Chapter 5
Planar Local Graph Limits

5.1 Local Convergence of Graphs and Maps

In order to study large random graphs it is mathematically natural and appealing to introduce an infinite limiting object and study its properties. In their seminal paper, Benjamini and Schramm [11] introduced the notion of locally convergent graph sequences, which we now describe.

We will consider random variables taking values in the space $\mathcal{G}_\bullet$ of locally finite connected rooted graphs viewed up to root preserving graph isomorphisms. That is, $\mathcal{G}_\bullet$ is the space of pairs (G, ρ) where G is a locally finite graph (which may be finite or infinite) and $\rho \in V(G)$ is a vertex of G and two elements $(G_1, \rho_1), (G_2, \rho_2)$ are considered equivalent if there is a graph isomorphism between them (that is, a bijection $\varphi : V(G_1) \to V(G_2)$ such that $\varphi(\rho_1) = \varphi(\rho_2)$ and $\{v_1, v_2\} \in E(G_1)$ if and only if $\{\varphi(v_1), \varphi(v_2)\} \in E(G_2)$). We remark that throughout this book our graphs will almost entirely be connected. In the rare case when G is not connected, we impose the convention that (G, ρ) refers to $(G[\rho], \rho)$ where $G[\rho]$ is the connected component of ρ in G. This way $(G, \rho) \in \mathcal{G}_\bullet$ even when G is disconnected (this will only be relevant in Chap. 6, and in particular in Lemma 6.11 and its usage).

In a similar fashion we define $\mathcal{M}_\bullet$ to be the set of equivalence classes of rooted maps; in this case we require the graph isomorphism to preserve additionally the cyclic permutations of the neighbors of each vertex, that is, it is a **map isomorphism**. Let us describe the topology on $\mathcal{G}_\bullet$ and $\mathcal{M}_\bullet$. For convenience we discuss $\mathcal{G}_\bullet$ but every statement in the following holds for $\mathcal{M}_\bullet$ as well.

Given an element (G, ρ) of $\mathcal{G}_\bullet$, the finite graph $B_G(\rho, R)$ is the subgraph of (G, ρ) rooted at ρ spanned by the vertices of distance at most R from ρ. We provide $\mathcal{G}_\bullet$ with a metric d_{loc}

$$d_{\mathrm{loc}}((G_1, \rho_1), (G_2, \rho_2)) = 2^{-R},$$

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where R is the largest integer for which $B_{G_1}(\rho_1, R)$ and $B_{G_2}(\rho_2, R)$ are isomorphic as graphs. This is a separable topological space (the finite graphs form a countable base for the topology) and is easily seen to be complete, thus it is a Polish space. The distances are bounded by 1 but the space is not compact. Indeed, the sequence G_n of stars with n leaves emanating from the root ρ has no converging subsequence.

Since $\mathcal{G}_\bullet$ is a Polish space, we can discuss convergence in distribution of a sequence of random variables $\{X_n\}_{n=1}^{\infty}$ taking values in $\mathcal{G}_\bullet$. We say that X_n **converges in distribution** to a random variable X, and denote it by $X_n \xrightarrow{d} X$, if for every bounded continuous function $f : \mathcal{G}_\bullet \to \mathbb{R}$ we have that $\mathbb{E}(f(X_n)) \to \mathbb{E}(f(X))$. We will be focused here on the particular situation in which X_n is a *finite* rooted random graph (G_n, ρ_n) such that given G_n, the root ρ_n is uniformly distributed among the vertices of G_n. It is a very common setting and justifies the following definition.

Definition 5.1 Let $\{G_n\}$ be a sequence of (possibly random) finite graphs. We say that G_n **converges locally** to a (possibly infinite) random rooted graph $(U, \rho) \in \mathcal{G}_\bullet$, and denote it by $G_n \xrightarrow{\text{loc}} (U, \rho)$, if for every integer $r \geq 1$,

$$B_{G_n}(\rho_n, r) \xrightarrow{d} B_U(\rho, r),$$

where ρ_n is a uniformly chosen vertex from G_n.

