decide whether he or she is ready for a certain type of secondary education. Monitoring tests serve a different purpose. They are, as mentioned, primarily meant to steer teaching efficiently towards students' abilities. However, as ministerial policy on secondary school advice changes, there is a danger that monitoring tests could be given a different, more serious function than their original purpose. In 2015, both the time set for administering the end of primary school test and the aims of this test changed. Before 2015, it was meant to be an objective test, indicating a direction for a student along with teacher advice. The test was administered in February, before students had to choose a secondary school. A large number of secondary schools required a minimum score on the end of primary school test for admission. This made this test a very important one for students and their parents. To avoid misuse of the test, the government decided to move the time for this test to after students have registered for secondary school. Now, teacher advice is the primary factor in secondary school choice, and students can change their choice only when they get a higher than expected score on the end of primary school test. The result of this is that some secondary schools now require minimum scores on the students' monitoring test progress results—again, a misuse, in this case of the monitoring tests instead of the end of school tests. Schools should use the monitoring tests only as a means of diagnosis and not as a selection tool. A positive development is that the results of the monitoring tests are no longer part of the inspectorate's evaluation framework. As a result, the emphasis in schools moves to the primary goal: namely, identifying students' capabilities and challenges.

16.5.3.5 One Test, Different Functions

On a related point, attention should be given to the use of a test for more than one purpose. Different types of tests each have their own goal and contribute to the quality of Dutch mathematics instruction in their own way. One is aimed at informing teachers and schools about student ability, while another test provides information about the school as a whole, and again other tests aim to determine the national level. As can be seen in Table 16.1, many tests are used for more than one purpose. In order to facilitate accurate assessments, each test should have its own goal(s) and, moreover, that goal (or those goals) should be clear for all parties involved. Only in this way can tests be used for the intended purpose, that is, as a means of improving education. Ultimately, all tests serve this purpose. Whether a test is meant to tune teaching to the needs of students or to determine the quality of teaching, all tests should finally contribute to the best possible education to prepare students for their future as much as possible.

Appendix A

Sample of items from the Cito End Primary School Test and the Cito Student Monitoring System Primary School (LOVS).

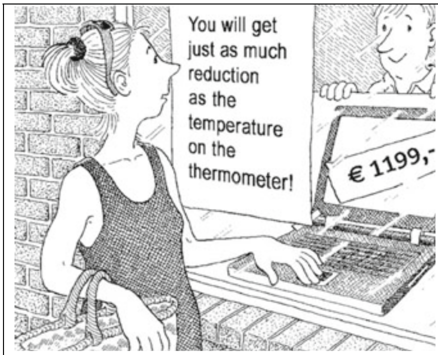


Jean and Peter are walking towards the village of Driepas.

How many metres do they have to walk from this sign to Driepas?

A 61 metres C 6100 metres
B 610 metres D 61000 metres

Cito End Primary School Test (2007)

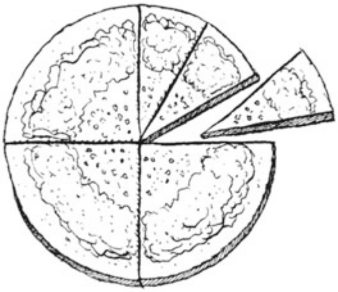


The thermometer indicates 24.9°C. So Wanda will get 24.9% reduction on this comp

How much reduction will Wanda get, roughly speaking?

A € 50.– C € 300.–
B € 250.– D € 400.–

Cito End Primary School Test (2007)



Mother takes a piece of pie.

Which fraction of the pie is this?

A $\frac{1}{12}$

B $\frac{1}{6}$

C $\frac{1}{5}$

D $\frac{1}{3}$

Cito Student Monitoring System, Grade 4 (2009)

Romy wins first prize. She gives her four best friends €100 each.

How many Euros are left?

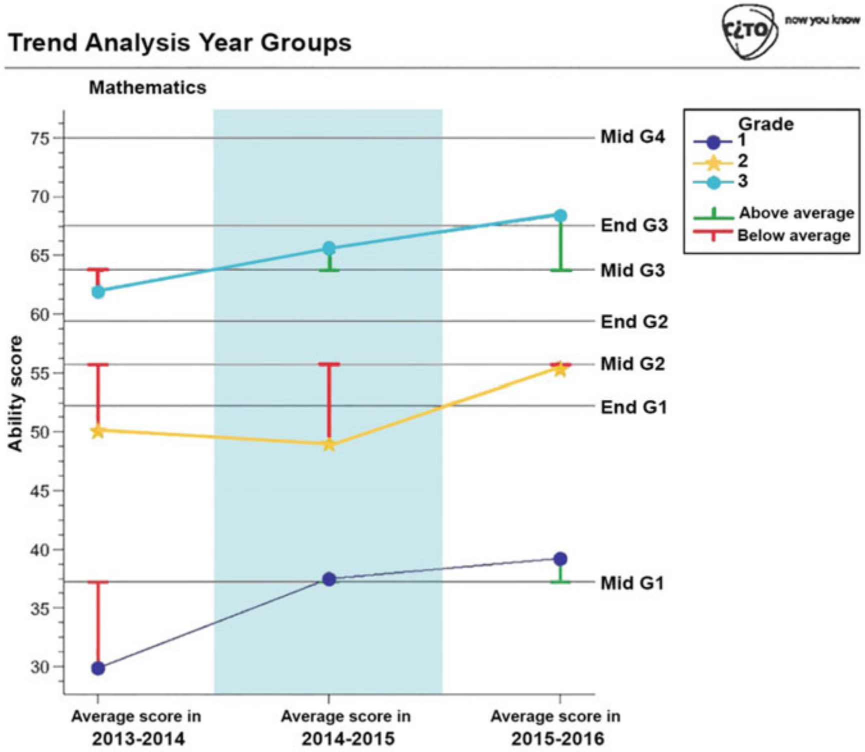
€ ______________

Cito Student Monitoring System, Grade 4 (2009)

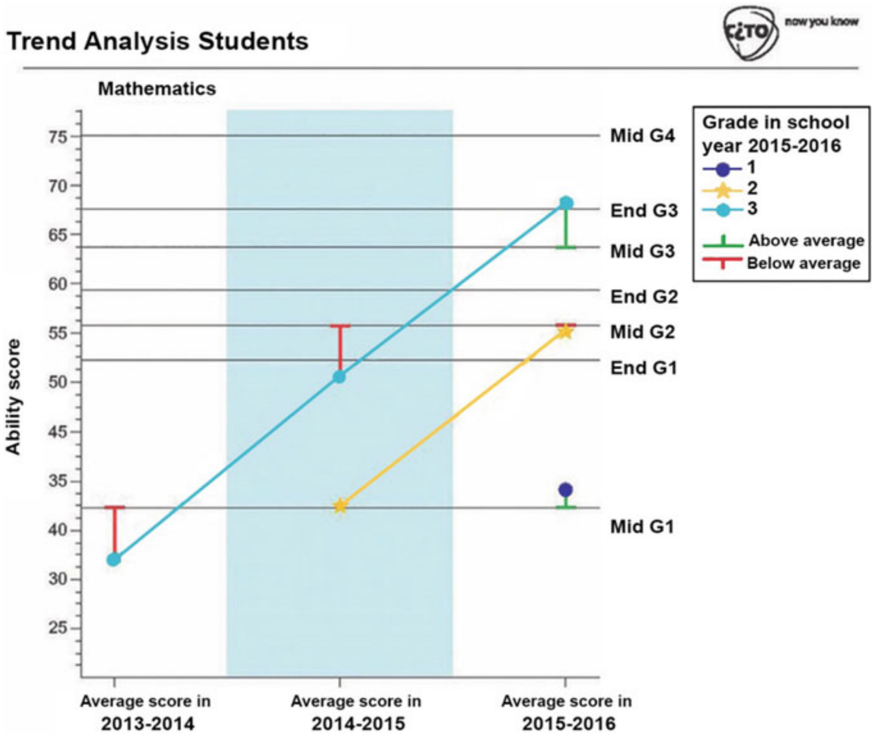
Appendix B

School level analysis based on the scores in the Cito Student Monitoring System Primary School (LOVS).

