Chapter 4
Parabolic and Hyperbolic Packings

4.1 Infinite Planar Maps

In this chapter we discuss countably infinite connected simple graphs that are **locally finite**, that is, the vertex degrees are finite. In a similar fashion to the previous chapter, an infinite planar graph is a connected infinite graph such that there exists a drawing of it in the plane. We recall that a *drawing* is a correspondence sending vertices to points of $\mathbb{R}^2$ and edges to continuous curves between the corresponding vertices such that no two edges cross. An **infinite planar map** is an infinite planar graph equipped with a set of cyclic permutations $\{\sigma_v : v \in V\}$ of the neighbors of each vertex v, such that there exists a drawing of the graph which respects these permutations, that is, the clockwise order of edges emanating from a vertex v coincides with σ_v.

Unlike the finite case, one cannot define faces as the connected components of the plane with the edges removed since the drawing may have a complicated set of accumulation points. This is the reason that we have defined faces in Sect. 3.1 combinatorially, that is, based solely on the edge set and the cyclic permutation structure. This definition makes sense in both the finite and infinite case. In the latter case we may have infinite faces.

A (finite or infinite) planar map is a **triangulation** if each of its faces has exactly 3 edges. Given a drawing of a triangulation, the Jordan curve theorem implies that the edges of each face bound a connected component of the plane minus the edges. We will often refer to the faces as these connected components. A triangulation is called a **plane triangulation** if there exists a drawing of it such that every point of the plane is contained in either a face or an edge and any compact subset of the plane intersects at most finitely many edges and vertices. The term **disk triangulation** is also used in the literature and means the same with the unit disk taking the place of the plane in the previous definition. Of course these two definitions are equivalent since the plane and the open disk are homeomorphic. For example, take the product of the complete graph K_3 on 3 vertices with an infinite ray $\mathbb{N}$ and add a diagonal edge

A. Nachmias, *Planar Maps, Random Walks and Circle Packing*, Lecture Notes in Mathematics 2243, https://doi.org/10.1007/978-3-030-27968-4_4

in each face that has 4 edges; this is a plane triangulation. However, the product of K_3 with a bi-infinite ray $\mathbb{Z}$ together with the same diagonals is a triangulation but not a plane triangulation, since it cannot be drawn in the plane without an accumulation point.

It turns out that there is a combinatorial criterion for a triangulation to be a plane/disk triangulation. We say that an infinite graph is **one-ended** if the removal of any finite set of its vertices leaves exactly one infinite connected component.

Lemma 4.1 *An infinite triangulation is a plane triangulation if and only if it is one-ended.*

Proof Suppose $G = (V, E)$ is a plane triangulation and consider a drawing of the graph with no accumulation points in the plane such that every point of the plane belongs to either an edge or a face. Let $A \subseteq V$ be a finite set of vertices and take $B \subset \mathbb{R}^2$ to be a ball around the origin which contains every vertex of A, every edge touching a vertex of A and every face incident to such an edge. Let $u \neq v$ be two vertices drawn outside of B and take a continuous curve γ between them in $\mathbb{R}^2 \setminus B$. By definition of B, this path only touches faces and edges that are not incident to the vertices of A and hence one can trace a discrete path from u to v in the graph that "follows" γ and avoids A. Since B intersects only finitely many edges and vertices, we learn that $G \setminus A$ has a unique infinite component.

Conversely, assume now that G is one-ended and consider a drawing of G in the plane. By the stereographic projection we project the drawing to the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$. Denote by $\mathcal{I}$ the complement in $\mathbb{S}^2$ of the union of all faces and edges. Since G is an infinite triangulation this union is an open set, hence $\mathcal{I}$ is a closed set and its boundary $\partial\mathcal{I}$ is precisely the set of accumulation points of the drawing. Since $\mathcal{I}$ is closed, each connected component of $\mathcal{I}$ must be closed as well and hence contain at least one accumulation point. Since G is one-ended $\mathcal{I}$ cannot have more than one connected component, since otherwise we would be able to separate the two components by a finite set of edges and obtain two infinite connected components. Now choose a point $p \in \mathcal{I}$ and rotate the sphere so that p is the north pole. Project back the rotated sphere to the plane and consider the drawing in the plane. In this drawing the union of all faces and edges must be a simply connected set. By the Riemann mapping theorem this set is homeomorphic to the whole plane, and we deduce that the triangulation is a plane triangulation. □

4.2 The Ring Lemma and Infinite Circle Packings

The circle packing theorem Theorem 3.5 is stated for finite planar maps. However, it is not hard to argue that any infinite map also has a circle packing. To this aim we will prove what is known as Rodin and Sullivan's *Ring Lemma* [70]; we will use it many times throughout this book. Given circles $C_0, C_1, \ldots, C_M$ with disjoint interiors, we say that $C_1, \ldots, C_M$ completely surround C_0 if they are all tangent to C_0 and C_i is tangent to C_{i+1} for $i = 1, \ldots, M$ (where C_{M+1} is set to be C_1).

