Chapter 11
Heuristic Strategies as a Toolbox in Complex Modelling Problems

Peter Stender

Abstract   The support of students who are working on realistic modelling issues is a complex process, especially if it is intended, that the students work as autonomously as possible. In the underlying research project, actions of tutors were analysed as they were fostering students who were working on complex, realistic, authentic modelling problems over three days. The tutors were prepared previously in seminars. The whole process was videotaped and analysed afterwards, looking for examples of successful teacher interventions especially interventions based on the idea of strategic assistance (Zech in Grundkurs Mathematikdidaktik. Beltz, Weinheim, 1996). The findings in the research led to the insight that heuristic strategies developed within problem solving could be identified in the modelling process and are an appropriate concept to formulate strategic interventions. This is shown here by examples based on the analysis of observed student solutions and a standard solution.

Keywords   Heuristic strategies · Modelling problem · Modelling activities · Teacher interventions

11.1   Theoretical Framework

In this chapter, modelling is understood as follows: A problem from outside of mathematics, in the "Rest of the World" (Pollak 1979), occurs, is simplified and then translated into a mathematical problem that is worked on. The solution found is translated back into the real world and it is validated whether the result answers the primary problem adequately. If this is not the case, the modelling cycle is run through again until a satisfactory solution is produced. The modelling cycle shown in Fig. 11.1 allows use with students in the classroom as well as being complex enough to illustrate the *two* steps from real world situation into the mathematics that occur, if the modelling problem itself is complex. The modelling cycle itself is a model of

P. Stender (✉)
University of Hamburg, Von-Melle-Park, 20146 Hamburg, Germany
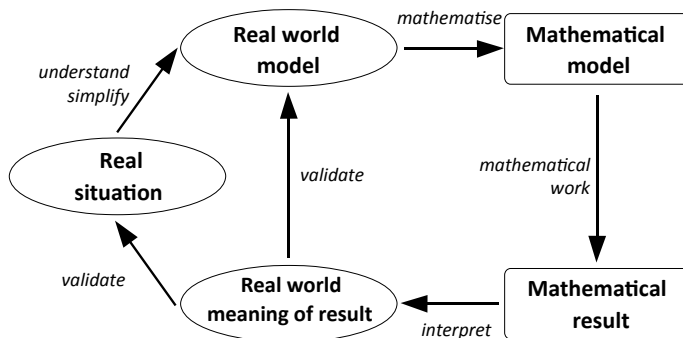e-mail: Peter.Stender@uni-hamburg.de

Fig. 11.1 Modelling cycle (Kaiser and Stender 2013, p. 279)

the modelling process so in reality modellers do not follow this cycle strictly but go back and forth as necessary as shown in Borromeo Ferri (2011).

11.1.1 Teacher Activities to Promote Independent Student Action

In the German discussion (e.g. Zech 1996; Leiss 2007), the concept of an *adaptive intervention* is used for the support of students in modelling and problem-solving situations with the aim of independent student activity: "Adaptive teacher interventions are defined as that assistance of the teacher to the student, which supports the individual learning and problem-solving process of students minimally, so that students can work at a maximum independent level" Leiss (2007, p. 65, own translation). As guidelines for the teacher supporting students who need support in their work, Zech (1996) proposed a step by step approach on five different levels. In this approach, the teacher should in the first two levels only motivate the students (first motivate: "You will make it"; second feedback "Go on like this!"). Only if this is not sufficient to enable the students to go on in the work on the task, should the next steps be done. In these steps, *strategic help* is given first, and then increasingly more assistance is given that relies more and more on the content of the task (e.g. calculations needed are explicitly shown). Strategic help provides students with support that relies on mathematical or other methods and strategies that regulate the work, not on the steps to fulfil these strategies. "Formulate an equation and then solve it!" is an example of strategic help, as long as the teacher does not explicitly show which kind of equation is appropriate or which are the single steps to build the equation or to solve it. While working on modelling problems, a reference to the modelling cycle can be used as a strategic help: "Simplify the situation!" "Try to bring this into a formula!" "What does the mathematical result mean in the real world?" "Does the result answer the real-world situation meaningfully?" Strategic help that is provided only when nec-

essary supports the independent work of students in the best way, as the students are only supported to find a way to go on, but the solution itself must still be developed by the students themselves.

If the strategic help does not enable the students to go on in the solution of the problem*, content related strategic help* should be given. This means that additional content related information around the strategic help is provided. This could be a hint about what kind of equation would be appropriate or what kind of formulas could be used. Only if this content related strategic help is not successful is more content related help provided, for example direct support formulating an equation or in the mathematical process. Between the phases of intervention, the students need time to think about what they could do and to try different approaches, so they have the chance to solve the problem or the next step as independently as possible.