The Cito LOVS contains a digital module providing a school analysis. This module produces different reports with an overview of the results on school level. Two different types of trend analysis can be made: trend analysis of year groups and of students.



The trend analysis of year groups answers the question: How are the results of this year's Grade 3 (or 1, 2, etc.) compared to the results of Grade 3 (or 1, 2, etc.) in previous years? In this analysis comparisons are made of different groups of students. Answering this question can be effective for monitoring the effect of a change in teaching approach, like an increase in attention for mathematical fluency. If the results of this year's Grade 3 are better than those of previous years, the school can confirm that the change has been effective. When the results of Grade 3 were above average this year and below average last year (horizontal lines indicate the average score), the school has an indication that mathematics education needs more attention. The school can then search for an explanation for this downturn. Has there been a change in student population? Or is a change of teacher a possible (part of the) explanation?



The trend analysis of the students' scores answers the question: How are the results of the students in Grade 3 compared to the results of these students when they were in Grade 2 and in Grade 1? This analysis follows the same group of students. If a group starts in Grade 1 and their scores are above average, one can also expect above average scores in Grades 2 to 4, assuming an average growth. The graph in Trend Analysis Students, shows a group of students scoring below average in Grades 1 and 2 and above average in Grade 3. This shows the growth of this is above average. School can use this information to identify successful factors in their education. What are causes of this growth? Is it due to more attention and time to mathematics in Grade 3? Or is it the effect of a teacher-training programme? Identifying the cause of successes helps schools identify the strengths in their educational approach. The trend analyses identify weak and strong points in education, providing ideas to (further) improve school quality.

Appendix C

Example from national examination Mathematics A (Pre-university Track of Secondary Education, 2014).

The population of Uganda

In 2012, Wali published a study into the population size of the African country of Uganda. According to Wali, this size can be described by a model of the form:

$$U_W = \frac{a}{1 + b \cdot g^t}$$

Here, U_W is the number of inhabitants of Uganda and t is the time in years with $t = 0$ in 1980. Wali used the values $a = 295{,}267{,}612$, $b = 22.78367259$ and $g = 0.965$. In the table, you can see that for the years 1980–2010 his model produced values for U_W that matched surprisingly well with the actual values.

Years	Actual population	Calculated population	Years	Actual population	Calculated population
1980	12,414,719	12,414,719	1996	21,248,718	21,266,298
1981	12,725,252	12,845,405	1997	21,861,011	21,980,197
1982	13,078,930	13,290,330	1998	22,502,140	22,716,074
1983	13,470,393	13,749,915	1999	23,227,669	23,474,471
1984	13,919,514	14,224,592	2000	23,955,822	24,255,934
1985	14,391,743	14,714,799	2001	24,690,002	25,061,014
1986	14,910,724	15,220,984	2002	25,469,579	25,890,262
1987	15,520,093	15,743,605	2003	26,321,962	26,744,234
1988	16,176,418	16,283,127	2004	27,233,661	27,623,485
1989	16,832,384	16,840,024	2005	28,199,390	28,528,571
1990	17,455,758	17,414,779	2006	29,206,503	29,460,048
1991	18,082,137	18,007,881	2007	30,262,610	30,418,471
1992	18,729,453	18,619,830	2008	31,367,972	31,404,390
1993	19,424,376	19,251,129	2009	32,369,558	32,418,352
1994	20,127,590	19,902,293	2010	33,398,682	33,460,902
1995	20,689,516	20,573,841			

Some people were impressed by the degree of agreement between the two series of numbers. “Nowhere does the model deviate more than 2% from reality”, one of them said.

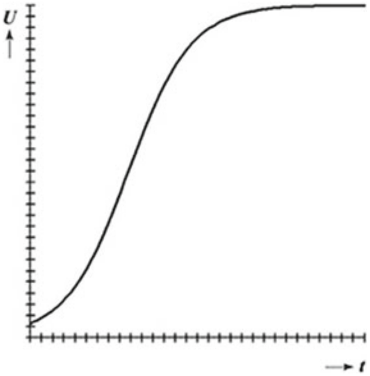
Question 1
Using a calculation, demonstrate that this statement is incorrect by giving a year in which the deviation exceeds 2%.
It is not practical when the constants of a model have many digits in front of or after the decimal point. In the sequel of this problem, we therefore work with the following model:

$$U = \frac{300}{1 + 22.8 \cdot 0.965^t}$$

Here, U is the number of inhabitants of Uganda in millions and t is the time in years with $t = 0$ in 1980.

In the figure, you can see that the model predicts a limit value for the population size of Uganda. The horizontal axis runs from 1980 to 2280.

Figure



Question 2

Explain, without substituting numbers into the formula, which limit value goes with this model.
For the derivative of U one has:

$$\frac{dU}{dt} \approx \frac{244 \cdot 0.965^t}{(1 + 22.8 \cdot 0.965^t)^2}$$

Question 3

Demonstrate this.

Question 4

With the aid of the derivative, investigate in which year the population of Uganda increases fastest according to the model.

The population of Uganda – marking scheme

1	maximum score 3	
	• A calculation of a percentage greater than 2, for example for 1983: $\frac{13749915-13470393}{13470393} \cdot 100\% \approx 2.1\%$	2
	• In 1983 the model deviates by more than 2%, hence the statement is incorrect	1
2	maximum score 3	
	• For large t, 0.965^t approaches (arbitrarily close to) 0	1
	• Then the denominator approaches 1	1
	• Then U approaches to 300 million	1
3	maximum score 4	
	• $[0.965^t]' = 0.965^t \cdot \ln(0.965)$	1
	• $\frac{dU}{dt} = \frac{(1+22.8 \cdot 0.965^t) \cdot 0 - 300 \cdot 22.8 \cdot 0.965^t \cdot \ln(0.965)}{(1+22.8 \cdot 0.965^t)^2}$	2
	• $\frac{dU}{dt} = \frac{-300 \cdot 22.8 \cdot 0.965^t \cdot \ln(0.965)}{(1+22.8 \cdot 0.965^t)^2} \approx \frac{244 \cdot 0.965^t}{(1+22.8 \cdot 0.965^t)^2}$	1
	or	
	• $U = 300(1 + 22.8 \cdot 0.965^t)^{-1}$	1
	• $[0.965^t]' = 0.965^t \cdot \ln(0.965)$	1
	• $\frac{dU}{dt} = -300(1 + 22.8 \cdot 0.965^t)^{-2} \cdot 22.8 \cdot 0.965^t \cdot \ln(0.965)$	1
	• $\frac{dU}{dt} = \frac{-300 \cdot 22.8 \cdot 0.965^t \cdot \ln(0.965)}{(1+22.8 \cdot 0.965^t)^2} \approx \frac{244 \cdot 0.965^t}{(1+22.8 \cdot 0.965^t)^2}$	1
4	maximum score 4	
	• The maximum of the derivative needs to be determined	1
	• Describing how with the GC can be determined for which t this derivative is maximal	1
	• $t \approx 87.8$	1
	• The answer: in 2067 (or 2068)	1

Appendix D

Example from national examination Mathematics B (Higher Secondary Education, 2014).

Two Functions

The functions f and g are given by $f(x) = (x+2)\sqrt{x+2}$ and $g(x) = x(x+2)$

The graphs of f and g intersect in the points A and B.

1 Determine the x-coordinates of A and B exactly.

Point C lies on the graph of f. The tangent line to the graph of f at C has slope 6.

