Chapter 1
State of the Art on Modelling in Mathematics Education—Lines of Inquiry

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Abstract This chapter provides a brief overview of the state of the art in research and curricula on mathematical modelling and applications of mathematics in education. Following a brief illustration of the nature of mathematical modelling in educational practice, research in real-world applications and mathematical modelling in mathematics curricula for schooling is overviewed. The theoretical and empirical lines of inquiry in mathematics education research related to teaching and learning of mathematical applications and mathematical modelling regularly in classrooms are then selectively highlighted. Finally, future directions are recommended.

Keywords Mathematical applications · Mathematical modelling · Theoretical lines of inquiry · Empirical lines of inquiry

1.1 What Is Mathematical Modelling?

Mathematical modelling conceived as real-world problem solving is the *process* of applying mathematics to a real-world problem with a view to understanding it (Niss et al. 2007). It is *more than applying mathematics* where we also start with a real-world problem, apply the necessary mathematics, but after having found the solution we no longer think about the initial problem except to check if our answer makes sense (Stillman 2004). With mathematical modelling the use of mathematics is more *for understanding the real-world problem/situation*. The modeller also poses the problem(s) and questions (Christou et al. 2005; Stillman 2015). To illustrate what this means in educational practice, a modelling task from a university teacher education course follows.

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1.1.1 An Example from Teacher Education

This task was used in a university mathematics unit for primary pre-service teacher education students. It was one of three choices (the others being dust storms and the spread of HIV/AIDS). The students had 4 weeks to work on the task independently out of class. The task is about the felling of a eucalypt forest on the edge of the freeway between Melbourne and Ballarat. The trees were not particularly old and not mature enough for harvesting. This context was used to ask students to pose a problem based on the logging of the forest as a modelling task. There was little to no information in the local press and the local council was less than helpful to students who enquired as to why the forest was removed. The following task stimulus was given to the students. All students were in the first semester of the first year of a 4-year education degree to become primary school teachers (teaching Preparatory year to Year 6).

The Task—Harvesting the Eucalypt Forest: Those of you who drive the Western Freeway between Ballarat and Ballan will have noticed that a large plantation of Eucalypts has been felled and the logs transported away. Using mathematical modelling pose a problem related to removal of the forest that can be mathematised and solved. [The task was accompanied by several photographs taken before, during and after the felling of the trees.]

Many mathematically tractable problems were posed by the students who worked on the task individually in their own time. An example from one student, Hannah (a pseudonym), follows:

> I will be researching and investigating the effects of human logging and deforesting of the Eucalypt forest on the Western freeway between Ballarat and Ballan.
>
> The problem I pose is this: At what rate would replanting need to occur for it to be sustainable with the rate of deforestation, and what percentage of the forest needs to remain 'untouched', either entirely or for a period of time, to maintain a viable habitat to creatures it may be home to?
>
> In order to come up with a reliable conclusion I will need to research the following:
>
> What was the original size of the forest?
>
> Why and for what purpose is it being logged?
>
> What age does the timber need to be for it to be commercially useful?
>
> Growth rate of the Eucalypt? [from Hannah's Modelling Task Report]

To begin she needed to know the initial number of trees. To work this out she firstly determined the area of the forest. Using a Google map aerial view, she divided the forest into four common shapes to best cover the entire area (Fig. 1.1). The shaded green in the top right corner is where trees had already been felled. This area was also included to determine how many trees were in the forest to begin with. Using scaling and area formulae she determined the forested area was 1,587,000 m^2. Assuming

Fig. 1.1 Finding area of original forest beside highway near Ballan (used with permission)

trees could be planted at the rate of 1000 per hectare this gave 158,700 trees as the size of the original forest.

Next she assumed a growth rate of 1.2 m per year and that the trees were being harvested with 15 years growth of useable timber, that is, trees with 18 m useable logs.

To transport the logs from the site she used 5 B-double logging trucks for 5 days for 46 weeks per year (allowing for 6 weeks holiday/annual leave). Each truck consisted of two trailers that could carry twenty-two 6 m logs in each. This meant that the trees were cut into three 6 m logs and 366.66 trees trucked per week (16,866.66 annually). If the trees were logged continually at this rate and not replenished, the forest planation would be removed within 9.4 years of commencement of logging.

