



Towards a Structural Proof Theory of Probabilistic μ -Calculi

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Abstract. We present a structural proof system, based on the machinery of hypersequent calculi, for a simple probabilistic modal logic underlying very expressive probabilistic μ -calculi. We prove the soundness and completeness of the proof system with respect to an equational axiomatisation and the fundamental cut-elimination theorem.

1 Introduction

Modal and temporal logics are formalisms designed to express properties of mathematical structures representing the behaviour of computing systems, such as, e.g., Kripke frames, trees and labeled transition systems. A fundamental problem regarding such logics is the *equivalence problem*: given two formulas ϕ and ψ , establish whether ϕ and ψ are semantically equivalent. For many temporal logics, including the basic modal logic K (see, e.g., [BdRV02]) and its many extensions such as the *modal μ -calculus* [Koz83], the equivalence problem is decidable and can be answered automatically. This is, of course, a very desirable fact. However, a fully automatic approach is not always viable due to the high complexity of the algorithms involved. An alternative and complementary approach is to use *human-aided* proof systems for constructing *formal proofs* of the desired equalities. As a concrete example, the well-known equational axioms of Boolean algebras together with two axioms for the \diamond modality:

$$\diamond \perp = \perp \qquad \diamond(x \vee y) = \diamond(x) \vee \diamond(y)$$

can be used to construct formal proofs of all valid equalities between formulas of modal logic using the familiar deductive rules of *equational logic* (see Definition 3). The simplicity of equational logic is a great feature of this kind of system but sometimes comes at a cost because even seemingly trivial equalities often require significant human ingenuity to be proved.¹ The problem lies in

¹ Example: the law of idempotence $x \vee x = x$ can be derived from the standard axioms of Boolean algebras (i.e., complemented distributive lattices) as: $x \vee x = (x \vee x) \wedge \top = (x \vee x) \wedge (x \vee \neg x) = x \vee (x \wedge \neg x) = x \vee \perp = x$.

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the *transitivity rule* ($a = b \ \& \ b = c \Rightarrow a = c$) which requires to guess, among infinitely many possibilities, an interpolant formula b to prove the equality $a = c$.

The field of *structural proof theory* (see [Bus98]), originated with the seminal work of Gentzen on his *sequent calculus* proof system LK for classical propositional (first-order) logic [Gen34], investigates proof systems which, roughly speaking, require less human ingenuity. The key technical result regarding the sequent calculus, the *cut-elimination theorem*, implies that when searching for a proof of a statement, only certain formulas need to be considered: the so-called *sub-formula property*. This simplifies significantly, in practice, the *proof search* endeavour. The original system LK of Gentzen has been extensively investigated and generalised and, for example, it can be extended with rules for the \diamond modality and becomes a convenient proof system for modal logic [Wan96]. Furthermore, it is possible to extend it with rules for dealing with (co)inductive definitions and it becomes a proof system for the modal μ -calculus (see, e.g., [Stu07]). Research on the structural proof theory of the modal μ -calculus is an active area of research (see, e.g., recent [Dou17]).

Probabilistic Logics and the Riesz Modal Logic. Probabilistic logics are temporal logics specifically designed to express properties of mathematical structures (e.g., Markov chains and Markov decision processes) representing the behaviour of computing systems using probabilistic features such as random bit generation. Unlike the non-probabilistic case, the equivalence problem for most expressive probabilistic logics (e.g., *pCTL* [LS82,HJ94], see also [BK08,BBLM17]) is not known to be decidable. Hence, human-aided proof systems are currently the only viable approach to establish equalities of formulas of expressive probabilistic logics. To the best of our knowledge, however, all the proof systems proposed in the literature (see, e.g., [DFHM16] for the logic pCTL, [BGZB09,Hsu17] for pRHL and [Koz85] for pPDL) are not entirely satisfactory because they include rules, such as the transitivity rule discussed above, violating the sub-formula property.

Another line of work on probabilistic logics has focused on *probabilistic μ -calculi* ([MM07, HK97, DGJP00, dA03, MS17, Mio11, Mio12a, Mio14]). These logical formalisms are, similarly to Kozen's modal μ -calculus, obtained by extending a base *real-valued* modal logic with (co)inductively defined operators. Recently, in [MFM17], a base real-valued modal logic called *Riesz modal logic* (\mathcal{R}) has been defined and a sound and complete equational axiomatisation has been obtained (see Definition 2). Importantly, the logic \mathcal{R} extended with (co)inductively defined operators is sufficiently expressive to interpret most other probabilistic logics, including pCTL [Mio12b, Mio18, MS13a]. Hence, the Riesz modal logic appears to be a convenient base for developing the theory of probabilistic μ -calculi and, more generally, probabilistic logics.

Contributions of This Work. This work is a first step towards the development of the structural proof theory of probabilistic μ -calculi. We introduce a *hypersequent calculus* called MGA (read *modal GA*) for a version of the Riesz modal logic (the

scalar-free fragment, see Sect. 2 for details) and by proving the cut-elimination theorem. Formally we prove:

Theorem 1. *The hypersequent calculus MGA is sound and complete with respect to the equational axioms of Fig. 1 and the CUT rule is eliminable.*

The machinery of hypersequent calculi has been introduced by Avron in [Avr87] and, independently, by Pottinger in [Pot83]. Our calculus extends the hypersequent calculus GA of Metcalfe, Olivetti and Gabbay [MOG05] (see also the book [MOG09] and the related [CM03] and [DMS18]) which is a sound and complete structural proof system for the equational theory of lattice-ordered abelian groups (axioms (1) in Fig. 1, see [Vul67] for an overview). The main contributions of this work are:

1. The careful extension of the system GA of [MOG05] with appropriate proof rules for the modality (\diamond) and the proof of soundness and completeness.
2. The non-trivial adaptation of the proof-technique used in [MOG09, §5.2] to prove the cut-elimination theorem for GA.
3. The formalisation using the theorem prover Agda of our key technical results: Theorems 4 and 9. The code is freely available at [Agd].

In particular, the last point above guarantees the correctness of the proofs of all our novel technical results which, as it is often the case in proof theory, involve complex and long induction arguments. Given the availability of formalised proofs, in this work we focus on illustrating the main ideas behind our arguments rather than spelling out all technical details.

