1 Introduction

It is a mistake to think about a mathematical model as if it were the reality. In the physical sciences, where the model often fits reality very well, this may be a convenient way of thinking that causes little harm. But in the social sciences, models are often little better than caricatures.

Ian Stewart
In Pursuit of the Unknown (page 127)

1.1 Stochastics in Finance Theory

Anyone who is occupied with modern financing theory will soon come across terms such as Brownian motion,[1] random processes, measure, and Lebesgue integral.[2] Based on the many years of experience we have gained in university teaching, we claim that some readers do not have sufficient knowledge in this field, unless they have studied mathematics. Therefore, they may not know what is meant by probability measures, Brownian motions, and similar terms.

Various Random Processes Time series of share prices generally look very different from price developments of bonds which can be explained (among other reasons) by the fact that bonds—in contrast to equities—have a limited term. As the remaining time to maturity becomes shorter, bond prices always approach their nominal value,[3] while with stocks it is extremely rarely observed that their prices to stabilize, as shown in Fig. 1.1. The development of the base interest rate of the European Central Bank in the period between 2009 and 2015 gives a different

[1]Robert Brown (1773–1858, British botanist).

[2]Henri Léon Lebesgue (1875–1941, French mathematician).

[3]We talk about the “Pull-to-par” phenomenon.

A. Löffler, L. Kruschwitz, *The Brownian Motion*, Springer Texts in Business and Economics, https://doi.org/10.1007/978-3-030-20103-6_1

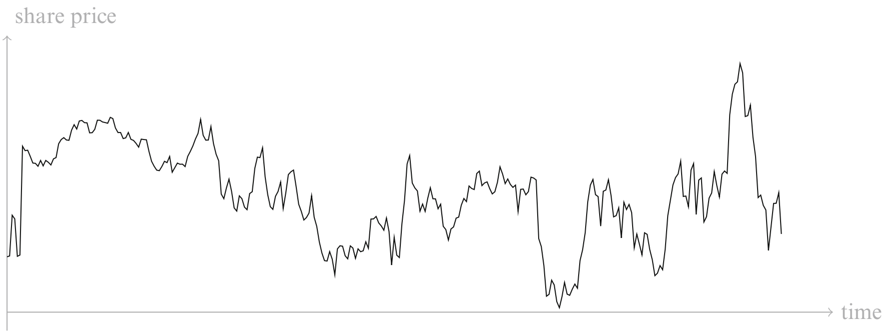

Fig. 1.1 Conceivable share price development

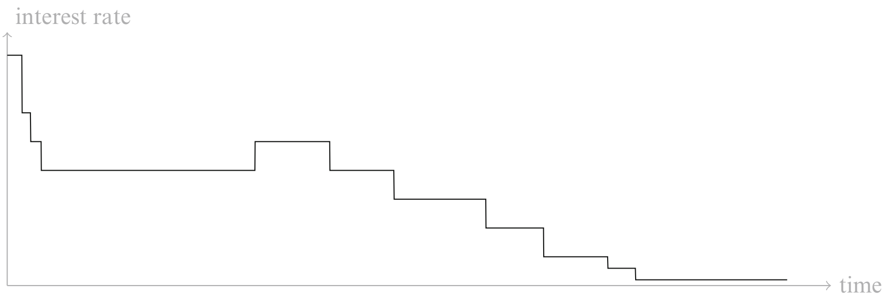

Fig. 1.2 Development of the ECB's base interest rate from 2009 to 2015. Source: www.finanzen.net/leitzins/@historisch

picture in every respect (see Fig. 1.2). In both cases, however, we are dealing with processes that would undoubtedly be described as random. While the first process seems to be in constant motion, the second process remains stable over longer periods of time and jumps up or down at irregular intervals the extent of which seems unpredictable.

If one now wants to do justice to the developments shown in these illustrations with the help of mathematical random processes, one has to resort to different models. The theory of random processes provides a comprehensive set of instruments. Mathematicians speak of stochastic processes and distinguish between Markov, Gauss, and Feller processes, each with several variants. Brownian motions, belonging to the class of Gaussian processes, are particularly prominent in the literature on finance theory.[4]

[4]Carl Friedrich Gauß (1777–1855), German mathematician.