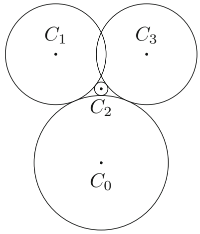

Fig. 4.1 C_2 is small, but both C_1 and C_3 are large

Lemma 4.2 (Ring Lemma, Rodin and Sullivan [70]) *For every integer $M > 0$ there exists $A > 0$ such that if C_0 is a circle completely surrounded by M circles $C_1, \dots, C_M$, and r_i is the radius of C_i for every $i = 0, 1, \dots, M$, then $r_0/r_i \le A$ for every $i = 1, \dots, M$.*

Proof We may scale the picture so that $r_0 = 1$. Assume that the radius of C_2 is small and consider the circles C_1 and C_3 to its left and right. It cannot be that both C_1 and C_3 have large radii compared to C_2 since in this case they will intersect; see Fig. 4.1. Hence, one of them has to be small as well. Assume without loss of generality that it is C_3. By similar reasoning, one of C_1 and C_4 has to be small. We continue this argument this way and get a path of circles of small radii; thus, for the circles $C_1, \dots, C_M$ to completely surround C_0 we learn that M must be large. □

For a circle packing P and a vertex v, denote by C_v the circle corresponding to v, by $\mathrm{cent}(v)$ the center of that circle, and by $\mathrm{rad}(v)$ its radius. We write $G(P)$ for the tangency graph of the packing P, that is, the graph in which each vertex is a circle of P and two such circles form an edge when they are tangent.

Claim 4.3 Let G be an infinite simple planar map. Then there exists a circle packing P such that $G(P)$ is isomorphic to G as planar maps.

Proof If G is not a triangulation, then it is always possible to add in each face new vertices and edges touching them so the resulting graph is a planar triangulation (in an infinite face we have to put infinitely many vertices). After circle packing this new graph, we can remove all the circles corresponding to the added vertices and remain with a circle packing of G. Thus, we may assume without loss of generality that G is a triangulation.

Fix a vertex x, and let G_n be the graph distance ball of radius n around x. Apply the circle packing theorem to G_n to obtain a packing P_n, and scale and translate it so that $\mathrm{rad}(x) = 1$ and $\mathrm{cent}(x)$ is the origin.

Consider a neighbor y of x. By the Ring Lemma (Lemma 4.2), there exists a constant $A = A(x, y) > 0$ such that $A^{-1} \le \mathrm{rad}(y) \le A$. By compactness there exists a subsequence of packings P_{n_k} for which $\mathrm{rad}_{n_k}(y)$ and $\mathrm{cent}_{n_k}(y)$ both converge. By taking further subsequences for the rest of x's neighbors, and then for the rest of the graph's vertices, it follows by a diagonalization argument that there exists a subsequence such that the radii and centers of all vertices converge. The limiting packing P_∞ satisfies that $G(P_\infty)$ is isomorphic to G. □

4.3 Statement of the He–Schramm Theorem

Given a circle packing P of a triangulation G, we define the **carrier** of P, denoted $\mathrm{Carrier}(P)$, to be the union of the closed discs bounded by the circles of P together with the spaces bounded between any three circles that form a face (i.e., the interstices). When P is a circle packing of an infinite one-ended triangulation, the argument in Lemma 4.1 shows that $\mathrm{Carrier}(P)$ is simply connected.

We say that G **is circle packed in** $\mathbb{R}^2$ when $\mathrm{Carrier}(P) = \mathbb{R}^2$. Denote by $\mathbb{U}$ the disk $\{z \in \mathbb{R}^2 : |z| < 1\}$; we say that G **is circle packed in** $\mathbb{U}$ when $\mathrm{Carrier}(P) = \mathbb{U}$. See Fig. 4.2.

Let G be a plane triangulation. Then G can be drawn in the plane $\mathbb{R}^2$ or alternatively in the disk $\mathbb{U}$ (since they are homeomorphic), but can it be *circle packed* both in $\mathbb{R}^2$ and in $\mathbb{U}$? A celebrated theorem of He and Schramm [40] states that this cannot be done: each plane triangulation can be circle packed in either the plane or the disk, but not both. In fact, the combinatorial property of G that determines on which side of the dichotomy we are is the recurrence or transience of the simple random walk on G (assuming also that G has bounded degrees, that is, $\sup_{x \in V(G)} \deg(x) < \infty$). This is the content of the He–Schramm theorem, which we are now ready to state.

Theorem 4.4 (He and Schramm [40]) *Let G be an infinite simple plane triangulation with bounded degrees.*

1. *If G is recurrent, then there exists a circle packing P of G such that* $\mathrm{Carrier}(P) = \mathbb{R}^2$.
2. *If G is transient, then there exists a circle packing P of G such that* $\mathrm{Carrier}(P) = \mathbb{U}$.

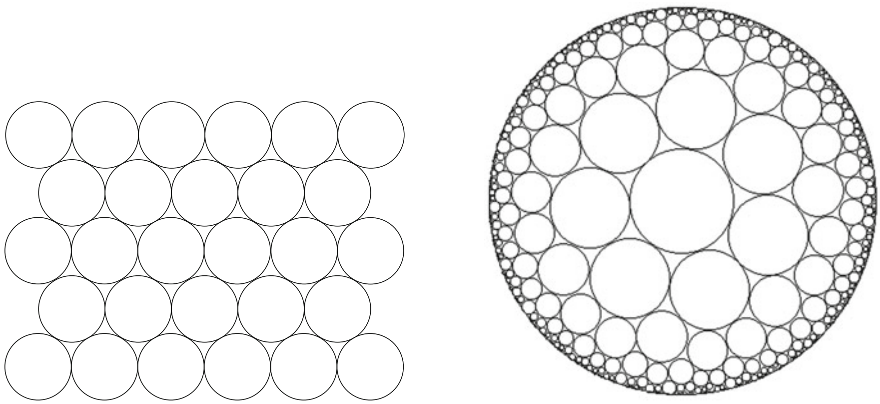

Fig. 4.2 Two circle packings with carriers $\mathbb{R}^2$ (left) and $\mathbb{U}$ (right)

