Pricing a Digital Services Marketplace Under Asymmetric Information

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Abstract. This paper addresses the pricing problem of a digital services marketplace under asymmetric information. An example is an online learning platform such as Coursera that provides courses from service providers (in this case, universities) to learners. We focus on the matching of digital services to the consumers of these services using partially-observable consumer and service attributes. We develop the optimal pricing policies of the marketplace and show that when the distributions of unobservable valuations are exponential, the marketplace sets a single matching fee (avoiding price-discrimination across providers) which is levied on the less price-sensitive side of the marketplace.

Keywords: Online marketplaces · Digital services · Pricing · Search

1 Introduction

An increasing share of the economy is managed through platforms that leverage technology to match consumers looking for digital services with service providers. For example, an online learning marketplace such as Coursera matches learners with courses that have been developed by service providers. Such online marketplaces are enabled by the Internet, which facilitates the aggregation of information and the efficient matching of consumers and service providers.

In this paper we focus on the pricing of the matching function of a marketplace for digital services. As matching generates value to the participants, how should the marketplace be compensated for creating this value? This question is particularly timely as digital services platforms are transitioning from a customer acquisition phase, where they are free or semi-free, to a monetization phase, where they need to make money. Online learning marketplaces such as Coursera, for example, used to be entirely free. Coursera, however, is funded by venture capitalists with a profit motive, and is transitioning to a pricing regime—first for courses with certificates, then for specializations, and it will probably implement a more robust pricing scheme in the future (tellingly, Coursera's free options are increasingly hidden during the registration process). Assuming that Coursera wants to maximize profits just like any other business, how should it

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price its services? Further, to what extent should a digital matching platform engage in price discrimination to maximize its profit?

Digital matching platforms such as Coursera maintain rich data which it can exploit when it prices its services. For example, if a student is looking for a job in the data science field, she may be willing to pay more for an advanced data science course than for a philosophy course. Should Coursera charge that student more for matching her to a data science course? When thinking about this issue, it is important to realize that some information (e.g., a customer's profile and past transaction information) is available to the digital matching platform, whereas other information remains private and is not shared by the platform (e.g., the student may be more interested in a given service provider or subject area for sentimental reasons). More formally, the value of a prospective match is driven by attributes of both consumers and service providers. Some of these attributes are private and cannot be exploited by the marketplace. Other attributes are observable and may be exploited by marketplace pricing to extract more of the value created by a match. How should the marketplace price each potential match given the observable attributes of both counterparties and the probability distributions of their unobservable attributes? And, how should it allocate its fees between consumers and service providers? These are some key research questions addressed in this paper.

We focus on digital services marketplaces with virtually unlimited supply. Our model fits such marketplaces particularly well as it addresses services that can be replicated at virtually zero cost. Our model incorporates agents' heterogeneity and asymmetric information and derive the platform's optimal pricing policies.

Pricing problems of this type are complex and are often intractable. Nevertheless, we are able to derive the optimal prices for general random (unobservable) valuations. We then specialize our results to the case where the random valuations are exponential. We find that in this case, it is optimal to charge the *same* fee for matches with *different* observable provider attributes, and this fee should be levied in its entirety on the less elastic (more price-sensitive) side of the marketplace, up to a threshold point.

In what follows, we present our model in Sect. 2, solve it in Sect. 3, and offer our concluding remarks in Sect. 4. We illustrate the application of our approach and consider structural results for a few special cases in Subsects. 3.1 and 3.2.

2 Model

A consumer a arrives at the digital services marketplace, seeking a service provider (for example, a learner may seek an appropriate course on Coursera). The consumer submits to the marketplace a request identifying the service he or she is looking for and provides other relevant information. In particular, the marketplace may manage a profile with relevant consumer information. The consumer's request remains live in the system for an exponentially-distributed period of time with mean $1/\tau_A$ and then expires.

In our model, responding service providers, whose observable characteristics can be summarized by an index b, arrive following a Poisson stream with rate $1/\tau_B$. Each arriving provider examines the consumer's known attributes (from the service request and the consumer's profile) and decides whether to respond to the request. Once the provider responds, both parties reveal their unobservable attributes to determine the actual value of a potential match. A match occurs if sufficient value is generated for both the provider and the consumer, after subtracting the marketplace fees. If this is not the case, the provider moves on and the process is repeated with subsequent suppliers until either a mutually-acceptable match is found or the consumer's request expires. We model the problem for a consumer at a time since digital services can be replicated without a supply limit. In the special case of Coursera, the provider's attributes are fully known to the marketplace and judgment is exercised only by the consumer based on her private information.