It is straightforward to see that this definition is equivalent to saying that the random variables (G_n, ρ_n) converge in distribution to (U, ρ). Note that this definition is consistent whether G_n is a deterministic finite graph or is a random variable drawn from some probability measure. In both cases $B_{G_n}(\rho_n, r)$ is a random variable taking values in $\mathcal{G}_\bullet$ and we clarify that ρ_n is drawn uniformly *conditioned* on G_n.

Examples

- The sequence $\{G_n\}$ of paths of length n converges locally to the graph $(\mathbb{Z}, 0)$ (note that the root vertex can be chosen to be any vertex of $\mathbb{Z}$ since $(\mathbb{Z}, i)$ and $(\mathbb{Z}, j)$ are equivalent for all $i, j \in \mathbb{Z}$).
- The sequence $\{G_n\}$ of the $n \times n$ square grid converges locally to the graph $(\mathbb{Z}^2, \mathbf{0})$ (again the root can be chosen to be any vertex of $\mathbb{Z}^2$).
- Let $\lambda > 0$ be fixed and let $\{G(n, \frac{\lambda}{n})\}$ be the sequence of random graphs obtained from the complete graph K_n by retaining each edge with probability $\frac{\lambda}{n}$ and erasing it otherwise, independently for all edges. This is known as the Erdös-Rényi random graph. One can verify that this sequences converges locally to a branching process with progeny distribution Poisson(λ). See exercise 1 of this chapter.

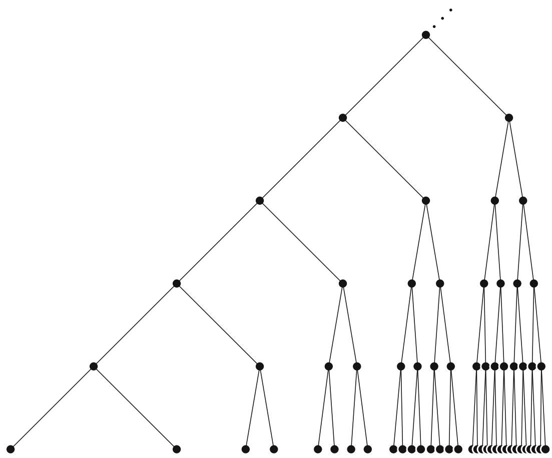


Fig. 5.1 A part of the **canopy tree**

- Let G_n be the binary tree of height n. Perhaps surprisingly, its local limit is *not* the infinite binary tree. Instead, it is the following so-called **canopy tree** depicted in Fig. 5.1 and the root is at distance $k \geq 0$ from the leaves with probability 2^{-k-1}. Note that the distance of the root from the leaves determines the isomorphism class of the rooted graph. It is easy to see that the canopy tree is not isomorphic to the infinite binary tree, for example, it has leaves; furthermore, unlike the infinite binary tree it is recurrent.
- Consider G_n to be a path of length n, glued via one of its leaves into a $\sqrt{n} \times \sqrt{n}$ grid. The local limit of G_n is (U, ρ), where (U, ρ) is $(\mathbb{Z}, 0)$ with probability $1/2$, and $(\mathbb{Z}^2, \mathbf{0})$ otherwise.

Our goal in this chapter is to prove the following pioneering result.

Theorem 5.2 (Benjamini–Schramm [11]) *Let $M < \infty$ and let G_n be finite planar maps (possibly random) with degrees almost surely bounded by M such that $G_n \stackrel{\text{loc}}{\longrightarrow} (U, \rho)$. Then (U, ρ) is almost surely recurrent.*

For instance, a local limit of planar maps cannot be the 3-regular infinite tree (however, the 3-regular infinite tree can be obtained as a local limit of uniformly random 3-regular graphs). The bounded degree assumption in Theorem 5.2 is necessary. Indeed, suppose we start with a binary tree of height n and replace each edge (u, v) that is at distance $k \geq 0$ from the leaves by 2^k parallel edges. By the same reasoning of the local convergence of binary trees to the canopy tree, the modified graph sequence converges locally to a modified canopy tree in which an edge at distance k from the leaves is replaced with 2^k parallel edges. Using the parallel law it is immediate to see that this graph is transient, and that the effective resistance from a leaf to ∞ is at most 2 (in fact it equals 2). See Fig. 5.2.