She then re-assessed her modelling as she had yet to incorporate sustainability. She realised that she had to determine the rate of logging to achieve her goal, not use existing rates. She decided that she would log 158,700 trees over 16 years so at the rate of 9919 trees annually and this would use 3 B-double trucks a day. She would then, at the same time, need to be planting 9919 trees annually and harvesting these after they had produced 15 years growth of useable timber. She did not continue on to answer other parts of her question posed.

The task and Hannah's modelling is an example of *descriptive modelling*, the most common form of modelling (Niss 2015). The purpose of the mathematical modelling was to analyse an existing real world situation (the felling of a forest) as a means of answering a practical question (what rate to (log and) replant so as to sustain the forest). Both mathematical and extra-mathematical knowledge were needed to answer this question. This is also an example of using *modelling as content* "empowering students to become independent users of their mathematics" (Galbraith 2015a, p. 342) rather than as a means to serve other curricular requirements such as teaching mathematical content (i.e. *modelling as vehicle*).

1.2 Real-World Applications and Mathematical Modelling in Curricula

Uptake and implementation of real-world applications and mathematical modelling in curricula in school and university vary widely. At ICME-7 in Quebec in 1992, Blum lamented in Working Group 14 on *Mathematical Modelling in the Classroom*,

> there is still a substantial gap between the forefront of research and development in mathematics education, on the one hand, and the mainstream of mathematics instruction, on the other hand. In most countries, modelling (in the broad and, even more so, in the strict sense) still plays only a minor role in everyday teaching practice at school and university. (1993, p. 7)

Fortunately, there has been some change in the intervening years with Maaß (2016) noting at ICME-13 in Hamburg:

> Nowadays in Germany Mathematical Modelling is part of the national standards of mathematics education and in consequence is part of many professional development courses, also addressing topics like differentiation and assessment when modelling. Textbooks include modelling tasks (to a different degree) and many teachers (though maybe not the majority) do include modelling in their mathematics classes. Of course, this has not always been the case.

Most implementations in individual mathematics subjects align with expressed goals of modelling and/applications in curriculum documents but this is not always borne through in the overall structure of the curriculum where there are alternative mathematical offerings or alternative pathways (e.g. academic versus vocational) (Smith and Morgan 2016). The goals are roughly equivalent to the five arguments that Blum and Niss (1991) present as those given for support of real world applications and mathematical modelling in curricula. In the following, research and evaluation studies where the particular curricular goal underpins the approach taken to modelling are shown in brackets. From a *mathematical point of view* such goals could be:

- To be a vehicle to teach mathematical concepts and procedures (e.g. Lamb and Visnovska 2015);
- To teach mathematical modelling and ways of applying mathematics as mathematical content (i.e. as an essential part of mathematics) (e.g. Didis et al. 2016; Tekin Dede 2019; Widjaja 2013);
- To promote mathematics as a human activity answering problems of a different nature giving rise to emergence of mathematical concepts, notions and procedures (e.g. Rodríguez Gallegos 2015).

From an *informed citizenry perspective,* goals include:

- To provide experiences that contribute to education for life after school such as looking at social problems (e.g. Yoshimura 2015);
- To promote values education (e.g. Doruk 2012);
- To question the role of mathematical models in society and the environment (e.g. Biembengut 2013; Ikeda 2018);
- To ensure or advance "the sustainability of health, education and environmental well-being, and the reduction of poverty and disadvantage" (Niss et al. 2007, p. 18) (e.g. Luna et al. 2015; Rosa and Orey 2015; Villarreal et al. 2015).

Smith and Morgan (2016) reviewed curriculum documents in 11 education jurisdictions identifying three main rationales in orientations of curricula to use of real-world contexts in mathematics, namely:

(1) "mathematics as a *tool* for everyday life,
(2) the real world as a *vehicle* for learning mathematics, and
(3) engagement with the real-world as a *motivation* to learn mathematics" (p. 40).

In Australia, they examined state curricula in Queensland where there has been mathematical modelling and applications in the senior curriculum for many years and New South Wales where there is no modelling and a very traditional mathematics curriculum. In Canada, they looked at curricula in Alberta and Ontario where modelling was reported in the latter as "embedded as a system-wide focus in secondary school mathematics education" (Suurtamm and Roulet 2007, p. 491). Other curricula examined came from Finland, Japan, Singapore, Hong Kong, Shanghai and the USA southern states of Florida and Mississippi.