We denote a match between consumer a and a provider of type b (hereafter, "provider b") by $i = (a, b)$. Upon a successful match i, the marketplace charges consumer a a matching fee f_i^a and provider b a matching fee f_i^b. Each consumer valuation, u_i^a, is the sum of an observed valuation v_i^a and a random, unobservable valuation ϵ_i^a. Similarly, each provider valuation u_i^b is the sum of an observed valuation v_i^b and the random unobserved valuation ϵ_i^b. Consumers (providers, respectively) have an outside option value (opportunity cost) of v_0^A (v_0^B). It follows that the match will be successful if for both the consumer and the provider, the value of the match net of marketplace matching fees exceeds the value of the outside option: $u_i^a - f_i^a > v_0^A$ and $u_i^b - f_i^b > v_0^B$. The objective of the marketplace is to find the pricing policy $f_i = \left(f_i^a, f_i^b\right)$ that maximizes its expected profit.

The valuations u_i^a and u_i^b have both observable (v) and unobservable (ϵ) components: $u_i^a = v_i^a + \epsilon_i^a$, $u_i^b = v_i^b + \epsilon_i^b$. We assume that the unobserved components (ϵ_i^a,ϵ_i^b) may be correlated with distributions and arrival rates that may differ across pair types i. In what follows, we'll consider both the more general formulation and the special case of exponential random valuations. We also assume $v_i^a \leq v_0^A,\ v_i^b \leq v_0^B$ so that probabilities are well-defined for $f_i^a, f_i^b \geq 0$.

2.1 Related Literature

Research in the area of marketplace platforms is extremely broad and space limitations prevent us from reviewing this entire literature. In the economics literature, several papers study the pricing of two-sided platforms that are subject to network effects. This literature aims to find static equilibrium structures, typically assuming a linear relationship between value and number of agents (cf. [1–4]). Within that literature, some researchers study price discrimination (e.g., [5]) but they do not base prices on agents' observable attributes; rather, they offer price schedules that induce agents to reveal their private information by self-selecting into designated tiers. [6] study the effects of market thickness on the efficiency of matching in a holiday rental platform, finding that contrary to the dictum of most network effects models, increased thickness was associated with a significant decline in the matching probability due to a deadline effect. Our model does not assume the existence of network effects, and it does consider the effects of time constraints on the consumer's search. It starts from the micro level to find the optimal personalized prices based on agents' observable attributes.

Our model is also related to the task-assignment problem in the online mechanism design literature [7–10]. This literature proposes algorithms for dynamically posting prices for suppliers who bid for specific tasks. However, these papers do not derive closed-form pricing solutions because of their different objectives. In addition to their different model structures, the algorithms analyzed in these papers do not allow for price-discrimination based on observable attributes. Our approach is closer to the classic dynamic stochastic settings of [11–13]. [14] study a more complex problem where demand types are dynamically matched to supply types to maximize total reward. Like the other papers in this stream, they do not consider the informational and pricing issues studied here. Overall, our model combines timing dynamics, information asymmetry and participants' heterogeneity to find tractable results and their implications.

3 Solving the Pricing Problem

Importantly, for any pair i, the fee vector f_i affects the matching probabilities of all pairs. In particular, the higher the fees for one particular match, the more likely are other matches to be successful. This interaction among matching probabilities and fees creates a $2 \times N$-dimensional problem, where N is the number of different match types and 2 is the number of sides to be priced. However, we show below how to reduce the problem to N two-dimensional problems, which allows us to compute the solution in closed-form.

The marketplace objective function is given by

$$\sum_{i=1}^{N} E\left[v_i \middle| \text{pair } i \text{ matched}\right] Pr\left(\text{pair } i \text{ matched}\right) \equiv \sum_{i=1}^{N} v_i \phi_i,$$

where $v_i = f_i^a + f_i^b$ is the marketplace profit from a successfully matched pair i. If arrivals follow a Poisson process, then the probability ϕ_i that the pair $i = (a, b)$ is successfully matched is given by the following.

Proposition 1. *Let u_i^a be the value that the consumer a derives from being matched with provider b, and i simultaneously indexes provider b and the pair $i = (a, b)$. Similarly, u_i^b is the value that provider b derives upon being matched with the consumer a, and λ_i is the arrival rate of type-b providers (corresponding to $i = (a, b)$). Finally, let λ_{N+1} be the arrival rate of the outside option: if the outside option "arrives" before the consumer request is matched, the consumer gives up the search and exits the marketplace, leaving no revenue to the marketplace. Then the probability that the match will be successful and of type b is given by*

$$\phi_i\left(f_i^a, f_i^b\right) = \begin{cases} \frac{\lambda_i Pr\left(u_i^a \geq f_i^a, u_i^b \geq f_i^b\right)}{\lambda_{N+1} + \sum \lambda_j Pr\left(u_j^a \geq f_j^a, u_j^b \geq f_j^b\right)} & , \textit{if } i = 1, \ldots, N \\ \frac{\lambda_{N+1}}{\lambda_{N+1} + \sum \lambda_j Pr\left(u_j^a \geq f_j^a, u_j^b \geq f_j^b\right)} & , \textit{if } i = N+1. \end{cases} \tag{1}$$

For simplicity of exposition, we assume that prices are non-negative, although this assumption is not needed in the derivation of the optimal prices. The results below provide a way to identify optimal prices in closed form in a general marketplace setting.