2 Determine the x-coordinate of C exactly.

Two functions – marking scheme

1	maximum score 4	
	• From $x(x+2) = (x+2)\sqrt{x+2}$ it follows that $x = -2$ or $x = \sqrt{x+2}$	1
	• $1x = \sqrt{x+2}$ gives $x^2 = x + 2$ (with $x \geq 0$)	1
	• Describing how the equation $x^2 = x + 2$ (with $x \geq 0$) can be solved exactly	1
	• (The x-coordinates of A and B are) $x = -2$ and $x = 2$	1
	Remark: If $x = -1$ is named as a solution of the equation, award a score of no more than 3 points.	
2	maximum score 5	
	• $f(x) = (x+2)^{1\frac{1}{2}}$	1
	• $f'(x) = 1\frac{1}{2}(x+2)^{\frac{1}{2}}$ (or a comparable form)	1
	• For the x-coordinate of C one has $1\frac{1}{2}(x+2)^{\frac{1}{2}} = 6$	1
	• From this it follows that $(x+2)^{\frac{1}{2}} = 4$ (i.e., $\sqrt{x+2} = 4$)	1
	• This gives $x + 2 = 16$ hence $x = 14$	1

Example from national examination Mathematics B (VWO National Examination, 2014).

Fractional Trigonometric Function

For every a with $a \neq 0$ the function f_a is given by: $f_a(x) = \frac{\sin(ax)}{1-2\cos(ax)}$

1 Determine for which values of a the line with equation $x = \pi$ is a vertical asymptote of the graph of f_a

2 Prove that the graph of f_2 is symmetric about the point $(\frac{1}{2}\pi,\ 0)$

Fractional trigonometric function—marking scheme

1	maximum score 4	
	• One must have: $1 - 2\cos(a\pi) = 0$, hence $\cos(a\pi) = \frac{1}{2}$	1
	• This gives $a\pi = \frac{1}{3}\pi + k \cdot 2\pi$ or $a\pi = -\frac{1}{3}\pi + k \cdot 2\pi$ (for integer k)	1
	• Hence $a = \frac{1}{3} + k \cdot 2$ of $a = -\frac{1}{3} + k \cdot 2$ (for integer k)	1
	• For these values of a one has $\sin(a\pi) \neq 0$ (hence, for these values of a the line with equation $x = \pi$ is a vertical asymptote of the graph of f_a)	1
	Remark: If only the solutions $\frac{1}{3}$ and $-\frac{1}{3}$ are found, award a maximum of 2 score points for this question.	
2	maximum score 5	
	• One needs to prove that $f_2(\frac{1}{2}\pi - p) = -f_2(\frac{1}{2}\pi + p)$ (for every p)	2

(continued)

(continued)

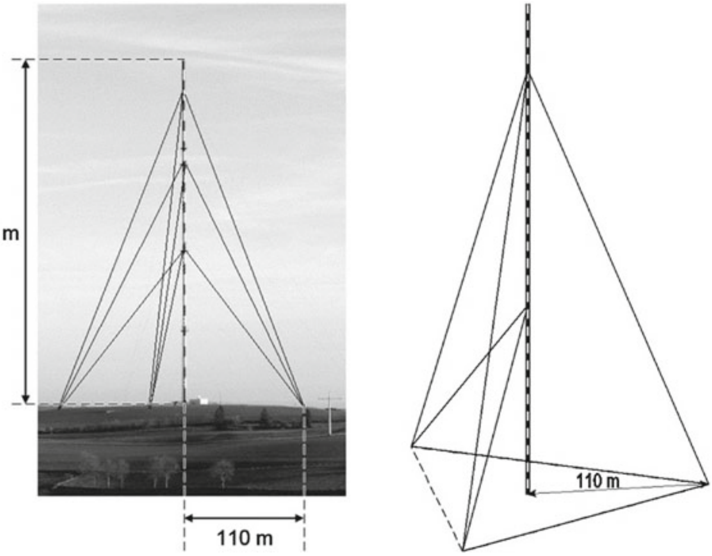
	• $f_2(\frac{1}{2}\pi - p) = \frac{\sin(\pi-2p)}{1-2\cos(\pi-2p)}$ and $f_2(\frac{1}{2}\pi + p) = \frac{\sin(\pi+2p)}{1-2\cos(\pi+2p)}$	1
	• ($\sin(\pi - 2p) = \sin(2p)$ and $\sin(\pi + 2p) = -\sin(2p)$, hence) $\sin(\pi - 2p) = -\sin(\pi + 2p)$	1
	• ($\cos(\pi - 2p) = -\cos(2p)$ and $\cos(\pi + 2p) = -\cos(2p)$, hence) $\cos(\pi - 2p) = \cos(\pi + 2p)$ (hence $f_2(\frac{1}{2}\pi - p) = -f_2(\frac{1}{2}\pi + p)$ for every p)	1
	Remark: If a value is substituted for p, do not award any score points for this question.	

Appendix E

Example from national examination mathematics (Lower Secondary Education, 2014).

Radio Mast

The radio mast of Radio Luxembourg is located in Hosingen, Luxembourg. The radio mast is 300 m tall.



The radio mast is held up from three sides by three guy-wires from each side. See the picture.

The three guy-wires from one side are all anchored to the ground at one and the same point. The distance from the foot of the radio mast to this point is 110 m. In the drawing on the right you see the radio mast together with the highest and the lowest guy-wires.

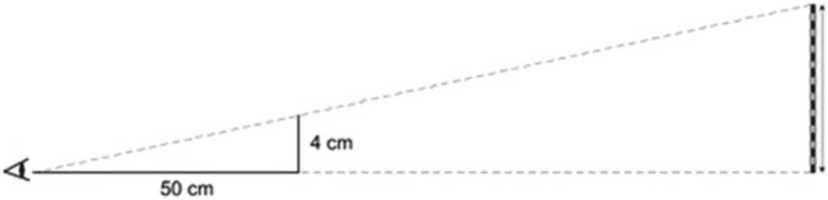
1 The highest guy-wire is attached to the radio mast at a height of 270 metres.
→ Determine how many metres the length of the highest guy-wire is. Write down your calculation. Round your answer to the nearest whole number.

2 The lowest guy-wire is attached to the radio mast at a height of 120 metres.
→ Determine how many degrees the angle between the lowest guy-wire and the ground is. Write down your calculation.

3 The guy-wires are anchored to the ground at equal distances from the radio mast.
On the worksheet a map of the three anchor points is shown.
→ Indicate the place of the radio mast with a dot on the map. Show how you have obtained your answer.

4 The radio mast can be seen from miles around. Agatha wants to know the distance to the radio mast. She stretches her arm and indicates the size of the radio mast with her fingers.

The distance from her eye to her fingers is 50 cm. The height she indicates with her fingers is 4 cm. You see a sketch of the situation.



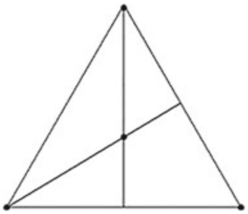
→ Determine the distance from Agatha to the radio mast in whole metres. Write down your calculation.

Radio mast—marking scheme

1	maximum score 3	
	• The length of the highest wire is equal to $\sqrt{270^2 + 110^2} = 291.5\ldots$ (m)	2
	• This is equal to 292 (m) when rounded to the nearest whole number	1
2	maximum score 3	
	• tan angle = $\frac{120}{110}$	2
	• The answer: 47(°) (or more accurately)	1
3	maximum score 3	
	• For example, two medians drawn correctly	2

(continued)

(continued)

<table>
<tr><td></td><td>• The place of the radio mast has been indicated in the correct position.
</td><td>1</td></tr>
<tr><td>4</td><td>maximum score 2</td><td></td></tr>
<tr><td></td><td>• A ratio table such as
<table><tr><td>horizontal distance</td><td>50</td><td>…</td></tr><tr><td>height</td><td>4</td><td>300</td></tr></table></td><td>1</td></tr>
<tr><td></td><td>• The distance is equal to 3750 (m).</td><td>1</td></tr>
</table>

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Chapter 17

There Is, Probably, No Need for Such an Institution—The Freudenthal Institute in the Last Two Decades of the Twentieth Century

Jan de Lange

Abstract In the 1970s, IOWO became well-known in the mathematics education community. IOWO was an institute for the development of mathematics education, with Professor Hans Freudenthal as flag bearer and source of inspiration. For purely political reasons the government decided that there was no need for such an institution in the 1980s, and that all collaborators should move to SLO, the institute in the Netherlands that is responsible for curriculum development. Most people refused to accept this offer. Many letters were written by our international colleagues in order to let IOWO survive. The politicians found a very creative solution: five people were allowed to carry on within the university as researchers (only). In this chapter, I describe how the remaining people took back what was 'stolen' from them. Within ten years the government found that a new very successful institute had been established, and even was 'proud' of this institute for its innovative ideas, and practical uses, based on developmental research.

