Chapter 3
The Influence of the Richardson Arms Race Model

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Abstract This chapter reviews the Richardson arms race model: a pair of differential equations which capture an action reaction process. Whereas many of Richardson's equations were quite specific about what they referred to, the arms race model was not. This lack of specificity was both a strength and a weakness. Its strength was that with different interpretations it could be applied as an organising structure in a wide variety of contexts. Its weakness was that the model could not be estimated or tested without some auxiliary interpretation. The chapter considers the impact of these issues in interpretation and empirical application on the influence of the Richardson arms race model.

3.1 Introduction

There are many definitions of arms races, but for the purpose of this chapter they can be thought of as enduring rivalries between pairs of hostile powers which prompt competitive acquisition of military capability. Two approaches to modelling arms races have been particularly influential. One is as a two-person game, in particular the Prisoner's dilemma, where the choices are to arm or not to arm, and the dominant strategy, for both to arm, is not Pareto optimal. The other, which is the focus of this chapter, is the Richardson model of the arms race as an action-reaction process, represented by a pair of differential equations.

Just as the two supply and demand equations have structured thought about the dynamics of markets for most economists, the two Richardson equations have structured thought about the dynamics of arms races for most subsequent analysts. Not only did he develop the model, he attempted to test it using data on military expenditure prior to World War I. One of the strengths of the model is that it has prompted a range of questions, many of which Richardson himself posed. This chapter reviews the influence of the Richardson arms race model on the subsequent literature through these questions, which include: What are the characteristics of the

This is a revised version of a paper presented at the Richardson Panel at ISA 2018, Saturday 7 April 2018. I am grateful to participants of the panel for comments.

N. P. Gleditsch (ed.), *Lewis Fry Richardson: His Intellectual Legacy and Influence in the Social Sciences*, Pioneers in Arts, Humanities, Science, Engineering, Practice 27, https://doi.org/10.1007/978-3-030-31589-4_3

solution to this model? What are the variables and actors? What is the time dimension? How do arms races relate to wars? How should the parameters be interpreted? How can the model allow for strategic factors and budget constraints? How should the model be related to the data? How do you stop arms races?

Given the variety of ways that his arms race model has been used, I was tempted to call this piece 'variations on a theme by Richardson', but specifying the theme precisely proved problematic. The voluminous arms race literature that arose from his work has many themes. In trying to identify the themes, I found the papers in the collection edited by Gleditsch & Njølstad (1990), hereafter G&N, very useful. G&N provides an overview that lies roughly half way between the publication of Richardson (1960a) which brought his arms race model to a wider audience and the present day. With the end of the Cold War, interest in arms races declined somewhat and many of the themes that are in that book remain central. It is difficult to say anything new in this area, and Wiberg (1990) makes many of the same points as I make below. There is a more technical discussion of many of the issues mentioned here in Dunne & Smith (2007).

As most readers of this chapter will probably know, Lewis Fry Richardson (1881–1953) was a Quaker physicist, a Fellow of the Royal Society, who made major contributions in the mathematics of meteorology, turbulence and psychology as well as his work on quantifying conflict. The significance of the work on conflict was only widely recognised posthumously with the publication of Richardson (1960a), *Arms and Insecurity*, which introduced the arms race model, and Richardson (1960b) *Statistics of Deadly Quarrels,* which looked at the distribution of conflict deaths.

Richardson was a very careful scientist. When he was investigating the hypothesis that the probability of war between two countries was a function of the length of their common border, he double-checked the data and noticed that adjoining countries gave different lengths for their common border: the smaller country tending to think the border longer than the larger. This was because the measured length was a function of the size of the ruler or scale of the map used; small countries tended to use smaller rulers and larger scale maps. Richardson's subsequent studies on this phenomenon introduced the idea of non-integer dimensions and prompted Mandelbrot's work on fractals. A common border may increase the probability of war but, as Richardson recognised, it also tends to increase the amount of trade, which may have a pacifying effect.