Theorem 1. *Consider the optimization problem*

$$\max_{f_1,\ldots,f_N} \sum_{i=1}^{N} v_i(f_i)\phi_i(f_1,\ldots,f_N), \qquad s.t.\ f_i \in \mathcal{F}_i. \tag{2}$$

where $v_i \equiv f_i^a + f_i^b$, $\phi_i(f_1,\ldots,f_N)$ *are given by (1), and* $P_i \equiv Pr\left(u_i^a \geq f_i^a, u_i^b \geq f_i^b\right)$ *are twice continuously-differentiable for* $i = 1, 2, \ldots, N$.

Also let $T_i^a(f_i^a) = \frac{\partial\left((f_i^a + f_i^b)P_i\right)/\partial f_i^a}{\partial P_i/\partial f_i^a}$ *and* $T_i^b(f_i^b) = \frac{\partial\left((f_i^a + f_i^b)P_i\right)/\partial f_i^b}{\partial P_i/\partial f_i^b}$. *Then, if problem (2) has an optimal solution, its value is given by the largest root of the scalar equation*

$$V = \frac{\sum_{i=1}^{N}(f_i^a(V) + f_i^b(V))\lambda_i P_i(f_i^a(V), f_i^b(V))}{\lambda_{N+1} + \sum_{i=1}^{N} \lambda_i P_i(f_i^a(V), f_i^b(V))}$$

among all potential roots V *obtained by plugging in all possible combinations* $(f_1^a(V), f_1^b(V), \ldots, f_N^a(V), f_N^b(V)$, *where* $f_i^a(V)$ *is either on the boundary of* $\mathcal{F}_i^a$ *or it solves* $T_i^a(f_i^a, f_i^b) = V$ *and* $f_i^b(V)$ *is either on the boundary of* $\mathcal{F}_i^b$ *or it solves* $T_i^b(f_i^a, f_i^b) = V$. *The vector* $(f_1^a(V^*), f_1^b(V^*), \ldots, f_N^a(V^*), f_N^b(V^*))$ *that yields the highest* V^*, *is the optimal fee vector.*

Proposition 2. *Assume that the problem (2) has an optimal solution with value* V^*. *Then for* $i = 1, 2, \ldots, N$, *each component* $f_i^* = (f_i^a, f_i^b)^*$ *of the solution* $(f_1^*, \ldots, f_N^*)$ *to problem (2) can be represented as a solution to the i-th subproblem*

$$max_{f_i} P_i\left(f_i^a, f_i^b\right)\left(f_i^a + f_i^b - V^*\right) \qquad s.t.\ f_i \in \mathcal{F}_i. \tag{3}$$

That is, the optimal fee vector in each state maximizes the weighted deviation from the global optimal profit V^*.

3.1 Pricing with Exponentially-Distributed Private Valuations

We now solve for the case where the random valuations are exponentially distributed i.i.d. random variables and where the arrival rates λ_i are the same across provider types. We assume that the unobserved components, ϵ_i^a

and ϵ_i^b, come from i.i.d. exponential distributions: $\epsilon_i^a \sim Exp\left(1/v^A\right)$, $\epsilon_i^b \sim Exp\left(1/v^B\right)$. As a result, from the marketplace perspective, the matching compatibility indicators $I\,(a\ accepts\ b)$ and $I\,(b\ accepts\ a)$ become random variables. In particular, the probability of b being acceptable to a is $Pr\,(a\ accepts\ b) = Pr\left(u_i^a \geq f_i^a + v_0^A\right) = Pr\left(v_i^a + \epsilon_i^a > f_i^a + v_0^A\right) = exp\left(\frac{v_i^a - f_i^a - v_0^A}{v^A}\right)$. Similarly, the probability of b willing to serve a is $Pr\,(b\ accepts\ a) = Pr\left(u_i^b \geq f_i^b + v_0^B\right) = Pr\left(v_i^b + \epsilon_i^b \geq f_i^b + v_0^B\right) = exp\left(\frac{v_i^b - f_i^b - v_0^B}{v^B}\right)$.

To solve the pricing problem, we first derive the matching probabilities. Applying Proposition 1 to the exponential case, we get

Corollary 1. *Given the information available to all participants (and, in particular, the marketplace), the probability that the request results in a match* $i = (a, b)$*, is given by*

$$\phi_i = Pr\,(b\ serves\ a) = \phi \frac{\beta_i e^{-\alpha f_i}}{\sum_j \beta_j e^{-\alpha f_j}} = \frac{\beta_i e^{-\alpha f_i}}{\sum_j \beta_j e^{-\alpha f_j} + \frac{1}{k}}, \tag{4}$$

where $\alpha \equiv \left(\alpha^A, \alpha^B\right)$*,* $f_i \equiv \left(f_i^a, f_i^b\right)$*,* $\alpha f_i \equiv \alpha^A f_i^a + \alpha^B f_i^b$*.*

We interpret $\alpha^A \equiv \frac{1}{v^A}$ as the price sensitivity of demand and $\alpha^B \equiv \frac{1}{v^B}$ as the price sensitivity of supply.