Organisation of the Paper. In Sect. 2 we provide the necessary definitions about the Riesz modal logic from [MFM17, Mio18] and about the hypersequent calculus GA of [MOG05, MOG09]. In Sect. 3 we present our hypersequent calculus MGA and state the main theorems. In Sect. 4 we sketch the main ideas behind our proof of cut-elimination. Lastly, in Sect. 5 we discuss some directions for future work.

2 Technical Background

2.1 The Riesz Modal Logic and Its Scalar-free Fragment

The Riesz modal logic \mathcal{R} introduced in [MFM17] is a probabilistic logic for expressing properties of discrete or continuous Markov chains. We refer to [MFM17] for a detailed introduction. Here we just restrict ourselves to the purely *syntactical* aspects of this logic: its syntax and its axiomatisation.

Definition 1 (Syntax). *The set of formulas of the Riesz modal logic is generated by the following grammar: $\phi, \psi ::= x \mid 0 \mid 1 \mid \phi + \psi \mid r\phi \mid \phi \sqcup \psi \mid \phi \sqcap \psi \mid \diamond\phi$ where r , called a scalar, ranges over the set \mathbb{R} of real numbers. We just write $-\phi$ in place of $(-1)\phi$.*

A main result of [MFM17] is that two formulas ϕ and ψ are semantically equivalent if and only if the identity $\phi = \psi$ holds in all *modal Riesz spaces*.

Definition 2. A modal Riesz space is an algebraic structure R over the signature $\Sigma = \{0, 1, +, r, \sqcup, \sqcap, \diamond\}_{r \in \mathbb{R}}$ such that the following set \mathcal{R} of axioms hold:

1. $\{R, 0, +, r, \sqcup, \sqcap\}_{r \in \mathbb{R}}$ is a Riesz space (see, e.g., [LZ71]), i.e.,
 - $(R, 0, +, r)_{r \in \mathbb{R}}$ is an \mathbb{R} -vector space,
 - (R, \sqcup, \sqcap) is a lattice,
 - the lattice order $(x \leq y \Leftrightarrow x \sqcap y = x)$ is compatible with addition, i.e.:
 - (a) $x \leq y$ implies $x + z \leq y + z$ (i.e., $(x \sqcap y) + z = ((x \sqcap y) + z) \sqcap (y + z)$),
 - (b) $x \geq 0$ implies $rx \geq 0$ (i.e., $0 = 0 \sqcap r(x \sqcup 0)$) for every $r \in \mathbb{R}_{\geq 0}$,
2. $0 \leq 1$ (i.e., $0 = 0 \sqcap 1$),
3. the \diamond operation is linear, positive and 1-decreasing, i.e.:
 - $\diamond(x + y) = \diamond(x) + \diamond(y)$ and $\diamond(rx) = r\diamond(x)$,
 - if $x \geq 0$ then $\diamond(x) \geq 0$ (i.e., $0 = 0 \sqcap \diamond(x \sqcup 0)$),
 - $\diamond(1) \leq 1$ (i.e., $\diamond 1 = \diamond 1 \sqcap 1$).

Note that the definition of modal Riesz spaces is purely equational: all axioms of Riesz spaces (1) can be expressed equationally and so can the axioms (2) and (3). This means, by Birkhoff completeness theorem, that two formulas are semantically equivalent if and only if the identity $\phi = \psi$ can be derived using the familiar deductive rules of equational logic, written as $\mathcal{R} \vdash \phi = \psi$.

Definition 3 (Deductive Rules of Equational Logic). Rules for deriving identities from a set \mathcal{A} of equational axioms:

$$\frac{(t_1 = t_2) \in \mathcal{A}}{\mathcal{A} \vdash t_1 = t_2} Ax \qquad \frac{}{\mathcal{A} \vdash t = t} refl \qquad \frac{\mathcal{A} \vdash t_2 = t_1}{\mathcal{A} \vdash t_1 = t_2} sym \qquad \frac{\mathcal{A} \vdash t_1 = t_2}{\mathcal{A} \vdash C[t_1] = C[t_2]} ctxt$$

$$\frac{\mathcal{A} \vdash t_1 = t_2 \quad \mathcal{A} \vdash t_2 = t_3}{\mathcal{A} \vdash t_1 = t_3} trans \qquad \frac{\mathcal{A} \vdash f(\mathbf{s}, x, \mathbf{u}) = g(\mathbf{w}, x, \mathbf{z})}{\mathcal{A} \vdash f(\mathbf{s}, t, \mathbf{u}) = g(\mathbf{w}, t, \mathbf{z})} subst$$

where $C[\cdot]$ is a context and f, g are function symbols of the fixed signature.

In what follows we denote with $\mathcal{R} \vdash \phi \leq \psi$ the judgment $\mathcal{R} \vdash \phi = \phi \sqcap \psi$. The following elementary facts from the theory of Riesz spaces (see, e.g., [LZ71, §2.12]) will be useful.

Proposition 1. The following assertions hold:

- $\mathcal{R} \vdash \phi = \psi$ iff $\mathcal{R} \vdash \phi - \psi = 0$,
- $\mathcal{R} \vdash \phi = \psi$ iff $(\mathcal{R} \vdash \phi \leq \psi$ and $\mathcal{R} \vdash \psi \leq \phi)$.
- $\mathcal{R} \vdash r(x \sqcup y) = rx \sqcup ry$, $\mathcal{R} \vdash r(x \sqcap y) = rx \sqcap ry$.

The first point says that an equality $\phi = \psi$ can always be expressed as an identity with 0. The second point says that we can express equalities with inequalities and *vice versa*. The third point, together with the other axioms, implies that scalar multiplication distributes over all other operations $\{+, \sqcup, \sqcap, \diamond\}$.

For most practical purposes (when expressing properties of probabilistic models) the scalars in the Riesz modal logic can be restricted to be rational numbers.

Definition 4 (Rational and Scalar-free formulas). A formula ϕ is rational if all its scalars are rational numbers. Similarly, ϕ is scalar-free if its scalars are all equal to (-1) . Equivalently, the set of scalar-free formulas is generated by the following grammar: $A, B ::= x \mid 0 \mid 1 \mid A + B \mid -A \mid A \sqcup B \mid A \sqcap B \mid \diamond(A)$.