11.1.2 Heuristic Strategies

During the research described below hints occurred, that heuristic strategies which are a well-known concept in the problem-solving theory (e.g. Dörner 1976; Pólya 1973; Schoenfeld 1985), are also used when solving modelling problems and are a strong concept to formulate strategic help, which was already mentioned by Zech (1996).

From a theoretical point of view, a *heuristic strategy* is a possible approach to solve a problem (Dörner 1976; Schoenfeld 1985). To clarify this definition, one has to define what is meant by the word *problem*. Following Dörner (1976, p. 10, own translation), "A problem is a situation where you achieve a goal, but you don't know how to achieve the goal." Dörner points out that there is a barrier between the actual situation α and the goal ω. In this approach, a heuristic strategy is an approach to overcome the barrier between situation α and ω. To discriminate the concept from other situations, Dörner uses the concept of a *task* as a situation where no barrier exists. If you work on a task you know what to do, even if it may be difficult: if someone knows the Gaussian elimination, for example, it is a task to solve a 10 by 10 system even if it is a lot of work: you always know what to do as all steps are given by the Gaussian elimination procedure. Whether a situation is a problem or a task depends on the knowledge and the experience of the person, but if something is a problem there is no explicit answer for this person of how to solve it. That means heuristic strategies are always ideas you can try but maybe they are not successful, and you have to try another one. If you have no more ideas what to do, you could try a strategy out of a list of heuristic strategies. Therefore, the following list of heuristic strategies was gathered based on the problem-solving theory. The classification of these strategies was formulated to keep a better overview of the single heuristic strategies as there are too many to keep them all in mind (Stender 2018).

- *organise your material/understand the problem*: change the representation of the situation if useful, trial and error, use simulations with or without computers, discretize situations,
- *use the working memory effectively*: combine complex items to supersigns, which represent the concept of 'chunks', use symmetry, break down your problem into sub-problems,
- *think big*: do not think inside dispensable borders, generalise the situation,
- *use what you know*: use analogies from other problems, trace back new problems to familiar ones, combine particular cases to solve the general case, use algorithms where possible,
- *functional aspects*: analyse special cases or extreme cases, in order to optimise you have to vary the input quantity,
- *organise the work*: work backwards and forwards, keep your approach—change your approach—both at the right moment.

The use of these heuristic strategies is shown below within a complex realistic modelling problem. Using this example, the single strategies used in the specific modelling problem are explained in more detailed.

In addition, examples are displayed for using heuristic strategies to formulate strategic interventions. The connection is obvious: if you want to provide strategic interventions, you need to know the appropriate strategy in the specific situation and these are often heuristic strategies.

11.2 The Study

The aim of the research project is to find appropriate strategic interventions in situations where teachers are tutoring students who are solving complex, realistic, authentic modelling problems. As the research environment, "modelling days" were established. The empirical research led to two assumptions: heuristic strategies are used while working on complex modelling problems and they can be used to create strategic help for students. To proof the first assumption, descriptions of the modelling process were analysed due to the underlying use of heuristic strategies. In order to see whether it is possible to create strategic help with these strategies corresponding teacher interventions were formulated. Whether these are effective had not yet been an object of the empirical research but is an assumption based on Zech (1996). A summary of the whole research project is shown here and a detailed analysis of one modelling process according to the use of heuristic strategies is presented.

11.2.1 Modelling Days

"Modelling days" is a learning environment (see also Kaiser et al. 2013 for other examples), where students of grade 9 (15 years old) work for three full days in a school on only one single modelling problem. The modelling days were held several times, and are still held, at a school for higher achieving students in Hamburg (Germany). For the students from grade 9, three modelling problems were presented from which each student chose one (see Stender 2018, for use of heuristic strategies in The Bus Stop Problem). Then groups of four to six students were formed, so that in each group students worked together on the same problem, supervised by tutors.

The tutors were future teacher students studying for their master's degree. They were prepared in a university seminar on modelling. In the seminar, they worked on the three modelling problems that were the choice for the modelling days, they learnt about the theory of mathematical modelling and the theory of teacher interventions and scaffolding. Heuristic strategies were also content of the seminar.

### 11.2.2 Modelling: Roundabout Versus Traffic Light

In the research project presented here, students worked on the problem: "At which kind of intersection (a roundabout or a intersection with traffic lights) can more cars pass a crossing?"

