

Chapter 7

Uniform Spanning Trees of Planar Graphs



7.1 Introduction

Let G be a finite connected graph. A **spanning tree** T of G is a connected subgraph of G that contains no cycles and such that every vertex of G is incident to at least one edge of T . The set of spanning trees of a given finite connected graph is obviously finite and hence we may draw one uniformly at random. This random tree is called the **uniform spanning tree** (UST) of G . This model was first studied by Kirchhoff [49] who gave a formula for the number of spanning trees of a given graph and provided a beautiful connection with the theory of electric networks. In particular, he showed that the probability that a given edge $\{x, y\}$ of G is contained in the UST equals $\mathcal{R}_{\text{eff}}(x \leftrightarrow y; G)$; we prove this fundamental formula in Sect. 7.2 (see Theorem 7.2).

Is there a natural way of defining a UST probability measure on an infinite connected graph? It will soon become clear that we have set the framework already in Sect. 2.3 to answer this question positively. Let $G = (V, E)$ be an infinite connected graph and assume that $\{G_n\}$ is a finite exhaustion of G as defined in Sect. 2.5. That is, $\{G_n\}$ is a sequence of finite graphs, $G_n \subset G_{n+1}$ for all n , and $\cup G_n = G$. Russell Lyons conjectured that the UST probability measure on G_n converges weakly to some probability measure on subsets of E and in his pioneering work Pemantle [68] showed that it is indeed the case.

More precisely, denote by \mathcal{T}_n a UST of G_n , then it is shown in [68] that for any two finite subset of edges A, B of G the limit

$$\lim_{n \rightarrow \infty} \mathbf{P}(A \subset \mathcal{T}_n, B \cap \mathcal{T}_n = \emptyset), \quad (7.1)$$

exists and does not depend on the exhaustion $\{G_n\}$. The proof is a consequence of Rayleigh's monotonicity (Corollary 2.29) and will be presented in Sect. 7.3. This together with Kolmogorov's extension theorem [24, Theorem A.3.1] implies that there exists a unique probability measure on infinite subsets of E for which a sample

of \mathfrak{F} satisfies

$$\mathbf{P}(A \subset \mathfrak{F}, B \cap \mathfrak{F} = \emptyset) = \lim_{n \rightarrow \infty} \mathbf{P}(A \subset \mathcal{T}_n, B \cap \mathcal{T}_n = \emptyset),$$

for any two finite subsets of edges A and B of G . Thus, the law of \mathfrak{F} is determined and we denote it by μ^F . The superscript F stands for *free* and will be explained momentarily. Let us explore some properties of μ^F that are immediate from its definition.

Since every vertex of G is touched by at least one edge of \mathcal{T}_n with probability 1 when n is large enough (so that G_n contains the vertex), we learn that the edges of \mathfrak{F} almost surely touch every vertex of G , that is, \mathfrak{F} is almost surely *spanning*. Similarly, the probability that the edges of a given cycle in G are contained in \mathcal{T}_n (once n is large enough so that G_n contains the cycle) is 0. Since G has countably many cycles we deduce that almost surely there are no cycles in \mathfrak{F} . By a similar reasoning we deduce that almost surely any connected component of \mathfrak{F} is infinite. However, a moment's reflection shows that this kind of reasoning cannot be used to determine that \mathfrak{F} is almost surely connected.

It turns out, perhaps surprisingly, that \mathfrak{F} need not be connected almost surely. A remarkable result of Pemantle [68] shows that a sample of μ^F on \mathbb{Z}^d is almost surely connected when $d = 1, 2, 3, 4$ and almost surely disconnected when $d \geq 5$. Since it may be the case that a sample of μ^F is disconnected with positive probability, we call μ^F the **free uniform spanning forest** (rather than tree) of G , denoted henceforth FUSF_G . The term *free* corresponds to the fact that we have not imposed any boundary conditions when taking a limit. It will be very useful to take other boundary conditions, such as the *wired* boundary condition, see Sect. 7.3. The seminal paper of Benjamini et al. [12] explores many properties of these infinite random trees (properties such as number of components and connectivity in particular, size of the trees, recurrence or transience of the trees and many others) on various underlying graphs with an emphasis on Cayley graphs. We refer the reader to [12] and to [61, Chapters 4 and 10] for a comprehensive treatment.

The question of connectivity of the FUSF is therefore fundamental and unfortunately it is not even known that connectivity is an event of probability 0 or 1 on any graph G , see [12, Question 15.7]. In [44] the circle packing theorem (Theorem 3.5) is used to prove that FUSF_G is almost surely connected when G is a bounded degree proper planar map, answering a question of [12, Question 15.2]. Our goal in this chapter is to present a proof for a specific case where G is a bounded degree, transient, one-ended planar triangulations. Even though this is a particular case of a general theorem, the argument we present here contains most of the key ideas. We refer the interested reader to [44] for the general statement.

Theorem 7.1 ([44]) *Let G be a simple, bounded degree, transient, one-ended planar triangulation. Then FUSF_G is almost surely connected.*

The rest of this chapter is organized as follows. In Sect. 7.2 we discuss two basic properties of USTs on finite graphs. Namely, Kirchhoff's effective resistance

formula mentioned earlier and the spatial Markov property for the UST. In Sect. 7.3 we prove Pemantle's [68] result (7.1) showing that FUSF_G exists. We will also define there the *wired uniform spanning forest* which is obtained by taking a limit of the UST probability measures over exhaustions with wired boundary. We will also need some fairly basic notions of electric networks on infinite graphs that we have not discussed in Sect. 2.5. Next, in Sect. 7.4 we will restrict to the setting of planar graph and employ planar duality to obtain an extremely useful connection between the free and wired spanning forests which will be useful later. Using these tools we have collected we will prove Theorem 7.1 in Sect. 7.5.