Note how we have switched to the letters A and B to range over scalar-free formulas to highlight this distinction.

Proposition 2. Let ϕ be a rational formula. Then there exists a scalar-free formula A such that $\mathcal{R} \vdash \phi = 0$ iff $\mathcal{R} \vdash A = 0$.

Proof. Let $\{r_i\}_{i \in I}$ be the list of rational scalars in ϕ , with $r_i = \frac{n_i}{m_i}$ and let $d = \prod_i m_i$ be the product of all denominators. Since scalar multiplication distributes with all operations it is easy to show that $\mathcal{R} \vdash d\phi = \psi$, for a formula ψ whose scalars are all integers. We can then obtain A from ψ by inductively replacing any sub-formula of ψ the form nB with $(B + B + \dots + B)$ (n times) if n is positive, with $-(B + B + \dots + B)$ if n is negative and with 0 if $n = 0$. \square

For this reason in this work we restrict attention to scalar-free formulas and we consider the restricted set of axioms \mathbb{T} of Fig. 1. The axioms of Riesz spaces, when scalar multiplication is omitted, reduce to the axioms of *lattice ordered abelian groups* (see, e.g., [Vul67]). The axiom $0 \leq 1$ is unaltered and the axioms for the \diamond modality are naturally adapted. For these reasons we refer to these axioms as of those of *lattice-ordered modal abelian groups*.

1. **Axioms of Lattice-ordered abelian groups:**
 - Abelian Group: $x + (y + z) = (x + y) + z, x + y = y + x, x + 0 = x, x - x = 0$.
 - Lattice axioms: (associativity) $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z, x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$, (commutativity) $z \sqcup y = y \sqcup z, z \sqcap y = y \sqcap z$, (absorption) $z \sqcup (z \sqcap y) = z, z \sqcap (z \sqcup y) = z$, (idempotency) $x \sqcup x = x, x \sqcap x = x$.
 - Compatibility: $(x \sqcap y) + z = ((x \sqcap y) + z) \sqcap (y + z)$
2. **Axioms for the unit:** $0 = 0 \sqcap 1$,
3. **Modal axioms:**
 - $\diamond(x + y) = \diamond(x) + \diamond(y), \diamond(-x) = -\diamond(x)$ and $\diamond(0) = 0$,
 - $0 = 0 \sqcap \diamond(x \sqcup 0)$,
 - $\diamond 1 = \diamond 1 \sqcap 1$.

Fig. 1. Set of axioms \mathbb{T} of lattice-ordered modal Abelian groups.

Remark 1. Note that from the previous discussion it does not follow directly that $\mathcal{R} \vdash A = B$ implies $\mathbb{T} \vdash A = B$. We indeed conjecture that \mathcal{R} is a conservative extension of \mathbb{T} but we have not proved this fact so far. In any case, this is not required for results of this work.

The main contribution of this work is the design of a sound and complete hypersequent calculus for the theory \mathbb{T} and the proof of cut-elimination.

2.2 The Hypersequent Calculus GA

Our starting point is the hypersequent calculus GA of [MOG05,MOG09] for the theory of lattice-ordered abelian groups (set of axioms (1) in Fig. 1).

Definition 5 (Formulas, Sequents and hypersequents). A formula A is a term built from a set of variables (ranged over by x, y, z) over the signature $\{0, +, -, \sqcap, \sqcup\}$. A sequent S is a pair of two (possibly empty) multisets of formulas $\Gamma = A_0, \dots, A_n$ and $\Delta = B_0, \dots, B_m$, denoted as $\Gamma \vdash \Delta$. A hypersequent G is a nonempty multiset S_1, \dots, S_n of sequents, denoted as $S_1 | \dots | S_n$.

Following [MOG05,MOG09], with some abuse of notation, we denote with S both the sequent and the hypersequent consisting of only the sequent S . The system GA is a deductive system for deriving hypersequents consisting of the rules of Fig. 2. The system GA without the CUT rule is denoted by GA^* .

Another convention we adopt from [MOG05,MOG09] is to write $d \vDash_{\text{GA}} G$ to express the fact that d is a valid GA-derivation of the hypersequent G . We write $\vDash_{\text{GA}} G$ to express the existence of a GA-derivation d such that $d \vDash_{\text{GA}} G$. Similarly, we write $d \vDash_{\text{GA}^*} G$ and $\vDash_{\text{GA}^*} G$ when referring to the subsystem GA^* .

Axioms:	
$\overline{\vdash} \Delta\text{-ax}$	$\overline{A \vdash A} \text{ID-ax}$
Structural rules:	
$\frac{G}{G \Gamma \vdash \Delta} \text{Weakening (W)}$	$\frac{G \Gamma \vdash \Delta \Gamma \vdash \Delta}{G \Gamma \vdash \Delta} \text{Contraction (C)}$
$\frac{G \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}{G \Gamma_1 \vdash \Delta_1 \Gamma_2 \vdash \Delta_2} \text{Split (S)}$	$\frac{G \Gamma_1 \vdash \Delta_1 \quad G \Gamma_2 \vdash \Delta_2}{G \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \text{Mix (M)}$
Logical rules:	
$\frac{G \Gamma \vdash \Delta}{G \Gamma, 0 \vdash \Delta} 0_L$	$\frac{G \Gamma \vdash \Delta}{G \Gamma \vdash \Delta, 0} 0_R$
$\frac{G \Gamma, A, B \vdash \Delta}{G \Gamma, A + B \vdash \Delta} +_L$	$\frac{G \Gamma \vdash \Delta, A, B}{G \Gamma \vdash \Delta, A + B} +_R$
$\frac{G \Gamma \vdash \Delta, A}{G \Gamma, -A \vdash \Delta} -_L$	$\frac{G \Gamma, A \vdash \Delta}{G \Gamma \vdash \Delta, -A} -_R$
$\frac{G \Gamma, A \vdash \Delta \quad G \Gamma, B \vdash \Delta}{G \Gamma, A \sqcup B \vdash \Delta} \sqcup_L$	$\frac{G \Gamma \vdash \Delta, A \Gamma \vdash \Delta, B}{G \Gamma \vdash \Delta, A \sqcup B} \sqcup_R$
$\frac{G \Gamma, A \vdash \Delta \Gamma, B \vdash \Delta}{G \Gamma, A \sqcap B \vdash \Delta} \sqcap_L$	$\frac{G \Gamma \vdash \Delta, A \quad G \Gamma \vdash \Delta, B}{G \Gamma \vdash \Delta, A \sqcap B} \sqcap_R$
CUT rule:	
$\frac{G \Gamma_1 \vdash \Delta_1, A \quad G \Gamma_2, A \vdash \Delta_2}{G \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \text{Cut}$	

Fig. 2. Inference rules of the hypersequent system GA of [MOG05].