In seven of these eleven educational jurisdictions, alternative pathways were offered, with more [mathematically] advanced pathways having less emphasis on real-world contexts. Such findings raise questions for those charged with overseeing curriculum implementation to consider in relation to the espoused goals of curricular embedding:

- If mathematics is seen as a tool for everyday life—why is the real-world given less emphasis for students studying more advanced mathematics?
- If the purpose was as a vehicle for learning, or motivation, why is there less focus on real-world contexts in the years of schooling prior to pathway options?

Changing the emphasis for different year levels or by nature of mathematics studied *conflicts* with all three of the espoused rationales.

1.3 What Do We Know?

Since the late 1960s, researchers in mathematics education have increasingly focussed on ways to change mathematics education in order to include mathematical applications and mathematical modelling regularly in teaching and learning in classrooms. This was in response to the dominance in many parts of the world of the school mathematics curricula by an abstract approach to teaching focusing on the

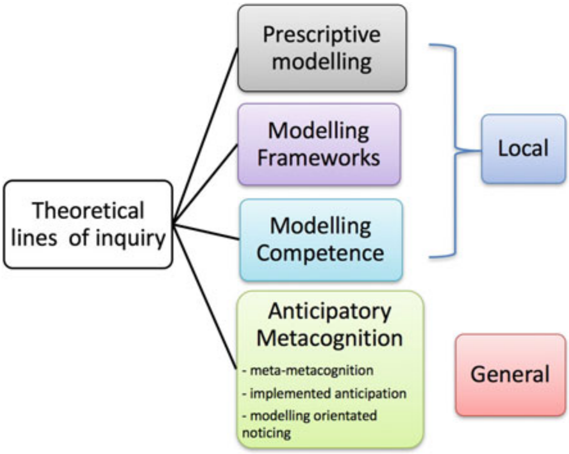

Fig. 1.2 Focuses of theoretical lines of inquiry

teaching of algorithms divorced from any applications in the real world. The focus of this research has been both theoretical and empirical. Within mathematical modelling and applications educational research, there has been an on-going building of analytical theories establishing foundational concepts and categories and interpretative models and theories for interpreting and explaining observed structures and phenomena which have been organized into stable, consistent and coherent systems of interpretation (Niss 1999). Constructs from these are claimed to meet particular theoretical or empirical evidence. This has led to many viable lines of inquiry over the years and the purpose of this chapter is to highlight some of these that are current within the field. To select examples I have surveyed the literature in the more recent books in the ICTMA series and the major mathematics education research journals.

1.3.1 Theoretical Focuses—Lines of Inquiry

In research into the teaching and learning of mathematical modelling there is a strong emphasis on developing "home grown theories" where the focus is on "particular *local theories*" such as the modelling cycle and modelling competencies rather than *general theories* from outside the field (Geiger and Frejd 2015). As the extent of theoretical developments in this field is extensive, four examples of current theoretical lines of inquiry—three local theories (prescriptive modelling, modelling frameworks/cycles and modelling competencies) and one general line of inquiry (anticipatory metacognition)—will be used to give a flavour of current thinking and work (Fig. 1.2). Some of these have been the subject of empirical testing or confirmation whilst others await such work.

1.3.1.1 Prescriptive Modelling

The first local theory is *prescriptive modelling*. The terms *descriptive model* and *prescriptive model* have been used previously by Meyer (1984) to describe models used for different modelling purposes: "A descriptive model is one which describes or predicts how something actually works or how it will work. A prescriptive model is one which is meant to help us choose the best way for something to work" (p. 61). According to Niss (2015), the modelling cycles used in theoretical and empirical research are limited with regards to adequately capturing all processes involved in prescriptive modelling. Descriptive modelling is usually the focus of practice as it is used to *understand* an existing part of the world. However, it is not the modelling cycle as such that is different in prescriptive modelling. What has happened is that historical development in keeping with the types of problems used has coupled the modelling cycle with descriptive modelling, so that features of descriptive modelling have become misleadingly assigned as intrinsic to the modelling cycle. In contrast, what happens within different phases of the cycle can differ stemming from the differing purposes of prescriptive and descriptive modelling.