7.2 Basic Properties of the UST

Kirchhoff's Effective Resistance Formula

Theorem 7.2 (Kirchhoff [49]) *Let G be a finite connected graph and denote by \mathcal{T} a uniformly drawn spanning tree of G . Then for any edge $e = (x, y)$ we have*

$$\mathbf{P}(e \in \mathcal{T}) = \mathcal{R}_{\text{eff}}(x \leftrightarrow y).$$

Proof Let $a \neq z$ be two distinct vertices of G (later we will take $a = x$ and $z = y$) and note that any spanning tree of G contains precisely one path connecting a and z . Thus, a uniformly drawn spanning tree induces a random path from a to z . By Claim 2.46 we obtain a unit flow θ from a to z . To be concrete, for each edge e we have that $\theta(\vec{e})$ is the probability that the random path from a to z traverses \vec{e} minus the probability that it traverses $\bar{\vec{e}}$. We will now show that θ satisfies the cycle law (see Claim 2.14), so it is in fact the unit current flow (see Definition 2.19).

Let $\vec{e}_1, \dots, \vec{e}_m$ be a directed cycle in G . Our goal is to show that

$$\sum_{i=1}^m \theta(\vec{e}_i) = 0. \quad (7.2)$$

Denote by $T(G)$ the set of spanning trees of G . Expanding the sum on the left hand side with the definition of θ we get that it equals

$$|T(G)|^{-1} \sum_{t \in T(G)} \sum_{i=1}^m f_i^+(t) - |T(G)|^{-1} \sum_{t \in T(G)} \sum_{j=1}^m f_j^-(t),$$

where $f_i^+(t)$ equals 1 if the unique path from a to z in t traverses \vec{e}_i and 0 otherwise, and similarly, $f_j^-(t)$ equals 1 if this path traverses $\bar{\vec{e}}_j$ and 0 otherwise.

For $1 \leq i \leq m$ we denote by T_i^+ the set of pairs (t, i) for which $f_i^+(t) = 1$. Similarly define T_j^- as the set of pairs (t, j) for which $f_j^-(t) = 1$. To prove (7.2) it

suffices to show that

$$|\cup_{i \in \{1, \dots, m\}} T_i^+| = |\cup_{j \in \{1, \dots, m\}} T_j^-|.$$

Let $(t, i) \in T_i^+$. The graph $t \setminus \{e_i\}$ has two connected components. Let \vec{e}_j be the first edge after \vec{e}_i , in the order of the cycle $\vec{e}_1, \dots, \vec{e}_m$, that is incident to both connected components and consider the spanning tree $t' = t \cup \{e_j\} \setminus \{e_i\}$. Note that the unique path in t' from a to z traverses \vec{e}_j , so $(t', j) \in T_j^-$. This procedure defines a bijection from $\cup_i T_i^+$ to $\cup_j T_j^-$. Indeed, given (t', j) from before, we can erase e_j and go on the cycle in the opposite order until we reach e_i which has to be the first edge incident to the two connected components of $t' \setminus \{e_j\}$. This shows (7.2) and concludes the proof. \square

Spatial Markov Property of the UST

We would like to study the UST probability measure conditioned on the event that some edges are present in the UST and others not. It turns out that sampling from this conditional distribution amounts to drawing a UST on a modified graph.

Let $G = (V, E)$ be a finite connected graph and let A and B be two disjoint subsets of edges. We write $(G - B)/A$ for the graph obtained from G by erasing the edges of B and contracting the edges of A . We identify the edges of $(G - B)/A$ with the edges $E \setminus B$. Denote by \mathcal{T}_G and $\mathcal{T}_{(G-B)/A}$ a UST on G and $(G - B)/A$, respectively, and assume that

$$\mathbf{P}(A \subset \mathcal{T}_G, B \cap \mathcal{T}_G = \emptyset) > 0.$$

This assumption is equivalent to $G - B$ being connected and that A contains no cycles.

Then, conditioned on the event that \mathcal{T}_G contains the edges A and does not contain any edge of B the distribution of \mathcal{T}_G is equal to the union of A with $\mathcal{T}_{(G-B)/A}$. In other words, for a set \mathcal{A} of spanning trees of G we have that

$$\mathbf{P}(\mathcal{T}_G \in \mathcal{A} \mid A \subset \mathcal{T}_G, B \cap \mathcal{T}_G = \emptyset) = \mathbf{P}(A \cup \mathcal{T}_{(G-B)/A} \in \mathcal{A}). \quad (7.3)$$

The proof of (7.3) follows immediately from the observation that the set of spanning trees of G not containing any edge of B is simply the set of spanning trees of $G - B$. Similarly, the set of spanning trees of G containing all the edges of A is simply the union of A to each spanning tree of G/A , and (7.3) follows.

7.3 Limits over Exhaustions: The Free and Wired USF

Let G be an infinite connected graph and let $\{G_n\}$ be a finite exhaustion of it. In this section we will show that (7.1) holds and that the UST measures with *wired* boundary conditions also converge. Let us first explain the latter. Denote by G_n^* the graph obtained from G by identifying the infinite set of vertices $G \setminus G_n$ to a single vertex z_n and erasing the loops at z_n formed by this identification. We say that $\{G_n^*\}$ is a **wired finite exhaustion** of G .