3. *If P is a circle packing of G with* $\mathrm{Carrier}(P) = \mathbb{R}^2$*, then G is recurrent.*
4. *If P is a circle packing of G with* $\mathrm{Carrier}(P) = \mathbb{U}$*, then G is transient.*

Remark 4.5 Schramm [73] proved that a circle packing P of a triangulation $G(P)$ with $\mathrm{Carrier}(P) = \mathbb{R}^2$ is uniquely determined up to dilations, rotations and translations. If $\mathrm{Carrier}(P) = \mathbb{U}$ the same holds up to Möbius transformations of $\mathbb{U}$ onto itself (see also [37]). Hence the packings guaranteed to exist in Theorem 4.4 (1) and Theorem 4.4 (2) are unique in this sense.

Corollary 4.6 *Any bounded degree plane triangulation can be circle-packed in* $\mathbb{R}^2$ *or* $\mathbb{U}$*, but not both.*

Remark 4.7 In fact, it is proved in [40] that the corollary above holds without the assumption of bounded degree. Furthermore, in [40] Theorem 4.4 (1) and Theorem 4.4 (4) are proved without the bounded degrees assumption, but the other two statements require this assumption.

The following example demonstrates why the bounded degree condition is necessary for Theorem 4.4 (2) and Theorem 4.4 (3).

Example 4.8 Let P be a triangular lattice circle packing (as in Fig. 4.3), and let $C_0, C_1, C_2, \ldots$ be an infinite horizontal path of circles in P going (say) to the right. In the upper face shared by C_n and C_{n+1}, draw 2^n circles which form a vertical path and each of them tangent both to C_n and C_{n+1}; the last circle of these is also tangent to the upper neighbor of C_n and C_{n+1}. See Fig. 4.3.

The resulting graph is a plane triangulation and the carrier of the packing is $\mathbb{R}^2$. However, it is an easy exercise to verify that the tangency graph of this circle packing is transient.

In the rest of this chapter we prove Theorem 4.4. We begin by proving parts 3 and 4, in which a circle packing is given and we use its geometry to estimate

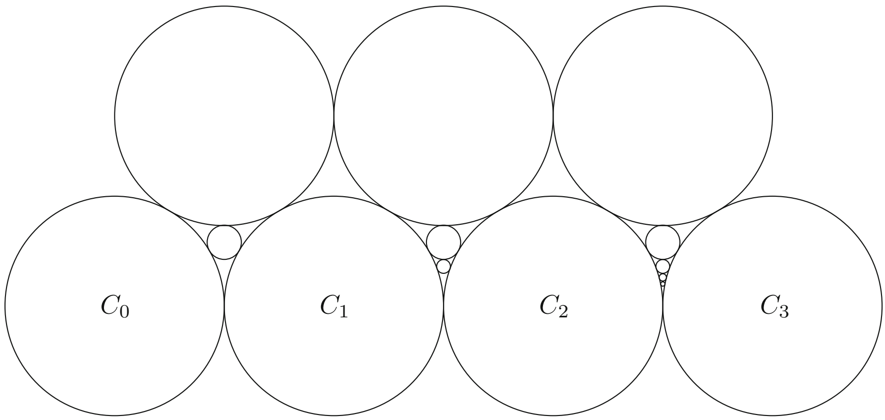

Fig. 4.3 Unbounded degree transient triangulation circle packed in $\mathbb{R}^2$

certain effective resistances. Afterwards we prove parts 1 and 2, in which we use the electrical estimates to deduce facts about the geometry of the circle packing.

4.4 Proof of the He–Schramm Theorem

Proof of Theorem 4.4 (3)

Denote the circle packing $P = \{C_v\}_{v \in V}$ where V is the vertex set of G and C_v denotes the circle corresponding to the vertex v. Write Δ for the maximum degree of G and fix a vertex v_0. By scaling and translating we may assume that C_{v_0} is a radius 1 circle around the origin. For a real number $R > 0$, let $V_R = V_{B(\mathbf{0},R)}$ denote set of vertices v for which $\mathrm{cent}(v)$ is in the Euclidean ball of radius R around the origin.

Lemma 4.9 *There exist $C = C(\Delta) > 1$ and $c = c(\Delta) > 0$ such that for every $R \geq 1$ we have*

(i) There are no edges between V_R and $V \setminus V_{CR}$, and
(ii) $\mathcal{R}_{\mathrm{eff}}(V_R \leftrightarrow V \setminus V_{CR}) \geq c$.

Proof We begin with part (i). For every $v \in V_R$ it holds that $\mathrm{rad}(v) \leq R$ since C_{v_0} is centered at the origin. By the Ring Lemma (Lemma 4.2), there exists $A = A(\Delta)$ such that $\mathrm{rad}(u) \leq AR$ for every $u \sim v$, and therefore $|\mathrm{cent}(u)| \leq (A+2)R$. Hence (i) holds with $C = A + 2$.