An example comes from Galbraith (2009, pp. 58–62) where he worked on the question: Is the method for scoring points in the heptathlon fair? 'Fairness' was interpreted with respect to strengths in track (e.g. 100 m hurdles) or field (e.g. javelin) events. Galbraith began to answer this question by evaluating the outcome of an earlier unknown (to him) modelling process by looking first at existing formulae and their implications for fairness. The modelling develops from there. A major difference is the essential role of sensitivity testing within the evaluation of prescriptive modelling. This ensures a cyclic dimension to the modelling process as it involves assessing the impact of changes in assumptions (e.g. world records in all contributory events should have similar weighting on the respective points scored in an event) or changing parameter values (e.g. a 1% increase in performance at the 1000 point mark of excellence in the different events) on the initial solution.

Niss (2015) points out that prescriptive modelling has little purchase in mathematics education, rarely being a focus. It would therefore follow that mathematics educators are less interested in modelling to take action based on decisions resulting from mathematical considerations so as to *change* the world. Niss (2015) advocates strongly for a greater focus in both theoretical and empirical research on prescriptive modelling in mathematics education using tasks of higher complexity than have been used in the limited work in this area to date.

1.3.1.2 Modelling Frameworks/Cycles

On the other hand, much work has been done on the second local theory to be highlighted—various modelling frameworks/cycles. Borromeo Ferri (2006), Czocher (2013), Doerr et al. (2017), and Perrenet and Zwaneveld (2012), amongst others, provide overviews of exemplars of these theoretical lines of inquiry in more recent years.

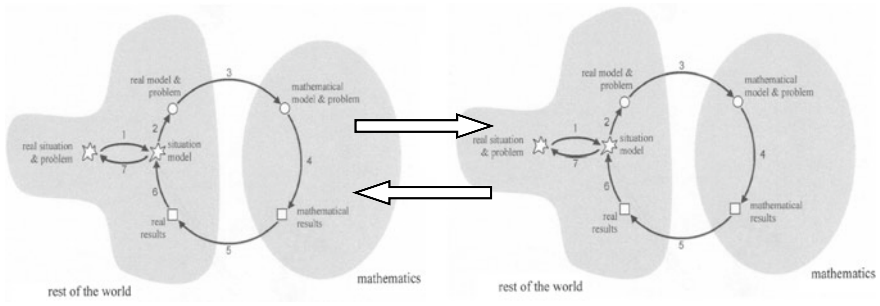

Fig. 1.3 Dual modelling cycle framework (Saeki and Matsuzaki 2013, p. 91)

The cycles/frameworks serve the researchers' purposes as is illustrated in the following example. A recent Japanese development in this area is the Dual Modelling Cycle Framework (Fig. 1.3) which combines two representations of the modelling cycle as depicted by Blum and Leiß (2007).

Sometimes, when modellers are unable to anticipate a model or solve a modelling task, they imagine models from a similar task in their prior experience to help progress the solution of the first task. Saeki and Matsuzaki (2013) used this idea to design two similar tasks that could be used in teaching to scaffold such a process for struggling modellers. By solving the analogous second task using a second modelling cycle, the modellers are, theoretically at least, able to apply the results to the location on the modelling cycle for the first task where they were struggling, forming linked dual modelling cycles (see Fig. 1.3). This theoretical work has been the subject of empirical testing and confirmation with both Japanese students (e.g. Kawakami et al. 2015) and Australian students (Lamb et al. 2017).

Fundamentally, the modelling cycle is a logical progression of problem-solving stages as the mathematical model, for example, cannot be solved before it has been formulated or the interpretation of outputs from the mathematical work before it has been done, etcetera. It is a theoretical description of what real-world modelling involves. Empirical data confirm its global structure; they do not give rise to it. Both the Blum and Leiß (2007) and the Saeki and Matsuzaki (2013) approaches elaborate this essential cycle with enhanced pedagogy in mind but not all cycles have been constructed with the logic of the modelling process in mind. Do we really need separate cycles for modelling with technology, say? Why would we expect the process to be different? Isn't the logic of the use of technology in these circumstances driven by the logic of the modelling process?

