Chapter 1
Introduction

1.1 The Circle Packing Theorem

A planar graph is a graph that can be drawn in the plane, with vertices represented by points and edges represented by non-crossing curves. There are many different ways of drawing any given planar graph and it is not clear what is a canonical method. One very useful and widely applicable method of drawing a planar graph is given by Koebe's 1936 *circle packing theorem* [51], stated below. As we will see, various geometric properties of the circle packing drawing (such as existence of accumulation points and their structure, bounds on the radii of circles and so on) encode important probabilistic information (such as the recurrence/transience of the simple random walk, connectivity of the uniform spanning forest and much more). This deep connection is especially fruitful to the study of random planar maps. Indeed, one of the main goals of these notes is to present a self-contained proof that the so-called *uniform infinite planar triangulation* (UIPT) is almost surely recurrent [31].

A **circle packing** is a collection of discs $P = \{C_v\}$ in the plane $\mathbb{C}$ such that any two distinct discs in P have disjoint interiors. That is, distinct discs in P may be tangent, but may not overlap. Given a circle packing P, we define the **tangency graph** $G(P)$ of P to be the graph with vertex set P and with two vertices connected by an edge if and only if their corresponding circles are tangent. The tangency graph $G(P)$ can be drawn in the plane by drawing straight lines between the centers of tangent circles in P, and is therefore planar. It is also clear from the definition that $G(P)$ is **simple**, that is, any two vertices are connected by at most one edge and there are no edges beginning and ending at the same vertex. See Fig. 1.1.

We call a circle packing P a circle packing of a planar graph G if $G(P)$ is isomorphic to G.

Theorem 1.1 (Koebe 1936) *Every finite simple planar graph G has a circle packing. That is, there exists a circle packing P such that $G(P)$ is isomorphic to G.*

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Fig. 1.1 A planar graph and a circle packing of it

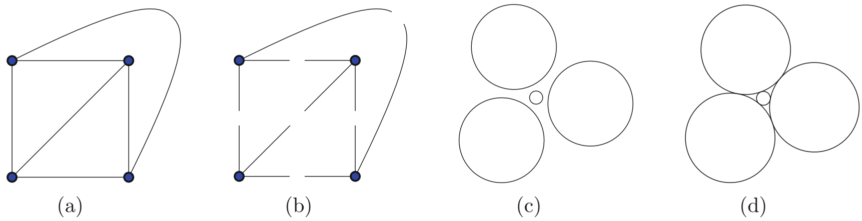

Fig. 1.2 A sketch of how to obtain circle packings using Koebe's extension of the Riemann mapping theorem to finitely connected domains, which states that every domain $D \subseteq \mathbb{C} \cup \{\infty\}$ with at most finitely many boundary components is conformally equivalent to a *circle domain*, that is, a domain all of whose boundary components are circles or points. (**a**) Step 1: We begin by drawing the finite simple planar graph G in the plane in an arbitrary way. (**b**) Step 2: If we remove the 'middle ε' of each edge, then the complement of the resulting drawing is a domain with finitely many boundary components. (**c**) Step 3: Finding a conformal map from this domain to a circle domain gives an 'approximate circle packing' of G. (**d**) Step 4: Taking the limit as $\varepsilon \downarrow 0$ can be proven to yield a circle packing of G

One immediate consequence of the circle packing theorem is Fáry's Theorem [25], which states that every finite simple planar graph can be drawn so that all the edges are represented by straight lines.

The circle packing theorem was first discovered by Koebe [51], who established it as a corollary to his work on the generalization of the Riemann mapping theorem to finitely connected domains; a brief sketch of Koebe's argument is given in Fig. 1.2. The theorem was rediscovered and popularized in the 1970s by Thurston [82], who showed that it follows as a corollary to the work of Andreev on hyperbolic polyhedra (see also [63]). Thurston also initiated a popular program of understanding circle packing as a form of *discrete complex analysis*, a viewpoint which has been highly influential in the subsequent development of the subject and which we discuss in more detail below (see [79] for a review of a different form of discrete complex analysis with many applications to probability). There are now many proofs of the circle packing theorem available including, remarkably, four distinct proofs discovered by Oded Schramm. In Chap. 3 we will give an entirely combinatorial proof, which is adapted from the proof of Thurston [63, 82] and Brightwell and Scheinerman [13].