To prove part (ii) we define

$$h(v) = \begin{cases} 0 & v \in V_R, \\ 1 & v \in V \setminus V_{CR}, \\ \frac{|\mathrm{cent}(v)| - R}{(C-1)R} & \text{otherwise.} \end{cases}$$

Recall from Lemma 2.32 that $\mathcal{R}_{\mathrm{eff}}(V_R \leftrightarrow V \setminus V_{CR}) \geq \mathcal{E}(h)^{-1}$. By the triangle inequality, for an edge $\{x, y\}$ with both endpoints in $V_{CR} \setminus V_R$ we have

$$|h(x) - h(y)| \leq \frac{|\mathrm{cent}(x) - \mathrm{cent}(y)|}{(C-1)R} = \frac{\mathrm{rad}(x) + \mathrm{rad}(y)}{(C-1)R},$$

and it is straightforward to check that the same bound holds also when one of the edge's endpoints is in V_R or $V \setminus V_{CR}$. Thus, using the Ring Lemma's (Lemma 4.2) constant $A = A(\Delta)$ from part (i),

$$\mathcal{E}(h) \leq \sum_{x \in V_{CR} \setminus V_R} \sum_{y : y \sim x} \frac{((A+1)\,\mathrm{rad}(x))^2}{(C-1)^2 R^2} \leq \frac{\Delta(A+1)^2}{\pi(C-1)^2 R^2} \cdot \sum_{x \in V_{CR} \setminus V_R} \mathrm{area}(C_x),$$

where $\text{area}(C_x)$ is the area that C_x encloses (that is, $\pi\, \text{rad}(x)^2$). We have that $\sum_x \text{area}(C_x) \le \text{area}(B(\mathbf{0}, 2CR)) = 4\pi C^2 R^2$, hence if $C = A + 2$, then

$$\mathcal{E}(h) \le 4\Delta C^2,$$

and the result follows for $c = (4\Delta C^2)^{-1}$. □

Proof of Theorem 4.4 (3) Consider the unit current flow I from v_0 to ∞ and fix any $R \ge 1$. Restricting this flow to the edges which have at least one endpoint in the annulus $V_{CR} \setminus V_R$ gives a unit flow from V_R to $V \setminus V_{CR}$, by part (i) of Lemma 4.9. Hence, by part (ii) of that lemma and by Thomson's principle (Theorem 2.28), the energy contributed to $\mathcal{E}(I)$ from these edges is at least c. In the same manner, the edges which have at least one endpoint in the annulus $V_{C^{2k+1}R} \setminus V_{C^{2k}R}$ contribute at least c to $\mathcal{E}(I)$. Part (i) of Lemma 4.9 implies that all these edge sets are disjoint, hence $\mathcal{E}(I) = \infty$ and we learn that G is recurrent (Corollary 2.39). □

Proof of Theorem 4.4 (4)

We will use the given circle packing of G to create a random path to infinity with finite energy. This gives transience by Claim 2.46. This proof strategy is similar to that of Theorem 2.47.

Proof of Theorem 4.4 (4) Let v_0 be a fixed vertex of the graph, and apply a Möbius transformation to make the circle of P corresponding to v_0 be centered at the origin $\mathbf{0}$. We now use Claim 2.46 to construct a flow θ from v_0 to ∞ by choosing a uniform random point $\mathbf{p}$ on $\partial\mathbb{U}$, taking the straight line from $\mathbf{0}$ to $\mathbf{p}$ and considering the set of all circles in the packing P that intersect this line in the order that they are visited; this set forms an infinite simple path in the graph which starts at v_0.

To bound the energy of the flow, we claim that there exists some constant C (which may depend on the graph G and the packing P) such that the probability that the random path uses the vertex v is bounded above by $C\, \text{rad}(v)$. Indeed, since there are only finitely many vertices with centers at distance at most $1/2$ from $\mathbf{0}$, we may assume that the center of v is of distance at least $1/2$ from $\mathbf{0}$. In this case, in order for v to be included in the random path the circle of v must intersect the line between $\mathbf{0}$ and $\mathbf{p}$. By the Ring Lemma (Lemma 4.2) the neighbors of v have circles of radii comparable to $\text{rad}(v)$ and so the probability of the line touching them is at most $C\, \text{rad}(v)$. Since the vertex degree is bounded by Δ and $\sum_{v \in V} \pi\, \text{rad}(v)^2$ is at most the area of $\mathbb{U}$, we find that

$$\mathcal{E}(\theta) \le C\Delta \sum_{v \in V} \text{rad}(v)^2 \le C\Delta\,.$$

Hence G is transient by Corollary 2.39 □

Proof of Theorem 4.4 (1)

We apply Claim 4.3 to obtain a circle packing P of G and prove that $\mathrm{Carrier}(P) = \mathbb{R}^2$. Fix some vertex v and rescale and translate so that $P(v)$ is the unit circle $\partial\mathbb{U}$. Assume by contradiction that $\mathrm{Carrier}(P) \neq \mathbb{R}^2$ and let $p \in \mathbb{R}^2 \setminus \mathrm{Carrier}(P)$ be a point not in the carrier. Rotate the packing so that $p = R$ for some real number $R > 1$. Let $U \in [-1, 1]$ and consider the circle $C_U = \{z : |z - p| = R - U\}$. We traverse C_U from the point U counterclockwise and consider all the circles of P which intersect C_U. These circles form a simple path in the graph G starting from v. Since $\mathrm{Carrier}(P)$ is simply connected by Lemma 4.1 and $p \notin \mathrm{Carrier}(P)$ it cannot be that $C_U \subset \mathrm{Carrier}(P)$. Thus, as we traverse C_U counterclockwise we must hit the boundary of $\mathrm{Carrier}(P)$. We conclude that the path in G we obtained in this manner is an infinite simple path starting at v.