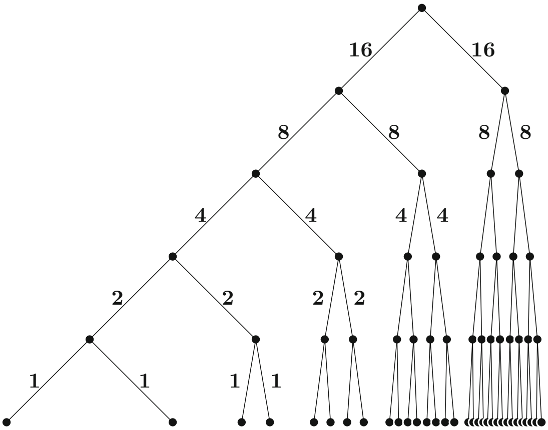


Fig. 5.2 A part of a **transient canopy tree**. Numbers on edges are conductances of those edges after applying the parallel law

5.2 The Magic Lemma

Suppose $C \subseteq \mathbb{R}^2$ is finite. For each $w \in C$, define

$$\rho_w = \min\{|v - w| : v \in C \setminus \{w\}\}.$$

We call ρ_w the **isolation radius** of w. Given $\delta \in (0, 1)$, $s \geq 2$ and $w \in C$, we say that w is (δ, s)**-supported** if in the disk of radius $\delta^{-1}\rho_w$ around w there are at least s points of C outside any given disk of radius $\delta\rho_w$. In other words, w is (δ, s)**-supported** if

$$\inf_{p \in \mathbb{R}^2} \left| C \cap B\left(w, \delta^{-1}\rho_w\right) \setminus B(p, \delta\rho_w) \right| \geq s.$$

The proof of Theorem 5.2 is based on the following lemma, which has been dubbed "the Magic Lemma".

Lemma 5.3 ([11]) *There exists $A > 0$ such that for every $\delta \in (0, 1/2)$, every finite $C \subseteq \mathbb{R}^2$ and every $s \geq 2$, the number of (δ, s)-supported points in C is at most*

$$\frac{A|C|\delta^{-2}\ln(\delta^{-1})}{s}.$$

Remark 5.4 We prove the lemma for $\mathbb{R}^2$, but it holds for $\mathbb{R}^d$ or any other doubling metric space. In fact, a metric space for which the lemma holds must be doubling; see [29].

Proof of Lemma 5.3

Let $k \geq 3$ be an integer (later we will take $k = k(\delta)$). Let G_0 be a tiling of $\mathbb{R}^2$ by 1×1 squares, rooted at some point p, and for every $n \in \mathbb{Z}$, let G_n be a tiling of $\mathbb{R}^2$ by $k^n \times k^n$ such that each square of G_n is tiled by $k \times k$ squares of G_{n-1}. We may choose p so that none of the points of C lies on the edge of a square.

We say that a square $S \in G_n$ is s**-supported** if for every smaller square $S' \in G_{n-1}$ we have that $|C \cap (S \setminus S')| \geq s$.

Claim 5.5 For any $s \geq 2$ the total number of s-supported squares, in $G = \bigcup_{n \in \mathbb{Z}} G_n$, is at most $2|C|/s$.

Proof Define a "flow" $f : G \times G \to \mathbb{R}$ as follows:

$$f(S', S) = \begin{cases} \min(s/2, |S' \cap C|) & S' \subseteq S,\ S' \in G_n,\ S \in G_{n+1}, \\ -f(S, S') & S \subseteq S',\ S \in G_n,\ S' \in G_{n+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Let us make two initial observations. First we have that

$$\sum_{S' \in G} f(S', S) \geq 0\,, \tag{5.1}$$

by splitting into the two cases depending on whether there exists a square $S' \subseteq S$ such that $f(S', S) = s/2$ or not. Secondly, if S is a s-supported square

$$\sum_{S' \in G} f(S', S) \geq \frac{s}{2}\,, \tag{5.2}$$

by splitting into cases depending on whether the number of squares $S' \subseteq S$ such that $f(S', S) = s/2$ is at most one or at least two.