In the exponential case, the profit that the marketplace generates from each request is given by

$$\sum_{i=1}^{N} \left(f_i^b + f_i^a\right) \phi_i = \sum_{i=1}^{N} \left(f_i^b + f_i^a\right) \frac{\beta_i e^{-\alpha^A f_i^a - \alpha^B f_i^b}}{\sum_j \beta_j e^{-\alpha^A f_j^a - \alpha^B f_j^b} + \frac{1}{k}}.$$

Thus, the reward in state i is given by

$$v_i\left(f_i^a, f_i^b\right) = \begin{cases} \left(f_i^b + f_i^a\right) & \text{if } i = 1, \ldots, N \\ 0 & \text{if } i = N+1 \end{cases},$$

and the matching demand function is given by

$$\begin{cases} P_i\left(f_i^a, f_i^b\right) = \beta_i e^{-\alpha^A f_i^a - \alpha^B f_i^b} & \text{if } i = 1, \ldots, N \\ P_{N+1} = \frac{1}{k} & \text{if } i = N+1 \end{cases}$$

The profit maximization problem of the marketplace is thus

$$\max_{\left(f_i^a, f_i^b\right)} \sum_{i=1}^{N} \left(v_i P_i / \sum_{j=1}^{N+1} P_j \right)$$
$$s.t.\ \left(f_i^a, f_i^b\right) \geq \left(v_i^a - v_0^A, v_i^b - v_0^B\right).$$

The solution is given by the following Proposition.

Proposition 3. *The pricing policy maximizing marketplace profit is as follows:*

(a) If $\alpha^A > \alpha^B$, then

$$\begin{cases} f_i^{b^*} &= v^B \left[1 + W_0 \left(e^{-\alpha^A v_0^A - \alpha^B v_0^B - 1} \frac{\tau_A}{\tau_B} \sum_j e^{\alpha^A v_j^a + \alpha^B v_j^b - \left(\alpha^A - \alpha^B\right) f_j^{a^*}}\right)\right] - f_i^{a^*} \\ f_i^{a*} &= v_i^a - v_0^A \end{cases}$$

(b) If $\alpha^B > \alpha^A$, the result is symmetric to (a), i.e.,

$$\begin{cases} f_i^{a^*} &= v^A \left[1 + W_0 \left(e^{-\alpha^A v_0^A - \alpha^B v_0^B - 1} \frac{\tau_A}{\tau_B} \sum_j e^{\alpha^A v_j^a + \alpha^B v_j^b - \left(\alpha^B - \alpha^A\right) f_j^{b^*}}\right)\right] - f_i^{b^*} \\ f_i^{b*} &= v_i^b - v_0^B \end{cases}$$

(c) If $\alpha^A = \alpha^B$,

$$\begin{cases} f_i^{b^*} + f_i^{a^*} = v^A \left(1 + W_0 \left(e^{-\alpha^A v_0^A - \alpha^B v_0^B - 1} \frac{\tau_A}{\tau_B} \sum_j e^{\alpha^A v_j^a + \alpha^B v_j^b}\right)\right) \\ \\ f_i^{b^*} \geq v_i^b - v_0^B, \; f_i^{a^*} \geq v_i^a - v_0^A \end{cases}$$

(d) the optimal total price $f_i^{a^*} + f_i^{b^*}$ is the same for each matched pair.
$W_0(\cdot)$ is the Lambert function [15] which is the solution to the equation

$$z = W\left(ze^z\right), \; z \geq -1.$$

Part (d) of Proposition 3 is surprising: even though the marketplace is able to price-discriminate among providers (and matched pairs) based on the particular information it observes about them, it charges the same price to all. This does not mean, of course, that it does not take advantage of that information at all. In fact, an increase in valuations within a particular pair will increase the fees charged to all pairs. Further, the optimal price depends on the observable consumer attributes. What the marketplace does not do is differentiate among providers (and corresponding pairs) based on the differences in observed valuations. This result follows from the memoryless property of the exponential distribution: the expected ex-ante value of a match is given by $E\left[V + \epsilon \middle| V + \epsilon > v_0 + f\right] = E\epsilon + v_0 + f$, which is *independent* of the observable component V.

Intuitively, the optimal price structure is driven by the probability distribution of the random valuation components. The observable component of the valuations affects the matching probabilities: the higher the valuation, the higher the probability of a match. Our finding means that the marketplace is better off engaging in quantity (or probability) differentiation (higher expected probability of a match for higher valuations) than in price discrimination.