We now let U be a uniform random variable in $[-1, 1]$ and let μ denote the probability measure on random infinite paths starting at v we obtained as described above. Let θ be the flow induced by μ as in Claim 2.46. We wish to bound the energy $\mathcal{E}(\theta)$. Consider a vertex $w \in G$ and its corresponding circle C_w and let B be the Euclidean ball of radius $R+1$ around p. If C_w does not intersect B, it cannot be included in the random path by our construction. If it does intersect this ball, then the probability that the random path intersects it is bounded above by its radius. Thus as in the proof of Theorem 4.4 (4),

$$\mathcal{E}(\theta) \leq C\Delta \sum_{w: C_w \cap B \neq \emptyset} \mathrm{rad}(w)^2 ,$$

where Δ is the maximal degree of G and we have used the Ring Lemma (Lemma 4.2). We learn that $\mathcal{E}(\theta)$ is bounded above by a constant multiple of the area of all circles of P that intersect B. Since $p \notin \mathrm{Carrier}(P)$, by the Ring Lemma (Lemma 4.2), any circle of P that intersects B cannot have radius more than AR for some large $A \geq R$ (since otherwise, all the circles surrounding this vertex will have radius more than $R+1$, contradicting the fact that $p \notin \mathrm{Carrier}(P)$). We learn that all the circles counted in the sum above are contained in the Euclidean ball of radius $(A+1)R+1$ around p. Since these circles has disjoint interiors, the sum of their area is bounded above by the area of the Euclidean ball above. We conclude that $\mathcal{E}(\theta) < \infty$, hence G is transient by Corollary 2.39 and we have reached a contradiction. □

Proof of Theorem 4.4 (2)

We will use the following simple corollary of the circle packing theorem, Theorem 3.5. A **finite triangulation with boundary** is a finite connected simple planar map in which all faces are triangles except for a distinguished outer face whose boundary is a simple cycle.

Claim 4.10 Let G be a finite triangulation with boundary. Then, there is a circle packing P of G such that all circles of the outer face are internally tangent to $\partial\mathbb{U}$ and all other circles of P are contained in $\mathbb{U}$.

Proof Denote by $v_1, \ldots, v_m$ the vertices of the outer face ordered according to the cycle they form. Add a new vertex v^* to the graph and connect it to $v_1, \ldots, v_m$ according to their order. We obtain a finite triangulation G^*. Apply Theorem 3.5 to obtain a circle packing $P = \{C_v\}_{v \in V(G^*)}$. By translating and dilating we may assume that C_{v^*} is centered at the origin and has radius 1. Apply the map $z \mapsto \frac{1}{z}$ on this packing. Since this map preserves circles, the image of the circles $\{C_v\}_{v \in V(G^*)\setminus\{v^*\}}$ under this map is precisely the desired circle packing. □

Furthermore, we will require an auxiliary general estimate. Given a circle packing P and a set of vertices A, we write $\mathrm{diam}_P(A)$ for the Euclidean diameter of the union of all circles in P corresponding to the vertices of A.

Lemma 4.11 *Let P be a circle packing contained in $\mathbb{U}$ of a finite triangulation with boundary with maximum degree Δ, such that the circle of a chosen non-boundary vertex v_0 is centered at the origin and has radius r_0. Assume that $r_0 \geq r_{\min}$ for some constant $r_{\min} > 0$. Then there exists a constant $c = c(r_{\min}, \Delta) > 0$ such that for any connected set A of vertices,*

$$\mathcal{R}_{\mathrm{eff}}(v_0 \leftrightarrow A) \geq c \log \frac{1}{\mathrm{diam}_P(A)} . \tag{4.1}$$

If in addition all circles of the outer face are tangent to $\partial\mathbb{U}$ and A contains a vertex of the outer face, then

$$\mathcal{R}_{\mathrm{eff}}(v_0 \leftrightarrow A) \leq c^{-1} \log \frac{1}{\mathrm{diam}_P(A) \wedge \frac{1}{2}} . \tag{4.2}$$

Proof Write $\varepsilon = \mathrm{diam}_P(A)$ and let $z(A)$ denote the union of all circles corresponding to the vertices of A. We begin with the proof of (4.1), which goes along similar lines to the proof of Lemma 4.9. Let $z_0 \in \mathbb{R}^2$ be such that $z(A) \subset \{|z - z_0| \leq \varepsilon\}$. For any $r > 0$ denote by V_r the set of vertices whose corresponding circles have centers inside $\{|z - z_0| \leq r\}$, so that $A \subset V_\varepsilon$. Repeating the proof of Lemma 4.9 shows that there exists a constant $C = C(\Delta) > 0$ such that

(i) There are no edges between V_r and $V \setminus V_{Cr}$, and
(ii) $\mathcal{R}_{\mathrm{eff}}(V_r \leftrightarrow V \setminus V_{Cr}) \geq C^{-1}$, as long as V_r and $V \setminus V_{Cr}$ are non-empty.