Let $a \in \mathbb{Z}$ be such that each square in G_a contains at most 1 point of C so there are no s-supported squares in $\bigcup_{n \leq a} G_n$. It easily follows from the definition of f that

$$\sum_{S' \in G_a} \sum_{S \in G_{a+1}} f(S', S) = |C|, \tag{5.3}$$

and that for every $b \in \mathbb{Z}$

$$\sum_{S' \in G_b} \sum_{S \in G_{b+1}} f(S', S) \geq 0. \tag{5.4}$$

Now, using (5.3) and (5.4),

$$\sum_{n=a+1}^{b}\sum_{S\in G_n}\sum_{S'\in G} f(S',S) = \sum_{n=a+1}^{b}\sum_{S\in G_n}\left(\sum_{S'\in G_{n-1}} f(S',S) + \sum_{S'\in G_{n+1}} f(S',S)\right)$$
$$= \sum_{S\in G_{a+1}}\sum_{S'\in G_a} f(S',S) + \sum_{S\in G_b}\sum_{S'\in G_{b+1}} f(S',S) \le |C|.$$

Therefore, using (5.1) and (5.2), we deduce that there are at most $2|C|/s$ squares in $\bigcup_{b\ge n>a} G_n$ that are s-supported. Sending $b\to\infty$ finishes the proof. □

The above claim is very close to the statement of Lemma 5.3 which we are pursuing. However, we need to move from squares to circles. We choose $k = \lceil 20\delta^{-2}\rceil$ and let $\beta \sim \mathsf{Unif}([0,\ln k])$. Let G_0 be a tiling with side length e^{β}, based at the origin. Suppose we have defined G_n as a tiling of squares of side length $e^{\beta}k^n$; then G_{n+1} is a tiling of squares of side length $e^{\beta}k^{n+1}$ that is based uniformly at one of the k^2 possible points of G_n. Because the desired statement is invariant under translation and dilation of C, we may assume that C does not intersect the edges of G_n (for every n) and that $\rho_w \ge k$ for every $w\in C$. We call a point $w\in C$ a **city** in a square $S\in G$ if:

- the side length of S is in the interval $[4\delta^{-1}\rho_w, 5\delta^{-1}\rho_w]$, and
- the distance from w to the center of S is at most $\delta^{-1}\rho_w$.

Claim 5.6 The probability that any given $w\in C$ is a city is $\Omega(\ln^{-1}(\delta^{-1}))$.

Proof For the first item to hold, β needs to satisfy that there exists $n\in\mathbb{Z}$ such that $e^{\beta}k^n \in [4\delta^{-1}\rho_w, 5\delta^{-1}\rho_w]$, or $\beta + n\ln k \in \ln(\delta^{-1}\rho_w) + [\ln 4, \ln 5]$. Since $\beta\in\mathsf{Unif}([0,\ln k])$, the probability for that is $(\ln(5/4))/\ln k$, which is $\Omega(\ln^{-1}(\delta^{-1}))$ when $\delta\in(0,1/2)$.

As for the second item, it holds with positive probability (independent of δ) over the k^2 choices for basing G_n on top of G_{n-1}, given that β satisfies the requirement posed by the first item. □

Claim 5.7 If w is a city in S and is (δ,s)-supported, then S is s-supported.

Proof If $S\in G_n$ is as above, then any little square $S'\in G_{n-1}$ has side length at most

$$\frac{\delta^2}{20}\cdot\frac{5\rho_w}{\delta} = \frac{\delta\rho_w}{4}.$$

Hence, it is contained in a disk of radius $\delta\rho_w$. Thus, for every $S'\in G_{n-1}$ with $S'\subseteq S$ there exists a point p such that

$$|C\cap(S\setminus S')| \ge \left|C\cap\left(B\left(w,\delta^{-1}\rho_w\right)\setminus B(p,\delta\rho_w)\right)\right| \ge s\,,$$

where we have used the fact that $B\left(w, \delta^{-1}\rho_w\right) \subset S$. □

Now note that the expected number of pairs (w, S) such that S is s-supported, w is (δ, s)-supported, and w is a city, is at least $c \ln^{-1}(\delta^{-1})N$, where N is the number of (δ, s)-supported points. Also, no more than $c\delta^{-2}$ points of C can be cities in a single square S. It follows from Claim 5.5 that

$$N \leq \frac{A|C|\delta^{-2}\ln(\delta^{-1})}{s},$$

concluding the proof of Lemma 5.3. □

5.3 Recurrence of Bounded Degree Planar Graph Limits

Theorem 5.2 follows immediately from the following theorem which gives a quantitative estimate on the growth of the resistance in local limits of bounded degree planar maps. In particular, it states that the resistance grows logarithmically in the Euclidean distance of the corresponding circle packing.