Richardson approached the analysis as a physicist. He often used differential equations to characterise the dynamics, and tried to match the models to data, often using probabilistic techniques. His work provides excellent teaching material in applied mathematics. Students find the arms race model a nice motivation for a neat system of differential equations which has a range of interesting solutions. Korner (1996) makes pedagogical use of a number of examples of Richardson's work to motivate the applications of mathematics, as well as discussing Richardson's life and influence. The teaching aspect is also noted in a recent paper, Beckmann, Gattke, Lechner & Reimer (2016: 22–23), say about the Richardson equations: 'our objective was to see whether this old staple can be brought back from the world of teaching (where it serves as an example for solving systems of differential equations) into modern research on conflict dynamics.'

While serving in the Friends Ambulance Unit during World War I, Richardson began to try to describe the causes of war in systems of equations, which he published as Richardson (1919). Maiolo (2016: 1) says the term arms race originated in the 19th century and was commonly presented as one of the causes of World War I. He quotes Lord Grey, who had been Foreign Secretary when Britain went to war, as writing after the war that ‘Great armaments lead inevitably to war. If there are armaments on one side, there must be armaments on the other sides’ (Maiolo, 2016: 2). Richardson also cited Grey and thus his equations captured a common perception of the cause of that war. However, as Maiolo (2016: 4) also notes, the sporting metaphor can be misleading: in athletics races have clear start and finish lines, arms races do not.

Richardson was not alone in trying to develop mathematical models of conflict. About the same time, Lanchester (1916), based on articles published in *Engineering* in 1914, developed models of the evolution of different types of battle. The models examined the role of the quantity and quality of forces deployed. Lanchester also used a pair of differential equations though to different ends. One might distinguish a Lanchester tradition, in operational research, of mathematical modelling to win wars, from a Richardson tradition, in peace research, of mathematical modelling to stop wars. MacKay (2020, in this volume) discusses a combination of Richardson’s arms race equations with Lanchester’s attritional dynamics.

3.2 The Equations

The Richardson model describes the path over time, t, of the level of arms, x and y, of two countries, A and B.

$$dx/dt = ky - \alpha x + g$$
$$dy/dt = lx - \beta y + h$$

The rate of change of the arms of each country is the sum of a positive reaction to the arms of the other country, a negative reaction to the level of its own arms through a ‘fatigue’ factor and a constant component through a ‘grievance’ factor. Setting it up in this way prompts a set of questions which are internal to the mathematical structure of solving linear differential equations. Does an equilibrium exist? Is it unique? Is it stable? Are there boundary conditions, e.g. $x, y > 0$?

In equilibrium $dx/dt = dy/dt = 0$, so the equilibrium reaction functions are two straight lines

$$0 = ky - \alpha x + g$$
$$0 = lx - \beta y + h$$

If $\alpha\beta = kl$ the lines are parallel, otherwise they intersect once at an equilibrium, which may involve negative values. Because of linearity, if the equilibrium exists, it is unique, and one can then consider how stability varies as a function of the parameters. Here arms race stability refers to the nature of the solution to these equations. An unstable solution would diverge from an equilibrium, for instance exhibiting exponential growth by both countries. Arms race stability is not the same thing as strategic stability, which can itself have many meanings. Richardson related them by suggesting that exponential growth could lead to war, though in principle it could lead to bankruptcy. Diehl (2020, in this volume) discusses the links between arms races and war.

Again, within the internal mathematical structure it is natural to ask if the model generalises. What happens if there are three or more actors? What happens if one relaxes the assumption of linearity? There is a large literature on both these questions. Broadly, as in the three-body problem in physics, the neat simplicity of the conclusions is lost when the model is generalised and multiple equilibria may exist. For instance, among three countries, the equations for each pair of nations may be stable, but the triplet is unstable.

The model has an immediate common-sense plausibility as a description of an interaction between hostile neighbours. This is what makes it such a nice teaching tool. There are also historical examples of such reaction functions, for instance the British policy before World War I of having a fleet as large as the next two largest navies combined. But the model has no unambiguous interpretation. In the physical sciences, when Richardson used equations, for instance in fluid dynamics, he knew exactly what the variables were, what measures they corresponded to, and the time dimension of the dynamic processes involved. Little interpretation was needed. But in the social sciences the interpretation of mathematics is rarely unambiguous.