Regarding this proof, we note that it is possible that the set $\{|z - z_0| \leq r\}$ is not contained in $\mathbb{U}$ (unlike the proof of Lemma 4.9 when the carrier is all of $\mathbb{R}^2$), however, this only works in our favor. The proof of (4.1) now proceeds similarly to the proof of Theorem 4.4 (3). When ε is small enough (depending only on $r_{\min}$

and Δ), by the Ring Lemma (Lemma 4.2), the Euclidean distance between the circle corresponding to v_0 and A is at least some constant (which again depends only on $r_{\min}$ and Δ) so that $v_0 \notin V_{C^K\varepsilon}$ for some $K = \Omega(\log(1/\varepsilon))$. For each $k = 0, 2, 4, \ldots, K$ the sets of edges which have at least one endpoint in the annulus $V_{C^{k+1}\varepsilon} \setminus V_{C^k\varepsilon}$ are disjoint by (i). By (ii), each of these sets of edges contribute at least C^{-1} to the energy of the unit current flow from A to v_0, concluding the proof of (4.1) using Thomson's principle (Theorem 2.28).

For the proof of (4.2) we construct a unit flow from v_0 to A that has energy $O(\log(1/\varepsilon))$. The construction is in the same spirit as the proof of Theorem 4.4 (4), but there are some technical difficulties to overcome. Since A contains a vertex that is tangent to $\partial\mathbb{U}$, we choose $z_0 \in \partial\mathbb{U}$ that belongs to a circle of A. By rotating the packing we may assume that $z_0 = e^{i\varepsilon/4}$.

We now treat two cases separately. In the first case we assume that there exists z_1 in $z(A)$ such that $\arg(z_1) \in [0, \varepsilon/2]$ and $|z_1| \le 1 - \varepsilon/2$ such that the path in $z(A)$ from z_0 to z_1 remains in the sector $\arg(z) \in [0, \varepsilon/2]$. Consider the points

$$x_0 = -r_0 \qquad x_1 = r_0 \qquad y_1 = 1 - \varepsilon/3 \qquad y_0 = 1\,,$$

and note that x_0, x_1 are the leftmost and rightmost points on the circle of v_0. Let C_0 and C_1 be the upper half plane semi-circles in which x_0, y_0 and x_1, y_1 are antipodal points, respectively. The choice of y_0, y_1 is made so that the path between z_0 to z_1 in $z(A)$ must cross the region bounded by C_0, C_1 and the intervals $[x_0, x_1], [y_1, y_0]$, by our assumption on z_1 as long as ε is small enough. See Fig. 4.4, left.

For each $t \in [0, 1]$ write C_t for the upper half plane semi-circle in which $ty_1 + (1-t)y_0$ and $tx_1 + (1-t)x_0$ are antipodal points, so that C_t continuously interpolates between C_0 and C_1. See Fig. 4.4, left. Choose $t \in [0, 1]$ uniformly at random and consider the random path γ which traces C_t from left to right. This random path

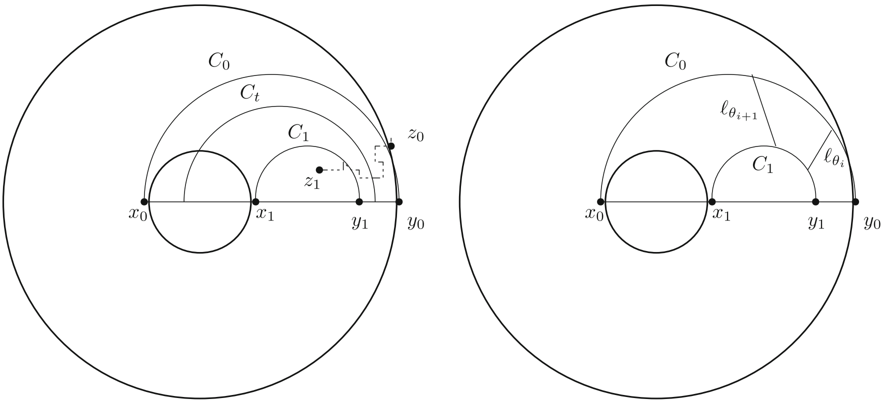

Fig. 4.4 Left: for any $t \in [0, 1]$ the semi-circle C_t must intersect the path in A between z_0 and z_1. Right: the quadrilateral Q_i is bounded between ℓ_{θ_i}, $\ell_{\theta_{i+1}}$, C_0 and C_1

starts at the circle of v_0 and must hit the path between z_0 and z_1 by our previous discussion. Hence, the circles of P that intersect γ must contain a path in the graph from v_0 to A. By Claim 2.46 we obtain a flow I from v_0 to A whose energy $\mathcal{E}(I)$ we now bound.

For an angle $\theta \in [0, \pi]$ we denote by $w_\theta(t)$ the point at angle θ, seen from the center of C_t, on the semi-circle C_t . It is an exercise to see that the set of points $\{w_\theta(t) : t \in [0, 1]\}$ form a straight line interval ℓ_θ. Furthermore, when t is chosen uniform in [0, 1], the intersection of C_t and ℓ_θ is a uniformly chosen point on ℓ_θ. Set $\theta_0 = 0$ and $\theta_i = 2^{i-1}\varepsilon$ for $i = 1, \dots, K-1$ where $K = O(\log(1/\varepsilon))$ such that $\theta_{K-1} \in [\pi/4, \pi/2]$ and set $\theta_K = \pi$. We will obtain the bound $\mathcal{E}(I) = O(K)$ by bounding from above by a constant the contribution to $\mathcal{E}(I)$ coming from edges which intersect the quadrilateral Q_i of $\mathbb{R}^2$ bounded by $\ell_{\theta_i}, \ell_{\theta_{i+1}}, C_0, C_1$; see Fig. 4.4, right. The random path γ restricted to Q_i can be sampled by choosing a uniform random point on ℓ_{θ_i}, setting $t \in [0, 1]$ to be the unique number such that C_t intersects ℓ_{θ_i} at the chosen point, and tracing the part of C_t from ℓ_{θ_i} to $\ell_{\theta_{i+1}}$. The lengths of the four curves bounding Q_i are all of order $2^i\varepsilon$ and so we deduce that if v corresponds to a circle of radius $O(2^i\varepsilon)$ which intersects Q_i, then the probability that it is visited by γ is $O(\mathrm{rad}(v)/2^i\varepsilon)$. Since the sum of $\mathrm{rad}(v)^2$ over such v's is at most the area of Q_i up to a multiplicative constant (note that some of these circles need not be contained in Q_i) it is at most $O(2^{2i}\varepsilon^2)$. Since the degrees are bounded we deduce that the contribution to the energy from edges touching such v's is $O(1)$. Lastly, if v corresponds to a larger circle, then we bound its probability of being visited by γ by 1 and note that there can only be $O(1)$ many such v's whose circles intersects Q_i. Thus the contribution from these is another $O(1)$. Since there are $O(\log(1/\varepsilon))$ such i's we learn that $\mathcal{E}(I) = O(\log(1/\varepsilon))$ finishing our proof in this case using Thomson's principle (Theorem 2.28).