1.3.1.3 Modelling Competence/Competencies

The last local theory to be dealt with is related to one of the most important goals for student modellers in any curricular implementation which is to develop "modelling competence" (Blomhøj and Højgaard Jensen 2003) or "modelling competency" (Niss et al. 2007). “Competence is someone’s insightful readiness to act in response to the challenges of a situation” (Blomhøj and Højgaard Jensen 2007, p. 47) and was introduced in the context of the Danish KOM project (Niss 2003) which focussed on mathematical competencies and the learning of mathematics and created a platform for in-depth reform of Danish mathematics education at all levels. Readiness to act is not the same as the ability to act on this readiness, however. Modelling competency, on the other hand, refers to an individual’s ability to perform required or desirable actions in modelling situations to progress the modelling (Niss et al. 2007). Kaiser (2007) would call this “modelling abilities” and would insist modelling competency includes a willingness to want to work out real world problems through mathematical modelling.

Each of the following *modelling competencies* based on phases in the modelling cycle can be subdivided into lists of sub-competencies:

- competencies to understand real-world problems and to construct a reality model;
- competencies to create a mathematical model out of a real-world model;
- competencies to solve mathematical problems within a mathematical model;
- competency to interpret mathematical results in a real-world model or a real situation
- competency to challenge solutions and, if necessary, to carry out another modelling process (Kaiser 2007, p. 111)

In addition, *metacognitive modelling competencies* have been proposed by both Maaß (2006) and Stillman (1998). However, metacognition was linked to modelling much earlier by McLone (1973) and Lambert et al. (1989). Competence in modelling would thus involve an ability to orchestrate a set of sub-competencies in a variety of modelling situations.

Several aspects of theoretical work in the area of modelling competence and modelling competencies are currently the subject of empirical testing and confirmation. Kaiser and Brand (2015) provide an insightful overview of the main theoretical lines of inquiry within the International Conferences on the Teaching of Mathematical Modelling (ICTMA) research community since the 1980s. Further work in this area is described in Kaiser et al. (2018).

1.3.1.4 Anticipatory Metacognition

Metacognition is considered important by several researchers in the research and practice of mathematical modelling especially reflection on actions when addressing a real world problem (Blum 2015; Vorhölter 2018). In reality metacognition is essential to properly conducted modelling as evaluation of the partially complete model(s) should be occurring through verification and the final model needs to be validated against the problem situation to see if it produces acceptable answers to the question posed. The focus of the reflection on actions is on the mathematics employed and the modelling undertaken. A new development in this area is anticipatory metacognition. *Anticipatory metacognition* is about reflection that points *forward* to actions yet to

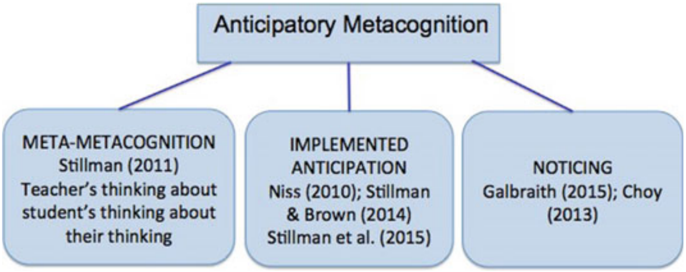

Fig. 1.4 Proposed dimensions of anticipatory metacognition

be undertaken, that is, noticing possibilities of potentialities. These reflections can arise from prior progress or lack of it. Anticipatory metacognition encompasses three distinct dimensions (see Fig. 1.4): meta-metacognition, implemented anticipation, and modelling oriented noticing (Galbraith et al. 2017).

Meta-metacognition results from teachers thinking about, that is, reflecting on, the appropriateness or effectiveness of their students' metacognitive activity during mathematical modelling and subsequently acting bearing this in mind (see Stillman 2011). Implemented anticipation is Niss's notion (2010) of successful implementation of anticipating in ideal mathematisation of a modelling situation. It results from the successful use of foreshadowing and feedback loops to govern actions in decision making during mathematisation (Stillman et al. 2015).