Another important result is that the fee split between the provider and the consumer depends only on their respective price sensitivities and is always obtained at an extreme point: the side with the greater price sensitivity is charged the lowest price needed to attract any participants to the marketplace. In particular, when prices are non-negative, the less price-sensitive side of the market pays the entire marketplace fee while the other side pays nothing.

3.2 Generalizations

To what extent do our structural results for the exponential case generalize to other distributions? The following Corollary shows that prices remain the same across provider matches as long as the matching demand functions $P_i\left(f_i^a, f_i^b\right)$ are proportional across the different providers.

Corollary 2. *If the matching demand functions $P_i\left(f_i^a, f_i^b\right)$ are proportional to each other for a subset of types I, i.e.,*

$$P_i\left(f_i^a, f_i^b\right) = C_i P\left(f_i^a, f_i^b\right),\ i \in I,$$

then the optimal fees charged by the marketplace to different pairs within the set I are the same.

This result directly follows from Proposition 2, as the optimization subproblems solved for each match $i \in I$ are the same (if there are multiple optima, one of them will have the same fees).

Next consider the allocation of the total marketplace fee between the consumer and the provider. Using our aggregate matching demand formulation, the price sensitivity of the consumer (provider, respectively) to its marketplace fee is $\frac{\partial P_i}{\partial f_i^a}$ ($\frac{\partial P_i}{\partial f_i^b}$, respectively). Transforming our variables to $\left(f_i^a, f_i^\Sigma\right)$, where $f_i^\Sigma \equiv f_i^a + f_i^b$, in each subproblem i, the partial derivative of the objective function with respect to f_i^a is $\left(f_i^\Sigma - V^*\right)\left(\frac{\partial P_i}{\partial f_i^a} - \frac{\partial P_i}{\partial f_i^b}\right)$, and at the optimum, $V^* < \left(f_i^\Sigma\right)^*$. Thus, the fee split is determined by the sign of $\left(\frac{\partial P_i}{\partial f_i^a} - \frac{\partial P_i}{\partial f_i^b}\right)$. In particular, if $\left(-\frac{\partial P_i}{\partial f_i^a} > -\frac{\partial P_i}{\partial f_i^b}\right)$ for all $\left(f_i^a, f_i^b\right) \in \mathcal{F}_i$, then the entire fee should be paid by the supplier and the consumer pays nothing, and if $\left(-\frac{\partial P_i}{\partial f_i^a} < -\frac{\partial P_i}{\partial f_i^b}\right)$ for all $\left(f_i^a, f_i^b\right) \in \mathcal{F}_i$, then the entire fee is paid by the consumer and the provider pays nothing. This directly generalizes the results we obtained in the exponential case. A similar analysis may be performed using the underlying preferences of marketplace participants. Here,

$$P_i\left(f_i^a, f_i^b\right) \equiv Pr\left(v_i^a + \epsilon_i^a \geq v_0^A + f_i^a, v_i^b + \epsilon_i^b \geq v_0^B + f_i^b\right)$$

For subproblem i, in transformed coordinates $\left(f_i^a, f_i^\Sigma\right)$, the partial derivative of the objective function with respect to f_i^a is $\left(f_i^\Sigma - V^*\right)\left(\frac{\partial Pi}{\partial f_i^a} - \frac{\partial P_i}{\partial f_i^b}\right)$. Thus, similar to the above analysis, the fee split is determined by the sign of

$\left(\frac{\partial P_i}{\partial f_i^a} - \frac{\partial P_i}{\partial f_i^b}\right)$: if $\left(-\frac{\partial P_i}{\partial f_i^a} > -\frac{\partial P_i}{\partial f_i^b}\right)$ for all $\left(f_i^a, f_i^b\right) \in \mathcal{F}_i$, then the entire marketplace fee is paid by the supplier, and if $\left(-\frac{\partial P_i}{\partial f_i^a} < -\frac{\partial P_i}{\partial f_i^b}\right)$ for all $\left(f_i^a, f_i^b\right) \in \mathcal{F}_i$, the entire marketplace fee is paid by the consumer[1].

Since we allowed for different matching demand functions for different matches, it is of course possible that unlike the more restrictive exponential case, one provider will be uniformly *more* price sensitive than the consumer whereas another will be uniformly *less* price sensitive than the consumer. In this case, the former provider will pay zero whereas the latter will pay the entire marketplace fee.

Our formulation also allows for more general fee splits. For example, if the matching demand functions are log-linear of the form $P_i\left(f_i^a, f_i^b\right) = C_i\left(A - f_i^a\right)^\alpha \left(B - f_i^b\right)^\beta$ for $f_i^a \in [0, A], f_i^b \in [0, B]$, then $\frac{\partial P_i}{\partial f_i^a} = -\alpha C_i\left(A - f_i^a\right)^{\alpha-1}\left(B - f_i^b\right)^\beta$ and $\frac{\partial P_i}{\partial f_i^b} = -\beta C_i\left(A - f_i^a\right)^\alpha \left(B - f_i^b\right)^{\beta-1}$, implying that $\frac{A-(f_i^a)^*}{B-\left(f_i^b\right)^*} = \frac{\alpha}{\beta}$. Thus, the optimal fees are the same for all i (Proposition 2), and their ratio is inversely related to their elasticity coefficients, consistent with our intuition.