In Germany, as in many other countries, many intersections have been reconstructed as roundabouts for several reasons. In other countries, a similar question to compare a traffic light and a four-way stop might be more appropriate. One aspect of the discussion is whether a roundabout can manage more traffic than an intersection with a traffic light. A sketched idea of a solution is presented here for a better understanding of the problem and the appearing complexity. Ideas that are more detailed are shown below connected to heuristic strategies.

In a first approach, it makes sense to assume the maximum possible symmetry in the situation: from all directions the same number of cars should come, and the drivers want to go in all directions with equal probability, with velocities and accelerations also being the same for all cars. The crossings are a simple four-road intersection, where the traffic light is green only for one direction at a time. The restrictive assumptions can be reduced during the modelling process, to arrive at a more sophisticated solution, but within the time available this did not occur in the modelling days. For the students, it took mostly a longer work process to arrive at these assumptions.

Once these real models (one for the roundabout, one for the traffic light) were formulated, the students had to calculate the time a whole line of cars needs to start, when the traffic light switches to green. This way one finds the number of cars that can enter the intersection during one green phase. In this calculation, one has to deal with constant and accelerated movements and the time a car has to wait until the

Fig. 11.2 Material for
simulation of the roundabout
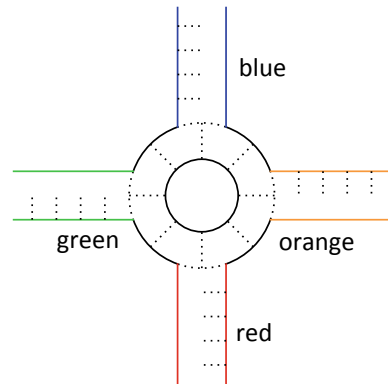
necessary distance to the car before occurs. One also needs to consider that the cars
at the end of the line drive with constant speed according to the speed limit after a
phase of acceleration.

The processes in the roundabout are more complex than those occurring at the
traffic light, since the probability to enter the roundabout depends on the situation
in the roundabout itself. If the roundabout is completely full of cars, a new car can
only enter the roundabout if a car from inside the roundabout left the roundabout
previously.

This process can be simulated with the help of a game, as shown in Fig. 11.2.
A simulation with a computer is also possible, but that was usually beyond the
capabilities of the students involved. The access roads shown in the figure are drawn
in different colours (blue, green, orange, red). Now pieces of paper in the same colours
are distributed representing cars on the streets, in a way that from each direction, say
21 "cars", approach in random order. The drivers of blue cars have the goal to drive
in the direction of the blue street. In the line in the blue street there are obviously
normally no blue cars. One turn of the simulation consists of the following steps: At
first all cars in the roundabout that are at the right exit leave the roundabout. In the
second step, all other cars in the roundabout drive one step ahead, which leads to
free places at the entrance to the streets, where a car left the roundabout previously.
Now cars from the waiting lines drive into the roundabout.

This simulation leads to a deeper understanding of the roundabout-process and to
the probability of 50% for a car to enter the roundabout in one turn of the simulation.
Afterwards it has to be calculated how long a single turn in the simulation lasts in
reality. This leads to calculations similar to traffic light ones.

The results for the capacity of the two designs of the intersection depend on the
values for velocities and accelerations used. A clear answer to the initial question
cannot be given without clearing the parameters depending on the size of the inter-
section. It is therefore useful for the students to visit an intersection and do some
measurements while working on the problem. These measurements could also be
considered for the evaluation of the results of calculations. The results will always

be based on the corresponding dimensions of intersections, but can be generalized by further calculations.

11.2.3   Empirical Survey

During the modelling days ten groups of students were videotaped in five rooms, with one camera for each group. The video-recordings include six hours of modelling activities for two days and a few hours on the third day. The phases during which the tutor communicated with the students and some minutes before and after every such communication was transcribed. In total, 238 contacts between teacher and individual groups were examined. The transcribed text passages were analysed and coded using qualitative content analysis (Mayring 2010). Three types of variables were used relying on the time before the intervention, during the intervention itself, and on the time after the intervention. Thus, the success of the interventions could be determined based on the coding. Findings were presented previously in Stender and Kaiser (2015) and Stender (2016).

While analysing single interventions that were not successful or delivered too much content related help, I tried to formulate alternative strategic interventions for further projects. In doing so, formulations using heuristic strategies seemed appropriate. This led to the next step in the research project to find out whether it is possible to provide evidence of the use of heuristic strategies in the modelling process and whether strategic help can be created based on the heuristic strategies that were used.