17.1 Introduction

In 1980 the institute named IOWO[1] was threatened in its existence. The Ministry of Education had concluded that "there was no need for such an institution." Many letters from colleagues all over the world convinced our government that they should insert the word 'probably'. So, five researchers, supported by three administrative staff, were relocated in a small institute, OW&OC,[2] as part of the Faculty of Mathematics of Utrecht University.

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[1] Instituut voor de Ontwikkeling van het Wiskunde Onderwijs (Institute for the Development of Mathematics Education).

[2] Onderzoek Wiskundeonderwijs en Onderwijs Computercentrum (Mathematics Education Research and Educational Computer Centre).

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Hans Freudenthal, who was instrumental in forming IOWO, was extremely disappointed. His expectations were low, at best. But he decided to stay with the ‘sinking ship’, as long as there was hope. He lived long enough to see how the institute, in 1991 re-named as Freudenthal Institute (FI), not only survived, but blossomed and grew reaching more than eighty mailboxes at its highpoint early in the 21st century.

Just two years ago, I was invited to reflect on my career as an educational designer. It resulted in a talk titled ‘There is, probably, no need for this presentation’ (De Lange, 2016). It took me lots of reflection, but I doubt whether the effort was worth the case. So, it must be because of my ripe age and wisdom that again I am invited to reflect. This time on my ‘leadership’ of the FI. That leadership started in 1981 with being appointed as a coordinator and culminated in becoming professor/director of the institute in 1989, ending with my ‘retirement’ in 2005.

The reflection in this chapter will be rather impressionistic, but with the best intentions. I will address:

- The mission: innovation in mathematics education.
- By means of connecting research and practice (developmental research).
- In teams of talented people, ‘organised’ in ways that let them shine.
- Working in a flat, informal maybe even somewhat chaotic, organisational structure.
- Connecting all players, politicians, scientists, practitioners, textbook authors, using a variety of dissemination methods.
- By powerful and relevant new ideas.
- Provocative and innovative with vision.
- Reaching out internationally to validate theories.
- Having fun.

17.2 The Mission: Innovation in Mathematics Education

Even before IOWO was founded, Freudenthal did not hesitate to formulate its mission ‘to teach mathematics as to be useful’ (Freudenthal, 1968). This was a very relevant question at that time, because of the rise of New Math.

In 1959 a seminar was held in France (Royaumont) with a great impact on mathematics education over the following decades. According to the report of this seminar, insight into the structure of mathematics is of fundamental importance for systematically directed education. Dieudonné, the famous French mathematician, was very influential, and proposed to offer the students a completely deductive theory, starting right from basic axioms. Freudenthal later admitted that not attending Royaumont was one of the two big mistakes he made in his professional life. The other mistake he refused to mention.

Many people in and outside the Netherlands had similar feelings about the mission of the FI and contributed to it. We name a few.

> The structure of mathematics is a beautiful edifice, but I do not think there was one student who shared that opinion (Vredenduin, in Goffree, 1985).

Vredenduin made this remark after designing a course as intended by Royaumont, which failed in the classroom.

> To know mathematics means to be able to do mathematics: to use mathematical language with some fluency, to do problems, to criticize arguments, to find proofs, and, what be the most important activity, to recognize a mathematical concept in, or to extract it from, a given concrete situation (Ahlfors et al., 1962, p. 8).
>
> The problem is not what kind of mathematics, but how mathematics has to be taught. In its first principles mathematics means mathematizing reality, and for most of its users this is the final aspect, too (Freudenthal, 1968, p. 7).
>
> What didactical phenomenology can do is prepare the following approach: starting from those phenomena that beg to be organized and from that starting point teaching the learner to manipulate these means of organizing (Freudenthal, 1983, p. 28).

If we look back, it is clear for me (joining the institute in 1976) that the mission can be phrased as:

> To develop theories about how to teach mathematics as to be useful, and develop materials that fit these theories and allow teachers and learners to learn mathematics in this way.

17.3 By Means of Connecting Research and Practice (Developmental Research)

As a consequence of the just mentioned mission, it seems logical that the methodology should be the approach of developmental research (Freudenthal, 1991; Treffers, 1987). It is research with an important development component. It is not merely established how things are in existing education, but much more how things should be, and one develops education that suits these findings in a theoretical and practical sense (Treffers, 1993). One can also change the order; if one wants innovations in education, the process starts somewhere with design fitting the existing theoretical basis, but also ready to adjust these theories as experiments and experiences dictate. In both cases development and research take place in an integrated, iterative cyclic process (Gravemeijer & Cobb, 2006).

Educational design and development and research is a genre of research in which the iterative development of solutions to practical and complex educational problems also provides the context for empirical investigation, which yields theoretical understanding that can inform the work of others (McKenney & Reeves, 2012). At the FI the big problem in this respect is the place of correct and fitting methodology. As a developmental researcher, you often know ‘for sure’ that something is really happening. And you have at least proof of its existence. And there is political pressure as well.

When the new discipline Mathematics A was being developed, a careful experiment was designed within the strict boundary restrictions of the Ministry of Education. The project started with pre-experimental design experiments at classroom scale, went into the next phase with two schools, and thereafter to 10 and 40 schools.

It may have been declared a success already when working with two schools. The professionalisation based on these experiments formed the start for the teachers of the 40 and all remaining schools. Experimental teachers became teacher-trainers, co-designers and sometimes ended up at the Freudenthal as colleagues, or with Cito, the Netherlands national institute for educational measurement. Research was carried out on assessment issues and attitudes of students, but whether or not existing methodological criteria were met, remains a bit vague.

We tried to make the experiments more serious by disseminating results often and transparently to the teachers' media, and the commercial publishers. The *Nieuwe Wiskrant*, a journal for mathematics teachers was the rather glossy magazine of the FI, that at its peak reached a large percentage of the target group.

In 1989, as a New Year's gimmick, I wrote a story *De Kamerronde* (De Lange, 1989) for our Freudenthal people, by describing a virtual walk through the institute, and peeking into some rooms. In the remaining part of this chapter I will return to this walk to illustrate the work at the FI. I will start in Room 5.

Room 5

Four gentlemen, varying in age from medium aged to really very old. The old man distinguished himself from the rest by wearing a butterfly tie and looking like a real professor in every aspect. Almost without saying it seems natural that he is the centre of the discussion. The subject of discussion is a new article written by him for his newest (and latest) book with the working title China Lectures. The discussion has two points as its focus: what is common sense and in which respect is mathematics distinct and different from physics?

Iron feels colder than wood. The earth is flat. Is that common sense? The sun sets and rises again. Common sense, or bare reality seen from the perspective of the observer? According to the present text of the draft article, mathematics education needs to be built on common sense, while in physics education you often have to battle common sense because it is an obstacle in the conceptual growth of physical concepts. And what to think about chance and probability in this respect? That is mathematics as pure as it gets, right? But it often collides with common sense. Although, what actually is common sense?