4 Concluding Remarks

The Internet has spawned new forms of economic activity and gave rise to the development and growth of online services marketplaces which in turn created new research challenges for both academics and practitioners. This paper derives optimal pricing policies for a matching marketplace platform for digital services. We show that with exponentially-distributed random participants' valuations, it is optimal to charge a constant total fee across provider matches, and this fee should be levied on the less elastic side of the market up to a threshold. For a marketplace such as Coursera, this means that the profit-maximizing matching fee would be constant across providers. Further, if learners are more price-sensitive than providers, Coursera would charge its entire matching fee to its content providers.

Our results shed light on how marketplaces for digital services may be compensated for their matching function, and it will be useful to consider them in conjunction with more elaborate specifications of particular vertical marketplaces.

[1] If the fees can be negative, the more price-sensitive side of the market will pay the lowest possible fee, and the less price-sensitive side will pay the highest possible fee.

5 Proofs

Proof of Proposition 1

The successful requests of type $i = (a, b)$ arrive as a marked Poisson process with the arrival rate $\lambda_i Pr(u_i^a \geq f_i^a,\ u_i^b \geq f_i^b)$, where $u_i^a \geq f_i^a,\ u_i^b \geq f_i^b$ is the condition that "marks" the match between the consumer and supplier successful. The outside option and the successful requests of other types arrive with the Poisson rates of λ_{N+1} and $\sum_{-i} \lambda_j Pr(u_j^a \geq f_j^a,\ u_j^b \geq f_j^b)$. As long as the first arrival occurs from the first stream, i.e., the one with the rate of $\lambda_i Pr(u_i^a \geq f_i^a,\ u_i^b \geq f_i^b)$, the request will be served by the provider of type i.

The probability of such an event is given by $Pr(t_i \leq min(t_{-i}, t_{N+1}))$, where t_i is the arrival time of the successful match of type i, t_{-i} – the arrival time of the successful match of any type other than i and t_{N+1} is the arrival time of the outside option. Since all the arrival streams are Poisson, and, hence, the respective arrival times are exponentially distributed, including $min(t_{-i}, t_{N+1})$, therefore

$$\phi_i\left(f_i^a, f_i^b\right) = \begin{cases} \frac{\lambda_i Pr\left(u_i^a \geq f_i^a, u_i^b \geq f_i^b\right)}{\lambda_{N+1} + \sum \lambda_j Pr\left(u_i^a \geq f_i^a, u_i^b \geq f_i^b\right)} & \text{, if } i = 1, \ldots, N \\ \frac{\lambda_{N+1}}{\lambda_{N+1} + \sum \lambda_j Pr\left(u_i^a \geq f_i^a, u_i^b \geq f_i^b\right)} & \text{, if } i = N+1 \end{cases}$$

Proof of Theorem 1:

The first-order condition w.r.t. f_i^a is given by

$$\frac{\frac{\partial}{\partial f_i^a}\left((f_i^a + f_i^b)\lambda_i P_i\right)\left(\lambda_{N+1} + \sum_{i=1}^N \lambda_i P_i\right) - \frac{\partial}{\partial f_i^a}\left(\lambda_i P_i\right)\sum_{i=1}^N\left((f_i^a + f_i^b)\lambda_i P_i\right)}{\left(\lambda_{N+1} + \sum_{i=1}^N \lambda_i P_i\right)^2} = 0,$$

which implies

$$\frac{\partial}{\partial f_i^a}\left((f_i^a + f_i^b)\lambda_i P_i\right) = \frac{\sum_{i=1}^N (\lambda_i P_i)(f_i^a + f_i^b)}{\left(\lambda_{N+1} + \sum_{i=1}^N \lambda_i P_i\right)} \frac{\partial}{\partial f_i^a}\left(\lambda_i P_i\right) = V^* \frac{\partial}{\partial f_i^a}\left(\lambda_i P_i\right),$$

where $V^* = V\left(f_1, \ldots, f_N\right)$ is the optimal profit.