Modelling oriented noticing involves 'noticing' how mathematicians as well as educators act when operating within the field of modelling, from both *mathematical* and *pedagogical* points of view (Galbraith 2015b). It provides a way to study aspects central to modelling, for example, problem finding and problem posing as well as conducting modelling. For both there is cognitive involvement. Modelling oriented noticing also facilitates study of task design and study of support for student activity by teachers.

From a teaching viewpoint, to carry out tasks successfully requires more than just observing. Discernment of the *relevance* of what is observed is essential, followed by appropriate action. The term 'noticing' as employed in Galbraith (2015b) encapsulates these components. Choy (2013) came up with the notion of *productive mathematical noticing* by combining the notion of mathematics teacher noticing, involving the generating of new knowledge through selective attending and knowledge-based reasoning to develop a repertoire of alternative strategies, with Sternberg and Davidson's (1983) processes of insight. The latter are selective encoding, selective comparison and selective combination. By extending this idea to modellers (who can be students), Galbraith et al. (2017) proposed the notion of *productive Modelling Oriented Noticing* (pMON). For modellers, pMON involves the processes of (a) sifting through information to notice what is relevant and what is irrelevant (i.e. selective encoding), (b) comparing and relating relevant information with prior experiences and knowledge (i.e. selective comparison), and (c) combining the relevant infor-

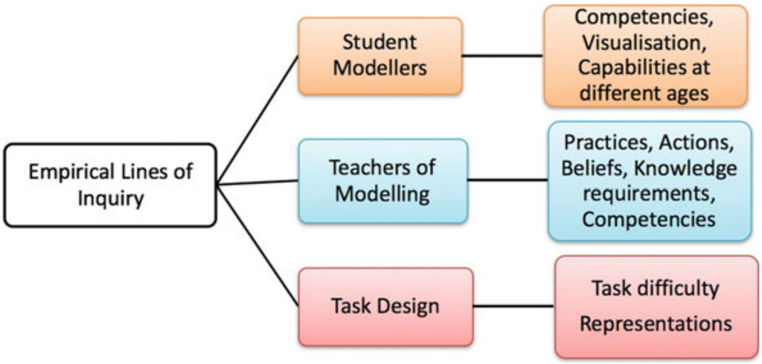

Fig. 1.5 Focuses of empirical lines of inquiry

mation (i.e. selective combination) to *generate productive alternatives* for decision making when responding to events as they carry out a modelling activity.

Aspects of the theoretical dimensions of anticipatory metacognition have been, or are currently, the subject of empirical testing and confirmation (Geiger et al. 2018; Stillman and Brown 2014).

1.3.2 Empirical Lines of Inquiry

Focuses of empirical lines of inquiry in mathematical modelling research are many and varied. Given the space available, I will focus on just three: student modellers, teachers of modelling and task design (Fig. 1.5). Within each of these foci, a small subset of exemplar studies and the major findings from these will be overviewed.

1.3.2.1 Empirical Results of Studies Focusing on Student Modellers

Prominent lines of inquiry focussing on students concern their modelling and mathematical competencies, visualisation and their capabilities at different ages.

Quantitative research by Kaiser and Brand (2015) has confirmed that modelling competency of student modellers appears to consist of a global overarching modelling competency and several sub-competencies, namely, simplifying/mathematising, working mathematically and interpreting/validating. Overall modelling competency was defined in Brand's study (2014) as the ability to solve complete modelling tasks as well as use metacognitive abilities to monitor the modelling process. Fifteen classes of Year 9 students from 4 higher-track and 2 comprehensive secondary schools in Hamburg, Germany, took part. However, these results need to be replicated in other contexts to show they are independent of the examples, intervention approach and test instruments used by Brand. In contrast, when Zöttl et al. (2011) applied Rasch modelling (Rasch 1960) to data in the KOMMA research project[1] in an attempt to capture the essential components of modelling competency (in keeping with Kaiser 2007) of Year 8 students, a sub-dimensional model proved superior to a uni-dimensional model. Thus, the results related to structure of the modelling competency differ with respect to the role played by the overall modelling competency from those of Kaiser and Brand (2015). The role of a global overarching modelling competency remains an open question. Further work on the conceptualisation of modelling competency and sub-competencies is presented in this volume by Hankeln et al.