Alternatives in Dealing with a New Scientific Terrain If you want to enter a previously unknown field of knowledge, you inevitably will be confronted with terms and contexts you have never been exposed to before. There are various possibilities to cope with the situation. Two typical options are as follows:

A thorough method is to put aside the text that is currently of interest and search for special sources dealing with the previously unknown terms and concepts. This can be very time-consuming and students of economics in particular cannot or do not always want to afford this approach.

Alternatively one can continue studying the material in the hope to gain some sort of intuitive understanding of the new terms and concepts. This approach is inevitably superficial. Nevertheless, it may be adequate if the authors are experienced textbook-writers. However, they usually do not provide sufficient details. After all, one wants to keep the reader in line and not expect him to specialize in a peripheral field. The latter approach also has its shortcomings.

1.2 Precision and Intuition in the Valuation of Derivatives

At this point we want to give our readers a first glimpse of how careful you have to be if you want to be logically consistent with Brownian motions in finance theory.

dt and Δt To this end, we start with a discrete model that describes the development of a share price. We look at any point in time t and ask how we could describe the change of the share price after the period $\Delta t > 0$. For example, we can imagine Δt being a day. If we call the change of the current share price ΔS, this amount could be modeled by

$$\Delta S = \mu\, S\, \Delta t + \sigma\, S\, \Delta z\,, \tag{1.1}$$

where S is the current share price. The parameters μ and σ should be any positive numbers at first.[5] Δt is—as already mentioned—the change in time, i.e., 1 day. The variable Δz not yet explained should be the change of a random number during the time interval Δt. For example, you could imagine a coin being flipped at the end of each day: Δz will be $+2\%$ if heads appear and -1% otherwise. None of the variables on the right side of Eq. (1.1) is especially "exciting" and therefore does not require much attention. It should be emphasized, however, that it would be entirely unproblematic to divide the equation by Δt, because mathematically Δt is a real number. With objects such as the real numbers you can perform many other mathematical operations without having to be particularly careful. For real numbers certain axioms apply which the mathematical layperson usually is not aware of. But

[5]We could make the coefficients μ and σ time-dependent which would not change anything decisive in our remarks.

it follows from the axioms that these objects can be used to perform operations known as addition, subtraction, multiplication, and division even mathematical laypersons are quite familiar with.[6]

However, all this changes as soon as we turn to a continuous-time model. If we call dt a change in time approaching zero, and if dz describes the change in a random number within such a vanishing interval, and finally if dS is to reflect the change of the share price, then it is obvious to express dS as

$$dS = \mu\, S\, dt + \sigma\, S\, dz\,. \tag{1.2}$$

Of course, we can realize that dt will never be exactly zero, otherwise time would come to a standstill. But what should we imagine when it comes to changing a random variable within a vanishingly small interval of time? Such a change (i.e., dz) can be small, but it could also be relatively large or even disappear entirely if chance would have it. Under no circumstances should this dz be ignored.

Let us now focus on the object dt. We have stated above that it is of infinitesimally small size. Which mathematical operations may be performed with it? The layperson can hardly imagine that a real number Δt could lose the property of being a real number simply because it gets smaller and smaller and is therefore called dt. However, if the above property was true Eq. (1.2) might not simply be divided by dt. And in fact, dt is not a real number.[7]

A First Encounter with Wiener[8]-Processes We will show what problems can arise if Eq. (1.2) is treated superficially. To this end, we first write (1.2) in a slightly different form

$$dS = \mu\, S\, dt + \sigma\, S\, dW \tag{1.3}$$

with dW taking the role of dz. dW is a very special random process known as *Wiener* process or Brownian motion. If you want to learn a little more about

[6]Therefore, an expression of type $\sum_{i=1}^{\infty} \Delta t$ also makes sense. And if $\Delta t > 0$ is valid the sum is infinite because the continued addition of positive real numbers (regardless of their amount) leads to an infinitely large positive value. We will return to this expression in the next footnote.

[7]A mathematical layperson can, for example, realize this by trying to evaluate the computation rule $\sum_{i=1}^{\infty} dt$. Does the expression go towards zero because the objects dt are infinitely small? Or does it go towards infinity because you add infinitely many of these objects? The solution is simpler than the layperson might assume. It comes down to the fact that the question was pointless, because the dt are simply not real numbers. The operation for which the result is asked is purely not allowed. This expression is as pointless as x^{dt} or $\frac{1}{dt}$.