Multisets of formulas, sequents and hypersequents are interpreted as a single formula as follows:

Definition 6 (Interpretation). *A multiset of formulas $\Gamma = \phi_1, \dots, \phi_n$ is interpreted as the formula $\llbracket \Gamma \rrbracket = \phi_1 + \phi_1 + \dots + \phi_n$ if $n \geq 1$ and as $\llbracket \Gamma \rrbracket = 0$ if $\Gamma = \emptyset$. A sequent $S = \Gamma \vdash \Delta$ is interpreted as the formula $\llbracket S \rrbracket = \llbracket \Delta \rrbracket - \llbracket \Gamma \rrbracket$. Finally, a hypersequent $G = S_0 \mid \dots \mid S_n$ is interpreted as the formula $\llbracket G \rrbracket = \llbracket S_0 \rrbracket \sqcup \dots \sqcup \llbracket S_n \rrbracket$.*

Example 1. Consider the hypersequent $G = (0 \sqcup x, y \vdash y) \mid (-y \vdash)$ consisting of two sequents. Then $\llbracket G \rrbracket = (y - ((0 \sqcup x) + y)) \sqcup (0 - (-y))$.

The soundness and completeness of the hypersequent system GA with respect to the theory of lattice-ordered abelian groups (axioms (1) of Fig. 1, written as $\mathbb{T}_{(1)}$) is expressed by the following theorem.

Theorem 2 ([MOG05]). *For all formulas A and hypersequents G :*

Soundness: if $\vDash_{GA} G$ then $\mathbb{T}_{(1)} \vdash \llbracket G \rrbracket \geq 0$.

Completeness: if $\mathbb{T}_{(1)} \vdash A \geq 0$ then $\vDash_{GA} (\vdash A)$

Proof. The proofs presented in [MOG05] exploit the following well-known fact (see, e.g., [Vul67]): the equality $A = B$ holds in all lattice-ordered abelian groups if and only if it holds in $(\mathbb{R}, 0, +, -, \max, \min)$ under any interpretation of the variables as real numbers. In other words, \mathbb{R} generates the variety of lattice-ordered abelian groups. □

The main result of [MOG05] regarding GA is that the CUT rule is eliminable.

Theorem 3 (Cut-elimination [MOG05]). *Any GA-derivation of a hypersequent G can be effectively transformed into a GA^* -derivation of G .*

3 The Hypersequent System MGA

In this section we introduce our hypersequent calculus system MGA, a modal extension of the GA system of [MOG05]. The system MGA deals with formulas over the signature of modal lattice-ordered abelian groups (see Fig. 1) thus including the constant 1 and the unary modality \diamond .

Definition 7 (Formulas of MGA). *A formula A is a term built from a set of variables (ranged over by x, y, z) over the signature $\{0, 1, +, -, \square, \sqcup, \diamond\}$.*

The definitions of sequents and hypersequents are given exactly as for the system GA in Definition 5 of Sect. 2.2. Similarly, multisets of formulas, sequents and hypersequents are interpreted as formulas exactly as already specified in Definition 6 of Sect. 2.2 for the system GA. Before presenting the deduction rules of MGA, it is useful to introduce the following abbreviations.

- For $n \in \mathbb{N}_{\geq 0}$, we denote with nF the multiset of formulas F, F, \dots, F .
So for example we write $2A, 1B \vdash 0C, D$ to denote the sequent $A, A, B \vdash D$.
- Given a multiset of formulas $\Gamma = F_0, \dots, F_k$ and $n \in \mathbb{N}_{\geq 0}$ we denote with $n\Gamma$ the multiset of formulas nF_0, \dots, nF_k . If $\Gamma = \emptyset$ then also $n\Gamma = \emptyset$.
- Given a multiset of formulas $\Gamma = F_0, \dots, F_n$ we denote with $\diamond\Gamma$ the multiset of formulas $\diamond F_0, \dots, \diamond F_n$. Consistently, if $\Gamma = \emptyset$ then also $\diamond\Gamma = \emptyset$.

The rules of the system MGA consist of all rules of GA (see Fig. 2) together with the additional rules of Fig. 3.

Axiom for 1:	Rule for \diamond :
$\overline{\vdash 1}$ 1-ax	$\frac{\Gamma \vdash \Delta, n1}{\diamond\Gamma \vdash \diamond\Delta, n1}$ \diamond -rule

Fig. 3. Additional inference rules of the hypersequent system MGA

The axiom (1-ax) for the constant 1 is straightforward and it simply expresses the axiom $0 \leq 1$ from Fig. 1 (i.e., $\mathbb{T} \vdash \llbracket 1 \rrbracket \geq 0$).

The rule (\diamond -rule) for the modality is more subtle as it imposes strong constraints on the shape of its premise and conclusion. First, both the conclusion and the premise are required to be hypersequents consisting of exactly one sequent. Furthermore, in the conclusion, all formulas, except those of the form 1 on the right side, need to be of the form $\diamond C$ for some C .

The following is an illustrative example of derivation in the system MGA:

$$\begin{array}{c}
\overline{\vdash 1} \text{ ID-ax} \quad \overline{A \vdash A} \text{ ID-ax} \\
\frac{}{A, 1 \vdash 1, A} \text{ M} \\
\frac{}{A, 1, \neg(A) \vdash 1} \text{ } \\
\frac{}{A, 1 - A \vdash 1} \text{ } \\
\frac{}{A, A \vdash 1 \mid A, 1 - A \vdash 1} \text{ W} \\
\frac{}{A, A \sqcap (1 - A) \vdash 1} \text{ } \\
\frac{}{A, A \sqcap (1 - A) \vdash 1 \mid 1 - A, A \sqcap (1 - A) \vdash 1} \text{ W} \\
\frac{}{A \sqcap (1 - A), A \sqcap (1 - A) \vdash 1} \text{ } \\
\frac{}{\diamond((A \sqcap (1 - A))), \diamond((A \sqcap (1 - A))) \vdash 1} \text{ } \\
\frac{}{\diamond((A \sqcap (1 - A))) + \diamond((A \sqcap (1 - A))) \vdash 1} \text{ }
\end{array}$$

Our first theorem regarding MGA states its soundness and completeness with respect to the theory of modal lattice-ordered abelian groups (see Fig. 1). The proof of [MOG05] of Theorem 2 cannot be directly adapted here because, unlike the case for lattice-ordered abelian groups and \mathbb{R} , we are not aware of any simple modal lattice-order abelian group which generates the whole variety.