11.3   Results

11.3.1   Using Heuristic Strategies in Modelling Problems

In this section the process of modelling the problem "Roundabout versus Traffic Light" is analysed. Two materials are used: The first solution is that of the author, formulated while developing the modelling problem to make sure that there was a chance for the students to find a meaningful solution and to prepare the tutors for the modelling days. The second solution is a reconstruction of students' solution. This reconstruction is based on the presentation of the students at the end of the modelling days and on videotapes of several groups from the modelling days. For a single group only parts of the modelling process and the approaches of the students are visible on the videos as there were always phases in which they worked without visible communication or documentation. Therefore, the visible parts of the modelling process of different groups were connected to one students' solution.

Both solutions were examined step by step, analysing whether the heuristic strategies mentioned above occurred and how they were realized. From these heuristic

strategies, strategic interventions were formulated. The results are shown under the headlines of single heuristic strategies and not in the strict timeline of the modelling process, but the first strategies described occurred earlier in this modelling process

Break down your problem into sub-problems: Pólya (1961, p. 129), citing Descartes, states: "Divide each problem that you examine into as many parts as you can and as you need to solve them more easily". At the same place, Pólya also cites Leibnitz underlining the core problem connected with this strategy: "This rule of Descartes is of little use as long as the art of dividing … remains unexplained. … By dividing his problem into unsuitable parts, the unexperienced problem-solver may increase his difficulty" (p. 129). In the modelling process this strategy occurs as a basic approach to the process of looping through the modelling cycle several times (Pollak 1979, p. 20). In each loop, one single part of the modelling problem is (partly) solved and is the groundwork for the following steps. Besides this example, there are many situations in problem solving and modelling that have to be divided into sub-problems in order to access a solution.

During the modelling days students were faced with the challenge to convert one unit of velocity (km/h) into another (m/s). Some of the students knew the conversion number to be 3.6, but did not know how this was to be applied. Longer work phases on this problem with constantly decreasing motivation were observed without the students getting closer to the answer, as they always searched for a single step operation to perform the calculation. In the end, the tutor showed the students how the conversion is calculated step by step (which is not a strategic help!), after several interventions that gave only general information. The statement: "There are two units involved, so convert only one of them in the first run!" could have helped the students to develop the calculation on their own. This can also be formulated more concretely: "Convert only the km in the first step", or more generally "It won't work with one step, you have to make at least two steps!"

This observation led to the insight, for the author of this chapter, of the high relevance of heuristic strategies in the modelling process: if a student deals with a problem in the modelling process and there is a barrier (a "red flag situation" according to Goos 1998) that the student cannot overcome on his/her own, the teacher or the tutor has to analyse the next steps he or she would do himself/herself and then identify the underlying strategy of the own solving process. This strategy identified by metacognition, leads to the strategic intervention that could help the student. This process can then be a general method to formulate strategic help.

Build the real model as symmetric as possible: The most often used strategy in this modelling problem is the use of symmetry, as already mentioned above. "Try to treat symmetrically what is symmetrical, and do not destroy wantonly any natural symmetry" (Pólya 1973, p. 200). Pólya emphasized that symmetry is not only meant in the usual geometric meaning but also in a general, logical meaning: "Symmetry, in a general sense, is important for our subject. If a problem is symmetric in some ways we may derive some profit from noticing its interchangeable parts and it often pays to treat those parts which play the same role in the same fashion" (p. 199). The concept of symmetry is very broad. "In a more general acceptance of the word, a whole is termed symmetric if it has interchangeable parts. There are many kinds of

symmetry; they differ in the number of interchangeable parts, and in the operations which exchange the parts" (Pólya 1973, p. 199).

In the analysed modelling problem, the symmetry is not necessarily there from the beginning but has to be created by the modeller: a four-street crossing is examined, and most students picked an example nearby the school in the first run of the modelling process. Mostly a bigger road meets a smaller one in their examples. According to the modelling cycle, the first approach to a real model should simplify the situation as much as possible. This simplification implies to model the situation as symmetric as possible: all four streets meeting at the crossing should "play the same role" and thus should be "treated in the same fashion". That means, in detail, that from all four streets the same number of cars arrive per hour and from each street one third of the cars is going to turn right, one third turns left and one third goes straight ahead.

A further aspect that should be treated symmetrically are the cars. Usually cars have different sizes, different accelerations and drive with different speeds and the drivers keep different distances to the car in front. All these quantities should be identical in the real model and the mathematical model. For example, the cars are all 5 m long, accelerate with 2 m/s^2 after starting at the green light and waiting in the line all drivers keep a distance of 1 m to the car in front.

The strategic help for the students can be formulated as: "Form the situation as symmetrically as possible!" or more concrete if necessary: "Treat all the streets and the cars in the situation in the same way in the first approach!"