Uniqueness

We cannot expect a uniqueness statement in Theorem 1.1 (see Fig. 1.1; we may "slide" circles 5 and 6 along circle 2). However, when our graph is a *finite triangulation*, circle packings enjoy uniqueness up to circle-preserving transformations.

Definition 1.2 A planar **triangulation** is a planar graph that can be drawn so that every face is incident to exactly three edges. In particular, when the graph is finite this property must hold for the outer face as well.

Claim 1.3 If G is a finite triangulation, then the circle packing whose tangency graph is isomorphic to G is unique, up to Möbius transformations and reflections in lines.

The uniqueness of circle packing was first proven by Thurston, who noted that it follows as a corollary to Mostow's rigidity theorem. Since then, many different proofs have been found. In Chap. 3 we will give a very short and elementary proof of uniqueness due to Oded Schramm that is based on the maximum principle.

Infinite Planar Graphs

So far, we have only discussed the existence and uniqueness of circle packings of *finite* planar triangulations. What happens with infinite triangulations? To address this question, we will need to introduce some more definitions.

Definition 1.4 We say that a graph G is **one-ended** if the removal of any finite set of vertices leaves at most one infinite connected component.

Definition 1.5 Let $P = \{C_v\}$ be a circle packing of a triangulation. We define the **carrier** of P to be the union of the closed discs bounded by the circles of P together with the spaces bounded between any three circles that form a face (i.e., the interstices). We say that P is **in** D if its carrier is D.

See Fig. 1.3 for examples where the carrier is a disc or a square. The circle packing of the standard triangular lattice (see Fig. 4.2) has the whole plane $\mathbb{C}$ as its carrier. It is not too hard to see that if $G(P)$ is an infinite triangulation, then it is one-ended if and only if the carrier of P is simply connected, see Lemma 4.1.

It can be shown via a compactness argument that any simple infinite planar triangulation can be circle packed in *some* domain. Indeed, one can simply take subsequential limits of circle packings of finite subgraphs (the fact that such subsequential limits can be taken is a consequence of the so-called Ring Lemma, see Lemma 4.2). This is performed in Claim 4.3. However, this compactness argument does not give us any control of the domain we end up with as the carrier of our circle packing. The following theorems of He and Schramm [39, 40] give us much

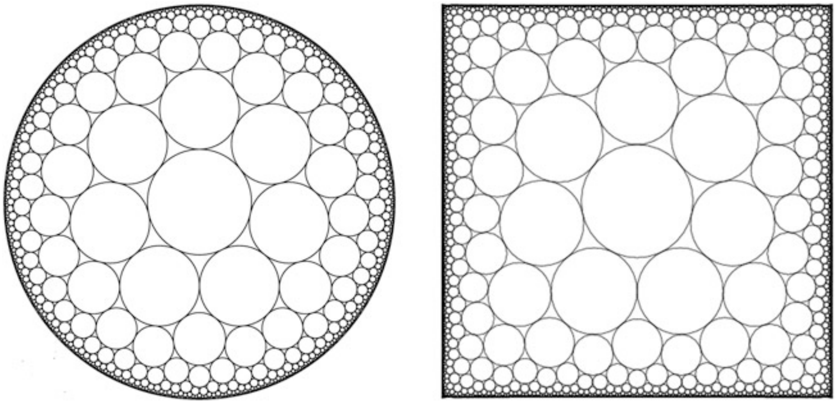

Fig. 1.3 The 7-regular hyperbolic tessellation circle packed in a disc and in a square

better control; they can be thought of as discrete analogues of the Poincaré-Koebe uniformization theorem for Riemann surfaces.