Theorem 5.8 *Let (U, ρ) be a local limit of (possibly random) finite planar maps with maximum degree at most D. Then, almost surely, there exist a constant $c > 0$ and a sequence $\{B_k\}_{k\geq 1}$ of subsets of U such that for each k we have*

1. $|B_k| \leq c^{-1}k$, *and*
2. $\mathcal{R}_{\text{eff}}(\rho \leftrightarrow U \setminus B_k) \geq c \log k$.

In particular, (U, ρ) is almost surely recurrent.

We write $B_{\text{euc}}(p, r)$ for the Euclidean ball of radius r around a point $p \in \mathbb{R}^2$. As before, for a subset $O \subset \mathbb{R}^2$ and a given circle packing we write V_O for the set of vertices in which the centers of the corresponding circles are in O. In order to prove Theorem 5.8, we will need the following immediate corollary of the Magic Lemma (Lemma 5.3):

Corollary 5.9 *Let G be a finite simple planar triangulation, and P a circle packing of G. Let ρ be a uniform random vertex and P' a dilation and translation of P such that the circle of ρ is a unit circle centered at the origin $\mathbf{0}$. Then, there exists a universal constant $A > 0$ such that in the packing P', for every real $r \geq 2$ and integer $s \geq 2$*

$$\mathbb{P}\left(\forall p \in \mathbb{R}^2 \quad \left|V_{B_{\text{euc}}(\mathbf{0},r)\setminus B_{\text{euc}}(p,\frac{1}{r})}\right| \geq s\right) \leq \frac{Ar^2\log r}{s}.$$

Proof Apply the Magic Lemma with $\delta = \frac{1}{r}$ and $s = s$, with the centers of circles of P' as the point set C. Note that there exists a constant $C > 0$ such that for all

$w \in V$ the isolation radius of w, ρ_w, satisfies $\mathrm{rad}(C_w) \le \rho_w \le C\,\mathrm{rad}(C_w)$ (without appealing to the Ring Lemma). $\square$

The following lemma provides the main estimate needed to prove Theorem 5.8. Once it has been shown, Theorem 5.8 will follow by a Borel-Cantelli argument.

Lemma 5.10 *Let G be a finite simple planar map with maximum degree at most D and let ρ be a uniform random vertex of G. Then, there exists a constant $C = C(D) < \infty$ such that for all $k \ge 1$,*

$$\mathbb{P}\left(\exists B \subseteq V,\ |B| \le Ck,\ \mathcal{R}_{\mathrm{eff}}(\rho \leftrightarrow V \setminus B) \ge C^{-1}\log k\right) \ge 1 - Ck^{-\frac{1}{3}}\log k\,,$$

where we interpret $\mathcal{R}_{\mathrm{eff}}(\rho \leftrightarrow V \setminus B) = \infty$ when $B = V$.

Proof We first assume that G is a triangulation and consider a circle packing of it where the circle of ρ is a unit circle centered at the origin $\mathbf{0}$. Applying Corollary 5.9 with $r = k^{\frac{1}{3}}$, $s = k$, we have that with probability at least $1 - Ak^{-\frac{1}{3}}\log(k)/3$, there exists $p \in \mathbb{R}^2$ with

$$\left|V_{B_{\mathrm{euc}}(\mathbf{0},r)\setminus B_{\mathrm{euc}}(p,\frac{1}{r})}\right| < k.$$