Is common sense a set of generally accepted agreements and trivialities that makes any further discussion unnecessary? If this is the case, then this is only valid within a certain group or at a certain time. Or is it more complex? What about the reasoning part of common sense? You literally say "the sun sets", but you know that is not really true. But it is true for young children.

Mathematics is rooted in common sense, the professor dictates. As an example, he mentions the natural number. Kids can acquire this concept within the overwhelming stream of physical and mental activities. Mathematics: just a sniff of common sense, some organisation, and the development continues, resulting in a better organised common sense. Your common sense reasons that $2 + 3$ *is 5 and the area of a rectangle is* $h \times b$. *Mathematics, without physics, gives security, trust your common sense.*

The discussion continues. Does $2 + 3 = 5$ *really constitute common sense? And area is length* × *width as well? And the theorem of Pythagoras? Is spatial orientation based on common sense?*

At the end, the participants make a sub-conclusion: common sense is local, both in time and place, and it includes reasoning.

The professor mumbles something. He will rewrite the draft. Will be continued.

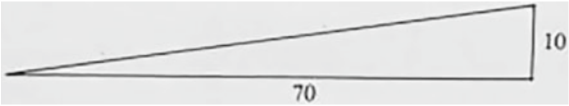

Fig. 17.1 Drawings by the average students

This discussion in Room 5 represents a snapshot of a discussion about the theory of mathematics education. The practice follows next: another snapshot, slightly longer, about a teacher, working also at the FI as a designer-researcher, who teaches rather 'lower-achieving children'.

Her question was simple: "Can you design something in the area of trigonometry?" The resulting booklet was *Vlieg Er Eens In: Goniometrie en Vektoren*[3] (De Lange, 1980a). We had tried it out with 'average' level students, but this promised to be something different. A student's reaction was: "You have to think, quite often." So, based on this experience our expectations were rather low.

Soon it became clear that we were too pessimistic. The so-called 'low achieving' students, who are often regarded as not being able to think, read or learn, were soon completely involved in the problems. The difference with the average group became quite clear. These students stayed within the context much longer than their higher scoring friends. An almost trivial example is the reaction of the students to the following problem: "Somebody jumps with a hang glider from a rock at 10 metres high and reaches 70 m horizontally. Draw this situation at scale." The average students drew something like what is shown in Fig. 17.1, while the lower achiever made more often drawings about the rock. Only later on, did these latter drawings become 'naked' triangles.

Another difference between the two groups was interesting as well. There were more students at the lower level who were 'willing' to think. They were very answer-oriented and only after discussions with their peers they accepted 'thinking' may be a part of the learning process. And in this respect, we noticed, quite surprisingly, that the lower achievers outperformed the average students, especially on more complex problems.

One aspect deserves special attention. How is the transfer from one context to another? It was one of those hot days in the past summer. The students wanted a lesson outside, of course. The teacher reacted as desired: "Okay folks, we're going outside to measure the height of buildings, towers, signs, lampposts etc." She gave the students simple angle measurement instruments, some paper and sandwiches and told them: "You have to be back in half an hour!"

My first thought was an exclamation "Good Lord", thinking of my own experiences as a student with outdoor lessons which were not very successful from the knowledge acquisition point of view. Although we certainly enjoyed eating lots of ice-cream, pushing girls in the pond, catching ducks and furthermore embarrassing our female teacher as much as we think was possible.

The teacher invited me to make a stroll around the school to observe the students in the wild. After a slight hesitation, I accepted the invitation. The small park alongside

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[3] Published in English as *Flying Through Math: Trigonometry and Vectors* (De Lange, 1991).

the school building proved, not surprisingly, to be a popular area for investigation. The height of the school, a tree, a lamppost, everything was measured and if the results were not according to expectations, a discussion followed. What a well-educated company!

A couple of streets onwards we found a girl, with beautiful blond curly hair, lying on the ground, neglecting the fact that her blouse was very white indeed. Her girlfriend was taking care of traffic around the girl with the white blouse. “Fifty-three degrees”, this latter girl told her friend. “And the distance was ten metres”, said her girlfriend, carefully watching cars passing by. From down under came the response: “Then we know the height.” She jumped up again, and asked if they were right. We both agreed with them. “Okay, let’s do another building or object then.” The teacher suggested: “Why don’t you measure the fire-brigade ladder.” ‘Oh’, reacted the girl immediately, “but then we need the cosine.”

At that moment, I almost became emotional. The girls went to the ladder, we to the school. The ladder measured 12 m. The firefighters had confirmed that the length was actually 15 m. “Right”, said the girls, “but we have measured to the edge of the roof, ignoring the piece that was above the edge.”

I am still thinking, after all these years, of the girl in the white blouse (De Lange, 1980b).

17.4 In Teams of Talented People, ‘Organised’ in Ways that Let Them Shine

When the institute was reinvented in 1981 there was an extremely small team, selected carefully by the successor of Freudenthal, Frederik van der Blij. A careful balance between primary and secondary education, between somewhat younger and older, between more theoretical and practical, between more mathematics and social sciences, to name a few. So, in one way or another these people were considered talented. But it was also clear that there was no clear scenario on how to make this handful into a driving force in mathematics education. Reflecting on this starting phase it was clear that the connections for the pre-1980 years were invaluable. There was almost no institution left, but the people were still out there, somewhere.

This network, including many teachers, was kept alive, including continuation of magazines and newsletters to let people know that ‘something’ was still alive and kicking. That the remaining people were talented, was taken as assumed. They were the ones that were ‘selected’ to continue the good work, albeit it under very different circumstances. The institute was now really part of Utrecht University. But it was still in the same boring office building out of reach of the university. The battle to stay out of the university’s bureaucracy was fought successfully for a very long time. The crown on this battle was the building next door that was really of the very best quality, and where we moved right after the completion of that building. Of course, this was much later when we really needed three floors for all those working at the FI.

The ‘organisation’ was not much of an organisation: primary, secondary and new technology. Not a breath-taking structure, but it reflected what we did. There was a lot of freedom and right from day one we had our first big project for five years: the design of a new Mathematics A curriculum. New money, new people hired, based on known and proven talents. This team had a daunting task as school experiments had to start in August 1981, just eight months after the reinvention of the institute.

Luck may have played a large role here. The bureaucratic work on the new project, the foundations, had been laid since 1978, including the participation of the ‘old’ pre-1980 institute. So, the announcement of the Ministry of Education that experiments would be carried out starting in 1980 and resulting in a new curriculum to be introduced in August 1985, nationwide came exactly at the right time (a detailed description can be found in De Lange, 1987).

It seemed that at least for some time the new institute, suffering under the name of OW&OC was alive and guaranteed a lifespan of at least five years.

Because of the perceived success of this project an extension became reality. A similar curriculum for a different student population followed, extending the lifespan to ten years. We follow the discussion in the starting phase of this project in Room 1.

Room 1

A buzzing room. Almost heated discussion. Four excited people in a small room. The subject of all the excitement: the content of the new curricula Mathematics A and B.

As almost usual for experiments with materials for new curricula the big problem is, that there is more than fits in a curriculum. To cut in an ideal program is difficult. And they agree on only one thing: there need to be cuts. The teachers say so, the development group (FI members with teachers), the experts (and there are many of those), and all others involved. So, the task is simple: what and where to cut?

The experiments started in 1987, which in itself is a small miracle as the Ministry of Education did not consider these experiments necessary as the new curriculum should be similar to the just newly introduced Mathematics A. But the people of FI knew better. To develop student materials, to professionalise teachers, to write articles to acquire ownership, to design high stake tests, to carry out attitude research are essential activities.

The discussion heats up even further. Matrices out? Not a good idea. It is a prime example of showing modelling aspects of mathematics. Maybe exponential growth out? No, that is not very wise, give the famous report of the Club of Rome. And how about statistics?

There is no agreement in sight. The discussion remains heated. The time pressure can be felt.