Theorem 1.6 (He and Schramm 1993) *Any one-ended infinite triangulation can be circle packed such that the carrier is either the plane or the open unit disk, but not both.*

This theorem will be proved in Chap. 4 (with the added assumption of finite maximal degree). The proofs in [39, 40] are based on the notion of *discrete extremal length*. We will present our own approach to the proof in Chap. 4 based on a very similar notion of electric resistance discussed in Chap. 2. This approach is somewhat more appealing to a probabilist and allows for quantitative versions of the He-Schramm Theorem that will be used later for the study of random planar maps in Chap. 6.

In view of Theorem 1.6, we call an infinite one-ended simple planar triangulation **CP parabolic** if it can be circle packed in $\mathbb{C}$, and call it **CP hyperbolic** if it can be circle packed in the open unit disk $\mathbb{U}$.

Theorem 1.7 (He and Schramm 1995) *Let T be a CP hyperbolic infinite one-ended simple planar triangulation and let $D \subsetneq \mathbb{C}$ be a simply connected domain. Then there exists a circle packing of T with carrier D.*

What about uniqueness? Theorem 1.7 shows that, in general, we have much more flexibility when choosing a circle packing of an infinite planar triangulation than we have in the finite case, see Fig. 1.3 again. Indeed, it implies that the circle packing of a CP hyperbolic triangulation is *not* determined up to Möbius transformations and reflections, since, for example, we can circle pack the same triangulation in both the unit disc and the unit square, and these two packings are clearly not related by a Möbius transformation. Fortunately, the following theorem of Schramm [73] shows that we recover Möbius rigidity if we restrict the packing to be in $\mathbb{C}$ or $\mathbb{U}$.

Theorem 1.8 (Schramm 1991) *Let T be a one-ended infinite planar triangulation.*

- *If T is CP parabolic, then its circle packing in $\mathbb{C}$ is unique up to dilations, rotations, translations and reflections.*
- *If T is CP hyperbolic, then its circle packing in $\mathbb{U}$ is unique up to Möbius transformations or reflections fixing $\mathbb{U}$.*

Relation to Conformal Mapping

A central motivation behind Thurston's popularization of circle packing was its role as a discrete analogue of conformal mapping. The resulting theory is somewhat tangential to the main thrust of these notes, but is worth reviewing for its beauty, and for the intuition it gives about circle packing. A more detailed treatment of this and related topics is given in [81].

Recall that a map $\phi : D \to D'$ between two domains $D, D' \subseteq \mathbb{C}$ is conformal if and only if it is holomorphic and one-to-one. Intuitively, we can think of the latter condition as saying that ϕ maps infinitesimal circles to infinitesimal circles. Thus, it is natural to wonder, as Thurston did, whether conformal maps can be approximated by graph isomorphisms between circle packings of the corresponding domains, which *literally* map circles to circles.

For each $\varepsilon > 0$, let $\mathbb{T}_\varepsilon = \{\varepsilon n + \varepsilon \frac{1+\sqrt{3}i}{2} m : n, m \in \mathbb{Z}\} \subseteq \mathbb{C}$ be the triangular lattice with lattice spacing ε, which we make into a simple planar triangulation by connecting two vertices if and only if they have distance ε from each other. This triangulation is naturally circle packed in the plane by placing a disc of radius ε around each point of $\mathbb{T}_\varepsilon$: this is known as the **hexagonal packing**. Now, let D be a simply connected domain, and take z_0 to be a marked point in the interior of D. For each $\varepsilon > 0$ let u_ε be an element of $\mathbb{T}_\varepsilon$ of minimal distance to z_0, and let $v_\varepsilon = u_\varepsilon + \varepsilon$ and $w_\varepsilon = u_\varepsilon + (1 + \sqrt{3}i)\varepsilon/2$. For each $\varepsilon > 0$, let $T_\varepsilon(D)$ be the subgraph of $\mathbb{T}_\varepsilon$ induced by the vertices of distance at least 2ε from ∂D (i.e., the subgraph containing all such vertices and all the edges between them), and let $T'_\varepsilon(D)$ be the component of $T_\varepsilon(D)$ containing u_ε. Finally, let $T''_\varepsilon(D)$ be the triangulation obtained from $T'_\varepsilon(D)$ by placing a single additional vertex ∂_ε in the outer face of $T'_\varepsilon(D)$ and connecting this vertex to every vertex in the outer boundary of $T'_\varepsilon(D)$.