Theorem 4. For all formulas A and hypersequents G :

Soundness: if $\models_{MGA} G$ then $\mathbb{T} \vdash \llbracket G \rrbracket \geq 0$.

Completeness: if $\mathbb{T} \vdash A \geq 0$ then $\models_{MGA} (\vdash A)$.

Proof. Soundness is proven by translating every MGA derivation d of G to a derivation in equational logic π of $\llbracket G \rrbracket \geq 0$. This is done by induction on the complexity of d . The difficult cases correspond to when d ends by applications of either the S-rule, the M-rule or the \sqcup_L rule. The formalised proof is implemented in the agda file `Syntax/Agda/MGA-Cut/Soundness.agda` in [Agd] and the type of the function is: `soundness : (G : HSeq) → (MGA G) → botAG ≤S [G]`.

Conversely, completeness is proven by translating every equational logic derivation π of $A = B$ to the MGA derivations d_1 and d_2 of the (hyper)sequents $A \vdash B$ and $B \vdash A$ respectively. The proof goes by induction on π . First, MGA derivations are obtained for all axioms of Fig. 1. For example, for the axiom $\diamond(x + y) = \diamond(x) + \diamond(y)$ we can derive the (hyper)sequent $\diamond(x + y) \vdash \diamond(x) + \diamond(y)$ as showed below (left-side). Translating applications of the rules *refl* and *sym* is simple. Translating applications of the *trans* rules is immediate using the *CUT* rule of MGA. To translate applications of the *ctxt* rule, it is sufficient to prove (by induction) a simple context-lemma that states that if $A \vdash B$ is MGA derivable then also $C[A] \vdash C[B]$ is MGA derivable. Similarly, to translate applications of the *subst* rule, it is sufficient to prove (by induction) a simple substitution-lemma stating that if G is MGA derivable then $G[A/x]$ is also derivable, where $G[A/x]$ is the hypersequent where every occurrence of x is replaced by A .

Note that $\mathbb{T} \vdash A \geq 0$ means that $\mathbb{T} \vdash 0 = 0 \sqcap A$. By the translation method outlined above, the (hyper)sequent $0 \vdash 0 \sqcap A$ is MGA derivable. We can then get a MGA derivation of $\vdash A$ as follows (right-side):

$$\begin{array}{c}
 \frac{}{x \vdash x} \text{ID-ax} \quad \frac{}{y \vdash y} \text{ID-ax} \\
 \frac{x, y \vdash x, y}{x + y \vdash x, y} \text{+L} \quad \text{M} \\
 \frac{}{\diamond(x + y) \vdash \diamond(x), \diamond(y)} \diamond \\
 \frac{}{\diamond(x + y) \vdash \diamond(x) + \diamond(y)} \text{+R}
 \end{array}
 \quad \Bigg| \quad
 \begin{array}{c}
 \frac{}{A \vdash A} \text{ID-ax} \quad \frac{}{0 \vdash 0 \sqcap A} \text{W} \quad \frac{}{\vdash 0} \Delta\text{-ax} \\
 \frac{}{0 \sqcap A \vdash A} \sqcap_L \quad \frac{}{0 \vdash 0 \sqcap A} \sqcap_R \quad \frac{}{\vdash 0} \text{cut} \\
 \frac{}{\vdash A} \text{cut}
 \end{array}$$

The file `Syntax/Agda/MGA-Cut/Completeness.agda` in [Agd] contains the formalised proof and the type of the function is: `completeness : (A : Term) → botAG ≤S A → MGA (head ([], [] :: A))`. □

Remark 2. The following natural looking variant of the (\diamond -rule), allowing hypersequents with more than one component, is unsound:

$$\frac{G \mid \Gamma \vdash \Delta, n1}{G \mid \diamond \Gamma \vdash \diamond \Delta, n1}$$

Our main theorem regarding the system MGA is the cut-elimination theorem. We denote with MGA^* the system without the CUT rule.

Theorem 5 (Cut-elimination). *Any MGA-derivation of a hypersequent G can be effectively transformed into a MGA^* -derivation of G .*

Theorems 4 and 5 imply the statement of Theorem 1 in the Introduction.

4 Overview of the Proof of the Cut-Elimination Theorem

In this section we illustrate the structure of our proof of the cut-elimination theorem. We first explain the main ideas behind the proof of cut-elimination for GA of [MOG09, §5.2]. We then explain why these ideas are not directly applicable to the system MGA. Lastly, we discuss our key technical contribution which makes it possible to adapt the proof method of [MOG09, §5.2] to prove the cut-elimination theorem for the MGA system.

4.1 The CAN-Elimination Theorem for the System GA

A key idea of [MOG09, §5.2] is to replace the CUT rule with an easier to handle rule called *cancellation* (CAN) rule. The CAN rule can derive the CUT rule in the basic cut-free system GA^* as follows (right-side):

$$\frac{G|\Gamma, A \vdash A, \Delta}{G|\Gamma \vdash \Delta} \text{ CAN} \quad \left| \quad \frac{\frac{d_1}{G|\Gamma_1, A \vdash \Delta_1} \quad \frac{d_2}{G|\Gamma_2 \vdash A, \Delta_2}}{G|\Gamma_1, \Gamma_2, A \vdash A, \Delta_1, \Delta_2} \text{ M}}{G|\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \text{ CAN}$$

The cut-elimination theorem is obtained in [MOG09, §5.2] by proving a CAN-elimination theorem expressed as: if $\vDash_{\text{GA}^*} G|\Gamma, A \vdash A, \Delta$ then $\vDash_{\text{GA}^*} G|\Gamma \vdash \Delta$.