Theorem 7.3 (Pemantle [68]) *Let G be an infinite connected graph, $\{G_n\}$ a finite exhaustion and $\{G_n^*\}$ the corresponding wired finite exhaustion. Denote by \mathcal{T}_n and \mathcal{T}_n^* USTs on G_n and G_n^* , respectively. Then for any two finite disjoint subsets $A, B \subset E(G)$ of edges of G we have that the limits*

$$\lim_{n \rightarrow \infty} \mathbf{P}(A \subset \mathcal{T}_n, B \cap \mathcal{T}_n = \emptyset),$$

and

$$\lim_{n \rightarrow \infty} \mathbf{P}(A \subset \mathcal{T}_n^*, B \cap \mathcal{T}_n^* = \emptyset),$$

exist and do not depend on the exhaustion $\{G_n\}$.

We postpone the proof for a little longer and first discuss some of its implications. As mentioned earlier, Theorem 7.3 together with Kolmogorov's extension theorem [24, Theorem A.3.1] implies that there exists two probability measures μ^F and μ^W on infinite subsets of the edges of E arising as the unique limits of the laws \mathcal{T}_n and \mathcal{T}_n^* . That is, the samples \mathfrak{F}^f and \mathfrak{F}^w of μ^F and μ^W satisfy

$$\mathbf{P}(A \subset \mathfrak{F}^f, B \cap \mathfrak{F}^f = \emptyset) = \lim_{n \rightarrow \infty} \mathbf{P}(A \subset \mathcal{T}_n, B \cap \mathcal{T}_n = \emptyset),$$

and

$$\mathbf{P}(A \subset \mathfrak{F}^w, B \cap \mathfrak{F}^w = \emptyset) = \lim_{n \rightarrow \infty} \mathbf{P}(A \subset \mathcal{T}_n^*, B \cap \mathcal{T}_n^* = \emptyset).$$

We call μ^F and μ^W the **free uniform spanning forest** and the **wired uniform spanning forest** and denote them by FUSF_G and WUSF_G respectively. We have seen earlier (one paragraph below (7.1)) that both \mathfrak{F}^f and \mathfrak{F}^w are almost surely spanning forests, that is, spanning graphs of G with no cycles and that every connected component of them is infinite. Thus μ^F and μ^W are supported on what are known as **essential spanning forests** of G , that is, spanning forests of G in which every component is infinite.

Are the probability measures FUSF_G and WUSF_G equal? Not necessarily. It is easy to see that on the infinite path \mathbb{Z} the $\text{WUSF}_{\mathbb{Z}}$ and the $\text{FUSF}_{\mathbb{Z}}$ are equal and are the entire graph \mathbb{Z} with probability 1. Conversely, it is not very difficult to see that

they are different on a 3-regular tree, see exercise 1 of this chapter. Pemantle [68] has shown that $\text{FUSF}_{\mathbb{Z}^d} = \text{WUSF}_{\mathbb{Z}^d}$ for any $d \geq 1$ and a very useful criterion for determining whether there is equality was developed in [12]. We refer the reader to [61, Chapter 10] for further reading.

Before presenting the proof of Theorem 7.3 let us make a few short observations regarding the effective resistance between two vertices in an infinite graph, extending what we proved in Sect. 2.5.

Effective Resistance in Infinite Networks

Let G be an infinite connected graph. We have seen in Sect. 2.5 that for any vertex v the electric resistance $\mathcal{R}_{\text{eff}}(v \leftrightarrow \infty)$ from v to ∞ is well defined as the limit of $\mathcal{R}_{\text{eff}}(a \leftrightarrow z_n; G_n^*)$ where $\{G_n^*\}$ is a wired finite exhaustion and z_n is the vertex resulting in the identification of the vertices $G \setminus G_n$.

To define the electric resistance between two vertices v, u of an infinite graph, one has to take exhaustions and specify boundary conditions since the limits may differ depending on them.

Claim 7.4 Let G be an infinite connected graph, $\{G_n\}$ a finite exhaustion and $\{G_n^*\}$ a wired finite exhaustion. Then for any two vertices u, v of G we have that the limits

$$\mathcal{R}_{\text{eff}}^F(u \leftrightarrow v; G) := \lim_n \mathcal{R}_{\text{eff}}(u \leftrightarrow v; G_n),$$

and

$$\mathcal{R}_{\text{eff}}^W(u \leftrightarrow v; G) := \lim_n \mathcal{R}_{\text{eff}}(u \leftrightarrow v; G_n^*),$$

exist and do not depend on the exhaustion $\{G_n\}$.

Proof For the first limit we note that by Rayleigh's monotonicity (Corollary 2.29), the sequence $\mathcal{R}_{\text{eff}}(u \leftrightarrow v; G_n)$ is non-increasing and non-negative since $G_n \subset G_{n+1}$, hence it converges. A sandwiching argument as in the proof of Claim 7.4 shows that the limit does not depend on the exhaustion $\{G_n\}$.

For the second limit, since G_n can be obtained by gluing vertices of G_{n+1} we deduce by Corollary 2.30 that the sequence $\mathcal{R}_{\text{eff}}(u \leftrightarrow v; G_n^*)$ is non-decreasing and bounded (by the graph distance in G between u and v for instance), hence it converges. The limit does not depend on the exhaustion by an identical sandwiching argument. \square

We call $\mathcal{R}_{\text{eff}}^F(u \leftrightarrow v; G)$ and $\mathcal{R}_{\text{eff}}^W(u \leftrightarrow v; G)$ the **free effective resistance** and **wired effective resistance** between u and v respectively.