Now, if $|V_{B_{\mathrm{euc}}(p,\frac{1}{r})}| \le 1$, we set $B = V_{B_{\mathrm{euc}}(\mathbf{0},r)}$. We then have $|B| \le k$ and by applying $\Omega(\log k)$ times Lemma 4.9 together with the series law (Claim 2.24) we get that $\mathcal{R}_{\mathrm{eff}}(\rho \leftrightarrow V \setminus B) \ge c\log k$ for some $c = c(D) > 0$. Else, if $|V_{B_{\mathrm{euc}}(p,\frac{1}{r})}| \ge 2$ then we take $B = V_{B_{\mathrm{euc}}(\mathbf{0},r)} \setminus V_{B_{\mathrm{euc}}(p,\frac{1}{r})}$. By the Ring Lemma, there exists a $c' = c'(D) > 0$ such that $|p| \ge 1 + c'$. Since $|V_{B_{\mathrm{euc}}(p,\frac{1}{r})}| \ge 2$, we have a vertex in that set with radius at most r^{-1}. Therefore, $B_{\mathrm{euc}}(p, \frac{2}{r})$ contains at least one full circle C_v. Hence, by scaling and translating such that $C_v = \mathbb{U}$, we get (again, by Lemma 4.9) that

$$\mathcal{R}_{\mathrm{eff}}\left(V_{B_{\mathrm{euc}}(p,\frac{2}{r})} \leftrightarrow V \setminus V_{B_{\mathrm{euc}}(p,c'/2)}\right) \ge c_2\log k\,,$$

for some other constant $c_2 = c_2(D) > 0$. Since $\rho \in V \setminus V_{B_{\mathrm{euc}}(p,c'/2)}$ we obtain

$$\mathcal{R}_{\mathrm{eff}}\left(\rho \leftrightarrow V_{B_{\mathrm{euc}}(p,\frac{2}{r})}\right) \ge c_2\log k\,.$$

By Lemma 4.9 we also have that

$$\mathcal{R}_{\mathrm{eff}}\left(\rho \leftrightarrow V \setminus V_{B_{\mathrm{euc}}(\mathbf{0},r)}\right) \ge c_3\log k\,,$$

for some $c_3 = c_3(D) > 0$. By Claim 2.22 this means that

$$\mathbb{P}_\rho\left(\tau_{V \setminus V_{B_{\mathrm{euc}}(\mathbf{0},r)}} < \tau_\rho^+\right) \le \frac{1}{c_2 \log(k)} \quad \text{and} \quad \mathbb{P}_\rho\left(\tau_{V_{B_{\mathrm{euc}}(p,\frac{2}{r})}} < \tau_\rho^+\right) \le \frac{1}{c_3 \log(k)}\,.$$

By the union bound

$$\mathbb{P}_\rho\left(\tau_{V \setminus B} < \tau_\rho^+\right) \le \frac{2}{\min(c_2, c_3)\log(k)}\,,$$

hence by Claim 2.22 again

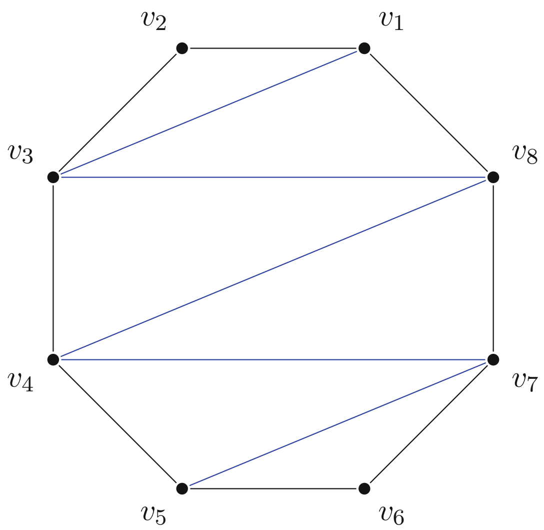
$$\mathcal{R}_{\mathrm{eff}}\left(\rho \leftrightarrow V \setminus B\right) \ge \min(c_2, c_3) D^{-1} \log(k)/2\,,$$

concluding the proof when G is a triangulation.

If G is not a triangulation, we would like to add edges to make it a triangulation while making sure that the maximal degree does not increase too much. We also have to ensure that the graph remains simple which may require us to add some additional vertices as well. Let f be a face of G with vertices $v_1, \ldots, v_k$. Suppose first that there are no edges between non-consecutive vertices of the face. In this case, we draw the edges in a zig-zag fashion, as in Fig. 5.3.

In the case where there are edges between non-consecutive vertices of the face exist, we draw a cycle $u_1, \ldots, u_k$ inside f. Then, we connect u_i to v_i and v_{i+1} for each $i < k$ and u_k to v_k and v_1. Finally, we triangulate the inner face created by the new cycle by zig-zagging as in the previous case (see Fig. 5.4).