A technology based study by Brown (2015) focussed on the visualisation tactics (i.e. employing either mental images or technology-generated images or both) of Year 11 Australian students attempting to solve a real world task involving platypus numbers in the wild. Unfortunately, students did not appreciate the cognitive role played by visualisation in supporting refinement of models and mathematisation in modelling. The potential of graphing technology to facilitate this process was thus not realised. In contrast, Villarreal et al. (2015, 2018) reported how pre-service education students in Argentina used the visual affordances of digital tools to represent their data in a visual manner to analyse the situation they were modelling and to communicate their results in an impactful manner.

English (2013) has shown that complex modelling tasks relating to engineering-based experiences can be handled by Years 7–9 Australian students. Such experiences target future competencies in the mathematical sciences, connecting learning across disciplines and involving the student modellers in planning, designing, constructing, testing and refining a life based model to solve problems of the built environment such as transport. In subsequent work, English and Watson (2018) have reported on how the statistical literacy of Year 6 students can be enhanced through modelling with data by developing shared problem spaces between mathematics and statistics.

1.3.2.2 Empirical Results of Studies Focusing on Teachers of Modelling

Empirical lines of inquiry that take teachers as their focus have focused on teacher practices, actions, beliefs, knowledge requirements and competencies, amongst other characteristics and influencing factors when planning, preparing, engaging in, assessing and reflecting on their facilitating of student modelling in and outside classrooms.

Two different approaches that teachers can take in the classroom in supporting the development of modelling competencies are *atomistic* where the focus is on mathematising processes and analysing models mathematically and *holistic* where the focus is on the modelling process as a whole with all phases expected to play a part. Further results from Kaiser and Brand (2015), for example, confirmed that both atomistic and holistic approaches fostered students' modelling competency in all sub-competencies mentioned above. The holistic approach promoted overall modelling competency more effectively. The hypothesis that the sub-competencies connected

[1] KOMMA was a research project funded by the German Federal Ministry for Education and Research (PLI3032).

to the sub-processes of the modelling cycle would be fostered more effectively by experiencing different modelling sub-tasks in an atomistic approach was not confirmed. However, as pointed out above, these results need further testing with broader samples of teachers and classes and tasks.

Czocher (this volume) raises the interesting question with respect to competencies for facilitating student modelling: How does a facilitator aid a student in moving from a nonmathematical interpretation of a problem situation to a mathematical interpretation of that same problem situation? In other words, how do teachers bring students to the realization that the crux of modelling is to reduce the complexity of a real-world situation so models can be applied or constructed, not to keep that complexity of the situation so the "model" is an exact image of reality?

A study by Kuntze et al. (2013) investigated Austrian teachers' self-perceptions of their pedagogical content knowledge with respect to diagnostic knowledge related to the modelling process and to providing modelling specific feedback. In particular, both pre-service and in-service teachers in the sample focussed on general suggestions rather than specific support related to the modelling process in reacting to potential difficulties students might experience when modelling. Results showed that these teachers needed professional development related to both modelling specific Pedagogical Content Knowledge and self-efficacy as teachers of modelling. Blomhøj (this volume) argues that there is also a need for the development of tools that allow teachers to make better use of theories of learning of mathematical concepts and to view modelling activities as a didactical means for supporting students' learning of mathematics not just to develop students' modelling competency.

1.3.2.3 Empirical Results of Studies Focusing on Task Design

Task design in educational modelling contexts is a fruitful area for research as specifications for suitable problems for the classroom need to be based on some sort of theoretical or empirical evidence. It seems wise that the essential elements of tasks used successfully in modelling implementations in research studies for different purposes be captured in design criteria that can be used for both classroom and research purposes in the future. However, it must be borne in mind how such tasks are implemented is a bigger issue than task design per se.

Reit and Ludwig (2015) have used simple modelling tasks in their work that are designed for an holistic approach to both teaching and assessment. The tasks were designed to meet the following criteria: authenticity of context, realistic numerical values, possession of a problem solving character, a naturalistic format for the question and openness of solution approaches. The degree of difficulty of these tasks was conjectured to be able to be determined using order of thought operations and cognitive demand from the perspective of cognitive load theory (Sweller 2010). Empirical results with Year 9 students confirmed thought structure complexity was related to solution rate with more sophisticated thought structure lowering solution rate of tasks.