3.3 Interpreting the Equations

The arms race equations prompted a number of questions about the interpretation. There were questions about how to interpret the measures of arming, x and y. In a symmetric arms race, they were the same variable, such as military expenditures or number of warships. In an asymmetric arms race, they could be different types of variable; historically there was an arms race between castle design and siege train technology. They might be quantitative, number of warheads, or qualitative, accuracy of the missiles. They might be given a more psychological interpretation as hostility or friendliness. There were many possibilities.

There was also a question about how to interpret the nature of the actors, A and B, and the motives for their actions. They might be countries, alliances, decision makers or non-state actors like terrorists. Their actions might be the result of rational calculations or bureaucratic rules of thumb and there were many possible sources of their hostility. Some, like Intriligator (1975), felt the need to motivate the equations with an explicit objective function for the actors. There were also questions about the time period, months or centuries, over which the interactions were taking place and the extent to which the parameters could be regarded as stable. Finally, from the policy perspective there was the crucial question: how might you stop the process?

This lack of specificity was both a strength and a weakness. Its strength was that the model could be applied in a wide variety of domains, by giving the variables x and y and the actors A and B different interpretations. As it was imported into a particular domain, other questions would arise. For instance, an economic interpretation would immediately prompt questions about the nature of the budget constraint. Economists tended to allow for the budget constraint by adding income as an extra variable, but there were many other ways, for instance Wiberg (1990: 366–367) assumes a fixed amount of resources available.

The weakness of the lack of specificity was that there was little clarity either about the precise predictions of the model or about the evidence that would falsify it. As a specific example, the parameters α and β could be interpreted as representing: (a) a measure of fatigue, as Richardson did: increased spending exhausts a country depressing the growth of arms; (b) the speed of adjustment towards a desired level in a stock adjustment model or (c) a measure of bureaucratic inertia; or perhaps some combination of the three. The form of the equation would be identical, but the story one told about the parameters would be different in each case. This was important, since in practice these parameters were estimated statistically and needed interpretation. If one does not know where the parameters came from or why they might differ, between the countries or over time, it is difficult to judge whether the statistical estimates are sensible.

Just as the term arms race is a metaphor, any model is a metaphor (the equations are interpreted as being like the world in some respects) and there is an issue as to how literally to take these equations. Some, like Beckmann et al. (2016) in a critique of the Richardson equations, treat them very literally. If they do not hold exactly, then the Richardson model is wrong or it is a different model. Others treat the model as being more loosely defined and are happy to label any set of equations involving action-reaction processes as a Richardson type model. Intriligator (1975) and Dunne & Smith (2007) take this approach. Economists, treating them like supply and demand curves, organising principles rather than exact specifications, seem inclined to take them less literally.

Of course, some do not accept the action-reaction description itself. Senghaas (1990: 15) rejects the explanation of the arms race as an other-directed reciprocal escalation spiral: ‘As much research on the biography of weapons systems has shown, the action-reaction scheme is at least highly dubious, if not completely false.’ Instead he sees it as inner-directed by the self-centred imperatives of national armaments policy. Gleditsch (1990: 8–9) lists a very large number of explanations for arms acquisitions, organised under four levels: (1) internal factors, such as particular interest groups; (2) actor characteristics such as being an alliance leader or authoritarian rule; (3) relational characteristics, such as action reaction or relations to allies and (4) system characteristics, such as upswings in long economic waves and technological imperatives. While the focus in this chapter is on action-reaction explanations of arms races, much of the work on other explanations was prompted by the desire to criticise the Richardson action-reaction explanation.

On whose behaviour they described, Richardson (1960a: 12) was enigmatic about the interpretation of the equations: 'the equations are merely a description of what people would do if they did not stop to think'. Intriligator (1975) derives Richardson type equations from optimising strategies in a nuclear war. Brito & Intriligator (1999) argue that new military technologies which imply increasing returns should mean the end of the Richardson paradigm with its implicit assumption of constant or declining returns to scale. The behaviour of participants and the research questions in increasing returns to scale systems are very different. For instance, multiple equilibria are possible, and arms control may have the potential to move the system from a high to low equilibrium. Increasing returns to scale increases the dominance of the dominant actor in its chosen technology, providing incentives for the non-dominant actors to choose alternative technologies such as terrorist attacks.