The CAN-elimination theorem for the system GA is proved in three steps:

Step A: proving the invertibility of all the logical rules ([MOG09, Lemma 5.18]). The invertibility states that if the conclusion of a logical rule (for instance, $G|\Gamma, A + B \vdash \Delta$ for the $+_L$ rule) is derivable without the CAN-rule, then all the premises (in this case $G|\Gamma, A, B \vdash \Delta$) are derivable too without the CAN-rule.

Step B: proving the atomic CAN-elimination theorem ([MOG09, Lemma 5.17]). This theorem deals with the special case of A being a variable and states that if $d \vDash_{\text{GA}^*} G|\Gamma, x \vdash x, \Delta$ then $\vDash_{\text{GA}^*} G|\Gamma \vdash \Delta$. This theorem is proven by induction on d and is mostly straightforward: the only difficult case is when d finishes with an application of the M-rule. A separate technical result ([MOG09, Lemma 5.16]) is used to take care of this difficult case.

Step C: proving the CAN-elimination theorem ([MOG09, Theorem 5.19]). The CAN-elimination theorem states that if $\vDash_{\text{GA}^*} G|\Gamma, A \vdash A, \Delta$ then $\vDash_{\text{GA}^*} G|\Gamma \vdash \Delta$. This proof is by induction on A :

- If A is a variable, we can conclude with the atomic CAN-elimination theorem.
- Otherwise we use the invertibility of the logical rules and we can conclude with the induction hypothesis. For instance, if $A = B + C$, then by invertibility of the $+_L$ and $+_R$ rules we have a GA^* -derivation of $\vDash_{\text{GA}^*} G|\Gamma, B, C \vdash \Delta, B, C$ and, from it, we can obtain a GA^* -derivation of $G|\Gamma \vdash \Delta$ by using twice the induction hypothesis, first on B then on C .

4.2 Issues in Adapting the Proof for the System MGA

The proofs of [MOG09] can be adapted to the context of MGA without much difficulty to perform the first two steps:

Theorem 6 (Invertibility of the logical rules). *All logical rules (including the \diamond -rule) are invertible in the system MGA*.*

Proof. The same proof technique used in [MOG09] works. The main idea is, in order to deal easily with the (S) and the (C) rules, to prove a slightly stronger statement about the invertibility of more general rules. For instance, the generalisation of the rule $+_L$ is:

$$\frac{[\Gamma_i, n_i A, n_i B \vdash \Delta_i]_{i=1}^k}{[\Gamma_i, n_i(A+B) \vdash \Delta_i]_{i=1}^n} \quad \square$$

Theorem 7 (Atomic CAN-elimination theorem). *If $\models_{MGA^*} \Gamma, x \vdash x, \Delta$ then $\models_{MGA^*} \Gamma \vdash \Delta$.*

The complication comes from the third and last Step C. We want to prove that if $\models_{MGA^*} G|\Gamma, A \vdash A, \Delta$ then $\models_{MGA^*} G|\Gamma \vdash \Delta$. An ordinary proof by induction on A could get stuck when $A = \diamond B$. For instance, if the hypersequent is $x, \diamond B \vdash \diamond B, x$, the invertibility of the \diamond -rule can not be used because of the syntactic constraints the \diamond -rule imposes on its conclusion. Indeed the invertibility of the \diamond -rule states that if $\models_{MGA^*} \diamond\Gamma \vdash \diamond\Delta$ then $\models_{MGA^*} \Gamma \vdash \Delta$, but $x, \diamond A \vdash \diamond A, x$ is not of this form because it contains the variable x .

For this reason, we deal with the case $A = \diamond B$ in a different way, using an induction argument on the derivation of $G|\Gamma, A \vdash A, \Delta$. In this argument, however, the M-rule is hard to deal with (as already remarked it is a main source of complications also on the proof of atomic CAN-elimination of [MOG09, §5.2]).

Our main technical result is that the M-rule can be eliminated from a simple variant of the system MGA called MGA-SR (which stands MGA with *scalar rules*). The system MGA-SR is obtained by modifying MGA as follows:

- The logical left-rules and right-rules for the connectives $\{0, -, +, \sqcup, \sqcap\}$ are generalised to deal with scalar coefficients (syntactic sugaring introduced in Sect. 3). For instance, the rules $+_L$ and \sqcup_L become:

$$\frac{G|\Gamma, nA, nB \vdash \Delta}{G|\Gamma, n(A+B) \vdash \Delta} +_L \quad \frac{G|\Gamma, nA \vdash \Delta \quad G|\Gamma, nB \vdash \Delta}{G|\Gamma, n(A \sqcup B) \vdash \Delta} \sqcup_L$$

- The axioms ID-ax and 1-ax are replaced by the rules

$$\frac{G|\Gamma \vdash \Delta}{G|\Gamma, nA \vdash nA, \Delta} \text{ID-rule} \quad \frac{G|\Gamma \vdash \Delta}{G|\Gamma \vdash \Delta, n1} \text{1-rule}$$

- All structural rules (C, W, S, M), the \diamond -rule and the CAN rule remain exactly as in MGA (see Fig. 2).

It is possible to verify that MGR and MGR-SR are equivalent, i.e., they can derive exactly the same hypersequents (Theorem 8 below). The first modification (scalar rules) is technically motivated because it simplifies several proofs: in fact scalar rules are also implicitly considered in several of the proofs of [MOG09] for the system GA. The second modification (ID-rule and 1-rule) is essential. Indeed in the system MGA (and also in GA) the (hyper)sequent $x, y \vdash x, y$ is not derivable without applying the M-rule. Hence M -elimination in MGA is impossible. On the other hand the (hyper)sequent $x, y \vdash x, y$ is easily derivable in MGA-SR without requiring applications of the M rule

$$\frac{\frac{\overline{\vdash} \Delta\text{-ax}}{\vdash y \vdash y} \text{ID-rule}}{x, y \vdash x, y} \text{ID-rule}$$

and, as we will prove (Theorem 12), it is indeed possible to eliminate all applications of the M-rule from MGA-SR.

As outlined above, the presence of the M-rule was the main source of complications in adapting Step C. Once the equivalence between MGA-SR and MGA-SR without the M-rule is established, most complications disappear and the CAN-elimination proof can be obtained by performing Steps A–B–C for the system MGA-SR.