In the modelling cycle using symmetry is one possible way to simplify the situation. A less specific strategic intervention, "Simplify as much as possible!" may help students with particular experience in modelling but others may need the idea, that creating the situation symmetrically is the appropriate approach to realise the simplification.

"*Here is a problem related to yours and solved before. This is good news; a problem for which the solution is known and which is connected with our present problem, is certainly welcome*" Pólya (1973, p. 110) with the related questions in the list: "Could you use it? Could you use its result? Could you use its method?" Everywhere in mathematics, it is an often-used method to apply the solution of solved problems in the form of proved theorems. In a modelling process, it also occurs that results from one-step of the modelling process can be used in further steps.

In the modelling problem analysed here, for both kinds of crossings there are cars waiting, then accelerating and driving through the crossing. When the traffic light switches to green, the first car accelerates, after a short time the second car starts and so on. To calculate, how many cars can pass the light in one green phase one has to model this process and thus to deal with the formulas $s = \frac{1}{2}at^2$ and $v = at$. Parts of the calculation for the first car can be used for the second car, one only has to add a delay as the second car starts later and drives a longer distance to pass the crossing. The calculation is almost completely similar for the following cars. Therefore, in the ongoing modelling process students can use what they have done before. Once they deal with the roundabout the calculations from the traffic light can be used again with slight changes. "You nearly had the same calculation before—adopt it to

this situation!" "Work similar to your foregoing calculation!" Pólya (1973, p. 37) emphasizes, "Analogy is a sort of similarity. Similar objects agree with each other in some respect, analogous objects agree in certain relations of their respective parts." According to this, a teacher could also formulate: "Work in analogy to your foregoing calculation!"

Generalization is described in detail by Pólya (1973, p. 108 ff.). Generalization means to leave certain restrictions of the problem and thus come to a more universal problem. Even though this new problem covers many more different cases than the original one, it might be easier to solve because it has fewer restrictions than a special problem. In the modelling process, the modeller is free to build a more or less general model, depending upon which gives access to a solution and if this occurs in the beginning of the modelling process or in an advanced state. A more general real model is mostly also a more abstract one, which is less complex but, due to the abstraction, often less accessible to some students. In this research study, it turned out that the students clung to more concrete real models that were often too complex for them to work on so the students were stuck.

The acceleration process mentioned above can be done by calculating everything for each single car with concrete numbers. This is of course the first approach of a modeller but in the long run, it pays off to calculate with variables instead of numbers. This way one calculates the acceleration process for a group of cars with one single calculation. Pólya (1973, p. 110) underlines: "Such a generalization may be very useful. Passing from a problem 'in numbers' to another one 'in letters' we gain access to new procedures; we can vary the data, and, doing so, we may check our results in various ways." In this modelling situation, generalization helps to realize how the calculation for one car can be transferred to the other cars and to the similar situation at the roundabout as mentioned above. So, in this situation two heuristic strategies come together. "After realizing the calculation with numbers try to use letters instead of numbers so you can easier transfer your results to the other cars!"

Extreme cases are particularly instructive (Pólya 1973, p. 192): "The allegedly general statement is concerned with a certain set of objects; in order to refute the statement, we specialize, we pick out from the set an object that does not comply with it. … If, however, we find that the general statement is verified even in the extreme case, the inductive evidence derived from this verification will be strong."

In our modelling problem, the question is: At which kind of intersection (a roundabout or an intersection with traffic lights) can more cars pass a crossing? To answer this question, one has to find out the *maximum possible number* of cars that can pass the crossing in a certain time. So, at the traffic light there have to be always enough cars waiting so that during the green phase the maximum possible number of cars can drive through the crossing (same for the roundabout). This is not obvious for the students in the beginning of the modelling process nor for more experienced people. In an experimental comparison between the roundabout and the American four-way-stop (Mythbusters 2013), the cars driving through a roundabout were counted but one could clearly see, that the number found in the experiment is too small as there were not enough cars involved. Students dealing with this modelling problem often started with assumptions like "from each direction there come 100 cars per hour" and

worked with this for a while. If they continue to use this assumption, the following advice is necessary for the students: "You have to calculate a situation where as many cars as possible go through the crossing!"