In the second case, we assume that there exists $z_1 \in z(A)$ such that $\arg(z_1) \notin [0, \varepsilon/2]$ and $|z_1| \geq 1 - \varepsilon$. It is clear that since $\mathrm{diam}_P(A) = \varepsilon$ either the first or the second case must occur. Denote $z_0' = |z_1|e^{i\varepsilon/4}$ and let x_0, x_1 be antipodal points on the circle of v_0 such that the straight line between x_0 and x_1 is parallel to the straight line between z_0' and z_1. The vertices z_0', z_1, x_0, x_1 form a trapezoid, see Fig. 4.5. We choose a uniform random point $t \in [0, 1]$ and stretch a straight line from $tx_0 + (1-t)x_1$ to $tz_0 + (1-t)z_1'$. We then continue it by a straight line from $tz_0 + (1-t)z_1'$ to $w \in \partial\mathbb{U}$ where $\arg(w) = \arg(tz_0 + (1-t)z_1')$. Denote the resulting path by γ_t and note that it starts inside the circle of v_0 and must hit the path between z_0 and z_1 in $z(A)$. Thus, the set of all circles which intersect γ_t form a path in the graph that starts at v_0 and ends at A; this random choice of γ_t gives us as usual a unit flow from v_0 to A by Claim 2.46. By repeating the same argument as in the previous case (that is, splitting the trapezoid into $O(\log(1/\varepsilon))$ many trapezoids of constant aspect ratio), we see that the contribution to the energy of the flow induced by the random path γ_t of the edges in the trapezoid is $O(\log(1/\varepsilon))$. Furthermore, the same argument gives that the edges in the quadrilateral formed by the vertices z_0, z_0', z_1 and $e^{i\arg(z_1)}$ also contribute at most a constant to the energy, concluding our proof for the second case by Thomson's principle again (Theorem 2.28). □

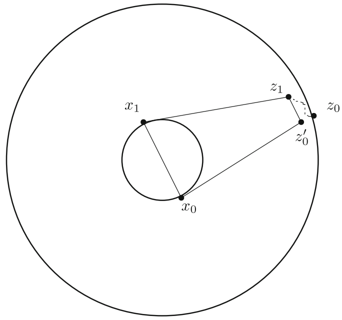

Fig. 4.5 The resistance across the trapezoid on vertices x_0, x_1, z'_0, z_1 is $O(\log(1/\varepsilon))$ when $|z'_0 - z_1| = \Theta(\varepsilon)$

Proof of Theorem 4.4 (2) Denote by $d_G(u, v)$ the graph distance between the vertices u, v of G. Fix some $v_0 \in V$ and let

$$B_j = \{v : d_G(v_0, v) \le j\},$$
$$V_j = B_j \cup \{\text{finite components of } V \setminus B_j\},$$
$$E_j = \{\text{edges induced by } V_j\}.$$

The graph $G_j = (V_j, E_j)$ with the map structure inherited from G is a finite triangulation with boundary. Indeed, it is straightforward to check that it is 2-connected (i.e., the removal of a single vertex does not disconnect the graph) which implies that the outer face forms a simple cycle, see [20, Proposition 4.2.5]. Furthermore, since G is one-ended and we have added all the finite components in $V \setminus B_j$ there cannot be a face with more than 3 edges except for the outer face which we denote by ∂G_j.

Thus G_j is an increasing sequence of finite triangulations with boundary such that $\cup_j G_j = G$. We apply Claim 4.10 to pack G_j inside the unit disk $\mathbb{U}$ such that the circles of ∂G_j are tangent to $\partial \mathbb{U}$. By applying a Möbius transformation from $\mathbb{U}$ onto $\mathbb{U}$, we may assume that the circle corresponding to v_0 is centered at the origin. We denote this packing by P_j and let r_0^j be the radius of v_0 in P_j.

Since G is transient it follows that there exists some $c = c(\Delta) > 0$ such that $r_0^j \ge c$ for all j by Corollary 2.39. Indeed, if $r_0^j \le \varepsilon$, we learn by Lemma 4.9 and the proof of Theorem 4.4 (3) that $\mathcal{R}_{\text{eff}}(v_0 \leftrightarrow \infty) \ge c' \log(\varepsilon^{-1})$ for some $c' = c'(\Delta) > 0$.