Applying the circle packing theorem to $T''_\varepsilon(D)$ and then applying a Möbius transformation or a reflection if necessary, we obtain a circle packing P_ε of $T''_\varepsilon(D)$ with the following properties:

- The boundary vertex ∂_ε is represented by the unit circle,
- the vertex u_ε is represented by a circle centered at the origin,
- the vertex v_ε is represented by a circle centered on the real line, and
- the vertex w_ε is represented by a circle centered in the upper half-plane.

The function sending each vertex of $T'_\varepsilon(D)$ to the center of the circle representing it in P_ε can be extended piecewise on each triangle by an affine extension. Call the resulting function ϕ_ε.

The following theorem was conjectured by Thurston and proven by Rodin and Sullivan [70].

Theorem 1.9 (Rodin and Sullivan 1987) *Let ϕ be the unique conformal map from D to $\mathbb{U}$ with $\phi(z_0) = 0$ and $\phi'(z_0) > 0$. Then ϕ_ε converge to ϕ as $\varepsilon \downarrow 0$, uniformly on compact subsets of D.*

See Fig. 1.4. The key to the proof of Theorem 1.9 was to establish that the hexagonal packing is the only circle packing of the triangular lattice, which is now a special case of Theorem 1.8.

Various strengthenings and generalizations of Theorem 1.9 have been established in the works [21, 36, 38, 41, 42, 80].

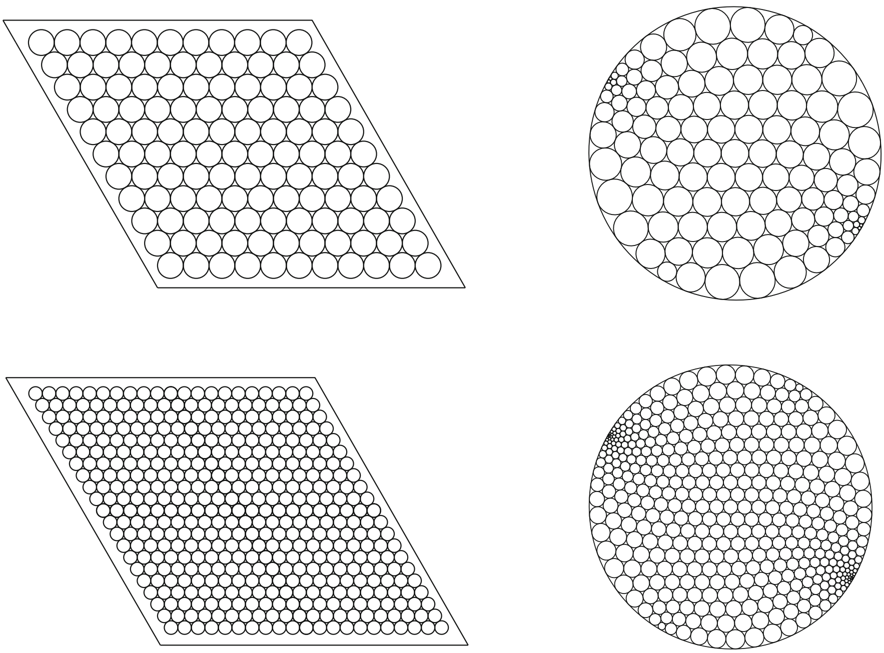

Fig. 1.4 Approximating the conformal map from a rhombus to the disc using circle packing, at two different degrees of accuracy