Proof of Theorem 7.3

We will prove the assertion regarding the first limit; the second is almost identical. Write $A = \{e_1, \dots, e_k\}$ and $e_i = (x_i, y_i)$ for each $1 \leq i \leq k$. Assume without loss of generality that G_n contains A for all n . As before, denote by \mathcal{T}_n a UST of G_n . By (7.3) and Theorem 7.2 we have that

$$\mathbf{P}(A \subset \mathcal{T}_n) = \prod_{i=1}^k \mathbf{P}(e_i \in \mathcal{T}_n \mid e_j \in \mathcal{T}_n \quad \forall j < i) = \prod_{i=1}^k \mathcal{R}_{\text{eff}}(x_i \leftrightarrow y_i; G_n / \{e_1, \dots, e_{i-1}\}).$$

Note that $\{G_n / \{e_1, \dots, e_{i-1}\}\}$ is a finite exhaustion of the infinite graph $G / \{e_1, \dots, e_{i-1}\}$ and so by Claim 7.4 we obtain that the limit

$$\lim_n \mathbf{P}(A \subset \mathcal{T}_n) = \prod_{i=1}^k \mathcal{R}_{\text{eff}}(x_i \leftrightarrow y_i; G / \{e_1, \dots, e_{i-1}\}),$$

exists and does not depend on the exhaustion.

Since we know this limit exists for all finite edge sets A , it follows by the inclusion-exclusion formula that $\mathbf{P}(A \subset \mathcal{T}_n, B \cap \mathcal{T}_n = \emptyset)$ converges for any finite sets A, B , concluding our proof. \square

It is now quite pleasant to see that the symbiotic relationship between electric network and UST theories continues to flourish in the infinite setting. Indeed, by combining Theorems 7.3 and Claim 7.4 we obtain the extension of Kirchhoff's formula for infinite connected graphs.

Theorem 7.5 *Let G be an infinite connected graph and denote by \mathfrak{F}^F and \mathfrak{F}^W a sample from FUSF_G and WUSF_G respectively. Then for any edge $e = (x, y)$ of G we have that*

$$\mathbf{P}(e \in \mathfrak{F}^F) = \mathcal{R}_{\text{eff}}^F(x \leftrightarrow y; G),$$

and

$$\mathbf{P}(e \in \mathfrak{F}^W) = \mathcal{R}_{\text{eff}}^W(x \leftrightarrow y; G).$$

7.4 Planar Duality

When G is planar there is a very useful relationship between FUSF_G and WUSF_G . Recall that given a planar map G , the **dual graph** of G is the graph G^\dagger whose vertex set is the set of faces of G and two faces are adjacent in G^\dagger if they share an edge in G . Thus, G^\dagger is locally-finite if and only if every face of G has finitely many edges.

To each edge $e \in E(G)$ corresponds a dual edge $e^\dagger \in E(G^\dagger)$ which is the pair of faces of G incident to e ; this is clearly a one-to-one correspondence.

When G is a finite planar graph, this correspondence induces a one-to-one correspondence between the set of spanning trees of G and the set of spanning trees of G^\dagger . Given a spanning tree t of G we slightly abuse the notation and write t^\dagger for the set of edges $\{e^\dagger : e \in t\}$, that is

$$e \in t \iff e^\dagger \notin t^\dagger.$$

If t^\dagger has a cycle, then t is disconnected. Furthermore, if there is a vertex G^\dagger not incident to any edge of t^\dagger , then all the edges of the corresponding face in G are present in t hence t contains a cycle. We deduce that if t is a spanning tree of G , then t^\dagger is a spanning tree of G^\dagger . The converse also holds since $(t^\dagger)^\dagger = t$ and $(G^\dagger)^\dagger = G$.

Now assume that G is an infinite planar maps such that G^\dagger is locally finite. Given an essential spanning forest \mathfrak{F} of G we similarly define \mathfrak{F}^\dagger as the set of edges $\{e^\dagger : e \in \mathfrak{F}\}$. A similar argument shows that \mathfrak{F}^\dagger is an essential spanning forest of G^\dagger . This raises the natural question: when \mathfrak{F} is a sample of FUSF_G , what is the law of \mathfrak{F}^\dagger ? The answer in general is an object known as the *transboundary uniform spanning forest* [44, Proposition 5.1]. However, when G is additionally assumed to be one-ended (in particular, in the setting of Theorem 7.1) it turns out that \mathfrak{F}^\dagger is distributed as WUSF_{G^\dagger} :

Proposition 7.6 *Let G be an infinite, one-ended planar map with a locally finite dual G^\dagger and let \mathfrak{F} be a sample of FUSF_G . Then the law of \mathfrak{F}^\dagger is WUSF_{G^\dagger} .*

Proof Let G_n be a finite exhaustion of G . Let F_n be a finite exhaustion G^\dagger defined by letting $f \in F_n$ if and only if every vertex of f in G belongs to G_n . Then G_n^\dagger is obtained from G^\dagger by contracting $G^\dagger \setminus F_n$ into a single vertex which corresponds to the outer face of G_n . Thus, G_n^\dagger is a wired exhaustion of G^\dagger and the statement follows. \square

We use to obtain an important criterion of connectivity of FUSF_G in the planar case.

Proposition 7.7 *Let G be an infinite, one-ended planar map with a locally finite dual G^\dagger . Then a sample of FUSF_G is connected almost surely if and only if each component of a sample of WUSF_G is one-ended almost surely.*

Proof By Proposition 7.6 it suffices to show that if \mathfrak{F} is an essential spanning forest of G , then \mathfrak{F} is connected if and only if every component of \mathfrak{F}^\dagger is one-ended. Indeed, if \mathfrak{F} is disconnected, then the boundary of a connected component of \mathfrak{F} induces an bi-infinite path in \mathfrak{F}^\dagger . Conversely, if \mathfrak{F}^\dagger contains a bi-infinite path, then by the Jordan curve theorem \mathfrak{F} is disconnected. \square

7.5 Connectivity of the Free Forest

Last Note on Infinite Networks

We make two more useful and natural definitions. Given two disjoint finite sets A and B in an infinite connected graph G we define the free and wired effective resistance between them $\mathcal{R}_{\text{eff}}^W(A \leftrightarrow B; G)$ and $\mathcal{R}_{\text{eff}}^F(A \leftrightarrow B; G)$ as the free and wired effective resistance between a and b in the graph obtained from G by identifying A and B to the vertices a and b .