Use a simulation! Computers are mostly available while modelling nowadays and it is often mentioned that simulations with computers can be very helpful during the modelling process (Greefrath 2011). In many situations, the computer skills of the students are not sufficient to implement an appropriate computer simulation on their own. Thus, for the roundabout problem a paper simulation was used to examine the roundabout using material shown in Fig. 11.2. This material was prepared before the modelling days and was already used in the tutor-training. The simulation material, as described above, was essential for the understanding of the situation and led to deeper understanding of the roundabout traffic. The simulation was introduced to the students by handing out the material to them without further instructions. Once they were using the material, hints for its usage were provided like: "Let all blue cars drive in the direction of the blue street, and do so with the other colours." This simulation material already involved other heuristic strategies.

Discretize the situation! Discretization is a core method in mathematics. In applied mathematics, continuous situations must be transformed into discrete ones for example while solving a differential equation using computers or in school using Cavalieri's principle. In pure mathematics, discretization occurs too but at the end of a proof the discrete situation is transformed back into a continuous situation using limits, for example in the definition of the Riemann integral. In the modelling process a discretization can be used to build the real model or while transferring the real model into the mathematical model.

In the reality of the roundabout, the cars drive with constant speed through a roundabout but in the simulation, they move in steps like on a board game. As the material for the simulation, as shown in Fig. 11.2, was handed over to the students, the discretization was already settled by the material. The students acted with this discretization without any problems due to the similarity to a board game. Problems occurred later while the students tried to interpret the simulation results. One result was, that 21 cars from each road can drive through the roundabout in 42 turns of the simulation. The problem for the students was to connect this to a particular time in reality. To get back to a continuous process and connect each turn of the simulation to a definite time was a barrier, maybe due to the fact, that they had not discretized the situation by themselves. So, they needed the help: "Now think again how cars drive through the roundabout in reality. A car needs five turns of the game to drive through the roundabout. How can you calculate how long this is in seconds in reality?"

Use an appropriate representation! In the simulation, there are four colours used for the different streets and the same colours for the cars. Similar to the discretization, this representation was delivered to the students with the material and helped to execute the simulation. In general, it is very helpful for solving a problem and equally a modelling problem to select a good representation of the situation. One aspect of this idea was described by Pólya (1973, p. 103) discussing the use of figures. Using figures is a very important representation dealing with modelling problems. So "Draw a figure!" is a very important heuristic help. The simulation led to a well-educated

Fig. 11.3 Calculating the probability
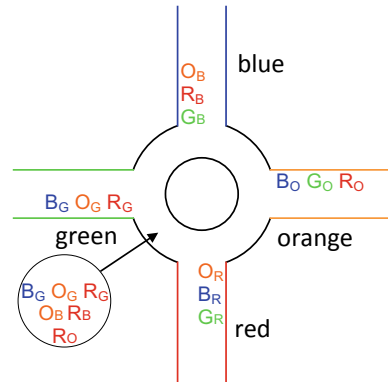
guess that on average a car from one certain street can enter the roundabout in every second turn of the simulation. To prove this supposition another representation of the roundabout is needed.

In this representation (Fig. 11.3), B_R means cars that drive from the red street (R) to the blue street (B). So obviously in the red street there are cars O_R (red to orange), B_R (red to blue) and G_R (red to green) and similarly in the other streets. The cars drive through the roundabout against the clockwise direction. So which sort of cars appear at the point marked by the arrow? Usually there should be no cars coming from the red street because that only happens if the driver missed the exit which should be the orange, blue or green street. Cars heading for the blue street should only come from the green street as cars from the red or orange street would have already passed the exit to the blue street at this point. With the same argument, cars heading for the orange street may come from the green or the blue street while cars that want to go into the red street may come from all three possible directions. Putting this information together, in the circle one can directly see that half of the cars passing the position marked by the arrow will drive in the red street. Now we have to switch back to Fig. 11.2 and realize that in the simulation the last statement means that in 50% of the turns of the game a car from the red road can enter the roundabout. The representation with two letters as shown (B_R etc.) leads to the insight, why, on average, in every second turn of the simulation a car can enter the roundabout. This works together with the use of colours and the drawing of the roundabout. The appropriate representation is the key to this result. Furthermore, the use of the symmetry, as mentioned above, is essential for this result and, in addition, allows the transfer of the result for the red street to the other streets. If students know a kind of representation that is helpful in the situation, a tutor can just mention it, for example "Draw a figure!" If the representation is new for the students, the tutor has to give a little bit more help: "Use O_R for cars driving from the red street to the orange one and examine, which sorts of cars pass a certain point of the roundabout!"

Supersigns are a concept that was introduced in the problem-solving discussion by Kießwetter (1983) based on the concept of chunking described by Miller (1956).