[8]The term "Wiener process" presumably does not go back to Norbert Wiener (see footnote 23 on page 48), but to the German mathematician and physicist Christian Wiener (1826–1896). He could prove in 1863 that Brownian motion is a consequence of the molecular movements of the liquid by disproving the biological causes Brown himself suspected.

this particular random process and restrict yourself to reading standard financial textbooks, you will learn that dW is a constantly evolving process for which

$$dW = \varepsilon \sqrt{dt} \quad \text{with } \varepsilon \sim N(0, 1) \tag{1.4}$$

applies.[9] This expresses that the change of the random variable during the infinitesimal small time interval dt results from the product of a standard normally distributed random number ε and $\sqrt{dt}$.

Value of a Derivative With the continued study of financial textbooks the change in the value of financial titles, depending on the development of a share price, is described by the so-called Itō lemma.[10] A value of a derivative $f(S)$ depending on the share price necessarily follows the stochastic process[11]

$$df = \left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} S^2 \frac{\partial^2 f}{\partial S^2} \sigma^2 \right) dt + \frac{\partial f}{\partial S} \sigma S \, dW. \tag{1.5}$$

While the reader may not be concerned with the development of (1.5), he may, however, be interested in its practical application.

Looking at Eqs. (1.3) and (1.5) from this perspective, one can see that the change in the stock price (dS) as well as the change in the value of the derivative (df) depend on the variables *time* (dt) and *randomness* (dW). If you now form a hedge portfolio by buying $\frac{\partial f}{\partial S}$ units of shares and selling one unit of the derivative, the random influences compensate each other and you actually hold a risk-free portfolio. If one proceeds this way, one can find a so-called fundamental equation[12] for each derivative from which the risk is entirely eliminated.

Itō-Lemma and Taylor Series There may be readers who want to understand the relations more precisely. Such readers do not merely take note of the Itō equation (1.5), but would like to be shown that this equation is correct. Then you have to get into the mathematical literature that is difficult to comprehend for readers having only an economic background. In the financial literature, however, we also like to show ways to understand the Itō lemma in an intuitive way.[13] This usually

[9]Here, once again, there is a certain carelessness in dealing with the infinitesimally small size. If you want to extract the root from a number, it must not be negative. Therefore, $dt \geq 0$ must apply. Of course the question arises why this relation should be fulfilled.

[10]Itō Kiyoshi (1915–2008, Japanese mathematician).

[11]For a European call option the payout function is $f(\cdot)$ depending on the share price, for example at an exercise price of K

$$f(x) = \max(x - K, 0).$$

[12]One also speaks of the Black–Scholes equation.

[13]For example, see Copeland et al. (2005, p. 964 f.).

happens in such a way that a function $f(S + \Delta S,\ t + \Delta t)$ will be approximated at $f(S, t)$ with the help of a Taylor series.[14] The result of such an exercise is

$$\Delta f \approx \left(\frac{\partial f}{\partial S}\,\mu S + \frac{\partial f}{\partial t} + \frac{1}{2} S^2 \frac{\partial^2 f}{\partial S^2}\,\sigma^2 \right) \Delta t + \frac{\partial f}{\partial S}\,\sigma S\,\Delta W\,. \tag{1.6}$$

The reader will easily realize that Eqs. (1.6) and (1.5) are not identical because a Taylor series usually ends with an approximation error. However, if the approximation formula (1.6) correctly describes the performance of a derivative, then the hedge portfolio would not really be risk-free at all, but at best approximately risk-free without knowing anything about the size of the approximation error. If this portfolio were now to yield risk-free interest an arbitrage opportunity could exist, which would nullify the decisive economic argument for deriving the Black–Scholes equation. The allegedly plausible derivation of the Black–Scholes equation is therefore anything but unproblematic.

1.3 Purpose of the Book

We want to give a reader, interested in questions of finance theory who has neither the time nor the interest to attend a complete mathematics course, an understandable introduction to the stochastic integration calculus or Brownian motion, which is correct (or at least acceptable) from a mathematician's perspective.

Many textbook authors make it too easy to deal with the Brownian motion through intuitive approaches.[15] Economic intuition may be important, but it cannot replace the engagement with mathematical formalism. Worse, pure intuition can even be economically flawed, as we have just shown.