Lastly, given a graph G , a wired finite exhaustion $\{G_n^*\}$ of G and two disjoint finite sets A and B we define

$$\mathcal{R}_{\text{eff}}(A \leftrightarrow B \cup \{\infty\}; G) := \lim_{n \rightarrow \infty} \mathcal{R}_{\text{eff}}(A \leftrightarrow B \cup \{z_n\}; G_n^*), \quad (7.4)$$

where the last limit exists since the sequence is non-increasing from n that is large enough so that G_n contains A and B . In the proof of Theorem 7.1 we will require the following estimate.

Lemma 7.8 *Let A and B be two finite sets of vertices in an infinite connected graph G . Then*

$$\mathcal{R}_{\text{eff}}^W(A \leftrightarrow B; G) \leq 3 \max[\mathcal{R}_{\text{eff}}(A \leftrightarrow B \cup \{\infty\}; G), \mathcal{R}_{\text{eff}}(B \leftrightarrow A \cup \{\infty\}; G)].$$

Proof For any three distinct vertices u, v, w in a finite network we have by the union bound that $\mathbf{P}_u(\tau_{\{v,w\}} < \tau_u^+) \leq \mathbf{P}_u(\tau_v < \tau_u^+) + \mathbf{P}_u(\tau_w < \tau_u^+)$. Hence by Claim 2.22 we get that

$$\mathcal{R}_{\text{eff}}(u \leftrightarrow \{v, w\})^{-1} \leq \mathcal{R}_{\text{eff}}(u \leftrightarrow v)^{-1} + \mathcal{R}_{\text{eff}}(u \leftrightarrow w)^{-1}.$$

Let $\{G_n^*\}$ be a wired finite exhaustion of G and assume without loss of generality that A and B are contained in G_n^* for all n . Then by the previous estimate

$$\mathcal{R}_{\text{eff}}(A \leftrightarrow B \cup \{z_n\}; G_n^*)^{-1} \leq \mathcal{R}_{\text{eff}}(A \leftrightarrow B; G_n^*)^{-1} + \mathcal{R}_{\text{eff}}(A \leftrightarrow z_n; G_n^*)^{-1}.$$

Denote by M the maximum in the statement of the lemma and take $n \rightarrow \infty$ in the last inequality. We obtain that

$$M^{-1} \leq \mathcal{R}_{\text{eff}}(A \leftrightarrow B \cup \{\infty\}; G)^{-1} \leq \mathcal{R}_{\text{eff}}^W(A \leftrightarrow B; G)^{-1} + \mathcal{R}_{\text{eff}}(A \leftrightarrow \infty; G)^{-1}.$$

Rearranging gives that

$$\mathcal{R}_{\text{eff}}(A \leftrightarrow \infty; G) \leq \frac{M \mathcal{R}_{\text{eff}}^W(A \leftrightarrow B; G)}{\mathcal{R}_{\text{eff}}^W(A \leftrightarrow B; G) - M}.$$

By symmetry, the same inequality holds when we replace the roles of A and B . We put this together with the triangle inequality for effective resistances (2.9) and get that

$$\mathcal{R}_{\text{eff}}^W(A \leftrightarrow B; G) \leq \mathcal{R}_{\text{eff}}(A \leftrightarrow \infty; G) + \mathcal{R}_{\text{eff}}(B \leftrightarrow \infty; G) \leq \frac{2M\mathcal{R}_{\text{eff}}^W(A \leftrightarrow B; G)}{\mathcal{R}_{\text{eff}}^W(A \leftrightarrow B; G) - M},$$

which by rearranging gives the desired inequality. \square

Method of Random Sets

We present the following weakening of the method of random paths as in Sect. 2.6. Let μ be the law of a random subset W of vertices of G . Define the *energy* of μ as

$$\mathcal{E}(\mu) = \sum_{v \in V} \mu(v \in W)^2.$$

Lemma 7.9 (Method of Random Sets) *Let A, B be two disjoint finite sets of vertices in an infinite graph G . Let W be a random subset of vertices of G and denote by μ its law. Assume that the subgraph of G induced by W almost surely contains a simple path starting at A that is either infinite or finite and ends at B . Then*

$$\mathcal{R}_{\text{eff}}(A \leftrightarrow B \cup \{\infty\}; G) \leq \mathcal{E}(\mu). \quad (7.5)$$

Proof Given W let γ be a simple path, contained in W , connecting A to B or an infinite path starting at A . We choose γ according to some prescribed lexicographical ordering. Then, letting ν be the law of γ ,

$$\mathcal{E}(\nu) \leq \sum_{\vec{e} \in E} \nu(\vec{e} \in \gamma)^2,$$

where by $\vec{e} \in \gamma$ we mean that the directed edge \vec{e} is traversed (in its direction) by γ , and by $\mathcal{E}(\nu)$ we mean the energy of the flow induced by γ , as in Claim 2.46.