Using supersigns means to chunk different items together to form a new idea (the supersign) in order to use the working memory more efficiently. The word "supersign" was used relying on information theory and it means a sign that represents several signs. Thus, the name of a mathematical set is a supersign but also a vector, a matrix, a function, an equivalence class and so on. Supersigns are used for structuring the situation in order to organise the material. In natural language, supersigns occur too, for example, to think about "a queue of cars" makes it possible to talk about 30 cars, say, without referring to each single car in the working memory.

In the roundabout simulation, each turn of the simulation includes many concrete steps. So, "one turn" is a supersign for which several times, in reality, are to be calculated. Several calculations that were done in the modelling process "dividing the problem into sub-problems" had to be combined to single ideas to work with them, which means to rebuild the supersign.

In order to gain insights from the problem-solving theory, building supersigns especially when using abstract patterns, is very challenging for students and it can only be expected from very gifted students to do this on their own. All others need support to formulate the supersign even if it is obvious to the tutor. So the tutor has to be conscious about the uses of supersigns. A teacher support using a supersign might appear when students focus on single steps and should combine them to a bigger pattern: "One turn in the simulation consists of several single steps. Build one number that describes all these steps together (for example the total time of one step in reality)."

11.3.2 Results Referring to the Modelling Cycle and Observations in the Empirical Research

Beside the strategic help relying on heuristic strategies, strategic interventions based on the modelling cycle (Fig. 11.1) were formulated in the tutor training and observed during the modelling days. The request to describe the situation precisely and then simplify it to build a real model, to transfer the real model into mathematics, to deal with the mathematics and then interpret the results in terms of the real model as with the request to validate the result with regard to the real-world situation or the real model were appropriate strategic interventions as they demand to do certain steps in the modelling process without giving specific support how these steps should be realised.

One very important strategic intervention observed during the modelling days was the request of the tutor to explain the work already done. This strategic help arose as a very powerful instrument as it has several advantages.

- For the tutors, it was very easy to apply this intervention that was part of the preparation seminar after the first observation. Even if the tutor did not know everything about the modelling problem and the solving process, he or she could ask this question.

- While the students answered the question, the tutor had time to ascertain the situation of the students in the modelling process and was able to analyse barriers or misconceptions. In other words, there was time for a good diagnosis for further interventions.
- The students are encouraged to reflect and structure their ideas. While answering, the students looked back on their own work and often realised thereby, what they have done and what went well or not. In the first approach, sometimes their arguments are poorly structured but asked to explain it again, because it is hard to understand, they rearranged the ideas and themselves gained more insight into their own results. There were situations, where the tutor only asked a group to explain their work when a group was stuck, and they started to explain and then shifted into a debate on their own work that enabled them to go on—while the tutor left the group without any other word.

Overall, strategic interventions turned out to be an appropriate approach to support students' work during complex modelling situations, but a tutor needed a deep insight into the modelling process, the modelling problem and possible solutions (see also Stender and Kaiser 2015).

So, tutoring students that are working on complex modelling problems needed a good preparation for the tutors. In the seminar for the tutors, they had to work on each modelling problem in groups. The process was accelerated a little bit compared with the setting in the modelling days, but the tutors overall worked three hours on each problem with phases of metacognition being the focus in the seminar in-between. This metacognition reflected the phases of the modelling cycle, possible assumptions and those made, and simplifications and expected problems in the modelling days and appropriate interventions. The three modelling problems were not solved in a row but there were seminar sessions that dealt with theory of modelling, strategic help and heuristic strategies. This way the metacognition of the tutors' own modelling processes included more and more theoretical aspects over the time.

11.4 Summary and Conclusions

Tutoring students who are working on complex modelling problems is a very complex challenge for the tutors. Essential for this work is good preparation of the tutors according to the special modelling problems and according to helpful theoretical background.

Heuristic strategies might be very helpful supporting the students but to apply them in concrete situations is not easy. One has to realize the barrier that prevents the students from undertaking the next steps, solve the problem to overcome this barrier and then find out the heuristic strategy one uses via metacognition. The last step needs a lot of experience in analysing solutions of modelling problems regarding the use of heuristic strategies and it needs time in the situation. This means that it is meaningful to prepare this kind of teacher intervention beforehand: tutors who

are going to supervise students while modelling should model the problem on their own. While doing this, tutors should analyse their own work via metacognition and identify possible barriers and strategies used in the way shown above. Then strategic interventions can be pre-formulated. Teacher trainings for modelling activities could use this approach.

Here only the Roundabout versus Traffic Lights Problem was analysed but in Stender and Kaiser (2016) and Stender (2018) the use of heuristic strategies in The Bus Stop Problem are shown too, so this approach is not limited to a single modelling process of one modelling problem.