The design of multiple choice items to test first year educational science students' ability to connect written descriptions of realistic situations to linear and almost linear models when presented in different representations (symbolic, tabular or graphical) underpins the study by Van Dooren et al. (2013). The representational mode in which an item was presented had a high impact on students' modelling accuracy and on the tendency to inappropriately connect non-linear situations to linear models. The students were proficient in connecting descriptions to models when the situation was linear. However, when the situation was almost linear they also connected these erroneously to a linear model. The authors point out that whilst the use of such testing can be for diagnosis and rectifying errors with respect to identifying suitable models, hopefully with the intended purpose of being able to do this in more in-depth modelling situations, it should also be interspersed with the use of more authentic real-world situations in tasks. Extrapolation of findings from insights obtained by use of multiple-choice items to modelling expertise to solve extended problems still presents as a credibility gap.

1.4 Future Directions

From this brief overview of current lines of inquiry in the field of mathematics education research related to teaching and learning of mathematical applications and mathematical modelling, a number of questions arise that could seed future research projects. Some of what challenges our current thinking in theoretical lines of inquiry are opportunities to advance knowledge. In particular, one might ask:

- What are the similarities, differences and relationships between descriptive and prescriptive modelling?

Similarly, issues that have arisen above with respect to particular theoretical frames or empirical studies give rise to a number of potential empirical lines of inquiry. Generative questions for these could be:

- How does activity within phases of a prescriptive modelling problem differ from its descriptive counterpart and what are the implications for scaffolding?
- What scaffolds would ensure meta-validation when prescriptive modelling is conducted? Fully?
- What is the role of a global overarching modelling competency in modelling?
- Should particular sub-competencies or global modelling competencies be the focus of teaching in regular classrooms?
- What is the role of anticipatory metacognition (especially pMON) by teachers and student modellers in ensuring technology is used in a transformative manner in modelling?
- How do teachers come to realise that by not offering young students challenging situations to model, we are not realising the potential of both students and teaching in the classroom?

- How is self-efficacy as a teacher of modelling different from self-efficacy as a teacher of mathematics? At secondary level? At primary level? At tertiary level?
- What is the structure of professional learning for teachers needed to enhance modelling specific Pedagogical Content Knowledge and self-efficacy as teachers of modelling?

It must be emphasised that this list is not meant to be exhaustive and is very much influenced by the particular selection of studies, within the categories, I have highlighted in the various lines of inquiry that have "caught my eye", as it were, in my surveying of the literature at the time of ICME-13.

1.5 Final Considerations

In this chapter a brief overview of the state of the art in curricula and research on mathematical modelling and applications of mathematics in education has been provided. The theoretical lines of inquiry in mathematics education research related to the teaching and learning of mathematical applications and mathematical modelling regularly in classrooms, selectively highlighted, have been the local theories of prescriptive modelling, modelling frameworks/cycles and modelling competencies and the potentially more general theory of anticipatory metacognition. Modelling frameworks/cycles and modelling competencies have received quite a deal of attention from scholars and researchers from both a theoretical and empirical perspective. The notions underpinning prescriptive modelling, on the other hand, have been in existence for some time but have not really been central to the modelling debate but Niss's (2015) drawing the attention of the field to them could arouse sufficient interest for them to be pursued further and brought to realization within classrooms and be the subject of future research. The ideas underpinning anticipatory metacognition have also been around for some time, albeit in other fields, but they have not been combined until now. Although some beginning work has been done with some of the dimensions of anticipatory metacognition, this is an area where there is a lot more empirical work to do.

The empirical lines of inquiry have taken as their focus student modellers, teachers of modelling and task design. This selection is in keeping with general major emphases in the field. The examples overviewed for lines of inquiry focussing on students concern their modelling and mathematical competencies, visualisation and their capabilities at different ages. All of these are fertile ground for further study. Prominent empirical lines of inquiry that take teachers as their focus concern teacher practices, actions, beliefs, knowledge requirements and competencies, The third area, task design, has been less of a focus at the time of surveying in studies of actual classroom practice and more of a focus for good instruments to assess modelling. This is however an area that changes emphases rapidly depending on who is researching in the field at the time of surveying the field.

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