Let γ' be an independent random path having the same law as γ . Then the sum above is precisely the expected number of directed edges traversed both by γ and γ' . Since these are simple paths, they each contain at most one directed edge emanating from each vertex $v \in W$. Thus, the expected number of directed edges used by both paths is at most the number of vertices used by both paths. Hence,

$$\mathcal{E}(\nu) \leq \sum_{v \in V(G)} \nu(v \in \gamma)^2 \leq \sum_{v \in V(G)} \mu(v \in W)^2 = \mathcal{E}(\mu),$$

and the proof is concluded by Thomson's principle (Theorem 2.28). \square

Proof of Theorem 7.1

In Theorem 7.1 we assume that $G = (V, E)$ is a bounded-degree, one-ended triangulation. Hence G^\dagger is a bounded degree (in fact, 3-regular), one-ended and transient planar map with faces of uniformly bounded size. We leave this verification as an exercise for the reader. To avoid carrying the \dagger symbol around, and with a slight abuse of notation, let $G = (V, E)$ be a graph satisfying these assumptions on G^\dagger , that is, we assume that G is a one-ended, transient, infinite planar map with bounded degrees and face sizes. We will prove under these assumptions that every component of WUSF_G is one ended almost surely which implies Theorem 7.1 by Proposition 7.7.

Let T be the bounded-degree one-ended triangulation obtained from G by adding a vertex inside each face of G and connecting it by edges to the vertices of that face according to their cyclic ordering. By Theorem 4.4 there exists a circle packing of T in the unit disc \mathbb{U} . We identify the vertices of T as the vertices $V(G)$ and faces $F(G)$ of G , and denote this circle packing as $P = \{P(v) : v \in V(G)\} \cup \{P(f) : f \in F(G)\}$.

Given $z \in \mathbb{U}$ and $r' \geq r > 0$ denote by $A_z(r, r')$ the annulus $\{w \in \mathbb{C} : r \leq |w - z| \leq r'\}$.

Definition 7.10 Write $V_z(r, r')$ for the set of vertices v of G such that either

- $P(v) \cap A_z(r, r') \neq \emptyset$, or
- $P(v) \subset \{w \in \mathbb{C} : |w| \leq r\}$ and there is a face f of G with $v \in f$ and $P(f) \cap A_z(r, r') \neq \emptyset$.

We emphasize that $V_z(r, r')$ contains only vertices of G ; no vertices of T that correspond to faces of G belong to it.

Lemma 7.11 *There exists a constant $C < \infty$ depending only on the maximal degree such that for any $z \in \mathbb{U}$ and any positive integer n satisfying $|z| \geq 1 - C^{-n}$ the sets*

$$V_z(C^{-i}, 2C^{-i}) \quad 1 \leq i \leq n,$$

are disjoint.

Proof By the Ring Lemma (Lemma 4.2) there exists a constant $B < \infty$ such that for any $C > 1$, any z satisfying $|z| \geq 1 - C^{-n}$ and any $1 \leq i \leq n$, if a circle of P intersects $A_z(C^{-i}, 2C^{-i})$ or is tangent to a circle that intersects $A_z(C^{-i}, 2C^{-i})$, then its radius is at most BC^{-i} . Hence, this set of circles is contained in the disc of radius $(2 + 4B)C^{-i}$ around z . Furthermore, since $|z| \geq 1 - C^{-n}$, by the Ring Lemma again there exists $b > 0$ such that any such circle must be of distance at least bC^{-i} from z . Hence, any fixed $C > \frac{4+4B}{b}$ satisfies the assertion of the lemma.

□

Lemma 7.12 *Let $z \in \mathbb{U}$ and $r > 0$. Let U be a uniform random variable in $[1, 2]$ and denote by μ_r the law of the random set $V_z(Ur, Ur)$ (as defined in Definition 7.10). Then there exists a constant $C < \infty$ depending only on the maximal degree such that*

$$\mathcal{E}(\mu_r) \leq C.$$

Proof For each vertex v , the event $v \in V_z(Ur, Ur)$ implies that the circle $\{w \in \mathbb{C} : |w - z| = Ur\}$ intersects the circle $P(v)$ or intersects $P(f)$ for some face f incident to v . The union of $P(v)$ and $P(f)$ over all such faces f is contained in the Euclidean ball around the center of $P(v)$ of radius $r(v) + 2 \max_{f: v \in f} r(f)$. Since T has finite maximal degree we have that $r(f) \leq Cr(v)$ for all f with $v \in f$ where $C < \infty$ depends only on the maximal degree by the Ring Lemma (Lemma 4.2). Hence,

$$\mu_r(v \in V_z(Ur, Ur)) \leq \frac{1}{r} \min \left(2r(v) + 4 \max_{f: v \in f} r(f), r \right) \leq \frac{C}{r} \min\{r(v), r\}. \quad (7.6)$$

We claim that

$$\sum_{v \in V_z(r, 2r)} \min\{r(v), r\}^2 \leq 16r^2. \quad (7.7)$$

Indeed, consider a vertex $v \in V_z(r, 2r)$ for which the corresponding circle $P(v)$ has radius larger than r . By Definition 7.10 this circle must intersect $\{w \in \mathbb{C} : |w - z| \leq 2r\}$. We replace each such $P(v)$ with a circle of radius r that is contained in the original circle and intersects $\{w \in \mathbb{C} : |w - z| \leq 2r\}$. The circles in this new set still have disjoint interiors and are contained in $\{w \in \mathbb{C} : |w - z| \leq 4r\}$. Therefore their area is at most $\pi 16r^2$ and (7.7) follows. The proof of lemma is now concluded by combining (7.6) and (7.7). \square

Proof of Theorem 7.1 Let \mathfrak{F} be a sample of \mathbf{WUSF}_G and given an edge $e = (x, y)$ we define \mathcal{A}^e to be the event that x and y are in two distinct infinite connected components of $\mathfrak{F} \setminus \{e\}$. It is clear that every component of \mathfrak{F} is one-ended almost surely if and only if

$$\mathbf{P}(e \in \mathfrak{F}, \mathcal{A}^e) = 0 \quad (7.8)$$

for every edge e of G . Consider the triangulation T described above Definition 7.10 and its circle packing P in \mathbb{U} . By choosing the proper Möbius transformation we may assume that the tangency point between $P(x)$ and $P(y)$ is the origin, and that the centers of $P(x)$ and $P(y)$ lie on the negative and positive real axis, respectively.