4.3 The System MGA-SR and the M-Elimination Theorem

In this subsection we introduce the system MGA-SR (MGA with *scalar rules*) for which we will prove the M-elimination theorem.

Definition 8 (MGA-SR). *The inference rules of MGA-SR are the rules of MGA modified as discussed previously. We denote by $MGA\text{-}SR^*$, $MGA\text{-}SR^\dagger$ and $MGA\text{-}SR^{\dagger*}$ the systems without the CUT rule, the M-rule and both the CUT and M-rules, respectively.*

Theorem 8. *The two systems MGA and MGA-SR are equivalent: $\models_{MGA} G$ if and only if $\models_{MGA\text{-}SR} G$.*

The two systems MGA^ and $MGA\text{-}SR^*$ are equivalent: $\models_{MGA^*} G$ if and only if $\models_{MGA\text{-}SR^*} G$.*

Proof. Translating MGA proofs to MGA-SR proofs is straightforward. All rules of MGA are specific instances of the scalar rules of MGA-SR (taking the scalar $n = 1$) and the the axioms 1-Axiom and ID-axioms are easily derivable in MGA-SR (without the need of the CAN rule) by using the id-rule and 1-rule (again, using the scalar $n = 1$). Translating MGA-SR to MGA is also mostly straightforward. Some care is needed to translate instances of the scalar-rules \sqcup_L and \sqcap_R from MGA-SR to MGA. This can be done by induction on the scalar n using the fact that the two premises $G|\Gamma, nA, B \vdash \Delta$ and $G|\Gamma, nB, A \vdash \Delta$ are derivable from $G|\Gamma, (n+1)A \vdash \Delta$ and $G|\Gamma, (n+1)B \vdash \Delta$. We remark that this derivation may require the usage of the M rule. \square

We now state our main technical contribution: the M-elimination theorem for the system MGA-SR.

Theorem 9 (M-elimination). *If $d_1 \vDash_{MGA-SR^\dagger} G_1 \mid \Gamma \vdash \Delta$ and $d_2 \vDash_{MGA-SR^\dagger} G_2 \mid \Sigma \vdash \Pi$ then $\vDash_{MGA-SR^\dagger} G_1 \mid G_2 \mid \Gamma, \Sigma \vdash \Delta, \Pi$.*

If $d_1 \vDash_{MGA-SR^{\dagger}} G_1 \mid \Gamma \vdash \Delta$ and $d_2 \vDash_{MGA-SR^{\dagger*}} G_2 \mid \Sigma \vdash \Pi$ then $\vDash_{MGA-SR^{\dagger*}} G_1 \mid G_2 \mid \Gamma, \Sigma \vdash \Delta, \Pi$.*

We now give a sketch of our proof argument. A formalised proof in Agda is available in [Agd] and is contained in the files `Syntax/MGA-SR/M-Elim.agda` and `Syntax/MGA-SR-CAN/M-Elim-CAN.agda`.

The general idea is to combine the derivations d_1 and d_2 in a *sequential way*. We first consider the case when no applications of the \diamond -rule appear in d_1 nor d_2 . First the proof d_1 is transformed into a pre-proof (i.e., where the derivation is left incomplete at some leaves) d'_1 of $G_1 \mid G_2 \mid \Gamma, \Sigma \vdash \Delta, \Pi$. The pre-proof d'_1 is structurally identically to d_1 and it essentially just ignores the G_2 , Σ and Π components of the hypersequent. While the leaves of d_1 are all of the form (\vdash) because Δ -ax is the only axiom of MGA-SR, the leaves of the pre-proof d'_1 are of the form $G_2 \mid n\Sigma \vdash n\Pi$ (the ignored part carried out until the end, which can get multiplied by applications of the C and S rules). We can now proceed with the second step and provide derivations for these leaves using (easily modified versions of) the proof d_2 .

When occurrences of the \diamond -rule appear in d_1 or d_2 the argument requires more care. Indeed an application of the \diamond -rule on d_1 acting on some hypersequent (necessarily) of the form:

$$\diamond \Gamma_1 \vdash \diamond \Delta_1, k1$$

cannot be turned into an application of \diamond -rule on:

$$G_2 \mid \Sigma, \diamond \Gamma_1 \vdash \diamond \Delta_1, k1, \Pi$$

because this hypersequent violates the structural constraints of the \diamond -rule. For this reason, we stop the construction of d'_1 at these points and, as a result, the leaves of the pre-proof d'_1 are generally of the form: $G_2 \mid n\Sigma, \diamond \Gamma_1 \vdash \diamond \Delta_1, k1, n\Pi$, for some Γ_1, Δ_1 and scalars n, k .

The idea now is, following the same kind of procedure, to modify the proof d_2 and turn it to a pre-proof d'_2 of $G_2 \mid n\Sigma, \diamond \Gamma_1 \vdash \diamond \Delta_1, k1, n\Pi$. Crucially, the previous issue disappears. Indeed proof steps in d_2 acting on hypersequents of the form:

$$\diamond \Sigma_1 \vdash \diamond \Pi_1, m1$$

using the \diamond -rule, can be turned into valid \diamond -rule steps for the extended hypersequent:

$$\diamond \Sigma_1, \diamond \Gamma_1 \vdash \diamond \Delta_1, k1, \diamond \Pi_1, m1$$

because the shape of the sequent is compatible with the constraint of the \diamond rule. Note that the hypersequent resulting from the application of the \diamond -rule is $\Sigma_1, \Gamma_1 \vdash \Gamma_1, k1, \Pi_1, m1$ and has a lower modal-depth than the starting one. Hence an inductive argument on modal-complexity can be arranged to

recursively reduce the general M-elimination procedure to the simpler case where d_1 and d_2 do not have occurrences of the \diamond -rule (Fig. 4).