Relating the equations to data

Richardson evaluated the model through an examination of the growth in military expenditures, 1908–14, of the two belligerent alliances, the Entente and Central powers, prior to World War I. He took the observed exponential growth as an indication of support for his models. He interpreted x and y as measuring military expenditures, A and B as coalitions, and the relevant time period as 7 years. However, he noted that other conflicts were not preceded by arms races.

As has been widely noted, e.g. by Gleditsch (1990: 9–10), there is an identification problem: quite different models can give observationally equivalent predictions. While one solution of the arms race model is exponential growth, exponential growth may equally well result from purely internal processes within each country, such as a military industrial complex, with no action-reaction component. Exponential growth may also result from both countries responding to a third country. Expectations further complicate the matter as discussed in Dunne & Smith (2007).

The empirical literature separated into a number of separate tracks. One track looked at whether arms races, suitably defined, preceded conflicts, again suitable defined. Diehl (2020) reviews this track. Another track looked at estimating the Richardson equations directly to see whether they showed action reaction features: significant coefficients for the arms of the other countries. This was usually done from time series though there are also some cross section and panel papers looking at arms race interactions.

To estimate the Richardson equations directly from time series data, they required various modifications. The equations had to be converted from continuous into discrete time, with corresponding judgements about the time-scale involved, how many lags were required and the interval over which one might expect the parameters to be stable. Typically, the lagged dependent variable, arms in the previous period, is a very strong predictor of the current value.

Specific measures had to be chosen for x and y. The logarithms of military expenditures and the shares of military expenditure in GDP were popular choices, but there were many other possibilities, including physical measures like number of warheads. Even when using military expenditures, the estimates could be quite sensitive to other measurement issues, such as the choice of exchange rate used to make them comparable. Of course, expenditures are an input rather than an output, capability, measure. Countries may differ in their efficiency, the amount of military expenditure required to achieve a particular level of capability.

The equations are deterministic and had to be supplemented by stochastic specification. Typically, 'well-behaved' error terms were added to the equations, but again there were many other possibilities, depending, for instance, on how one treated the endogeneity that resulted from the variables being jointly determined, serial correlation and heteroskedasticity. Supplemental variables might be added to control for other factors, e.g. GDP to allow for the budget constraint.

Given all these decisions, it could be difficult to judge what light these estimates threw on the Richardson equations. Firstly, as noted above, since Richardson provided little in the way of interpretation of the coefficients, it was not always clear whether the statistical estimates were consistent with his model or not. Secondly, there is the Duhem-Quine problem: any test involves a joint hypothesis. What is being tested is both the substantive hypothesis, the validity of the Richardson model in this case, and a set of auxiliary hypotheses, such as those about choice of measure, dynamics and functional form. One never knows whether it is the substantive or the auxiliary hypotheses that has led to rejection. McKenzie (1990) discusses the Duhem-Quine problem in the context of the sociology of nuclear weapons technologies. The converse of this problem is that since the Richardson model is not very specific, this allows great freedom for specification search over such things as measures for x and y; functional forms; dynamics; estimation methods; sample period and control variables included. This search can continue until one finds a specification that confirms one's prior beliefs.

Despite these qualifications most surveys of this literature including Dunne & Smith (2007) conclude that there is limited time-series evidence for stable equations, of the Richardson type, describing the interaction of quantitative measures of military expenditure or capability. That article discusses the case of India and Pakistan, where there had been more evidence of a stable Richardson type action-reaction process between constant dollar military expenditure, 1962–97, but it seemed to have broken down after 1997, about the time both powers went nuclear. Empirical estimates of Richardson type equations are sensitive to choice of measure of military expenditure and to many aspects of specification such as other covariates included and functional form used.