Fix now an arbitrary $\varepsilon > 0$ and let V_ε be all the vertices of G such that the center $z(v)$ of $P(v)$ satisfies $|z(v)| \leq 1 - \varepsilon$. Denote by $\mathcal{B}_\varepsilon^e$ the event that every connected

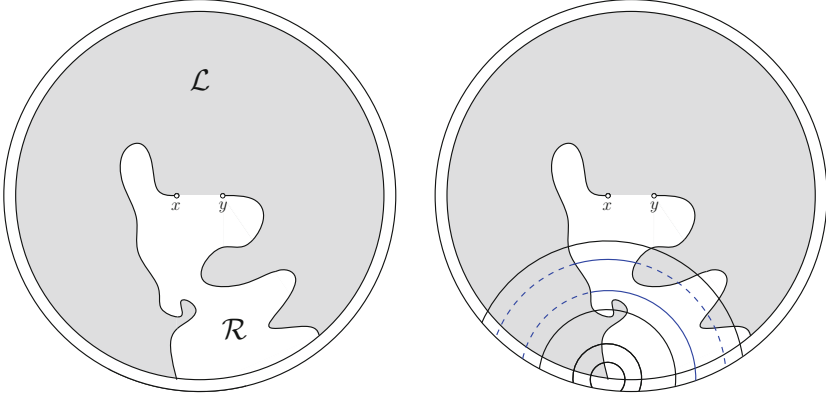


Fig. 7.1 Illustration of the proof. Left: On the event $\mathcal{A}_\varepsilon^e$, the paths η^x and η^y split V_ε into two pieces, \mathcal{L} and \mathcal{R} . Right: We define a random set containing a path (solid blue) from η^x to $\eta^y \cup \{\infty\}$ in $G \setminus K_c$ using a random circle (dashed blue). Here we see two examples, one in which the path ends at η^y , and the other in which the path ends at the boundary (i.e., at infinity)

component of $\mathfrak{F} \setminus \{e\}$ intersects $V \setminus V_\varepsilon$. Note that $\mathcal{A}^e \subset \cap_{\varepsilon>0} \mathcal{B}_\varepsilon^e$ but this containment is strict since it is possible that $e \notin \mathfrak{F}$ and x is connected to y in \mathfrak{F} inside V_ε .

Assume that $\mathcal{B}_\varepsilon^e$ holds. Let η^x be the rightmost path in $\mathfrak{F} \setminus \{e\}$ from x to $V \setminus V_\varepsilon$ when looking at x from y , and let η^y be the leftmost path in $\mathfrak{F} \setminus \{e\}$ from y to $V \setminus V_\varepsilon$ when looking at y from x . As mentioned above, the paths η_x and η_y are not necessarily disjoint. Nonetheless, concatenating the reversal of η^x with e and η^y separates V_ε into two sets of vertices, \mathcal{L} and \mathcal{R} , which are to the left and right of e (when viewed from x to y) respectively. See Fig. 7.1 for an illustration of the case when η_x and η_y are disjoint (when they are not, \mathcal{R} is a “bubble” separated from $V \setminus V_\varepsilon$).

On the event $\mathcal{B}_\varepsilon^e$, let K be the set of edges that are either incident to a vertex in \mathcal{L} or belong to the path $\eta_x \cup \eta_y$, and set $K = E$ off of this event. Note that the edges of K do not touch the vertices of \mathcal{R} . The condition that η^x and η^y are the rightmost and leftmost paths to $V \setminus V_\varepsilon$ from x and y is equivalent to the condition that K does not contain any open path from x to $V \setminus V_\varepsilon$ other than η^x , and does not contain any open path from y to $V \setminus V_\varepsilon$ other than η^y . We note that K can be explored algorithmically, without querying the status of any edge in $E \setminus K$, by performing a right-directed depth-first search of x ’s component in \mathfrak{F} and a left-directed depth-first search of y ’s component in \mathfrak{F} , stopping each search when it first leaves V_ε .

Denote by $\mathcal{A}_\varepsilon^e$ the event that η_x and η_y are disjoint, or equivalently, that K does not contain an open path from x to y (and in particular, no path starting at η_x and ending at η_y). The event $\mathcal{A}_\varepsilon^e$ is measurable with respect to the random set K and $\mathcal{A}^e = \cap_{\varepsilon>0} \mathcal{A}_\varepsilon^e$. Hence

$$\mathbf{P}(e \in \mathfrak{F}, \mathcal{A}^e) \leq \mathbf{P}(e \in \mathfrak{F} \mid \mathcal{A}_\varepsilon^e) = \mathbf{E}[\mathbf{P}(e \in \mathfrak{F} \mid \mathcal{A}_\varepsilon^e, K)]. \quad (7.9)$$

Denote by K_o the open edge of K (that is, the edge of K in \mathfrak{F}) and by K_c the closed edges of K (that is, the edges of K not belonging to \mathfrak{F}). In particular, η_x and η_y are contained in K_o . Then by the UST Markov property (7.3), conditioned on K and the event $\mathcal{A}_\varepsilon^e$, the law of \mathfrak{F} is equal to the union of K_o with a sample of the WUSF on $(G - K_c)/K_o$. In particular, by Kirchhoff's formula Theorem 7.5 we have that

$$\mathbf{P}(e \in \mathfrak{F} \mid \mathcal{A}_\varepsilon^e, K) \leq \mathcal{R}_{\text{eff}}^W(\eta_x \leftrightarrow \eta_y; G - K_c), \quad (7.10)$$

where in the last inequality we used the fact that gluing cannot increase the resistance (Corollary 2.30).