Our approach is a tightrope walk. We want to present the Brownian motion as precise as possible without overtaxing the reader with the methodology used in mathematics. If mathematicians deal with certain problems in one way or another, there are always good reasons for doing so which can also be explained vividly.

Our approach is not free of problems. We cannot and will not provide a mathematically precise text because such monographs already exist.[16] We do not concentrate on mathematical precision nor will we deliver extensive mathematical proofs. Instead we will present substantiated reasons why certain concepts must be defined or derived in this way and not in any other way. Of course, what *we* accept as factually justified is always subjective; and in this respect this text is also an experiment. In any case, we believe that there is no comparable book on the market for this type of presentation.

[14] Brook Taylor (1685–1731, British mathematician).

[15] In addition, what intuition means in scientific discourse is not at all clear, see Kruschwitz et al. (2010, p. 370 ff).

[16] See for example Karatzas and Shreve (1991), Huang (1989), Harrison (1990), Revuz and Yor (1999), Musiela and Rutkowski (2005).

When writing their scientific texts, economists want readers to understand why certain assumptions and definitions are formulated in this way and not differently. If one looks at texts written by mathematicians, on the other hand, corresponding efforts are usually lacking. It is often hard to understand why complex issues are developed in exactly this way and not in any other way. Our book deals with mathematical problems of interest to economists. Therefore, we want to try to increase the readability of our explanations for this target group by explaining why mathematicians often use quite complicated ways to arrive at certain results. For example, it is not immediately obvious why one has to deal with σ-algebras in order to be able to define the concept of measure reasonably. Nor is it possible to understand without further explanation why the point-by-point convergence of functions is not a particularly suitable candidate for the concept of convergence. In this book we want to present important issues in such a way that they can be understood by readers who are not immediately familiar with the subject.

We will briefly address several ideas which deserve a thorough examination.

Two Notations for a Brownian Motion We will begin with a statement that may surprise economists: Eq. (1.7) is nothing else but another representation of Eq. (1.3)

$$S(t) - S(0) = \int_0^t \mu\, S(s)\, ds + \int_0^t \sigma\, S(s)\, dW(s)\,. \tag{1.7}$$

Equations (1.3) and (1.7) are expressing just the same. Mathematicians like to speak of stochastic differential equations or also of stochastic integral equations in this context.[17] Let it be clear that "H_2O," "dihydrogenium oxide," and "water" are one and the same. However, when writing down chemical formulas, there are certain rules that prescribe how to deal with the chemical elements named H and O. Thus, "H" stands for a hydrogen atom, while "O" denotes an oxygen atom. The low-set number $_2$ also has a certain meaning. And it is not irrelevant whether this number is attached to the hydrogen atom or to the oxygen atom. However, we do not want to strain the comparison with chemical formulas here.[18]

[17] We will go into more detail on page 9.

[18] Our readers may know similar things from the field of mathematics. So you can either write

$$f'(x) = a$$

or

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = a$$

or

$$\frac{df(x)}{dx} = a\,.$$

It is always the same. But anyone who believes that the mathematically (mark you) perfectly correct equation

$$df(x) = a\,dx$$

We now return to the equivalence of Eqs. (1.3) and (1.7). Usually economists are not exposed to the form of (1.7). And that is precisely the reason why it is worth taking a closer look at this equation.

The Symbol $dW(s)$ The terms $dW(s)$ and dS are not objects with which you can easily carry out transformations. The "differential" $dW(s)$ is not defined as you define a derivative, a limit, or an integral. This expression is found in stochastic analysis exclusively in connection with equations of the form (1.3) or (what is the same) equations of type (1.7). If we want to make another comparison with chemical formulas, the low-set number $_2$ can prove helpful. This number only appears in chemical formulas and it will never be placed as the very first sign in such a representation. The reason is that the low-set number is always preceded by the chemical element in the molecule (representing the quantum of atoms). Without any chemical element the expression like $_2$ does not make any sense. Similarly, $dW(s)$ is inextricably linked to a stochastic integral (1.7).