1.2 Probabilistic Applications

Why should we be interested in circle packing as probabilists? At a very heuristic level, when we uniformize the *geometry* of a triangulation by applying the circle packing theorem, we also uniformize the *random walk* on the triangulation, allowing us to compare it to a standard reference process that we understand very well, namely Brownian motion. Indeed, since Brownian motion is conformally invariant and circle packings satisfy an approximate version of conformality, it is not unreasonable to expect that the random walk on a circle packed triangulation will behave similarly to Brownian motion. This intuition turns out to be broadly correct, at least when the triangulation has bounded degrees, although it is more accurate to say that the random walk behaves like a *quasi-conformal image* of Brownian motion, that is, the image of Brownian motion under a function that distorts angles by a bounded amount.

Although it is possible to make the discussion in the paragraph above precise, in these notes we will be interested primarily in much coarser information that can be extracted from circle packings, namely *effective resistance estimates* for planar graphs. This fundamental topic is thoroughly discussed in Chap. 2. One of the many definitions of the effective resistance $\mathcal{R}_{\text{eff}}(A \leftrightarrow B)$ between two disjoint sets A and B in a finite graph is

$$\frac{1}{\mathcal{R}_{\text{eff}}(A \leftrightarrow B)} = \sum_{v \in A} \deg(v) \mathbb{P}_v(\tau_B < \tau_A^+),$$

where $\mathbb{P}_v$ is the law of the simple random walk started at v, τ_B is the first time the walk hits B, and τ_A^+ is the first positive time the walk visits A. Good enough control of effective resistances allows one to understand most aspects of the random walk on a graph. We can also define effective resistances on infinite graphs, although issues arise with boundary conditions. An infinite graph is recurrent if and only if the effective resistance from a vertex to infinity is infinite.

The effective resistance can also be computed via either of two variational principles: *Dirichlet's principle* and *Thomson's principle*, see Sect. 2.4. The first expresses the effective resistance as a *supremum* of energies of a certain set of *functions*, while the second expresses the effective resistance as an *infimum* of energies of a certain set of *flows*. Thus, we can bound effective resistances from above by constructing flows, and from below by constructing functions. A central insight is that we can *use the circle packing* to construct these functions and flows. This idea leads fairly easily to various statements such as the following:

- The effective resistance across a Euclidean annulus of fixed modulus is at most a constant. If the triangulation has bounded degrees, then the resistance is at least a constant.

- The effective resistance between the left and right sides of a Euclidean square is at most a constant. If the triangulation has bounded degrees, then the resistance is at least a constant.

See for instance Lemma 4.9. We will use these ideas to prove the following remarkable theorem of He and Schramm [40], which pioneered the connection between circle packing and random walks.

Theorem 1.10 (He and Schramm 1995) *Let T be a one-ended infinite triangulation. If T has bounded degrees, then it is CP parabolic if and only if it is recurrent for simple random walk, that is, if and only if the simple random walk on T visits every vertex infinitely often almost surely.*

This has been extended to the multiply-ended cases in [32], see also Chap. 8, item 4.

Recurrence of Distributional Limits of Random Planar Maps

Random planar maps is a widely studied field lying at the intersection of probability, combinatorics and statistical physics. It aims to answer the vague question "what does a typical random surface look like?"

We provide here a very quick account of this field, referring the readers to the excellent lecture notes [58] by Le Gall and Miermont, and the many references within for further reading. The enumerative study of planar maps (answering questions of the form "how many simple triangulations on n vertices are there?") began with the work of Tutte in the 1960s [83] who enumerated various classes of finite planar maps, in particular triangulations. Cori and Vauquelin [18], Schaeffer [72] and Chassaing and Schaeffer [16] have found beautiful bijections between planar maps and labeled trees and initiated this fascinating topic in enumerative combinatorics. The bijections themselves are model dependent and extremely useful since many combinatorial and metric aspects of random planar maps can be inferred from them. This approach has spurred a new line of research: limits of large random planar maps.