We will show that the last quantity tends to 0 as $\varepsilon \rightarrow 0$ which gives (7.8). To that aim, let v^x be the endpoint of the path η^x and let z_0 be the center of the $P(v_x)$. On the event $\mathcal{A}_\varepsilon^e$, for each $1 - |z_0| \leq r \leq 1/4$, we claim that the set $V_{z_0}(r, r)$, as defined in Definition 7.10, contains a path in G from η^x to η^y that is contained in $\mathcal{R} \cup \eta^x \cup \eta^y$ or an infinite simple path starting at η_x that is contained in $\mathcal{R} \cup \eta^x$. Either of these paths are therefore a path in $G - K_c$.

To see this, consider the arc $\mathbf{A}'(z_0, r) = \{z \in \overline{\mathbb{U}} : |z - z_0| = r\}$ viewed in the clockwise direction and let $\mathbf{A}(z_0, r)$ be the subarc beginning at the last intersection of $\mathbf{A}'(z_0, r)$ with a circle corresponding to a vertex in the trace of η^x , and ending at the first intersection after this time of $\mathbf{A}'(z_0, r)$ with either $\partial\mathbb{U}$ or a circle corresponding to a vertex in the trace of η^y (see Fig. 7.1). Hence, if $\mathcal{A}_\varepsilon^e$ holds, then the set of vertices of T whose circles in P intersect $\mathbf{A}(z_0, r)$ contains a path in T starting at η^x and ending η^y or does not end at all, for every $1 - |z_0| \leq r \leq 1/4$. To obtain a path in G rather than T we divert the path counterclockwise around each face of G . That is, whenever the path passes from a vertex u of G to a face f of G and then to a vertex v of G , we replace this section of the path with the list of vertices of G incident to f that are between u and v in the counterclockwise order. By Definition 7.10 this diverted path is in $V_{z_0}(r, r)$ and so this construction shows that the subgraph of $G - K_c$ induced by the set $V_{z_0}(r, r)$ contains a path from η^x to η^y or an infinite path from η^x , as claimed.

Let $r_i = C^{-i}$ for $i = 1, \dots, N$ where $C < \infty$ the constant from Lemma 7.11 and $N = \lfloor \log_C(\varepsilon) \rfloor$. Assume without loss of generality that $C \geq 4$ so that $\varepsilon \leq r_i \leq 1/4$ for all $i = 1, \dots, N$. By Lemma 7.11 the measures μ_{r_i} defined in Lemma 7.12 are supported on sets that are contained in the disjoint sets $V_z(r_i, 2r_i)$. Thus, by Lemma 7.9 and Lemma 7.12 we have

$$\mathcal{R}_{\text{eff}}^W(\eta^x \leftrightarrow \eta^y \cup \{\infty\}; G \setminus K_c) \leq \mathcal{E}\left(\frac{1}{N} \sum_{i=1}^N \mu_{r_i}\right) = \frac{1}{N^2} \sum_{i=1}^N \mathcal{E}(\mu_{r_i}) \leq \frac{B}{\log(1/\varepsilon)},$$

where $B < \infty$ is a constant depending only on the maximum degree. By symmetry we also have

$$\mathcal{R}_{\text{eff}}^W(\eta^y \leftrightarrow \eta^x \cup \{\infty\}; G - K_c) \leq \frac{B}{\log(1/\varepsilon)}.$$

Applying Lemma 7.8 and (7.10) gives

$$\mathbf{P}(e \in \mathfrak{F} \mid \mathcal{F}_K, \mathcal{A}_\varepsilon^e) \leq \frac{3B}{\log(1/\varepsilon)}.$$

We plug this estimate into (7.10) and take $\varepsilon \rightarrow 0$, which together with (7.9) shows that (7.8) holds, concluding our proof. \square

7.6 Exercises

1. Use Theorem 7.5 to show that on the 3-regular infinite tree \mathbb{T}_3 the probability measures $\text{FUSF}_{\mathbb{T}_3}$ and the $\text{WUSF}_{\mathbb{T}_3}$ are distinct.
2. Let $(G; \{r_e\})$ be a tree with edge resistances $\{r_e\}$ such that $\sum_{n \geq 1} r(e_n) = \infty$ for any simple infinite path $\{e_n\}_{n \geq 1}$ in G . Show that the free and wired uniform spanning forests coincide if and only if $(G; \{r_e\})$ is recurrent.
3. Let L_n be the ladder graph, that is, the vertex set is $\{1, \dots, n\} \times \{a, b\}$ and the edges set is $\{(i, a), (i, b)\} : 1 \leq i \leq n\} \cup \{(i, a), (i+1, a)\} : 1 \leq i \leq n-1\} \cup \{(i, b), (i+1, b)\} : 1 \leq i \leq n-1\}$. Compute the limiting probability, as $n \rightarrow \infty$, that the edge $[(1, a), (1, b)]$ is in the UST of L_n .
4. Show that \mathbb{Z}^3 contains a transient subtree.

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