Repeating the above with respect to f_i^b yields

$$\begin{cases} \frac{\partial}{\partial f_i^a}\left((f_i^a + f_i^b)P_i\right) = V^* \frac{\partial}{\partial f_i^a} P_i, \\ \frac{\partial}{\partial f_i^b}\left((f_i^a + f_i^b)P_i\right) = V^* \frac{\partial}{\partial f_i^b} P_i, \end{cases} \tag{5}$$

Let $f^* = (f_1^*, \ldots, f_N^*)$ denote the optimal fee vector, then $\left(f_i^a\left(V^*\right)\right)^*$, $\left(f_i^b\left(V^*\right)\right)^*$ either solves the respective equation in (5) or is on the boundary of $\mathcal{F}_i$. As a result, there is a set of candidates $\left\{f^j\left(V\right)\right\}$ for an optimal fee vector f^*. For each j we plug $f^j\left(V\right)$ in the expression for the objective $V = \frac{\sum_{i=1}^N\left(f_i^a(V) + f_i^b(V)\right)\lambda_i P_i\left(f_i^j(V)\right)}{\lambda_{N+1} + \sum_{i=1}^N \lambda_i P_i\left(f_i^j(V)\right)}$ and solve for V. Then we arrive at the set of values $\left\{V^j\right\}$ corresponding to optimal action candidates $\left\{f^j\left(V\right)\right\}$. Obviously, the largest value $V^* \equiv V^{j^*} = \max_j V^j$ is the optimal profit value, and, hence, the corresponding vector $f^{j^*}\left(V^{j^*}\right)$ is the optimal fee vector.

Proof of Proposition 2:
If f_i^* is the solution to (3), then for all f_i

$$\left(v_i^* - V^*\right) h_i^* \geq \left(v_i - V^*\right) h_i, \tag{6}$$

where for $i = 1 \ldots N$, $v_i^* = \left(f_i^a + f_i^b\right)^*,\ h_i^* = \lambda_i P_i\left(f_i^*\right),\ v_i = f_i^a + f_i^b,\ h_i = \lambda_i P_i\left(f_i\right)$ and $v_{N+1} = 0,\ h_{N+1} = \lambda_{N+1}$.

Plugging in

$$V^* = \frac{v_i^* h_i^* + (VH)_{-i}^*}{h_i^* + H_{-i}^*},$$

where $(VH)_{-i} = \sum_{-i} v_j h_j,\ H_{-i} = \sum_{-i} h_j$, into (6) we get

$$\left(v_i^* - \frac{v_i^* h_i^* + (VH)_{-i}^*}{h_i^* + H_{-i}^*}\right) h_i^* \geq \left(v_i - \frac{v_i^* h_i^* + (VH)_{-i}^*}{h_i^* + H_{-i}^*}\right) h_i$$

Rearranging this expression yields

$$\left(\frac{v_i^* h_i^* + (VH)_{-i}^* h_i^*}{h_i^* + H_{-i}^*}\right) \geq \frac{v_i h_i^* h_i + v_i H_{-i}^* h_i - v_i^* h_i^* h_i + (VH)_{-i}^* h_i}{h_i^* + H_{-i}^*}$$

$$\left(v_i^* - v_i\right) h_i^* h_i + \left(v_i^* H_{-i}^* - (VH)_{-i}^*\right) h_i^* - \left(v_i H_{-i}^* - (VH)_{-i}^*\right) h_i \geq 0$$

$$\left(v_i^* h_i^* + (VH)_{-i}^*\right)\left(h_i + H_{-i}^*\right) \geq \left(v_i h_i + (VH)_{-i}^*\right)\left(h_i^* + H_{-i}^*\right)$$

$$\frac{\left(v_i^* h_i^* + (VH)_{-i}^*\right)}{h_i^* + H_{-i}^*} \geq \frac{\left(v_i h_i + (VH)_{-i}^*\right)}{h_i + H_{-i}^*}.$$

Thus, f_i^* is the optimal solution for (2) as well.

Proof of Proposition 3
We solve the problem in four steps.

Step 1: show that the optimal value V^* exists and is finite. The existence and finiteness of V^* follow from three facts: *(a)* the value function is linear in the marketplace fees, whereas *(b)* the matching probabilities decline exponentially with the marketplace fees, and *(c)* the marketplace can achieve zero revenue by rejecting all suppliers. It follows that there is an optimal solution with a finite, positive V^*.

The optimal prices are in a bounded set for the following reason. since V^* is finite, the gradient of the objective function with respect to at the optimal fee vector f^* is given by

$$\frac{\frac{\partial}{\partial f_i^a} h_i}{\sum_j h_j}\left[T\left(\begin{bmatrix} f_i^a \\ f_i^b \end{bmatrix}\right) - V^*\right] = \frac{\frac{\partial}{\partial f_i^a} h_i}{\sum_j h_j}\left[f_i^a + f_i^b - \begin{bmatrix} 1/\alpha^A \\ 1/\alpha^B \end{bmatrix} - V^*\right],$$

where $h_i = \lambda_i P_i$ for $i = 1, \ldots, N$.

Since $\frac{\frac{\partial}{\partial f_i^a} h_i}{\sum_j h_j} < 0$ uniformly, $\frac{\partial V}{\partial f_i^a} < 0$ for large enough f_i^a. Similarly, $\frac{\partial V}{\partial f_i^b} < 0$ for large enough f_i^b. Thus, the optimal prices are contained in a bounded set.