At the primary level, some very talented people in Room 3 were able to continue activities in which the institute was very instrumental in a facilitative way.

> *Room 3*
>
> *Somewhat concerned, the gentleman with the red-rimmed glasses looked ahead. He mumbles a bit hopelessly that he has no clue how to continue the Panama*[4] *project. In the chair opposite him sits a somewhat young lady who agrees that the situation is a bit foggy. A discussion at the Ministry of Education had brought clarity, but no transparency, a feeling that many people have after visiting the department.*
>
> *The Panama project started in 1981 as a collaboration between different parties with the goal of professionalisation of those who are working in and for primary school. The whole project was to be carried out by one person, with some administrative support. The way this project developed over time show what talented people, given the opportunity to shine, can do.*
>
> *A newsletter developed into a leading professional magazine. The conferences that were organised were always sold out and played a very important role in developments in primary education in the Netherlands. It was a truly national platform and offered the staff of the institute a platform to really shine and inspire.*
>
> *The gentleman with the trendy glasses knows that the institute and the other collaborators want the project to survive. But there are so many things to do, most of them of a complex nature. And there is so little money. And politics is so difficult. The Ministry of Education has been ordered not to fund 'outside' projects any more. The budget is going down, the partners in the project are in reorganisation, the institute is looking for possibilities and money. The new law on how to organise education-related institutions seems to make matters even more complex, but that, people say, was one of its intentions anyway.*
>
> *The shaking of heads makes place for frowns on the foreheads. But the sparkling eyes tell another story. The upcoming Panama conference has sold out once more.*

17.5 Working in a Flat, Informal, Maybe Even Somewhat Chaotic, Organisational Structure

> It's far better to rely upon a broad base of individuals and leaders who share a common set of values and feel personal ownership for the overall success of the organization. These responsible and empowered individuals will serve as much better watchdogs than any single, dominant leader or bureaucratic structure (Terri Kelly, cited by Kastelle, 2013).

The luxury that reflection offers is that it may make you clearer about what you did in your past, more or less intuitively, and just because there seemed to be no other way. If you start with a handful of people that you know very well, it seems a waste of time to think of a structure at all. But soon you will find out that however horizontal or flat, there needs to be a person who is somewhat central. In those years this person was called 'coordinator'. Leen Streefland was the first, but after a year he decided that this suited me more, not realising that this would become another challenge in itself. How to handle the monotone growth that is so typical for the first 25 years of the FI since 1980?

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[4] Panama stands for Pabo Nascholing Mathematische Activiteiten (Pedagogical Academy Training Mathematical Activities). Panama is the Dutch network of mathematics teacher educators for primary education. One of the activities of Panama is organising the annual Panama Conference.

To be honest, it never occurred to me that this was a problem. My focus as coordinator was on new bright ideas and opportunities, and putting talented people in charge of the actual execution of these plans after we found money. Somewhere. I think, looking back, that I agreed very much with Kevin O'Connor who stated in a blog: "All organisational structures are evil; but when you have to, align your organisation around markets" (O'Connor, 2012).

Well, the markets were our colleagues in research, but even more the teachers and students. Indeed, there was, with some good will, a weak organisational structure honouring the tri-partition primary, secondary and new media, but other partitions played a role as well: from practice to theory and vice versa, to name an important one. Only much later, when the institute had many more people involved in projects one could see the first steps to middle management. This of course was in part due to the fact that the institute was part of the university. And universities cannot be accused of embracing flat organisational structures. So, the university structure forced us to 'unflatten' the institute to at least a certain level.

So, it is comfortable to reflect at the initial very flat and informal structure. Small is flat, especially if you have been working together already for some time. The need for a coordinator emerged from this structure as something 'natural'.

More conscious was the battle within the university structure against the university. Let me explain this in a bit more detail. Our building in the early ages was shabby at best. And the rather chaotic (remember: flat) way we worked fit perfectly with that building. There were paper and boxes everywhere, and for visitors it was unclear at which moment they actually entered our offices. All of a sudden, they were in, if they had not returned already. But we liked our offices very much, because we were out-of-sight of any university office. We were very much aware that some of our salaries were taken care of, which was very nice indeed, but for the rest we looked more like a start-up business, as they are called today 'free as a bird'.

If we really did something based on an agenda, it was staying out of the bureaucracy. And until the end of my directorship, we succeeded quite well. Of course, we invested heavily in good contacts with the Faculty of Mathematics (trying to stay away from Social Sciences) and the Rector of the University. Especially after reaching out internationally, the executives at that level actually started to like us. Of course, our building was horrible. But just when the owner decided on restoration, another brand-new office building was erected next to the old building. The connections and appreciation with the rector and others ensured us a place in this new, fancy and very representative building. And renewed independence from the university.

We never forget the remark of one of the Secretaries of State for Education after a visit: "Jan, I really appreciate and am fond of the work of all of you, but you need a more representative building and entrance!"

Although we had a terribly good time working in these times, my fear of being eaten by the university was well grounded. Not only has this been proven true after retiring in 2005, but there is plenty of evidence from other sources. The growing bureaucracy, more middle management, more vertical structure, accountability, irrelevance of much work, make universities not really a sparkling, innovative, risk taking environment. I know, reflecting sometimes colours the image. All the better.

Room 9

The phone rings. The man in the room, seemingly deep in thought with stretched legs on his office table covered with lots of paper, awakes and grabs the telephone. "Bolivia", he mumbles. His legs sweep the table, cleaning it from all papers. His attention is on the phone. The line is garbled and the Bolivian English does not communicate well with Dutch English. They, on the other side, want more computers. We were warned by colleagues from the University of Agriculture in Wageningen: there are always needs for having some hardware added. For them it was not computers, but Jeeps.

In the meantime, the line before the open door was waiting patiently. They look around the corner, ever so friendly, but with the signal: we need you. The telephone call ended quite abruptly after promising some more computers.

The room was quickly taken over by the whole team of the project Being in Charge that is about how to become in charge of computers. There is a problem. The problem is simple, the professionalisation course and project are too successful. And now the question is how to deal with this luxury. We will contact the Ministry of Education.

The discussion switches to the Fair Share programme which is based on an intelligent tutoring system. This experimental half-product has been tried out successfully. And the question of how to proceed next is on the table.

The phone rings. All people look at the man in the room picking up the phone, standing this time. He listens and says: "Okay." "The Ministry of Education. They want to talk about the future of Fair Share".

17.6 Connecting All Players—Politicians, Scientists, Practitioners, Textbook Authors—Using a Variety of Dissemination Methods

"What chaos." Famous first words exclaimed by yet another Secretary of State for Education on entering the office building of the institute. She and her company crawled their way to the director's office, which was in a similar style, although the three chairs were made available for seating. Of course, an apology and explanation were required and offered. But the coffee helped a lot, and soon the discussion was about mathematics education and the expanding role of the institute in the world.

She was proud of the growing international role and projects in the United States. Even funds from the National Science Foundation came to the Netherlands. The question was, of course, whether we should accept it. One could easily argue that there was still more than enough work to be done in the Netherlands. But on the other hand, if one really wanted to validate the domain-specific instruction theory of Realistic Mathematics Education, one should reach outside our small country. And the higher regions at the university were very supportive as well. An institute like the FI needed to reach out internationally. Freudenthal himself expressed this point of view repeatedly when traveling across the borders in his favourite mode of transportation, a Land Rover Defender. He loved the Spartan jeep quite a bit, although comfort was lacking. But the discussion about the need for international contacts, and the desirable and fierce discussions resulting from these, was inspirational indeed.

As indicated before, teachers were also engaged in many aspects of the work of the FI. There were repeated interesting discussions with the university about the lack of academic qualifications of many of the people working at the FI. But the institute included teachers in more ways than only as colleagues.