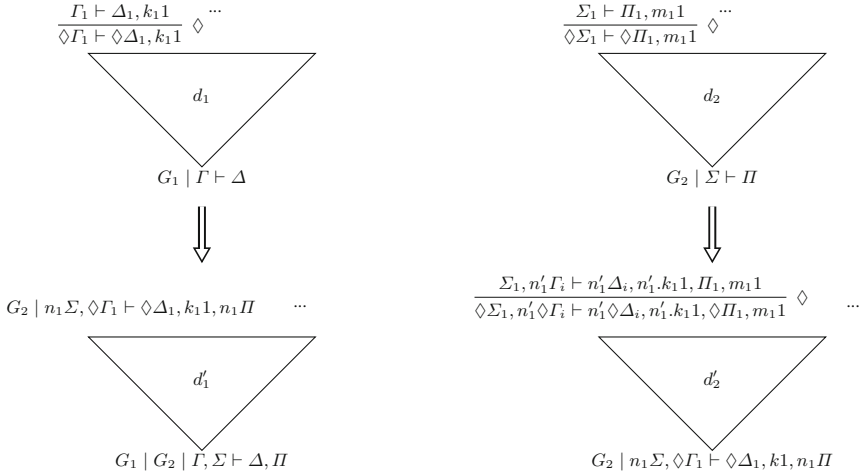


Fig. 4. Sequentially composing d_1 and d_2 in the M-elimination proof.

The following is a direct consequence Theorems 8 and 9.

Corollary 1. *The two systems MGA and MGA-SR † are equivalent: $\models_{MGA} G$ if and only if $\models_{MGA-SR^\dagger} G$.*

The two systems MGA and MGA-SR †* are equivalent: $\models_{MGA^*} G$ if and only if $\models_{MGA-SR^{\dagger*}} G$.*

4.4 Cut-Elimination Theorem for the System MGA

We have already remarked that the cut-elimination theorem for the system MGA follows from the CAN-elimination theorem. By Corollary 1, the CAN-elimination theorem for the system MGA-SR † implies the CAN-elimination for MGA. Since there is no M-rule in MGA-SR † , the proof of CAN-elimination can follow the three Steps A–B–C outlined in Subsect. 4.1. As for Step A, we need to prove the invertibility of the logical rules in the system MGA-SR †* .

Theorem 10 (Invertibility of the logical rules). *The logical rules of the system MGA-SR †* , $\{0_L, 0_R, +_L, +_R, \sqcup_L, \sqcup_R, \sqcap_L, \sqcap_R\}$, are invertible.*

Remark 3. We note that, just as in [MOG09, §5.2], it is in fact possible and indeed technically useful to prove the invertibility of generalised scalar rules dealing with scalar rules, as in the proof of Theorem 6.

As for Step B we prove the atomic CAN-elimination theorem. Following the previous remark, we prove the following stronger version of the statement.

Theorem 11 (Atomic CAN-elimination). *If $\vDash_{MGA-SR^{\dagger*}} [\Gamma_i, k_i x \vdash k_i x, \Delta_i]_{i=1}^n$ then $\vDash_{MGA-SR^{\dagger*}} [\Gamma_i \vdash \Delta_i]_{i=1}^n$.*

Since we removed the M-rule, there are no significant difficulties in the induction arguments, and the proof is quite straightforward.

We also need a technical lemma regarding the constant formula 1 which is provable by a simple induction on the length of derivations.

Lemma 1. *If $\vDash_{MGA-SR^{\dagger*}} [\Gamma_i, n_i 1 \vdash n_i 1, \Delta_i]_{i=1}^n$ then $\vdash_{MGA-SR^{\dagger*}} [\Gamma_i \vdash \Delta_i]_{i=1}^n$.*

We can now prove the CAN-elimination theorem for MGA-SR[†]. This, together with Corollary 1 implies the cut-elimination (Theorem 5) for MGA.

Theorem 12 (CAN-elimination). *If $d \vDash_{MGA-SR^{\dagger*}} G \mid \Gamma, A \vdash A, \Delta$ then $\vDash_{MGA-SR^{\dagger*}} G \mid \Gamma \vdash \Delta$.*

Proof. Again, it is convenient to prove the stronger statement: If $d \vDash_{MGA-SR^{\dagger*}} [\Gamma_i, k_i A \vdash, k_i A, \Delta_i]_{i=1}^n$ then $\vDash_{MGA-SR^{\dagger*}} [\Gamma_i \vdash \Delta_i]_{i=1}^n$. This is done by induction on the (lexicographical) complexity of the pair (A, d) :

- If A is a variable, we can conclude with Theorem 11.
- If $A = 1$, we can conclude with Lemma 1.
- If $A = \diamond B$, we look at d .
 - If d finished with the \diamond -rule, then the end hypersequent is necessarily of the form: $[\Gamma_i, k_i A \vdash, k_i A, \Delta_i]_{i=1}^n = \diamond \Gamma_1, n_1 \diamond B \vdash n_1 \diamond B, \diamond \Delta_1, k1$, and is derived from the hypersequent $\vDash_{MGA-SR^{\dagger*}} \Gamma_1, n_1 B \vdash n_1 B, \Delta_1, k1$. By induction hypotheses (B has smaller complexity than A), we have that $\vDash_{MGA-SR^{\dagger*}} \Gamma_1 \vdash \Delta_1, k1$. Hence we can derive $\vDash_{MGA-SR^{\dagger*}} \diamond \Gamma_1 \vdash \diamond \Delta_1, k1$ by application of the \diamond -rule.
 - Otherwise, the hypersequent is derived by application of some other rule (not active on $A = \diamond B$) from some premises. In this case, we simply apply the inductive hypothesis on the premises (the formula A is unchanged but the complexity of the premise derivation has decreased) and use the same rule to construct a derivation of the desired hypersequent.
- Otherwise, using the same argument of [MOG09, §5.2] discussed in Sect. 4.1, we make progress in the inductive proof (reducing the complexity of A) by using the invertibility of the logical rules (Theorem 10). \square

5 Conclusions and Future Work

We have presented a structural proof system called MGA for the scalar-free fragment of the Riesz modal logic. A natural direction of research is to extend the system MGA to deal with the full Riesz modal logic, thus handling arbitrary scalars $r \in \mathbb{R}$. The (integer-)scalar rules of the system MGA-SR could be naturally generalised to handle real-scalars but it is not clear, at the present moment, if the resulting system would satisfy a reasonable formulation of the sub-formula property. Another interesting topic of research is to consider extensions of MGA for fixed-point extensions of the Riesz modal logic (e.g., [MS17, Mio18]). In this direction, the machinery of *cyclic proofs* (see, e.g., [Stu07, MS13b, BS11, Dou17]) appears to be particularly promising.

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