Two natural notions of such limits come to mind: scaling limits and local limits. In the first notion, one takes a random planar map M_n on n vertices, scales the distances appropriately (in most models the correct scaling turns out to be $n^{-1/4}$), and aims to show that this random metric space converges in distribution in the Gromov-Hausdorff sense. The existence of such limits was suggested by Chassaing and Schaeffer [16], Le Gall [55], and Marckert and Mokkadem [62], who coined the term *the Brownian map* for such a limit. The recent landmark work of Le Gall [56] and Miermont [65] establishes the convergence of random p-angulations for $p = 3$ and all even p to the Brownian map.

The study of local limits of random planar maps, initiated by Benjamini and Schramm [11], while bearing many similarities, is independent of the study of scaling limits. The *local* limit of a random planar map M_n on n vertices is an infinite random rooted graph (U, ρ) with the property that neighborhoods of M_n around a random vertex converge in distribution to neighborhoods of U around ρ. The infinite random graph (U, ρ) captures the local behavior of M_n around typical vertices. We develop this notion precisely in Chap. 5.

In their pioneering work, Angel and Schramm [5] showed that the local limit of a uniformly chosen random triangulation on n vertices exists and that it is a one-ended infinite planar triangulation. They termed the limit as the *uniform infinite planar triangulation* (UIPT). The uniform infinite planar quadrangulation (UIPQ), that is, the local limit of a uniformly chosen random quadrangulation (i.e., each face has 4 edges) on n vertices, was later constructed by Krikun [52].

The questions in this line of research concern the almost sure properties of this limiting geometry. It is a highly fractal geometry that is drastically different from $\mathbb{Z}^2$. Angel [4] proved that the volume of a graph-distance ball of radius r in the UIPT is almost surely of order $r^{4+o(1)}$ and that the boundary component separating this ball from infinity has volume $r^{2+o(1)}$ almost surely. For the UIPQ this is proved in [16].

Due to the various combinatorial techniques of generating random planar maps, many of the metric properties of the UIPT/UIPQ are firmly understood. Surface properties of these maps are somewhat harder to understand using enumerative methods. Recall that a non-compact simply connected Riemannian surface is either conformally equivalent to the disc or the whole plane and that this is determined according to whether Brownian motion on the surface is transient or recurrent. Hence, the behavior of the simple random walk on the UIPT/UIPQ is considered here as a "surface property" (see also [30]).

As mentioned earlier, one of the main objectives of these notes is to answer the question of the almost sure recurrence/transience of the simple random walk on the UIPT/UIPQ. We provide a general statement, Theorem 6.1 of these notes, to which a corollary is

Theorem 1.11 ([31]) *The UIPT and UIPQ are almost surely recurrent.*

The proof heavily relies on the circle packing theorem and can be viewed as an extension of the remarkable theorem of Benjamini and Schramm [11] stating that the local limit of finite planar maps with finite maximum degree is almost surely recurrent. The maximum degree of the UIPT is unbounded and so one cannot apply [11]. A combination of the techniques presented in Chaps. 4–6 is required to overcome this difficulty.

Recently, there have been terrific new developments studying further surface properties of the UIPT/UIPQ. Lee [59] has given an exciting new proof of Theorem 1.11 based on a spectral analysis and an embedding theorem for planar maps due to [48]. His proof also yields that the spectral dimension of the UIPT/UIPQ is at most 2 and applies to local limits of sphere-packable graphs in higher dimensions as well. Gwynne and Miller [33] provided the converse bound showing that the spectral dimension of the UIPT equals 2 and calculated other exponents governing the behavior of the random walk. Their results are based on the deep work of Gwynne et al. [35] (see also Chap. 8, item 9).