There were professional magazines, conferences (Mathematics for All), regular conferences (Panama Conference, Nationale Wiskunde Dagen[5]), competitions (A-lympiad, B-day), school activities (Grote Rekendag[6]), professionalisation, key positions in organisations (CIEAEM, ICMI, Mathematical Sciences Education Board, PISA, the ISTRON Group on mathematical modelling, and so on) and scientific magazines, to name a few. The hundreds of small applets and the electronic work environment for mathematics education, and software can also be mentioned.

Commercial textbook publishers and writers were often suggested to use the material developed by the Freudenthal Institute, free of copyright. In the 1980s, De Jong (1986) published a report about the success achieved by this dissemination strategy for primary education. Ten years later it was not difficult to conclude that similar results were also found in secondary education.

Room 4

An empty room. The New Media project team that should do its work here according to the note on the doorpost, is on the road. The whole team, including a teacher is at another office, at the Ministery of Education, then housed in Zoetermeer.

The room is not too big, and is occupied in large part by a huge, circular, white table. In the corner opposite the entrance is a construction suggesting that we are dealing with new media here. Around the blessings of the advances in technology sits a team that looks at least at ease and relaxed. Three quarters of the full circle are occupied by people from the ministry, and other experts.

The atmosphere is tense. For the team the meeting is very important. Will the department be satisfied, at least, and maybe even excited? That would bring in more money of course.

The ultimate goal is to carry out professionalisation using new media in a way that would later became fashionable as blended learning.

You can feel the tension and excitement in the room. Now and then a simple small nod, sometimes even an affirmative slight smile. "Quite interesting indeed", says the obviously most important person at the three-quarter of a circle part of the table. Exactly on time the session is finished. Other meetings are waiting. Good sign. The most important person stays in the room a little longer to talk a bit more in detail. The hardware is being dismantled. New media are a handful indeed. That causes problems at the school level. Should every school have a system? Plenty of ideas for further development. But they need to leave quickly now, on to a school to implement and experiment with new media at school level.

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[5] National Mathematics Days.

[6] Big Mathematics Day.

17.7 By Powerful and Relevant New Ideas

Figure 17.2 shows a number of problems Freudenthal (1968) presented at the colloquium Why to Teach Mathematics as to Be Useful, held in Utrecht in 1967, to prove the fact that it is not so easy to learn that in all these situations the same arithmetical operation applies. He opened the panel discussion at this colloquium as follows:

> Ladies and Gentlemen. I open the panel discussion. First of all, I'll give a summary of what I learned these past two days about the ideas of those who dealt with the general theme. I got the impression that we all agree about fundamentals. We all are convinced, I suppose, that mathematics has to be taught in way that people can apply it. We are convinced that this goal cannot be reached by simply teaching applications of mathematics, but that mathematics has to be related to its applications in a much earlier state, in a closer and more fundamental way, and that the ability to apply mathematics can only be acquired by starting with students from situations that have to be mathematised. (see Freudenthal et al., 1968, p. 61)

This can be seen as expressing the philosophy of the very first years of IOWO (1971–1980), the successful institute that was considered not be useful anymore. It barely survived the early 1980s with a small team, as described before. It led eventually to the idea of Realistic Mathematics Education.

This idea was developed in a powerful way by the Wiskobas[7] group in the 1970s already, later to be followed by similar development in secondary schools, and parallel the development of a more computer-oriented approach in mathematics education. The plea for more applications in education starting in reality was facilitated through the projects mentioned earlier (the Mathematics A project), facilitated by politics and the Ministry of Education.

Another powerful idea that really 'made' the FI, was the developmental research approach. In this way, the research was fed by practice, brought on a higher scientific level and validated before 'descending' again to the practical level. In the meantime, all kinds of material were developed from complete curricula to beautiful micro designs, from software to applets, from tasks and tests for classroom assessment to high-stake tests.

The communication that is part of developmental research also had some relevant instruments, which were mentioned before.

If I have ten marbles and I give three away, how many are left?
If I have ten marbles, and john has three less, how many does he have?
If there are ten students in the room and three are girls, how many are boys?
If I'm ten years old now, how old was I three years ago?
If B is between A and C, B is at a distance of 7 miles from A, and C is at a distance of 10 miles from A, how far is B from C?

Fig. 17.2 Problems presented by Freudenthal

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[7] Wiskunde op de Basisschool (Mathematics in Primary School).

Room 7

A heavily gesticulating woman, speaking with an accent from down south in the Netherlands, tries to convince her more business-like colleagues. Not that the emotions rise particularly high, but the devil is in the details. Tomorrow an important presentation will take place, and the project needs to be brought out into the open in a very good way.

The research is about comparing what teachers think their students at the beginning of primary school are able to do, and what they actually can do.

The teachers were quite conservative in their expectations. The students fared much better than expected. This finding has great and important consequences, especially from the perspective of connecting to the reality of students. Do not start at zero if you know your students are much farther on the 'number line' of learning.

The woman is still gesticulating heavily. She is worried about her presentation. Is there time enough to present everything clear and transparent? Her counterpart, the gentleman, is less worried. His slides are well laid out, and not too many. He tries to calm down the lady. In vain, however.

17.8 Provocative and Innovative with Vision

Room 2

Three comfortable chairs, a lady and two gentlemen. They look as if they are thinking seriously about the task at hand. Indeed, the three have been invited to do a whole morning presentation at a large conference in the United States. Important points come up in the discussion: what kind of people will attend (mathematics educators), what do they want (just about everything, especially concrete examples), how deep will we go into the more theoretical aspects (not too deep), on which points are we different from U.S. society (in many more ways than 'they' think). But more down to earth matters, like plane tickets, also come along

They talk a minute or two about a side trip of one of the three to Princeton. There will be a brainstorm about a favourite subject: higher-order thinking skills. A remarkable initiative given the tradition of, and love affair with, the multiple-choice format. Multiple choice and the new experimental Mathematics A exams seem to be light years apart. Lots of text and context, visuals, open-ended questions, even long-answer-format questions. Okay, looks nice, maybe higher-order thinking skills, but what about validity and costs?

It is a recurrent theme: tests and tasks, both for primary (Van den Heuvel-Panhuizen, 1996) as well as for secondary education (De Lange, 1987). Consider the discussion about the end test in primary schools. Highly valued, taken by almost all children as the direction where to go in secondary education. Multiple choice. In a half-page interview in one of the quality newspapers in the Netherlands, the person at Cito who is responsible for this test, battles it out with me, the director of the FI. I argue that the test is misused as the sole measure of where to place a child within the Dutch tracking system. Research has shown that the teacher's advice is a better indicator for future success. There should be more real-world problem solving in the test and fewer fanciful illustrations to cheer up the kids. Moreover, I criticise the side effect that the last year at primary schools is dedicated to preparing the students to pass the test: “A whole year lost for test preparation!”

In yet another major article, the discussion focuses on the new Mathematics A curriculum. I argue: “The test tasks of Mathematics A connect to the student’s real world, and show mathematics as to be useful”. I also made it clear that I am concerned that the quality of the experimental exams will be difficult to maintain. The downward trend has already started. According to me: “The exam fails to meet the principles and philosophy we developed when developing Mathematics A. Insight in concepts and creative thinking are rarely present.”

A quotation from another interview makes clear that I am not the only one who is concerned: “The State Secretary for Education shared our concerns about the development of the high-stakes tests. She invited us at the Ministry of Education. ‘What to do?’ was her obvious question.” The problem was purely political. A new law, the same one that declared the FI ‘not needed’, made it virtually impossible to change practice. The State Secretary was not allowed to fund any project in that vein, because that task belonged to the Cito. That same law played a huge role in the first years of the FI as creativity of a purely political nature was the game to play.

It was deemed necessary to distinguish the institute by being different: sometimes provocative, often innovative, seeking free publicity excelling in communication. And having excellent ties with different levels at the Ministry of Education, that was carefully bombarded with an array of novel ideas: new media, new tests for primary school, comparing textbook results, new curricula (as to be useful), new software, computer science at school, mathematics for all, A-lympiads, collaboration with the Dutch association for mathematics teachers, graphing calculators (ready for the museum right now), cutting edge conferences, international collaboration. Or, in short: never a dull moment.