Step 2: decompose the global objective. By Proposition 2, we can decompose the problem into N optimization problems, one for each match i, $i = 1, 2, \ldots, N$:

$$\begin{aligned} &max_{f_i} P_i\left(f_i\right)\left(v_i\left(f_i\right) - V^*\right) \\ s.t.\ &\left(f_i^a, f_i^b\right) \geq \left(v_i^a - v_0^A, v_i^b - v_0^B\right), \end{aligned}$$

which yields the following first-order conditions:

$$f_i^a + f_i^b - \begin{bmatrix} 1/\alpha^A \\ 1/\alpha^B \end{bmatrix} = \begin{bmatrix} V^* \\ V^* \end{bmatrix}.$$

Step 3: filter the set of solution candidates. There are two candidate points for the consumer-side fee f_i^a – one interior solution $V^* + 1/\alpha^A - f_i^b$ and one corner solution $\left(v_i^a - v_0^A\right)$. Similarly, there are 2 corresponding candidates for the supply-side fee f_i^b: $V^* + 1/\alpha^B - f_i^a$ and $\left(v_i^b - v_0^B\right)$. Thus, the optimal candidate fees $\left(f_i^a, f_i^b\right)$ for state i constitute all four pairwise combinations of these individual candidates. Then, we have the following candidate pair types: $\left(v_i^a - v_0^A, v_i^b - v_0^B\right)$, $\left(V^* + 1/\alpha^A - f_i^b,\ V^* + 1/\alpha^B - f_i^a\right)$, $\left(v_i^a - v_0^A,\ V^* + 1/\alpha^B - \left(v_i^a - v_0^A\right)\right)$, and $\left(V^* + 1/\alpha^A - \left(v_i^b - v_0^B\right),\ v_i^b - v_0^B\right)$.

Rather than enumerate the results, we can eliminate some of the candidates upfront. For instance, both partial derivatives at $\left(v_i^a - v_0^A, v_i^b - v_0^B\right)$ are positive, hence this candidate be eliminated. The second candidate

$$\begin{bmatrix} f_i^a \\ f_i^b \end{bmatrix} = \begin{bmatrix} V^* + 1/\alpha^A - f_i^b \\ V^* + 1/\alpha^B - f_i^a \end{bmatrix}$$

can be eliminated if $\alpha^A \neq \alpha^B$. In fact, we can rewrite the expression as

$$\begin{bmatrix} f_i^a + f_i^b \\ f_i^a + f_i^b \end{bmatrix} = \begin{bmatrix} V^* + 1/\alpha^A \\ V^* + 1/\alpha^B \end{bmatrix},$$

which becomes inconsistent if $\alpha^A \neq \alpha^B$. Thus, if the price sensitivities are different, then we necessarily have a corner solution. Further, a variation in the fees $\left(\Delta f_i^a, \Delta f_i^b\right) = (\epsilon,\ -\epsilon)$, where $\epsilon > 0$, is profitable if $\alpha^B > \alpha^A$, that is, such variation increases the objective function value. Hence, if $\alpha^B > \alpha^A$, then the corner solution $\left(v_i^a - v_0^A,\ V^* + 1/\alpha^B - \left(v_i^a - v_0^A\right)\right)$ cannot be optimal since it permits the above variation. As a result, when $\alpha^A \neq \alpha^B$ we are left with only one solution candidate, namely,

$$\left(f_i^a, f_i^b\right) = \begin{cases} \left(v_i^a - v_0^A,\ V^* + 1/\alpha^B - \left(v_i^a - v_0^A\right)\right) & \alpha^A > \alpha^B \\ \left(V^* + 1/\alpha^A - \left(v_i^b - v_0^B\right),\ v_i^b - v_0^B\right) & \alpha^A < \alpha^B \end{cases}. \quad (7)$$

Finally, if $\alpha^A = \alpha^B$, the total price must be equal to $V^* + 1/\alpha^A$, and it may be allocated arbitrarily between the consumer and the service provider.

Step 4: plug the solution candidates into the equation for V^*. Plugging $\left(f_i^a, f_i^b\right)$ into the equation $V^* = \sum_{i=1}^{N+1} v_i \left(f_i^a, f_i^b\right) \frac{P_i\left(f_i^a, f_i^b\right)}{\frac{1}{k} + \sum_j P_i\left(f_i^a, f_i^b\right)}$, then solving for V^* and plugging it back into the expression (7) completes the solution. The exact formula is given in the statement of the proposition.

Frequently, the elimination of certain candidate solutions may come from additional constraints imposed by the nature or operating rules of the marketplace. For instance, for some platforms, charging the consumer (e.g., Yelp) or consumer and supplier (e.g., Stackoverflow) may be inappropriate, while the feasible price region for another side (e.g., advertisers) is unconstrained. In that case, constraints of the form of $f_i^a = 0$ or $f_i^b = 0$ may automatically eliminate a number of solution candidates.

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