In further research, this approach should be examined in more detail as up to now there is no empirical evidence that heuristic strategies really improve the work of students while working on complex modelling problems. This research should include the teacher training and an appropriate modelling environment. In other areas of mathematical work there is (unpublished) evidence that strategic interventions based on heuristic strategies are successful: this approach was used supporting mathematics teacher students in the first semester doing high level mathematics with a very good outcome. This indicates that there is a good chance that using heuristic strategies, as a generalized toolbox to describe students' work via metacognition and support students' work via strategic help, is a very promising approach in all parts of mathematics.

References

Borromeo Ferri, R. (2011). *Wege zur Innenwelt des mathematischen Modellierens: Kognitive Analysen zu Modellierungsprozessen im Mathematikunterricht*. Perspektiven der Mathematikdidaktik. Wiesbaden: Vieweg + Teubner.

Dörner, D. (1976). *Problemlösen als Informationsverarbeitung* (1. Aufl.). Kohlhammer-Standards Psychologie Studientext. Stuttgart: Kohlhammer.

Goos, M. (1998). "I don't know if I'm doing it right or I'm doing it wrong!": Unresolved uncertainty in the collaborative learning of mathematics. In C. Kanes, M. Goos, & E. Warren (Eds.), *Teaching mathematics in new times* (Vol. 1, pp. 225–232). Gold Coast: Mathematics Education Research Group of Australasia.

Greefrath, G. (2011). Using technologies: New possibilities of teaching and learning modelling—Overview. In G. Kaiser, W. Blum, R. Borromeo Ferri, & G. Stillman (Eds.), *Trends in teaching and learning of mathematical modelling* (pp. 301–304). Dordrecht: Springer.

Kaiser, G., Bracke, M., Göttlich, S., & Kaland, C. (2013). Authentic complex modelling problems in mathematics education. In A. Damlamian, J. F. Rodrigues, & R. Sträßer (Eds.), *Educational interfaces between mathematics and industry* (pp. 287–297). Cham: Springer.

Kaiser, G., & Stender, P. (2013). Complex modelling problem in cooperative, self-directed learning environments. In G. A. Stillman, G. Kaiser, W. Blum, & J. P. Brown (Eds.), *Teaching mathematical modelling: Connecting to research and practice* (pp. 277–294). Dordrecht: Springer.

Kießwetter, K. (1983). Modellierung von Problemloseprozessen - Voraussetzung und Hilfe für tiefergreifende didaktische Überlegungen. *Der Mathematikunterricht, 29*(3), 71–101.

Leiss, D. (2007). *"Hilf mir es selbst zu tun": Lehrerinterventionen beim mathematischen Modellieren* (Univ., Diss.–Kassel, 2007). Texte zur mathematischen Forschung und Lehre. Hildesheim: Franzbecker.

Mayring, P. (2010). *Qualitative Inhaltsanalyse: Grundlagen und Techniken* (11., aktualisierte und überarb. Aufl.). Beltz Pädagogik. Weinheim: Beltz.

Miller, G. A. (1956). The magical number seven, plus or minus two: Some limits on our capacity for processing information. *Psychological Review, 63,* 81–97.

Mythbusters. (2013). http://www.wimp.com/testroundabout/Video.

Pollak, H. O. (1979). The interaction between mathematics and other school subjects. In UNESCO (Ed.), *New trends in mathematics teaching* (Vol. 4, pp. 232–248). Paris: UNESCO.

Pólya, G. (1961). *Mathematical discovery: On understanding, learning, and teaching problem solving* (Vol. 1). New York: Ishi Press.

Pólya, G. (1973). *How to solve it. A new aspect of mathematical methods* (2nd ed.). Princeton: Princeton University Press.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando: Academic Press.

Stender, P. (2016). *Wirkungsvolle Lehrerinterventionsformen bei komplexen Modellierungsaufgaben*. Wiesbaden: Springer.

Stender, P. (2018). The use of heuristic strategies in modelling activities. *ZDM Mathematics Education, 50*(1–2), 315–326.

Stender, P., & Kaiser, G. (2015). Scaffolding in complex modelling situations. *ZDM Mathematics Education, 47*(7), 1255–1267.

Stender, P., & Kaiser, G. (2016). Fostering modeling competencies for complex situations. In C. Hirsch (Ed.), *Annual perspectives in mathematics education. Mathematical modeling and modeling mathematics* (pp. 107–115). Reston, VA: National Council of Teachers of Mathematics.

Zech, F. (1996). *Grundkurs Mathematikdidaktik* (8, Aufl.). Weinheim: Beltz.