A Known Integral What mathematical statement can be made of a stochastic differential equation in the form of (1.7)? To this end we will take a closer look at the two integrals on the right side of this equation. First we recognize the term

$$\int_0^t \mu\, S(s)\, ds. \tag{1.8}$$

This is a definite integral.[19] So if $\mu\, S(s)$ is a "normal" function, this integral describes the area under the function within the limits of the $[0, t]$ interval. In Fig. 1.3 we give a schematic representation for this integral. For a mathematician, this raises a host of other questions.[20] In the context of a conventional education in economics, these questions are dealt with shallowness such that the student may feel sufficiently safe to analyze economic problems adequately.

A Strange Integral It is much more complicated with the second term in Eq. (1.7)

$$\int_0^t \sigma\, S(s)\, dW(s). \tag{1.9}$$

can be obtained by simply multiplying the last equation by dx is wrong. It, too, is only another spelling of the identities mentioned, the so-called differentials. Someone who succumbs to such errors is also not immune from making serious mistakes when dealing with stochastic differential equations.

[19] We will talk about a Riemann integral later, see page 71.

[20] Examples are the following: under what conditions does this integral exist? Is the integral over a sum equal to the sum of the individual integrals? Can any continuous function be integrated?

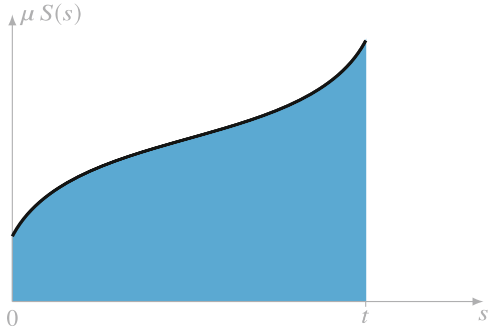

Fig. 1.3 The integral $\int_0^t \mu\, S(s)\, ds$ as the colored area under the function $\mu\, S(s)$ in the interval $[0, t]$

This expression looks like a definite integral, but we will immediately understand that it can no longer be interpreted by the area under a function as shown in Fig. 1.3.

In Fig. 1.3 we find the time s on the abscissa. This makes sense because s is a variable that can assume any value from zero to infinity. $\sigma\, S(s)$ is also a function that assigns a numerical value $\sigma\, S(s)$ to the time s between zero and infinity. It has to be emphasized that the function will not be integrated over time s! Instead, the integration now takes place, as it is formally called, "over a Brownian motion $W(s)$." For a non-mathematician this type of integration probably remains a great mystery.

An integration over a Brownian motion could only be understood as shown in Fig. 1.3 if the object $W(s)$ should be treated as a real number. Real numbers have the property that they can be arranged in ascending or descending order. If you look at the real numbers, you can use a real line. In Fig. 1.3 this real line plays an important role because it corresponds to the abscissa.

The Brownian motion $W(s)$ is anything but a real number. Rather, it is a very large—even infinitely large—set of continuous functions that can be represented graphically as (time-dependent) paths. To understand this in more detail, look at Fig. 1.4 which illustrates the development of Brownian paths. In the figure you see two possible paths. In order to establish the analogy to the classical integral, these paths had to be arranged on a real line. We would have to clarify which of the two paths is further to the left or further to the right. Obviously, this is not possible. Brownian paths simply cannot be arranged one after the other on a real line. There is also no "smallest Brownian motion," which could correspond to zero. It remains absolutely mysterious how one could illustrate the "abscissa" of a stochastic integral of the form (1.3) analogous to Fig. 1.3. We will address this mystery in this book.

As indicated on page 7 we will now address the terms "stochastic differential equation" and "stochastic integral equation." Equation (1.3) is called differential equation because it contains the term dW, while Eq. (1.7) is a stochastic integral equation. The statement that Eqs. (1.3) and (1.7) are equivalent in content must