And the institute was very lucky to find the press on its side. A whole page article about the institute stated the philosophy quite well, quoting Freudenthal about the usefulness of mathematics, and quoting a statement of myself: “What makes us different and innovative is developmental research. In this way, we try to develop a new educational reality by doing research. Not researching ‘what is’, but ‘what ought to be’”. And of course, the reporter was happy to quote the well-known American professor Romberg: “We are carrying out a ‘robbery’ on the Freudenthal Institute.” This was his way of announcing a big cooperative project between FI and the University of Wisconsin at Madison.

17.9 Reaching Out Internationally to Validate Theories

Three people played a major role in the internalisation of the institute, one of my personal key issues. The first one was Hans Freudenthal himself. Of course, he was well-known and famous all over the world. Less well known is the fact that he was very much in favour of internalisation, while many people within the institute insisted on ‘completing’ the national agenda, whatever that means.

There is the story, playing in the late 1970s, about an invitation to make a presentation in Brazil, an invitation from Ubiratan d'Ambrosio (no need to say more). As I had already travelled as far as Norway at the time, people were looking at me. I immediately said yes. There was a slight problem. The then administrative director did not approve at all. The coordinator of secondary education talked to Freudenthal and they concluded I should go anyway. But how to make that secret operation work?

It happened to be February, so snow abounded in the Alps. So officially I went skiing. But what about the money? A solution that we all found quite elegant was to give me more local travel expenses for as long as needed, until my 'skiing' costs were covered. So, I went 'under cover' to Brazil, a turning point in internationalisation.

But Freudenthal did more than just carry out and cover travel under the radar. Quite often when he was invited, especially often to Germany, he invited me to accompany him, and show examples of our design work. The problem was, in my opinion, that my car happened to be a crude Land Rover, that was used for a Sahara trip. For Freudenthal it seemed more like an attraction.

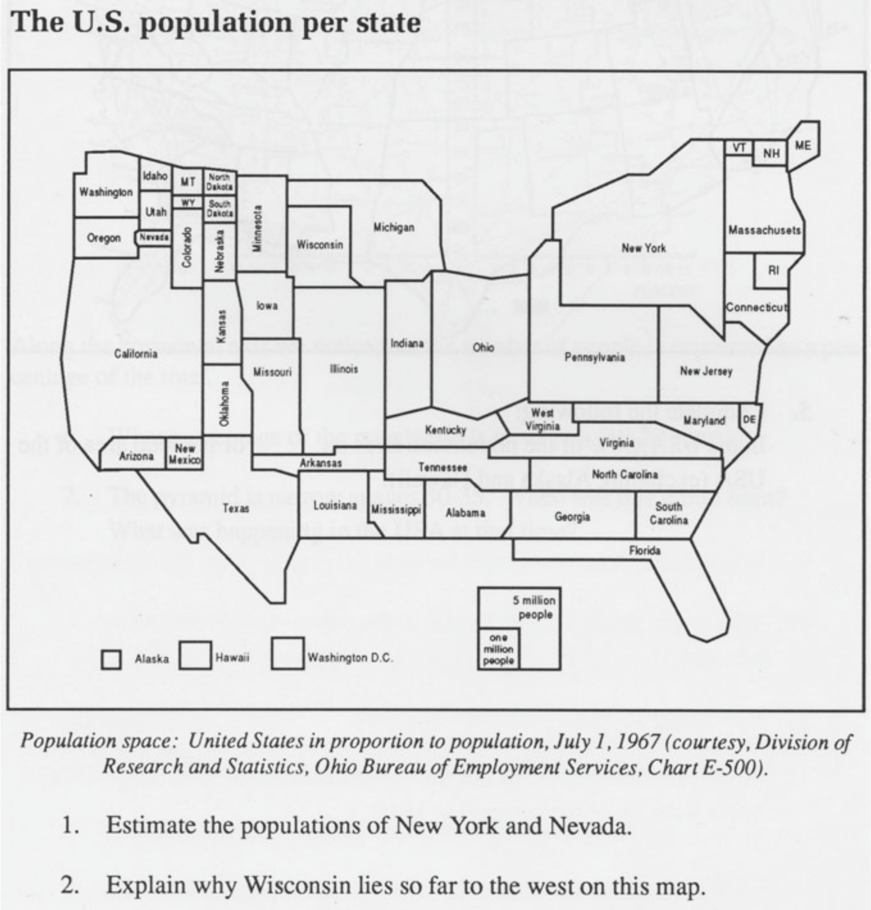
The other two very important people in establishing international projects, were two famous American scholars: Tom Romberg and Tom O'Brien. Almost at the same time they approached me for different activities. Tom O'Brien thought that what happened in the Netherlands was worth spreading in the United States. He acted as my impresario for an east-west coast tour of the United States, doing many presentations in places unknown to me.

It was at least quite interesting, and very enjoyable. I did presentations at schools, at universities, a school boards, at universities, for students and staff, superintendents and enjoyed diners just to get know important people like Marge Cappo. There were workshops, lectures, discussions and other formats. They lasted at least 45 min, but in one case 1.5 days. This was in Montana, where a real reception committee waited for me at the airport, and I barely survived with the couple of hundred slides that were to my avail. Rick Billstein and Johnny Lott were in charge of that incredible event.

Tom Romberg approached us more carefully. Not really the 'artists entrance' as the TO'B tour. Just exploring slowly and carefully to find out if it was worth investing in the Dutch. He was especially interested in developmental research and 'mathematics as to be useful'. He challenged us to design a little unit for use in an American High School, not far from Madison (meaning one hour by car). The teacher (Gail Burrill, later president of NCTM) selected the topic of data visualisation (see Fig. 17.3).

For design, flying tickets and observations we got $3000. We realised of course that this was almost nothing, but … The experiment was declared a success. So, this time Tom (and Gail) offered $6000 if we designed another unit. No surprise. It worked quite well. The next phase was a dinner with Tom at his golf club in Madison (my first time ever). He leisurely informed me that he was happy with the two tiny experiments: "How about participating in a multi-million dollar National Science Foundation project?"

It can be considered as the start of many projects, run by many people in a variety of countries. Not just in Madison, but all over the United States. In Bolivia, South Africa, Indonesia, Malaysia, to name a few, and participating in studies like TIMSS,



The U.S. population per state

Population space: United States in proportion to population, July 1, 1967 (courtesy, Division of Research and Statistics, Ohio Bureau of Employment Services, Chart E-500).

1. Estimate the populations of New York and Nevada.

2. Explain why Wisconsin lies so far to the west on this map.

Fig. 17.3 Page from the unit data visualization

NAEP and PISA. And participating in the Mathematical Sciences Education Board (MSEB) of the National Research Council, being Secretary of CIEAEM, organising PME Conferences and an increasing number of colleagues in boards of prestigious journals.

The argument we used for internationalisation was to validate the theory of Realistic Mathematics Education. What came free with the ride was the richness of the different cultures. And what there was to learn abroad. Little did we realise at that time that while the way we carry out discussions in our institute (seen as rather vibrant) may have been a bit frank within the Netherlands, it was sometimes a real culture shock to see the differences in culture in general and in education. Hopefully we learned, both ways.

17.10 Having Fun

What I liked most about the Freudenthal Institute, where I spent most of my professional life, was the fact that, in reflection, every day seemed to be a fun-day. During this actual writing activity, one could have seen me smiling most of the time. When Marja van den Heuvel-Panhuizen asked me to reflect on how the institute worked in the last two decades of the past century, it did not take much effort to convince me. Better even, at my ripe old age I decided to attend ICMI 13 in Hamburg. So, I will confront my reflections with reality of the present state of art on mathematics education. It will again be FUN!

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