As we did in Claim 4.3, we now take a subsequence in which the centers and radii of all vertices converge. Denote the resulting limiting packing by P_∞. This packing has all circles inside $\mathbb{U}$ and we therefore deduce that Carrier$(P_\infty) \subseteq \mathbb{U}$. It is a priori possible that Carrier(P_∞) is some strict subset of $\mathbb{U}$, i.e., that all the circles stabilize inside some strict subset of $\mathbb{U}$. We now argue that this is not possible.

Let Z be the set of accumulation points of Carrier(P_∞); it suffices to show that $Z \subset \partial \mathbb{U}$ since any simply connected domain $\Omega \subset \mathbb{U}$ for which $\partial \Omega \subset \partial \mathbb{U}$ must equal $\mathbb{U}$. Since Z is a compact set, let $z \in Z$ minimize $|z|$ among all $z \in Z$; it suffices to show that $z \in \partial\mathbb{U}$. Fix $\varepsilon > 0$ and put

$$U_\varepsilon(z) = \left\{v \in G : |\,\mathrm{cent}_{P_\infty}(v) - z| \leq \varepsilon\right\}.$$

The graph spanned on the vertices $U_\varepsilon(z)$ may be disconnected, yet by our choice of z it is clear that $U_\varepsilon(z)$ contains an infinite connected component. Indeed, one can draw a straight line from the origin to z without intersecting Z and consider the set of all circles intersecting this line; from some point onwards the vertices corresponding to these circles will reside in $U_\varepsilon(z)$.

Therefore, let $W_\varepsilon(z)$ be an infinite connected component of the graph spanned on $U_\varepsilon(z)$. Let $J = J(z, \varepsilon)$ be the first integer such that $V_J \cap W_\varepsilon(z) \neq \emptyset$. Since the G_j's are increasing finite sets and $W_\varepsilon(z)$ is an infinite connected set, we have that $\partial G_j \cap W_\varepsilon(z) \neq \emptyset$ for all $j \geq J$. Consider now any connected component A_j of the graph spanned on the vertices $V_j \cap W_\varepsilon(z)$.

Denote by P_∞^j the finite circle packing obtained from P_∞ by taking only the circles of V_j. It has the same adjacency graph as P_j but it is a different packing. Since $A_j \subset W_\varepsilon(z)$, it follows that $\mathrm{diam}_{P_\infty^j}(A_j) \leq 4\varepsilon$. By Lemma 4.11, Eq. (4.1), applied to the set A_j in the packing P_∞^j, we deduce that $\mathcal{R}_{\mathrm{eff}}(v_0 \leftrightarrow A_j; G_j) \geq c \log(1/\varepsilon)$. Since A_j is a connected component of $V_j \cap W_\varepsilon(z)$ and since $W_\varepsilon(z)$ is an infinite connected set of vertices in G, it follows that A_j must contain a vertex of ∂V_j. Thus, we may apply Lemma 4.11, Eq. 4.2, to the set A_j, this time in the packing P_j, to get that there exists some $c > 0$ such that

$$\mathrm{diam}_{P_j}(A_j) \leq \varepsilon^c. \tag{4.3}$$

Choose some $v_J \in \partial G_J \cap W_\varepsilon(z)$ so that $|\,\mathrm{cent}_{P_\infty}(v_J) - z| \leq \varepsilon$. For each $j \geq J$ choose $v_j \in \partial G_j \cap W_\varepsilon(z)$ so that v_j and v_J are in the same connected component A_j of the graph spanned on $V_j \cap W_\varepsilon(z)$. Since the circle of v_j in P_j touches $\partial\mathbb{U}$ we learn by (4.3) that the distance of the circle of v_J in P_j from $\partial\mathbb{U}$ is at most ε^c for all $j \geq J$. Since the circle corresponding to v_J in P_∞ is the limit of its circles in P_j we deduce that the distance of $\mathrm{cent}_{P_\infty}(v_J)$ from $\partial\mathbb{U}$ is at most ε^c. Hence the distance of z from $\partial\mathbb{U}$ is at most $\varepsilon + \varepsilon^c$. Since ε was arbitrary we obtain that $z \in \partial\mathbb{U}$, as required. □

4.5 Exercises

1. Let G be a triangulation of the plane with maximal degree at most 6. Prove that the simple random walk on G is recurrent.
2. Let G be a plane triangulation that can be circle packed in the unit disc $\{z : |z| < 1\}$. Show that the simple random walk on G is transient. (Note that G may have *unbounded* degrees)

3.(*) Let P be a circle packing of a finite simple planar map with degree bounded by D such that all of its faces are triangles except for the outerface. Assume that the carrier of P is contained in $[-11, 11]^2$, contains $[-10, 10]^2$ and that all circles have radius at most 1. Let h be the harmonic function taking the value 1 on all vertices with centers left of the line $\{-10\} \times \mathbb{R}$, taking the value 0 on all vertices with centers right of the line $\{10\} \times \mathbb{R}$, and is harmonic anywhere else. Assume x and y are two vertices such that their centers are contained in $[-1, 1]^2$ and that the Euclidean distance between these centers is at most $\epsilon > 0$. Show that

$$|h(x) - h(y)| \le \frac{C}{\log(1/\epsilon)},$$

for some constant $C = C(D) > 0$ independent of ϵ. [Hint: assume $h(x) < h(y)$ and consider the sets $A = \{v : h(v) \le h(x)\}$ and $B = \{v : h(v) \ge h(y)\}$].