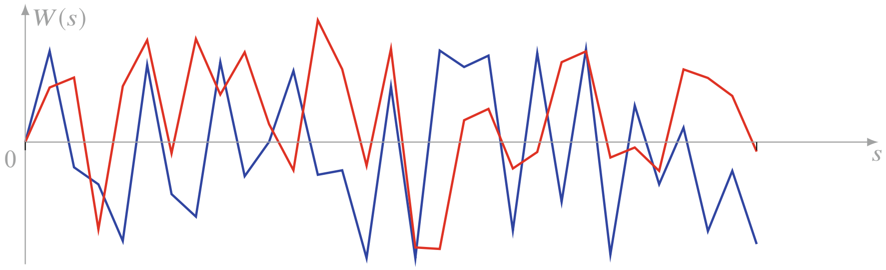

Fig. 1.4 Two realizations of a Brownian motion

irritate a non-mathematician, because it is difficult to accept that a differential equation is the same as an integral equation. But the irritation goes even further if one looks at the object dW and interprets it as the "differential of the Brownian motion." But what should be the differential of Brownian motion? As will be shown later a Brownian motion is an infinitely large set of continuous functions which can rarely be differentiated at any point.[21] The fact that equations such as (1.3) persist in the literature, although important terms are actually "mathematically absurd," can only be explained from the history of this theory. Often these equations were created by physicists and not by mathematicians. Although physicists usually manage to avoid fundamental mathematical errors, their crude procedures are frequently put on a solid mathematical foundation in later years. If they finally succeed the "wrong" spelling established long time ago will not be excluded from the everyday life of physics.[22]

Readers interested in the historical backgrounds of the Brownian motion are invited to refer to the Figs. 1.5 and 1.6.

[21] See page 95.

[22] A famous example is the distribution theory from physics. Before it could be represented mathematically error-free with the help of the Schwartz spaces, the calculations of the users (above all Oliver Heaviside) were notorious for their carelessness in formalism. Dirac wrote: "It seemed to me that when you're confident that a certain method gives the right answer, you didn't have to bother about rigour." Quoted from Peters (2004, p. 106).

5. *Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen;*
von A. Einstein.

In dieser Arbeit soll gezeigt werden, daß nach der molekularkinetischen Theorie der Wärme in Flüssigkeiten suspendierte Körper von mikroskopisch sichtbarer Größe infolge der Molekularbewegung der Wärme Bewegungen von solcher Größe ausführen müssen, daß diese Bewegungen leicht mit dem Mikroskop nachgewiesen werden können. Es ist möglich, daß die hier zu behandelnden Bewegungen mit der sogenannten „Brownschen Molekularbewegung" identisch sind; die mir erreichbaren Angaben über letztere sind jedoch so ungenau, daß ich mir hierüber kein Urteil bilden konnte.

Fig. 1.5 It was Albert Einstein (1878–1959), who was the first to publish a physical theory for the Brownian motion in 1905. An earlier piece of work by Louis Bachelier (1870–1946) from the year 1900, in which Brownian motions were applied to financial markets, remained entirely unnoticed for a long time

THE

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[NEW SERIES.]

SEPTEMBER 1828.

XXVII. *A brief Account of Microscopical Observations made in the Months of June, July, and August,* 1827, *on the Particles contained in the Pollen of Plants; and on the general Existence of active Molecules in Organic and Inorganic Bodies. By* ROBERT BROWN, *F.R.S., Hon. M.R.S.E. & R.I. Acad., V.P.L.S., Corresponding Member of the Royal Institutes of France and of the Netherlands, &c. &c.*

[We have been favoured by the Author with permission to insert the following paper, which has just been printed for private distribution.—ED.]

THE observations, of which it is my object to give a summary in the following pages, have all been made with a simple microscope, and indeed with one and the same lens, the focal length of which is about $\frac{1}{32}$nd of an inch*.

The examination of the unimpregnated vegetable Ovulum, an account of which was published early in 1826†, led me to attend more minutely than I had before done to the structure of the Pollen, and to inquire into its mode of action on the Pistillum in Phænogamous plants.

In the Essay referred-to, it was shown that the apex of the

* This double convex Lens, which has been several years in my possession, I obtained from Mr. Bancks, optician, in the Strand. After I had made considerable progress in the inquiry, I explained the nature of my subject to Mr. Dollond, who obligingly made for me a simple pocket microscope, having very delicate adjustment, and furnished with excellent lenses, two of which are of much higher power than that above mentioned. To these I have often had recourse, and with great advantage, in investigating several minute points. But to give greater consistency to my statements, and to bring the subject as much as possible within the reach of general observation, I continued to employ throughout the whole of the inquiry the same lens with which it was commenced.

† In the Botanical Appendix to Captain King's Voyages to Australia, vol. ii. p. 534. *et seq.*

Fig. 1.6 Facsimile of the original article by Brown (1828). It contains neither a drawing nor a formula