Fig. 5.3 Adding diagonals to a face in a zigzag fashion



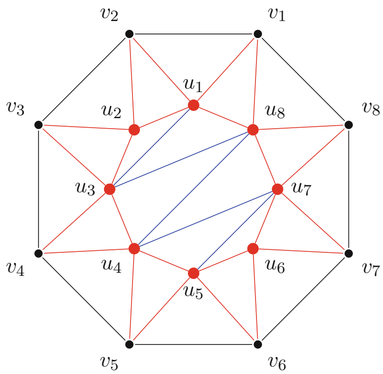


Fig. 5.4 Drawing an inner cycle and triangulating the new inner face

Since each vertex of the original graph is a member of at most D faces and for each face at most 2 edges are added, the maximal degree of the resulting graph is at most $3D$. Similarly, the number of vertices in the resulting graph is at most D times the number of vertices in the original graph hence the probability of a random vertex being a vertex of the original graph is at least D^{-1}. If this occurs then it is straightforward to see that the existence of a subset of vertices B in the new graph which satisfies the required conditions implies the existence of such a set in the old graph, concluding our reduction to the triangulation case and finishing our proof. $\square$

We are ready to deduce Theorem 5.8.

Proof of Theorem 5.8 Assume that G_n are finite planar maps with maximum degree at most D such that $G_n \xrightarrow{\text{loc}} (U, \rho)$. If $\{G_n\}$ are not simple graphs we erase self-loops and merge parallel edges into a single edge to obtain the sequence $\{G'_n\}$. It is immediate that $G'_n \xrightarrow{\text{loc}} (U', \rho')$ where (U', ρ') is distributed as (U, ρ) after removing from U all loops and merging parallel edges into a single edge. Since the maximum degree is bounded, U' is recurrent if and only if U is recurrent. Thus we may assume that G_n are simple graphs so the previous estimates may be used.

Denote by $\mathcal{A}_k$ the event

$$\mathcal{A}_k = \{\exists B \subseteq U,\ |B| \leq Ck,\ \mathcal{R}_{\text{eff}}(\rho \leftrightarrow V \setminus B) \geq c \log k\}\ ,$$

where $C = C(D) < \infty$ is the constant from Lemma 5.10. Therefore $\mathbb{P}(\mathcal{A}_k^c) \leq c^{-1}k^{-\frac{1}{3}}\log(k)$. Looking at the sequence $\{\mathcal{A}_{2^j}\}_{j\geq 1}$, by Borel-Cantelli, almost surely there exists j_0 such that for all $j \geq j_0$ the event $\mathcal{A}_{2^j}$ holds. Thus we have proved the required assertion for k which is a power of 2. To prove this for all k sufficiently large, let B_{2^j} be the set guaranteed to exist in the definition of $\mathcal{A}_{2^j}$, and take $B_k = B_{2^j}$ for the unique j for which $2^j \leq k < 2^{j+1}$. It is immediate that these sets satisfy the assertion of the theorem, concluding our proof. $\square$

5.4 Exercises

1. Let $G(n, p)$ be the random graph on n vertices drawn such that each of the $\binom{n}{2}$ possible edges appears with probability p independently of all other edges. Let $\lambda > 0$ be a constant, show that $G(n, \lambda/n)$ converges locally to a branching process with progeny distribution Poisson(λ).
2. For a graph G, let G^2 be the graph on the same vertex set as G so that vertices u, v form an edge if and only if the graph distance in G between u and v is at most 2. Show that if G has uniformly bounded degrees, then G is recurrent if and only if G^2 is recurrent.
3. Construct an example of a local limit (U, ρ) of finite planar graphs such that U is almost surely recurrent, but U^2 is almost surely transient.
4. Fix an integer $k \geq 1$. Construct an example of a sequence of finite simple planar maps G_n such that G_n converge locally to (U, ρ) with the property that $\mathbb{E}[\deg^k(\rho)] < \infty$ and U is almost surely transient.
5. (*) Suppose that G_n is a sequence of finite trees converging locally to (U, ρ). Show that U is almost surely recurrent. (Note that the degrees may be *unbounded*).