3.4 Other Arms Races

The arms race metaphor has spread beyond military interactions and a comparison with its use in another area is revealing. We changed the title of Dunne & Smith (2007) from the one we had been given 'The econometrics of arms races' to 'The econometrics of military arms races', because on putting the term 'arms races' into Google Scholar the top paper was Dawkins & Kreps (1979) 'Arms races between and within species', followed by many highly cited biological papers.

The comparison between Dawkins & Kreps and Richardson is interesting both for the similarity in the process and difference in approach: they are much more specific, much less metaphorical than Richardson. They do not cite Richardson but have a very similar process in mind: 'An adaptation in one lineage (e.g. predators) may change the selection pressure on another lineage (e.g. prey), giving rise to a counter-adaptation. If this occurs reciprocally, an unstable runaway escalation or 'arms race' may result' (Dawkins & Kreps, 1979: 489).

They begin using a military analogy and clarifying the time scales considered. 'Foxes and rabbits race against each other in two senses. When an individual fox chases an individual rabbit, the race occurs on the time scale of behaviour. It is an individual race, like that between a particular submarine and the ship it is trying to sink. But there is another kind of race, on a different time scale. Submarine designers learn from earlier failures. As technology progresses, later submarines are better equipped to detect and sink ships and later-designed ships are better equipped to resist. This is an 'arms race' and it occurs over a historical time scale. Similarly, over the evolutionary time scale the fox lineage may evolve improved adaptations for catching rabbits, and the rabbit lineage improved adaptations for escaping. Biologists often use the phrase 'arms race' to describe this evolutionary escalation of ever more refined counter-adaptations (Dawkins & Kreps, 1979: 489–490). They cite use of the term arms race in a biological context in a 1940 biology paper, though as noted above the term arms race goes back to the 19th century.

They are also specific about who is involved. 'In all of this discussion it is important to realize who are the parties that are 'racing' against one another. They are not individuals but lineages' (Dawkins & Kreps, 1979: 492). They distinguish between symmetric and asymmetric arms races, arguing that asymmetric arms races are more likely between species and symmetric ones within species, e.g. male-male competition for females. They propose the 'life-dinner principle': when a fox chases a rabbit, the fox is running for its dinner, the rabbit is running for its life. Thus, the incentives and the evolutionary selection pressures on the rabbit are greater. This principle has obvious military analogies in cases such as Vietnam where the weak defeat the strong, because the weak have more at stake. They do not have any equations in the paper, but much of the work they cite, such as by William D Hamilton and John Maynard Smith, is mathematical, often involving game theory, particularly evolutionary stable games.

Dawkins & Kreps are quite specific about the time scales, parties and mechanisms involved in their biological arms races. This is like Richardson's treatment of physical processes and statistics of deadly quarrels, but unlike his more metaphorical treatment of the mathematics of military arms races.

3.5 Conclusion

Military arms races are perceived as common and usually regarded as a bad thing. Wiberg (1990: 353) suggests that they are matters of concern because of the risk of war, the waste of resources, the threat to other states, and the danger that they can breed militarism. The two main tools we have for understanding the process have been game theory, particularly the Prisoner's dilemma, and the Richardson model. The Richardson model has motivated much more empirical work than game theory, where papers tend to just use illustrative historical examples to motivate the mathematics, rather than to attempt to test the theory.

The strength and the weakness of the Richardson arms race model was that it was not very specific. This was a strength in that by giving the variables and actors different interpretations, it could be applied in a wide variety of contexts and prompt a range of interesting questions. It was a weakness in that it made it difficult to evaluate the theory. Richardson's models for the distribution of conflict statistics, power laws for size and Poisson distributions for frequency, were more like physical results and have been widely replicated. It may be that arms races, representing historically specific human decisions, are not subject to systemic regularities, so being prompted to ask the right questions is helpful in itself.

As Gates, Gleditsch & Shortland (2016: 345) put it 'Richardson's formal dynamic model of arms races may not be very useful as a description of the data or as an explanation of conflict – indeed, no decision to use force per se appears in the model. Still it is clear that it has helped move the field ahead and stimulate new research and interest in formal models of conflict.'

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